Borell-Brascamp-Lieb type inequalities in spaces with bitriangular laws of composition and their applications

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Abstract

In this paper, we study Borell-Brascamp-Lieb inequalities and Brunn's concavity principle in spaces with bitriangular laws of composition. This kind of generalizations contains the particular interest example of the Heisenberg group $\mathbb{H}^n$. As an application, we prove some Brunn-Minkoski type inequalities in the Heisenberg group $\mathbb{H}^n$, especially an isomorphic version of the conjectured Brunn-Minkowski inequality in $\mathbb{H}^n$, which gives a positive answer to a modified conjecture.

1 Introduction

A basic inequality in convex geometry is the Brunn-Minkowski inequality, which provides a fundamental relation between volume and Minkowski addition in $\mathbb{R}^n$. The classical Brunn-Minkowski inequality states that for all Borel sets $A, B \subset \mathbb{R}^n$ and $t \in (0,1)$, it holds

$$|(1-t)A + tB|^{\frac{1}{n}} \geq (1-t)|A|^{\frac{1}{n}} + t|B|^{\frac{1}{n}}, \tag{1.1}$$

where $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}^n$. By the homogeneous of volume in Eucliean spaces $\mathbb{R}^n$, the Brunn-Minkowski inequality can be written as

$$|A + B|^{\frac{1}{n}} \geq |A|^{\frac{1}{n}} + |B|^{\frac{1}{n}}. \tag{1.2}$$

A far-reaching generalization of Brunn-Minkowski inequality in analysis is a family of functional inequalities, for example, Prékopa-Leindler inequality, and Borell-Brascamp-Lieb inequality. The classical Prékopa-Leindler inequality [11–13] states that for measurable functions $f, g, h : \mathbb{R}^n \to \mathbb{R}^+$ with satisfying that for some $\lambda \in (0,1)$,

$$h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^{\lambda}, \qquad \forall x, y \in \mathbb{R}^n,$$

we have

$$\int_{\mathbb{R}^n} h(z)dz \geq \left(\int_{\mathbb{R}^n} f(x)dx\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g(y)dy\right)^{\lambda}.$$

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Furthermore, the Borell-Brascamp-Lieb inequality was first proved (in a slightly different form) by Henstock and Macbeath [10] and by Dinghas [7], and was generalized by Brascamp and Lieb [6] and by Borell [5]. The Borell-Brascamp-Lieb inequality asserts that

Theorem 1.1. *Let $0 < \lambda < 1, -\frac{1}{n} \le p \le +\infty, 0 \le f, g, h \in L^1(\mathbb{R}^n)$, and for every $x, y \in \mathbb{R}^n$,*

$$h\left((1-\lambda)x + \lambda y\right) \ge M_p^\lambda\left(f(x), g(y)\right).$$

Then

$$\int_{\mathbb{R}^n} h \ge M^\lambda_{\frac{p}{pn+1}}\left(\int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g\right).$$

Clearly, the number $p/(pn+1)$ has to be interpreted in the extremal cases (namely, it is equal to $-\infty$ when $p = -1/n$, and to $1/n$ when $p = +\infty$). The quantity $M_q^\lambda(x, y)$ denotes the (λ-weighted) q-mean of x, y. That is, for $x, y \ge 0$ and $xy = 0$, we set $M_q^\lambda(x, y) = 0$ for every $q \in \mathbb{R} \cup \{\pm\infty\}$. For all $x, y > 0$ and $\lambda \in (0, 1)$, in the case $q \ne 0$ we set

$$M_q^\lambda(x, y) = \left((1-\lambda)x^q + \lambda y^q\right)^{1/q}, \tag{1.3}$$

and in the case $q = 0$, we set

$$M_0^\lambda(x, y) = x^{(1-\lambda)} y^\lambda. \tag{1.4}$$

Obviously, the Prékopa-Leindler inequality is the special case of Borell-Brascamp-Lieb inequality with $p = 0$.

Brunn's concavity principle plays a significant role in convex geometry, which supplies the first proof of the Brunn-Minkowski inequality, see [1].

Theorem 1.2. *Let K be a convex body in $\mathbb{R}^n$ and let F be a k-dimensional subspace. Then the function $f : F^\perp \to \mathbb{R}$ defined by*

$$f(x) = |K \cap (F + x)|^{\frac{1}{k}}$$

is concave on its support.

The Brunn-Minkowski inequality and its functional version have been generalized to many different spaces and used to solve various problems in convex geometry, see [1, 8, 16–19] for example. Especially, Bobkov [4] proved the Brunn-Minkowski inequality in spaces with bitriangular laws of composition.

In this paper, we shall extend Borell-Brascamp-Lieb inequality and Brunn's concavity principle to the case of spaces with bitriangular laws of composition. We denote by e_i the i-th unit coordinate vectors in $\mathbb{R}^n$ whose i-th component is 1, all others 0, for $i = 1, 2, \cdots, n$. We call a binary operation or composition

$$(x, y) = (x_1, \cdots, x_n; y_1, \cdots, y_n) \to x \oplus y \in \mathbb{R}^n \quad (x, y \in \mathbb{R}^n)$$

bitriangular if the coordinates of the "sum" $x \oplus y$ are

$$(x \oplus y)_k = x_k + y_k + \varphi_{k-1}(x_1, \cdots, x_{n-1}; y_1, \cdots, y_{n-1}), \tag{1.5}$$

for some functions $\varphi_{k-1} : \mathbb{R}^{k-1} \times \mathbb{R}^{k-1} \to \mathbb{R}, k = 1, \cdots, n$ (with the convention that $\varphi_0 = 0$). For $\lambda > 0$, we define λx by $(\lambda x)_k = \lambda x_k$ for $k = 1, \cdots, n$.

If functions $\varphi_k, k = 1, \cdots, n-1$, are continuous, and A and B are Borel measurable sets, the *Minkowski sum in spaces with bitriangular laws of composition* of A and B is defined by

$$A \oplus B = \{x \oplus y : x \in A, y \in B\}.$$

For a body K in $\mathbb{R}^n$, we say that K is a *$\oplus$-convex body*, if for every $x, y \in K$ and $\lambda \in (0,1)$, it holds $(1-\lambda)x \oplus \lambda y \in K$.

Our main results are the following three theorems. At first, we prove in Section 2 the Borell-Brascamp-Lieb inequality in spaces with bitriangular laws of composition.

Theorem 1.3. *Let $f, g, h : \mathbb{R}^n \to \mathbb{R}^+$ be measurable functions, and let $-\frac{1}{n} \leq p \leq +\infty$ and $\lambda \in (0,1)$. If f and g are integrable, and for every $x, y \in \mathbb{R}^n$,*

$$h\left((1-\lambda)x \oplus \lambda y\right) \geq M_p^{\lambda}\left(f(x), g(y)\right),$$

then

$$\int_{\mathbb{R}^n} h \geq M^{\lambda}_{\frac{p}{pn+1}}\left(\int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g\right).$$

Using Theorem 1.3, the Brunn's convexity principle will be extended to the spaces of $\mathbb{R}^n$ with respect to $\oplus$ in Section 3.

Theorem 1.4. *Let K be a $\oplus$-convex body in $\mathbb{R}^n$, and H be the k-dimensional space spanned by $\{e_{n-k+1}, e_{n-k+2}, \cdots, e_n\}$. Then the function $F : H^{\perp} \to \mathbb{R}$ defined by*

$$F(x) = |K \cap (x \oplus H)|^{1/k}$$

is concave on its support.

As applications, we will obtain some Brunn-Minkowski type inequalities in the Heisenberg group $\mathbb{H}^n$ in Section 4. It is especially interesting that an isomorphic version of the conjectured Brunn-Minkowski inequality in $\mathbb{H}^n$ will be shown, that is,

Theorem 1.5. *For all nonempty Borel sets A and B in the Heisenberg group $\mathbb{H}^n$,*

$$C|A \cdot B|^{\frac{1}{2n+2}} \geq |A|^{\frac{1}{2n+2}} + |B|^{\frac{1}{2n+2}},$$

where $C = 2^{\frac{1}{2n+2}}$.

Note that the conjectured Brunn-Minkowski type inequality in the Heisenberg group $\mathbb{H}^n$ comes from an isoperimetric problem in $\mathbb{H}^n$ with respect to Carnot-Carathéodory distance (or an equivalent gauge distance), which suggests that

$$|A \cdot B|^{\frac{1}{2n+2}} \geq |A|^{\frac{1}{2n+2}} + |B|^{\frac{1}{2n+2}}.$$

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It has been shown by Monti [15] that this conjecture is not true, even in the case $n = 1$ (see also [14]). Our theorem (Theorem 1.5) gives a positive answer to the next modified conjecture: whether there exists a constant C depending only on n such that

$$C|A \cdot B|^{\frac{1}{2n+2}} \geq |A|^{\frac{1}{2n+2}} + |B|^{\frac{1}{2n+2}}?$$

The rest of the paper is organized as follows. In Section 2, we will prove in Theorem 2.1, an extension of the Borell-Brascamp-Lieb inequality in spaces with bitriangular laws of composition. In Section 3, we give some preliminary lemmas, and then prove Brunn's convexity principle in spaces with bitriangular laws of composition, in Theorem 3.5. In the last section, we use the Borell-Brascamp-Lieb inequality in spaces with bitriangular laws of composition to obtain some Brunn-Minkowski type inequalities in the Heisenberg group $\mathbb{H}^n$.

2 Borell-Brascamp-Lieb inequality

In this section, we give an proof of the Borell-Brascamp-Lieb inequality with bitriangular laws of composition.

Theorem 2.1. *Let $f, g, h : \mathbb{R}^n \to \mathbb{R}^+$ be measurable functions, and let $-\frac{1}{n} \leq p \leq +\infty$ and $\lambda \in (0,1)$. We assume that f and g are integrable, and for every $x, y \in \mathbb{R}^n$,*

$$h\left((1-\lambda)x \oplus \lambda y\right) \geq M_p^\lambda\left(f(x), g(y)\right). \tag{2.1}$$

Then

$$\int_{\mathbb{R}^n} h \geq M^\lambda_{\frac{p}{pn+1}}\left(\int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g\right).$$

We note that the underlying functions φ_k in this theorem are allowed to depend on the parameter λ.

Clearly, in the case with $\varphi_k = 0, k = 0, \cdots, n-1$, the statement responds to the Borell-Brascamp-Lieb inequality, see [5, 6].

Proof. We will prove the theorem by induction on the dimension n. In dimension $n = 1$, $(1-\lambda)x \oplus \lambda y$ is the usual vector sum $(1-\lambda)x + \lambda y$, and Theorem 2.1 is the usual one-dimensional Borell-Brascamp-Lieb inequality, see [11–13].

Assume then $n \geq 2$ and the assertion of theorem is true in all dimension $k = \{1, 2, \cdots, n-1\}$. Let $x = (a, x_n)$ and $y = (b, y_n)$ with $a = (x_1, x_2, \cdots, x_{n-1})$, $b = (y_1, y_2, \cdots, y_{n-1})$. For fixed a, b, we define the three functions on $\mathbb{R}^n$,

$$f_a(x_n) = f(a, x_n), \quad g_b(y_n) = g(b, y_n)$$

and

$$h_{(1-\lambda)a\oplus\lambda b}(z_n) = h\left((1-\lambda)a \oplus \lambda b, z_n + \varphi_{n-1}((1-\lambda)a, \lambda b)\right),$$

where $(1-\lambda)a \oplus \lambda b$ is defined in $\mathbb{R}^n$ in the usual way for the collection $\varphi_0, \varphi_1, \cdot, \varphi_{n-2}$. Using the definition of bitriangular operation, we have

$$(1-\lambda)(a, x_n) \oplus \lambda(b, y_n) = \Big((1-\lambda)a \oplus \lambda b, M_1^\lambda(x_n, y_n) + \varphi_{n-1}((1-\lambda)a, \lambda b)\Big),$$

Then for $x_n, y_n \in \mathbb{R}$,

$$
\begin{aligned}
h_{(1-\lambda)a\oplus\lambda b}(M_1^\lambda(x_n, y_n)) &= h\left((1-\lambda)a \oplus \lambda b, M_1^\lambda(x_n, y_n) + \varphi_{n-1}((1-\lambda)a, \lambda b)\right) \\
&= h\left((1-\lambda)(a, x_n) \oplus \lambda(b, y_n)\right) \\
&\geq M_p^\lambda\left(f(a, x_n), g(b, y_n)\right) \\
&= M_p^\lambda\left(f_a(x_n), g_b(y_n)\right),
\end{aligned}
$$

by using definitions of $f_a, g_b, h_{(1-\lambda)a\oplus\lambda b}$ and (2.1). Thus the triple $(f_a, g_b, h_{(1-\lambda)a\oplus\lambda b})$ satisfies the condition (2.1) in the one-dimensional case, then we get

$$
\int_{\mathbb{R}} h_{(1-\lambda)a\oplus\lambda b}(z_n))dz_n \geq M^\lambda_{\frac{p}{p+1}}\left(\int_{\mathbb{R}} f_a(x_n)dx_n, \int_{\mathbb{R}} g_b(y_n)dy_n\right).
$$

Then we get

$$
\begin{aligned}
\int_{\mathbb{R}} h((1-\lambda)a \oplus \lambda b, z_n'))dz_n' &= \int_{\mathbb{R}} h((1-\lambda)a \oplus \lambda b, z_n + \varphi((1-\lambda)a, \lambda b)))dz_n \\
&= \int_{\mathbb{R}} h_{(1-\lambda)a\oplus\lambda b}(z_n))dz_n \\
&\geq M^\lambda_{\frac{p}{p+1}}\left(\int_{\mathbb{R}} f_a(x_n)dx_n, \int_{\mathbb{R}} g_b(y_n)dy_n\right).
\end{aligned}
$$

This means the three functions on $\mathbb{R}^n$

$$
F(a) = \int_{\mathbb{R}} f_a(x_n)dx_n, G(b) = \int_{\mathbb{R}} g_b(y_n)dy_n, H(c) = \int_{\mathbb{R}} h(c, z_n))dz_n
$$

satisfy (2.1) in dimension $n-1$. Using the induction hypothesis to (F, G, H), we have

$$
\begin{aligned}
\int_{\mathbb{R}^{n-1}} H(c)dc &\geq M^\lambda_{\frac{\frac{p}{(p+1)}}{\frac{p}{(p+1)}(n-1)+1}}\left(\int_{\mathbb{R}^{n-1}} F(a)da, \int_{\mathbb{R}^{n-1}} H(b)db\right) \\
&\geq M^\lambda_{\frac{p}{pn+1}}\left(\int_{\mathbb{R}^{n-1}} F(a)da, \int_{\mathbb{R}^{n-1}} H(b)db\right).
\end{aligned}
$$

Then the desired result follows from the Fubini theorem. □

In the case of $p = 0$, we have the Prékopa-Leindler type inequality, which was proved by Bobkov [4].

Corollary 2.2. *[4] Let $f, g, h : \mathbb{R}^n \to \mathbb{R}^+$ be measurable functions, and $\lambda \in (0, 1)$. We assume that f and g are integrable, and for every $x, y \in \mathbb{R}^n$,*

$$
h\left((1-\lambda)x \oplus \lambda y\right) \geq f(x)^{1-\lambda} g(y)^\lambda.
$$

Then

$$
\int_{\mathbb{R}^n} h \geq \left(\int_{\mathbb{R}^n} f\right)^{1-\lambda}\left(\int_{\mathbb{R}^n} g\right)^\lambda.
$$

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3 Brunn's convexity principle

In this section we show Brunn's convexity principle in spaces with bitriangular laws of composition. We shall need the definition for an α-concave function.

Definition 3.1. *Let K be a convex set in $\mathbb{R}^n$ and let $f : K \to \mathbb{R}^+$.*

(1). We say that f is α-concave for some $\alpha > 0$ if f^α is concave on K, equivalently, if $f^\alpha((1-\lambda)x_0 + \lambda x_1) \geq (1-\lambda)f^\alpha(x_0) + \lambda f^\alpha(x_1)$, for $0 < \lambda < 1$ and $x_0, x_1 \in K$.

(2). The function f is called log*-concave (0-concave) if $f((1-\lambda)x_0+\lambda x_1) \geq f^{(1-\lambda)}(x_0)f^\lambda(x_1)$, for $0 < \lambda < 1$ and $x_0, x_1 \in K$. Namely, $\log f$ is a concave function.*

(3). The function f is α-concave for some $\alpha < 0$ if f^α is convex on K, that is, if $f((1-\lambda)x_0 + \lambda x_1) \geq ((1-\lambda)f^\alpha(x_0) + \lambda f^\alpha(x_1))^{1/\alpha}$, for $0 < \lambda < 1$ and $x_0, x_1 \in K$.

We need extend α-concave functions to cases with bitriangular laws of composition.

Definition 3.2. *Let K be a $\oplus$-convex set in $\mathbb{R}^n$ and let $f : K \to \mathbb{R}^+$.*

(1). We say that f is α-concave with respect to $\oplus$ for some $\alpha > 0$ if f^α is concave on K, equivalently, if $f^\alpha((1-\lambda)x_0 \oplus \lambda x_1) \geq (1-\lambda)f^\alpha(x_0) + \lambda f^\alpha(x_1)$, for $0 < \lambda < 1$ and $x_0, x_1 \in K$.

(2). The function f is called log*-concave (0-concave) with respect to $\oplus$ if $f((1-\lambda)x_0\oplus\lambda x_1) \geq f^{(1-\lambda)}(x_0)f^\lambda(x_1)$, for $0 < \lambda < 1$ and $x_0, x_1 \in K$. Namely, $\log f$ is a concave function.*

(3). The function f is α-concave with respect to $\oplus$ for some $\alpha < 0$ if f^α is convex on K, that is, if $f((1-\lambda)x_0 \oplus \lambda x_1) \geq ((1-\lambda)f^\alpha(x_0) + \lambda f^\alpha(x_1))^{1/\alpha}$, for $0 < \lambda < 1$ and $x_0, x_1 \in K$.

We first prove a useful lemma.

Lemma 3.3. *Let K be a $\oplus$-convex body in $\mathbb{R}^n$, $\alpha, \beta > 0$, and let $f, g : K \to \mathbb{R}^+$. If f is $1/\alpha$-concave with respect to $\oplus$, g is $1/\beta$-concave with respect to $\oplus$, then fg is $1/(\alpha+\beta)$-concave with respect to $\oplus$.*

Proof. Let $x_0, x_1 \in K$ and $\lambda \in (0,1)$. Applying Hölder's inequality for the measure $(1-\lambda)\delta_{x_0} + \lambda\delta_{x_1}$ with $p = (\alpha+\beta)/\alpha$ and $q = (\alpha+\beta)/\beta$, we get

$$\begin{aligned}
&(1-\lambda)\left(f(x_0)g(x_0)\right)^{\frac{1}{\alpha+\beta}} + \lambda\left(f(x_1)g(x_1)\right)^{\frac{1}{\alpha+\beta}} \\
&\leq \left((1-\lambda)f(x_0)^{\frac{1}{\alpha}} + \lambda f(x_1)^{\frac{1}{\alpha}}\right)^{\frac{\alpha}{\alpha+\beta}} \left((1-\lambda)g(x_0)^{\frac{1}{\beta}} + \lambda g(x_1)^{\frac{1}{\beta}}\right)^{\frac{\beta}{\alpha+\beta}} \\
&\leq f((1-\lambda)x_0 \oplus \lambda x_1)^{\frac{1}{\alpha+\beta}} g((1-\lambda)x_0 \oplus \lambda x_1)^{\frac{1}{\alpha+\beta}}
\end{aligned}$$

where the latter inequality follows from the convexity of f and g with respect to $\oplus$. □

The above lemma will be used to show that a one-dimensional marginal of a function affects its level of convexity. Let K be a $\oplus$-convex body in $\mathbb{R}^n$, and $P_{e_n^\perp}K$ be the orthogonal projection of K to the subspace $e_n^\perp$. Given a continuous function $f : K \to \mathbb{R}^+$, we define $P_{e_n^\perp}f$ by

$$P_{e_n^\perp}f(y) = \int_{\{r: K\cap(y,r)\}} f(y,r)dr = \int_{\mathbb{R}} 1_K(y,t)f(y,t)dt,$$

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for $y \in P_{e_n^\perp} K$.

Theorem 3.4. *Let K be a $\oplus$-convex body in $\mathbb{R}^n$, $\alpha > 0$, and let e_n be the n-th unit coordinate vector. If $f : K \to \mathbb{R}^+$ is $1/\alpha$-concave with respect to $\oplus$, then $P_{e_n^\perp} f$ is $1/(\alpha+1)$-concave with respect to $\oplus$.*

In the case of indicator function of K, we have for all $\alpha > 0$, the length $|(y + \mathbb{R}e_n) \cap K|$ is $1/\alpha$-concave with respect to $\oplus$.

Proof. Let $F(y, s) = 1_K(y, s) f(y, s)$ for $y \in P_{e_n^\perp} K$ and $s \in \mathbb{R}$. The indicator function of $\oplus$-convex body K is constant on K, and hence it is $1/\beta$-concave with respect to $\oplus$ for every $\beta > 0$. Thus Lemma 3.3 implies that $F(y, s)$ is $1/(\alpha + \beta)$-concave with respect to $\oplus$ for every $\beta > 0$. Taking the limit of pointwise inequality which is satisfied for any $\beta > 0$, $F(y, s)$ is $1/\alpha$-concave with respect to $\oplus$. Therefore, for all $y, z \in P_{e_n^\perp} K,\ s, t \in \mathbb{R}$,

$$\begin{aligned} F\left((1-\lambda)y \oplus \lambda z, s + t + \varphi_{n-1}((1-\lambda)y, \lambda z)\right) &= F\left((1-\lambda)(y, s) \oplus \lambda(z, t)\right) \\ &\geq M_{1/\alpha}^{\lambda}\left(F(y, s), F(z, t)\right). \end{aligned}$$

Consider three functions on $\mathbb{R}$

$$f(s) = F(y, s), g(t) = F(z, t),$$

and

$$h(r) = F\left((1-\lambda)y \oplus \lambda z, r + \varphi_{n-1}((1-\lambda)y, \lambda z).\right)$$

The triple (f, g, h) satisfies the condition of Theorem 2.1, we have

$$\int_{\mathbb{R}} h(r) dr \geq M^{\lambda}_{\frac{1/\alpha}{(1/\alpha)+1}} \left(\int_{\mathbb{R}} f(s) ds, \int_{\mathbb{R}} g(t) dt \right).$$

Then we have

$$\begin{aligned} &\int_{\mathbb{R}} F\left((1-\lambda)y \oplus \lambda z, r\right) dr \\ &= \int_{\mathbb{R}} F\left((1-\lambda)y \oplus \lambda z, r + \varphi_{n-1}((1-\lambda)y, \lambda z)\right) dr \\ &\geq M^{\lambda}_{\frac{1}{\alpha+1}} \left(\int_{\mathbb{R}} F(y, s) ds, \int_{\mathbb{R}} F(z, t) dt \right). \end{aligned}$$

This proves the desired result. $\square$

As a consequence of Theorem 3.4, we prove the following Brunn's concavity principle for spaces with bitriangular laws of composition.

Theorem 3.5. *Let K be a $\oplus$-convex body in $\mathbb{R}^n$, and H be the k-dimensional space spanned by $\{e_{n-k+1}, e_{n-k+2}, \cdots, e_n\}$. Then the function $F : H^\perp \to \mathbb{R}$ defined by*

$$F(x) = |K \cap (x \oplus H)|^{1/k}$$

is concave on its support.

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Proof. Consider the function $f_1(x') = 1_K(x', x_{n-k+1}, \cdots, x_n)$, which is $1/\alpha$-concave with respect to $x' \in \mathbb{R}^{n-k}$ for every $\alpha > 0$. For the direction e_{n-k+1}, we denote by F_1 the subspace spanned by $H^{\perp} \cup \{e_{n-k+1}\}$. In the space H_1, Theorem 3.4 shows that the function

$$x' \to \int_{\mathbb{R}} 1_K(x', x_{n-k+1}, \cdots, x_n) dx_{n-k+1}$$

is $1/(\alpha+1)$-concave with respect to $\oplus$ for every $\alpha > 0$. Putting the same argument in the directions $e_{n-k+2}, \cdots, e_n$ successively, Theorem 3.4 implies that the function

$$x' \to \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} 1_K(x', x_{n-k+1}, \cdots, x_n) dx_{n-k+1} \cdots dx_n$$

is $1/(\alpha+k)$-concave with respect to $\oplus$ for every $\alpha > 0$. By Fubini's theorem,

$$\big|\{(x', z) : (x', z) \in K\}\big| = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} 1_K(x', x_{n-k+1}, \cdots, x_n) dx_{n-k+1} \cdots dx_n$$

is $1/(\alpha+k)$-concave with respect to $\oplus$ for every $\alpha > 0$. Taking the limit of pointwise inequality as $\alpha \to 0$, we have that it is satisfied in the limit, which means that

$$F(x') = \big|\{(x', z) : (x', z) \in K\}\big|^{1/k}$$

is concave on its support, as proved. $\square$

4 Brunn-Minkowski inequality in Heisenberg group

An interest example of the spaces with bitriangular law of composition is the Heisenberg group $\mathbb{H}^n$. The *Heisenberg group* is the space $\mathbb{C}^n \times \mathbb{R} \sim \mathbb{R}^{2n+1}$ with the noncommunicative multiplication

$$[z_1, \cdots, z_n, t] \cdot [z_1', \cdots, z_n', t'] = [z_1 + z_1', \cdots, z_n + z_n', t + t' + \varphi],$$

where z_i, z_i' are complex, t, t' are real, and

$$\varphi = 2 \sum_{i=1}^{n} \Im(z_i \overline{z_i'}),$$

where $\Im(x)$ denotes the imaginary parts of $x \in \mathbb{C}$. Hence this is the bitriangular law of composition in $\mathbb{R}^{2n+1}$ of form with $\varphi_0 = \cdots = \varphi_{2n-1} = 0$ and $\varphi_{2n} = \varphi$. Geometric inequalities in the sub-Riemannian setting of the Heisenberg group $\mathbb{H}^n$ have been attracted many author's interests, see [2, 3, 9] for examples.

When $f = \chi_A, g = \chi_B, h = \chi_{(1-\lambda)A \cdot \lambda B}$ in Theorem 2.1, we have the following corollary.

Corollary 4.1. *For all nonempty Borel sets A and B in the Heisenberg group $\mathbb{H}^n$, and $p \geq -\frac{1}{2n+1}$,*

$$|(1-\lambda)A \cdot \lambda B|^{\frac{1}{2n+1+1/p}} \geq (1-\lambda)|A|^{\frac{1}{2n+1+1/p}} + \lambda |B|^{\frac{1}{2n+1+1/p}}.$$

As a consequence, if $p = 1$, we have

Corollary 4.2. *For all nonempty Borel sets A and B in the Heisenberg group $\mathbb{H}^n$,*

$$|(1-\lambda)A \cdot \lambda B|^{\frac{1}{2n+2}} \geq (1-\lambda)|A|^{\frac{1}{2n+2}} + \lambda|B|^{\frac{1}{2n+2}}. \tag{4.1}$$

Note that in analogy with the classical isoperimetric problem, an isoperimetric problem in the Heisenberg group $\mathbb{H}^n$ with respect to Carnot-Carathéodory distance (or an equivalent gauge distance) suggests to the following conjectured Brunn-Minkowski type inequality

$$|A \cdot B|^{\frac{1}{2n+2}} \geq |A|^{\frac{1}{2n+2}} + |B|^{\frac{1}{2n+2}}. \tag{4.2}$$

In comparison with the equivalence of the classical Brunn-Minkowski inequalities (1.1) and (1.2), the inequality (4.1) is not equivalent to the inequality (4.2). In fact, the conjecture (4.2) cannot be true, as shown by Monti [15] that (4.2) is not true for $n = 1$ (see also [14]).

Since (4.2) is not true, a natural isomorphic question asks: whether there exists a constant C depending only on n such that

$$C|A \cdot B|^{\frac{1}{2n+2}} \geq |A|^{\frac{1}{2n+2}} + |B|^{\frac{1}{2n+2}}.$$

In the next corollary, we give an isomorphic version of inequality (4.2).

Corollary 4.3. *For all nonempty Borel sets A and B in the Heisenberg group $\mathbb{H}^n$,*

$$2^{\frac{1}{2n+2}}|A \cdot B|^{\frac{1}{2n+2}} \geq |A|^{\frac{1}{2n+2}} + |B|^{\frac{1}{2n+2}}.$$

Proof. Let $f = \chi_A, g = \chi_B, h = \chi_{(1-\lambda)A\oplus\lambda B}$, where $\oplus$ is with respect to $\overline{\varphi_0} = \cdots = \overline{\varphi_{2n-1}} = 0$ and $\overline{\varphi_{2n}} = 4\sum\limits_{i=1}^{n} \Im(z_i\overline{z_i'})$. Using Theorem 2.1 we have for $p > 0$, $\lambda \in (0,1)$,

$$|(1-\lambda)A \oplus \lambda B|^{\frac{1}{2n+1+1/p}} \geq (1-\lambda)|A|^{\frac{1}{2n+1+1/p}} + \lambda|B|^{\frac{1}{2n+1+1/p}}. \tag{4.3}$$

Now we claim that

$$\frac{1}{2}A \oplus \frac{1}{2}B = \frac{1}{2}(A \cdot B), \tag{4.4}$$

where the left-hand side is applied with the functions $\overline{\varphi_k}$, and the right-hand side, with respect to φ_k (the Heisenberg group case). It is clear that the k-th coordinates of the left-hand side are the same with that of the right-hand side, for $k = 1, \cdots, 2n$. Let us check the last coordinate. Consider $x \in A, y \in B$. The $(2n+1)$-th coordinate on the left-hand side is

$$\begin{aligned}\frac{1}{2}x_{2n+1} + \frac{1}{2}y_{2n+1} + \overline{\varphi_{2n+1}}\left(\frac{x}{2}, \frac{y}{2}\right) &= \frac{1}{2}x_{2n+1} + \frac{1}{2}y_{2n+1} + \sum_{i=1}^{n}\Im(z_i\overline{z_i'}) \\ &= \frac{1}{2}\left(x_{2n+1} + y_{2n+1} + 2\sum_{i=1}^{n}\Im(z_i\overline{z_i'})\right),\end{aligned}$$

which is just the $(2n+1)$-th coordinate on the right-hand side of (4.4).

Letting $p = 1$ and $\lambda = 1/2$, the claim combining with (4.3) implies the desired results. □

Note that although the conjectured inequality (4.2) is not true, Bobkov [4] proved a weaker version of (4.2) by using Corollary 2.2.

Corollary 4.4. *[4] For all nonempty Borel sets A and B in the Heisenberg group $\mathbb{H}^n$,*

$$|A \cdot B|^{\frac{1}{2n+1}} \geq |A|^{\frac{1}{2n+1}} + |B|^{\frac{1}{2n+1}}.$$

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