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DHAGE ITERATION METHOD FOR AN ALGORITHMIC APPROACH TO LOCAL SOLUTION OF THE NONLINEAR SECOND ORDER ORDINARY HYBRID DIFFERENTIAL EQUATIONS WITH MAXIMA

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ABSTRACT. In this paper, we establish a couple of approximation results for local existence and uniqueness of the solution of a IVP of nonlinear second order ordinary hybrid differential equations with maxima by using the Dhage monotone iteration method based on the hybrid fixed point theorems with maxima of Dhage (2022) and Dhage et al. (2022). An approximation result for Ulam-Hyers stability of the local solution of the considered hybrid differential equation with maxima is also established. Finally, our main abstract results are also illustrated with a couple of numerical examples.

1. Introduction

The differential equations with maxima contains the maximum value of the unknown function over the past interval of time, that is, the present state depends upon the maximum past value of the state variable and such equations occur in several natural and physical phenomena such as electricity signal model, disease model, growth and decay population, plant and uranium models etc., to mention a few. The differential equations with maxima have already been studied using integral inequalities of Belman and Grownwall [16] for different aspects of the solution. Similarly, they have also been studied for existence and uniqueness of solution by using functional analytic technique like Schauder and Banach fixed point theorems. See Bainov and Hristova [1], otrocol and Rus [18] and references therein. Furthermore, the Picard and Dhage iteration methods are also applied to nonlinear differential equations with maxima for proving the approxiation of solutions. See for example, Dhage and Dhage [7, 8, 9, 10, 11] and references therein. The approximation results along with existence give the algorithms for finding the approximate solution. Therefore, such results are very much useful to predict the behavior of the dynamic systems governed by the models of nonlinear differential equations with maxima. This is the motivation of the present work and here, we discuss the nonlinear second order ordinary differential equations with maxima via Dhage iteration method under certain monotonicity but without the usual strong Lipschitz condition on the nonlinear function involved in the equation.

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Given a closed and bounded interval $J=[t_0,t_0+a]$ in $\mathbb{R}$ for some $t_0,a\in\mathbb{R}$ with $a>0$, we consider the IVP of nonlinear second order hybrid ordinary differential equation (HDE),

$$\left.\begin{aligned} &x''(t)=f\big(t,x(t),M_x(t)\big),\ t\in J,\\ &x(t_0)=\alpha_0,\ x'(t_0)=\alpha_1, \end{aligned}\right\} \tag{1.1}$$

where α_0,α_2 are real constants, $M_x(t)=\max\limits_{\xi\in[t_0,t]}x(\xi)$ and the function $f:J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfies some hybrrid, that is, mixed hypotheses from algebra, analysis and topology to be specified later.

Definition 1.1. A function $x\in C(J,\mathbb{R})$ is said to be a *solution* of the HDE (1.1 if it satisfies the equations in (1.1) on J, where $C(J,\mathbb{R})$ is the space of continuous real-valued functions defined on J. If the solution x lies in a closed ball $\overline{B_r(x_0)}$ centered at some point $x_0\in C(J,\mathbb{R})$ of radius $r>0$, then we say it is a *local solution* or *neighborhood solution* (in short *nbhd solution*) of the HDE (1.1 on J.

Remark 1.1. We note that our notion of a loal solution is different from usual one that given in Coddington [2].

The HDE (1.1 is familiar in the subject of nonlinear analysis and can be studied for a variety of different aspects of the solution by using analytical methods from nonlinear functional analysis. The existence of local solution can be proved by using the Schauder fixed point principle, see for example, Coddington [2], Lakshmikantham and Leela [16], Granas and Dugundji [14] and references therein. The approximation result for uniqueness of solution can be proved by using the Banach fixed point theorem under a Lipschitz condition which is considered to be very strong in the subject of nonlinear analysis. But to the knowledge the present authors, the approximation results for the local existence and uniqueness theorems without using the Lipschitz condition or under its weaker form is not discussed in the literature as for the theory of nonlinear differential and integral equations. In this paper, we discuss the approximation results for local existence, uniqueness and Ulam-Hyers stability of solution to the HDE (1.1) under weaker Lipschitz condition but via construction of the algorithms based on Dhage iteration method and a hybrid fixed point theorem of Dhage [6]. Also see Dhage *et al.* [13] and references therein.

The rest of the paper is organized as follows. Section 2 deals with the auxiliary results and main hybrid fixed point theorems involved in the Dhage iteration method. The hypotheses and main approximation results for the local existence and uniqueness of solution are given in Section 3. The approximation of the Ulam-Hyer stability is discussed in Section 4 and a couple of illustrative examples are presented in Section 5. Finally, some concluding remarks are mentioned in Section 6.

2. Auxiliary Results

We place the problem of HDE (1.1) in the function space $C(J,\mathbb{R})$ of continuous, real-valued functions defined on J. We introduce a supremum norm $\|\cdot\|$ in $C(J,\mathbb{R})$ defined by

$$\|x\| = \sup_{t\in J} |x(t)|, \tag{2.1}$$

and an order relation $\preceq$ in $C(J,\mathbb{R})$ by the cone K given by

$$K = \{x \in C(J,\mathbb{R}) \mid x(t) \geq 0 \;\forall\, t \in J\}. \tag{2.2}$$

Thus,

$$x \preceq y \iff y - x \in K, \tag{2.3}$$

or equivalently,

$$x \preceq y \iff x(t) \leq y(t) \;\forall\, t \in J.$$

It is known that the Banach space $C(J,\mathbb{R})$ together with the order relations $\preceq$ becomes an ordered Banach space which we denote for convenience, by $\big(C(J,\mathbb{R}),K\big)$. We denote the open and closed spheres centered at $x_0 \in C(J,\mathbb{R})$ of radius r, for some $r > 0$, by

$$B_r(x_0) = \{x \in C(J,\mathbb{R}) \mid \|x - x_0\| < r\},$$

and

$$B_r[x_0] = \{x \in C(J,\mathbb{R}) \mid \|x - x_0\| \leq r\},$$

receptively. It is clear that $B_r[x_0] = \overline{B_r(x_0)}$. Let $M > 0$ be a real number. Denote

$$B_r^M[x_0] = \big\{x \in B_r[x_0] \,\big|\, |x(t_1) - x(t_2)| \leq M\,|t_1 - t_2| \text{ for } t_1, t_2 \in J\big\}. \tag{2.4}$$

Then, we have the following result.

Lemma 2.1. *The set $B_r^M[x_0]$ is compact in $C(J,\mathbb{R})$.*

Proof. By definition, $B_r[x_0]$ is a closed and bounded subset of the Banach space $C(J,\mathbb{R})$. Moreover, $B_r^M[x_0]$ is an equicontinuous subset of $C(J,\mathbb{R})$ in view of the condition (2.1). Now, by an application of Arzelá-Ascoli theorem, $B_r^M[x_0]$ is compact set in $C(J,\mathbb{R})$ and the proof of the lemma is complete. ☐

It is well-known that the hybrid fixed point theoretic technique is very much useful in the subject of nonlinear analysis for dealing with the nonlinear equations qualitatively. See Granas and Dugundji [14] and the references therein. Here, we employ the Dhage monotone iteration method or simply *Dhage iteration method* based on the following two hybrid fixed point theorems of Dhage [6] and Dhage *et al.* [13].

Theorem 2.1 (Dhage [6]). *Let S be a non-empty partially compact subset of a regular partially ordered Banach space $\big(E, \|\cdot\|, \preceq,\big)$ with every chain C in S is Janhavi set and let $\mathscr{T} : S \to S$ be a monotone nondecreasing, partially continuous mapping. If there exists an element $x_0 \in S$ such* *that* $x_0 \preceq \mathscr{T}x_0$ *or* $x_0 \succeq \mathscr{T}x_0$*, then the hybrid mapping equation* $\mathscr{T}x = x$ *has a solution* ξ^* *in* S *and the sequence* $\{\mathscr{T}^n x_0\}_0^\infty$ *of successive iterations converges monotonically to* ξ^**.*

Theorem 2.2 (Dhage [6])**.** *Let* $B_r[x]$ *denote the partial closed ball centered at* x *of radius* r*, in a regular partially ordered Banach space* $\big(E, \|\cdot\|, \preceq,\big)$ *and let* $\mathscr{T} : E \to E$ *be a monotone nondecreasing and partial contraction operator with contraction constant* q*. If there exists an element* $x_0 \in X$ *such that* $x_0 \preceq \mathscr{T}x_0$ *or* $x_0 \succeq \mathscr{T}x_0$ *satisfying*

$$\|x_0 - \mathscr{T}x_0\| \leq (1-q)r,$$

for some real number $r > 0$*, then* $\mathscr{T}$ *has a unique comparable fixed point* x^* *in* $B_r[x_0]$ *and the sequence* $\{x_n\}_{n=0}^\infty$ *of successive iterations converges monotonically to* x^**. Furthermore, if every pair of elements in* X *has a lower or upper bound, then* x^* *is unique.*

If a Banach X is partially ordered by an order cone K in X, then in this case we simply say X is ordered Banach space which we denote it by (X,K). Then, we have the following useful results proved in Dhage [4, 5].

Lemma 2.2 (Dhage [4, 5])**.** *Every ordered Banach space* (X,K) *is regular.*

Lemma 2.3 (Dhage [4, 5])**.** *Every partially compact subset* S *of an ordered Banach space* (X,K) *is a Janhavi set in* X*.*

As a consequence of Lemmas 2.2 and 2.3, we obtain the following hybrid fixed point theorem which we need in what follows.

Theorem 2.3 (Dhage [6] and Dhage *et al.* [13])**.** *Let* S *be a non-empty partially compact subset of an ordered Banach space* (X,K) *and let* $\mathscr{T} : S \to S$ *be a partially continuous and monotone nondecreasing operator. If there exists an element* $x_0 \in S$ *such that* $x_0 \preceq Tx_0$ *or* $x_0 \succeq Tx_0$*, then* $\mathscr{T}$ *has a fixed point* $x^* \in S$ *and the sequence* $\{x_n\}_{n=0}^\infty$ *of successive iterations converges monotonically to* x^**.*

Theorem 2.4 (Dhage [6])**.** *Let* $B_r[x]$ *denote the partial closed ball centered at* x *of radius* r *for some real number* $r > 0$*, in an ordered Banach space* $\big(X,K\big)$ *and let* $\mathscr{T} : (X,K) \to (X,K)$ *be a monotone nondecreasing and partial contraction operator with contraction constant* q*. If there exists an element* $x_0 \in X$ *such that* $x_0 \preceq \mathscr{T}x_0$ *or* $x_0 \succeq \mathscr{T}x_0$ *satisfying*

$$\|x_0 - \mathscr{T}x_0\| \leq (1-q)r, \tag{2.5}$$

for some real number $r > 0$*, then* $\mathscr{T}$ *has a unique comparable fixed point* x^* *in* $B_r[x_0]$ *and the sequence* $\{x_n\}_{n=0}^\infty$ *of successive iterations converges monotonically to* x^**. Furthermore, if every pair of elements in* X *has a lower or upper bound, then* x^* *is unique.*

The details of the notions of partial order, Janhavi set, regularity of ordered space, monotonicity of mappings, partial continuity, partial closure, partial boundedness, partial completeness, partial compactness and partial contraction etc. and related applications appear in Dhage [3, 4, 5], Dhage and Dhage [7, 8], Dhage *et al.* [13] and references therein.

3. Local Approximation Results

We consider the following set of hypotheses in what follows.

(H_1) The function f is continuous and bounded on $J\times\mathbb{R}\times\mathbb{R}$ with bound M_f.

(H_2) There exist constants $\ell_1>0$ and $\ell_2>0$ such that

$$0\le f(t,x_1,x_2)-f(t,y_1,y_2)\le \ell_1(x_1-y1)+\ell_2(x_2-y_2)$$

for all $x_1,x_2,y_1,y_2\in\mathbb{R}$ with $x_1\ge y_1$, $x_2\ge y_2$, where $(\ell_1+\ell_2)\,a^2<1$.

(H_3) $f(t,x,y)$ is nondecreasing in x and y for each $t\in J$.

(H_4) $f(t,\alpha_0,\alpha_0)\ge 0$ and $\alpha_1\ge 0$ for all $t\in J$.

Then, we have the following useful lemma.

Lemma 3.1. *If $h\in L^1(J,\mathbb{R})$, then the IVP of ordinary second order linear differential equation*

$$x''(t)=h(t),\ t\in J,\quad x(t_0)=\alpha_0,\quad x'(t_0)=\alpha_1, \tag{3.1}$$

is equivalent to the integral equation

$$x(t)=\alpha_0+\alpha_1(t-t_0)+\int_{t_0}^{t}(t-s)\,h(s)\,ds,\ ,t\in J. \tag{3.2}$$

Theorem 3.1. *Suppose that the hypotheses (H_1), (H_3) and (H_4) hold. Furthermore, if the inequalities $|\alpha_1|\,a+M_f\,a^2\le r$ and $|\alpha_1|+2M_f\,a\le M$ hold, then the HDE (1.1) has a solution x^* in $B_r^M[\alpha_0]$, where $x_0\equiv\alpha_0$, and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by*

$$\left.\begin{aligned} x_0(t)&=\alpha_0,\ t\in J,\\ x_{n+1}(t)&=\alpha_0+\alpha_1(t-t_0)+\int_{t_0}^{t}(t-s)\,f\big(s,x_n(s),M_x(s)\big)\,ds,\ t\in J, \end{aligned}\right\} \tag{3.3}$$

where $n=0,1,\ldots$; converges monotone nondecreasingly to x^*.

Proof. Set $X=C(J,\mathbb{R})$. Clearly, (X,K) is a partially ordered Banach space. Let x_0 be a constant function on J such that $x_0(t)=\alpha_0$ for all $t\in J$ and define a closed ball $B_r^M[x_0]$ in X defined by (2.3). By Lemma 2.1, $B_r^M[x_0]$ is a compact subset of X. By Lemma 3.1, the HDE (1.1) is equivalent to the nonlinear hybrid integral equation (HIE)

$$x(t)=\alpha_0+\alpha_1(t-t_0)+\int_{t_0}^{t}(t-s)\,f\big(s,x(s),M_x(s)\big)\,ds,\ t\in J. \tag{3.4}$$

Now, define an operator $\mathscr{T}$ on $B_r^M[x_0]$ into X by

$$\mathscr{T}x(t)=\alpha_0+\alpha_1(t-t_0)+\int_{t_0}^{t}(t-s)\,f\big(s,x(s),M_x(s)\big)\,ds,\ t\in J. \tag{3.5}$$

We shall show that the operator $\mathscr{T}$ satisfies all the conditions of Theorem 2.3 on $B_r^M[x_0]$ in the following series of steps.

Step I: *The operator $\mathscr{T}$ maps $B_r^M[x_0]$ into itself.*

Firstly, we show that $\mathscr{T}$ maps $B_r^M[x_0]$ into itself. Let $x \in B_r^M[x_0]$ be arbitrary element. Then,

$$\begin{aligned}
|\mathscr{T}x(t) - x_0(t)| &= |\alpha_1 (t-t_0) + \left| \int_{t_0}^{t} (t-s)\, f\big(s, x(s), M_x(s)\big)\, ds \right| \\
&\leq |\alpha_1|\, a + \int_{t_0}^{t} |t-s|\, \big|f\big(s, x(s), M_x(s)\big)\big|\, ds \\
&< |\alpha_1|\, a + M_f\, a \int_{t_0}^{t_0+a} ds \\
&= |\alpha_1|\, a + M_f\, a^2 \\
&\leq r,
\end{aligned}$$

for all $t \in J$. Taking the supremum over t in the above inequality yields

$$\|\mathscr{T}x - x_0\| \leq |\alpha_1|\, a + M_f\, a^2 \leq r$$

which implies that $\mathscr{T}x \in B_r[x_0]$ for all $x \in B_r^M[x_0]$. Next, let $t_1, t_2 \in J$ be arbitrary. Then,

$$\begin{aligned}
&\big|\mathscr{T}x(t_1) - \mathscr{T}x(t_2)\big| \\
&\quad \leq |\alpha_1|\, |t_1 - t_2| + \bigg| \int_{t_0}^{t} (t_1 - s)\, f\big(s, x(s), M_x(s)\big)\, ds \\
&\qquad\qquad - \int_{t_0}^{t_2} (t_2 - s)\, f\big(s, x(s), M_x(s)\big)\, ds \bigg| \\
&\quad \leq |\alpha_1|\, |t_1 - t_2| + \bigg| \int_{t_0}^{t_1} (t_1 - s)\, f\big(s, x(s), M_x(s)\big)\, ds \\
&\qquad\qquad - \int_{t_0}^{t_1} (t_2 - s)\, f\big(s, x(s), M_x(s)\big)\, ds \bigg| \\
&\qquad + \bigg| \int_{t_0}^{t_1} (t_2 - s)\, f\big(s, x(s), M_x(s)\big)\, ds \\
&\qquad\qquad - \int_{t_0}^{t_2} (t_2 - s)\, f\big(s, x(s), M_x(s)\big)\, ds \bigg| \\
&\quad \leq |\alpha_1|\, |t_1 - t_2| + \int_{t_0}^{t_1} |t_1 - t_2|\, \big|f\big(s, x(s), M_x(s)\big)\big|\, ds \\
&\qquad\qquad + \left| \int_{t_1}^{t_2} |t_2 - s|\, \big|f\big(s, x(s), M_x(s)\big)\big|\, ds \right| \\
&\quad \leq |\alpha_1|\, |t_1 - t_2| + \int_{t_0}^{t_0+a} |t_1 - t_2|\, M_f\, ds + \left| \int_{t_1}^{t_2} a M_f\, ds \right| \\
&\quad \leq |\alpha_1|\, |t_1 - t_2| + 2M_f\, a\, |t_1 - t_2| \\
&\quad = \big(|\alpha_1| + 2M_f\, a\big)\, |t_1 - t_2| \\
&\quad \leq M\, |t_1 - t_2|,
\end{aligned}$$

where, $|\alpha_1| + 2M_f a \leq M$. Therefore, from the definition of the closed set $B_r^M[x_0]$, it follows that $\mathscr{T}x \in B_r^M[x_0]$ for all $x \in B_r^M[x_0]$ As a result, we have $\mathscr{T}(B_r^M[x_0]) \subset B_r^M[x_0]$.

Step II: *$\mathscr{T}$ is a monotone nondecreasing operator.*

Let $x, y \in B_r^M[x_0]$ be any two elements such that $x \succeq y$. Then, from continuity of the function y we have an element $\xi^* \in [t_0, t]$ such that $y(\xi^*) = \max\limits_{\xi \in [t_0,t]} y(\xi)$. But $x(\xi^*) \geq y(\xi^*)$. Consequently, $M_x(t) \geq M_y(t)$ for each $t \in J$. Hence,

$$\begin{aligned}\mathscr{T}x(t) &= \alpha_0 + \alpha_1(t - t_0) + \int_{t_0}^{t} (t-s)\, f\big(s, x(s), M_x(s)\big)\, ds \\ &\geq \alpha_0 + \alpha_1(t - t_0) + \int_{t_0}^{t} (t-s)\, f\big(s, y(s), M_y(s)\big)\, ds \\ &= \mathscr{T}y(t),\end{aligned}$$

for all $t \in J$. So, $\mathscr{T}x \succeq \mathscr{T}y$, that is, $\mathscr{T}$ is monotone nondecreasing on $B_r^M[x_0]$.

Step III: *$\mathscr{T}$ is partially continuous operator.*

Let C be a chain in $B_r^M[x_0]$ and let $\{x_n\}$ be a sequence in C converging to a point $x \in C$. Then, $M_{x_n} \to M_x$ in view of the inequality

$$|M_{x_n}(t) - M_x(t)| \leq \|x_n - x\|$$

for all $t \in J$. Now, by dominated cnonvergence theorem, we have

$$\begin{aligned}\lim_{n\to\infty} \mathscr{T}x_n(t) &= \lim_{n\to\infty} \left[\alpha_0 + \int_{t_0}^{t} (t-s)\, f\big(s, x_n(s), M_{x_n}(s)\big)\, ds\right] \\ &= \alpha_0 + \alpha_1(t - t_0) + \lim_{n\to\infty} \int_{t_0}^{t} (t-s)\, f\big(s, x_n(s), M_{x_n}(s)\big)\, ds \\ &= \alpha_0 + \alpha_1(t - t_0) + \int_{t_0}^{t} (t-s)\, \left[\lim_{n\to\infty} f\big(s, x_n(s), M_{x_n}(s)\big)\right] ds \\ &= \alpha_0 + \alpha_1(t - t_0) + \int_{t_0}^{t} (t-s)\, f\big(s, x(s), M_x(s)\big)\, ds \\ &= \mathscr{T}x(t),\end{aligned}$$

for all $t \in J$. Therefore, $\mathscr{T}x_n \to \mathscr{T}x$ pointwise on J. As $\{\mathscr{T}x_n\} \subset B_r^M[x_0]$, $\{\mathscr{T}x_n\}$ is an equicontinuous sequence of points in X. As a reult, we have that $\mathscr{T}x_n \to \mathscr{T}x$ uniformly on J. Hence $\mathscr{T}$ is partially continuouus operator on $B_r^M[x_0]$.

Step IV: *The element $x_0 \in B_r^M[x_0]$ satisfies the relation $x_0 \preceq \mathscr{T}x_0$.*

Since (H$_4$) holds, one has

$$\begin{aligned}x_0(t) &= \alpha_0 + \alpha_1(t-t_0) + \int_{t_0}^{t}(t-s)f(s,x_0(s),M_{x_0}(s))\,ds\\ &\leq x_0(t) + \alpha_1(t-t_0) + \int_{t_0}^{t}(t-s)f(s,\alpha_0,\alpha_0)\,ds\\ &= \alpha_0 + \alpha_1(t-t_0) + \int_{t_0}^{t}(t-s)f(s,x_0(s),M_{x_0}(s))\,ds\\ &= \mathscr{T}x_0(t),\end{aligned}$$

for all $t \in J$. This shows that the constant function x_0 in $B_r^M[x_0]$ serves as to satisfy the operator inequality $x_0 \preceq \mathscr{T}x_0$.

Thus, the operator $\mathscr{T}$ satisfies all the conditions of Theorem 2.3, and so $\mathscr{T}$ has a fixed point x^* in $B_r^M[x_0]$ and the sequence $\{\mathscr{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotone nondecreasingly to x^*. This further implies that the HIE (3.4) and consequently the HDE (1.1) has a local solution x^* and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.3) is monotone nondecreasing and converges to x^*. This completes the proof. □

Next, we prove an approximation result for existence and uniqueness of the solution simultaneously under weaker form of Lipschitz condition.

Theorem 3.2. *Suppose that the hypotheses (H$_1$), (H$_2$) and (H$_4$) hold. Furthermore, if*

$$|\alpha_1|\,a + M_f a \leq \big[1-(\ell_1+\ell_2)\,a^2\big]r, \quad (\ell_1+\ell_2)a^2 < 1, \tag{3.6}$$

for some real number $r > 0$, then the HDE (1.1) has a unique solution x^* in $B_r[x_0]$ defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.3) is a monotone nondecreasing and converges to x^*.

Proof. Set $(X,K) = \big(C(J,\mathbb{R}),\preceq\big)$ which is a lattice w.r.t. the lattice *join* and *meet* operations defined by $x \vee y = \max\{x,y\}$ and $x \wedge y = \min\{x,y\}$, and so every pair of elements of X has a lower and an upper bound. Let $r > 0$ be a fixed real number and consider closed sphere $B_r[x_0]$ centred at x_0 of radius r in the partially ordered Banach space (X,K).

Define an operator $\mathscr{T}$ on X into X by (3.5). Clearly, $\mathscr{T}$ is monotone nondecreasing on X. To see this, let $x,y \in X$ be two elements such that $x \succeq y$. Then, by hypothesis (H$_2$),

$$\mathscr{T}x(t) - \mathscr{T}y(t) = \int_{t_0}^{t}(t-s)\,\big[f\big(s,x(s),M_x(s)\big) - f\big(s,y(s),M_y(s)\big)\big]\,ds \geq 0,$$

for all $t \in J$. Therefore, $\mathscr{T}x \succeq \mathscr{T}y$ and consequently $\mathscr{T}$ is monotone nondecresing on X.

Next, we show that $\mathscr{T}$ is a partial contraction on X. Let $x,y \in X$ be such that $x \succeq y$. Then, by hypothesis (H$_2$), we obtain

$$|\mathscr{T}x(t) - \mathscr{T}y(t)| = \left|\int_{t_0}^{t}(t-s)\,\big[f\big(s,x(s),M_x(s)\big) - f\big(s,y(s),M_y(s)\big)\big]\,ds\right|$$

$$
\begin{aligned}
&\leq \left| \int_{t_0}^{t} (t-s) \left[\ell_1 (x(s)-y(s)) + \ell_2 (M_x(s) - M_y(s)) \right] ds \right| \\
&= \int_{t_0}^{t} a \left[\ell_1 |x(s)-y(s)| + \ell_2 |M_x(s) - M_y(s)| \right] ds \\
&< a \int_{t_0}^{t_0+a} (\ell_1 + \ell_2) \|x-y\| \, ds \\
&= (\ell_1 + \ell_2)\, a^2 \, \|x-y\|,
\end{aligned}
$$

for all $t \in J$, where $(\ell_1 + \ell_2)\, a^2 < 1$. Taking the supremum over t in the above inequality yields

$$\|\mathscr{T}x - \mathscr{T}y\| \leq (\ell_1 + \ell_2)\, a^2 \, \|x-y\|$$

for all comparable elements $x, y \in X$. This shows that $\mathscr{T}$ is a partial contraction on X with contraction constant ka. Furthermore, it can be shown as in the proof of Theorem 3.1 that the element $x_0 \in B_r^M[x_0]$ satisfies the relation $x_0 \preceq \mathscr{T}x_0$ in view of hypothesis (H$_4$). Finally, by hypothesis (H$_1$) and condition (3.6), one has

$$
\begin{aligned}
\|x_0 - \mathscr{T}x_0\| &\leq |\alpha_1|\, a + \sup_{t \in J} \left| \int_{t_0}^{t} (t-s)\, f(s, \alpha_0, \alpha_0)\, ds \right| \\
&\leq |\alpha_1|\, a + \sup_{t \in J} \int_{t_0}^{t} |t-s|\, |f(s, \alpha_0, \alpha_0)|\, ds \\
&\leq |\alpha_1|\, a + M_f\, a^2 \\
&\leq \left[1 - (\ell_1 + \ell_2)\, a^2 \right] r,
\end{aligned}
$$

which shows that the condition (2.5) of Theorem 2.4 is satisfied. Hence $\mathscr{T}$ has a unique fixed point x^* in $B_r[x_0]$ and the sequence $\{\mathscr{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotone nondecreasingly to x^*. This further implies that the HIE (3.4) and consequently the HDE (1.1) has a unique local solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations is monotone nondecreasing and converges to x^*. This completes the proof. □

Remark 3.1. The conclusion of Theorems 3.1 and 3.2 also remains true if we replace the hypothesis (H$_4$) with the following one.

(H$_4$) The function f satisfies $f(t, \alpha_0, \alpha_0) \leq 0$ and $\alpha_1 \leq 0$ for all $t \in J$.

In this case, the HDE (1.1) has a local solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.3) is monotone nonincreasing and converges to the solution x^*.

Remark 3.2. If the initial condition in the equation (1.1) is such that $\alpha_0 > 0$, then under the conditions of Theorem 3.1, the HDE (1.1) has a local positive solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.3) converges monotone nondecreasingly to the positive solution x^*. Similarly, under the conditions of Theorem 3.2, the HDE (1.1) has a unique local positive solution x^* defined on J and the sequence of successive approximations defined by (3.3) $\{x_n\}_{n=0}^{\infty}$ converges monotone nondecreasingly to the unique positive solution x^*.

4. Local Approximation of Ulam-Hyers Stability

The Ulam-Hyers stability for various dynamic systems has already been discussed by several authors under the conditions of classical Schauder fixed point theorem (see Tripathy [20], Huang *et al.* [15] and references therein). Here, in the present paper, we discuss the approximation of the Ulam-Hyers stability of local solution of the HDE (1.1) under the conditions of hybrid fixed point principle stated in Theorem 2.4. We need the following definition in what follows.

Definition 4.1. The HDE (1.1) is said to be locally Ulam-Hyers stable if for $\varepsilon > 0$ and for each solution $y \in B_r[x_0]$ of the inequality

$$\left.\begin{aligned} \Big|y''(t) - f(t,y(t),M_y(t))\Big| \le \varepsilon,\ t \in J,\\ y(t_0) = \alpha_0,\ y'(t_0) = \alpha_1, \end{aligned}\right\} \tag{$*$}$$

there exists a constant $K_f > 0$ such that

$$\big|y(t) - \xi(t)\big| \le K_f\varepsilon, \tag{$**$}$$

for all $t \in J$, where $\xi \in B_r[x_0]$ is a local solution of the HDE (1.1) defined on J. The solution ξ of the HDE (1.1) is called Ulam-Hyers stable local solution on J.

Theorem 4.1. *Assume that all the hypotheses of Theorem 3.2 hold. Then the HDE (1.1) has a unique Ulam-Hyers stable local solution $x^* \in B_r[x_0]$ and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations given by (3.3) converges monotone nondecreasingly to x^*.*

Proof. Let $\varepsilon > 0$ be given and let $y \in B_r[x_0]$ be a solution of the functional inequality (4.1) on J, that is, we have

$$\left.\begin{aligned} \Big|y''(t) - f\big(t,y(t),M_y(t))\Big| \le \varepsilon,\ t \in J,\\ y(t_0) = \alpha_0,\ y'(t_0) = \alpha_1, \end{aligned}\right\} \tag{4.1}$$

By Theorem 3.2, the HDE (1.1) has a unique local solution $\xi \in B_r[x_0]$. Then by Lemma 2.1, one has

$$\xi(t) = x_o + \alpha_1(t-t_0) + \int_{t_0}^{t}(t-s)\,f\big((s,\xi(s),M_\xi(s)\big)\,ds,\ t \in J. \tag{4.2}$$

Now, by integration of (4.1) yields the estimate:

$$\left|y(t) - \alpha_0 - \alpha_1(t-t_0) - \int_{t_0}^{t}(t-s)\,f\big((s,y(s),M_y(s)\big)\,ds\right| \le \frac{a^2}{2}\varepsilon, \tag{4.3}$$

for all $t \in J$.

Next, from (4.2) and (4.3) we obtain

$$\begin{aligned}
&\big|y(t)-\xi(t)\big| \\
&\quad = \left|y(t)-\alpha_0-\alpha_1(t-t_0)-\int_{t_0}^{t}(t-s)\,f\big((s,\xi(s),M_\xi(s)\big)\,ds\right| \\
&\quad \leq \left|y(t)-\alpha_0-\alpha_1(t-t_0)-\int_{t_0}^{t}(t-s)\,f\big((s,y(s),M_y(s)\big)\,ds\right| \\
&\qquad + \left|\int_{t_0}^{t}(t-s)\,f\big(s,y(s),M_y(s)\big)\,ds-\int_{t_0}^{t}(t-s)\,f\big((s,\xi(s),M_\xi(s)\big)\,ds\right| \\
&\quad \leq \frac{a^2}{2}\,\varepsilon+\int_{t_0}^{t}|t-s|\,\big|f\big(s,y(s),M_y(s)\big)-f\big((s,\xi(s),M_\xi(s)\big)\big|\,ds \\
&\quad \leq \frac{a^2}{2}\,\varepsilon+(\ell_1+\ell_2)\,a^2(\|y-\xi\|).
\end{aligned}$$

Taking the supremum over t, we obtain

$$\|y-\xi\| \leq \frac{a^2}{2}\varepsilon+(\ell_1+\ell_2)\,a^2\|y-\xi\|,$$

or

$$\|y-\xi\| \leq \left[\frac{a^2\,\varepsilon/2}{1-\big(\ell_1+\ell_2\big)a^2}\right],$$

where, $(\ell_1+\ell_2)\,a^2<1$. Letting $K_f=\left[\dfrac{a^2}{2\big[1-(\ell_1+\ell_2)\,a^2\big]}\right]>0$, we obtain

$$\big|y(t)-\xi(t)\big| \leq K_f\varepsilon,$$

for all $t\in J$. As a result, ξ is a Ulam-Hyers stable local solution of the HDE (1.1) on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.3) is monotone nondecreasing and converges to ξ. Consequently the HDE (1.1) is a locally Ulam-Hyers stable on J. This completes the proof. $\square$

Remark 4.1. If the given initial condition in the equation (1.1) is such that $x_0>0$, then under the conditions of Theorem 4.1, the HDE (1.1) has a unique Ulam-Hyers stable local positive solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.3) converges monotone nondecreasingly to x^*.

5. The Examples

In this section we illustrate the hypotheses and main approximation result by giving a couple of numerical examples.

Example 5.1. Given a closed and bounded interval $J = [0,1]$ in $\mathbb{R}$, consider the IVP of nonlinear first order HDE,

$$x''(t) = \tanh x(t) + \tanh M_x(t),\ t \in [0,1]; \quad x(0) = \frac{1}{4},\ x'(0) = 1. \tag{5.1}$$

Here $\alpha_0 = \frac{1}{4}$, $\alpha_1 = 1$ and $f(t,x,y) = \tanh x + \tanh y$ for $(t,x) \in [0,1] \times \mathbb{R}$. We show that f satisfies all the conditions of Theorem 3.1. Clearly, f is bounded on $[0,1] \times \mathbb{R} \times \mathbb{R}$ with bound $M_f = 2$ and so the hypothesis (H_1) is satisfied. Also the function $f(t,x,y)$ is nondecreasing in x and $y$for each $t \in [0,1]$. Therefore, hypothesis (H_3) is satisfied. Moreover, $f(t,\alpha_0,\alpha_0) = f(t,\frac{1}{4},\frac{1}{4}) = \tanh(\frac{1}{4}) + \tanh(\frac{1}{4}) \geq 0$ and $\alpha_1 = 1 > 0$ for each $t \in [0,1]$, and so the hypothesis (H_4) holds. If we take $r = 2$ and $M = 5$, all the conditions of Theorem 3.1 are satisfied. Hence, the HDE (5.1) has a local solution x^* in the closed ball $B_2^5[\frac{1}{4}]$ of the Banach space $C(J,\mathbb{R})$ and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by

$$\begin{aligned} x_0(t) &= \frac{1}{4},\ t \in [0,1], \\ x_{n+1}(t) &= \frac{1}{4} + t + \int_0^t (t-s)\left[\tanh x_n(s) + \tanh M_{x_n}(s)\right] ds,\ t \in [0,1], \end{aligned}$$

converges monotone nondecreasingly to x^*.

Example 5.2. Given a closed and bounded interval $J = [0,1]$ in $\mathbb{R}$, consider the IVP of nonlinear first order HDE,

$$x''(t) = f_1\big(t,x(t),M_x(t)\big),\ t \in [0,1]; \quad x(0) = \frac{1}{4},\ x'(0) = 1, \tag{5.2}$$

where

$$f_1(t,,x,y) = \begin{cases} \dfrac{1}{4} \cdot \left[\dfrac{x}{1+x} + \dfrac{y}{1+y}\right], & \text{if } x \geq 0,\ y \geq 0, \\ \dfrac{1}{4} \cdot \dfrac{x}{1+x}, & \text{if } x \geq 0,\ y < 0, \\ \dfrac{1}{4} \cdot \dfrac{y}{1+y}, & \text{if } y \geq 0,\ x < 0, \\ 0, & \text{if } x < 0,\ y < 0. \end{cases}$$

Here $\alpha_0 = \dfrac{1}{4}$ and $\alpha_1 = 1$. We show that f_1 satisfies all the conditions of Theorem 3.2 on $[0,1] \times \mathbb{R} \times \mathbb{R}$. Clearly, f_1 is bounded on $[0,1] \times \mathbb{R}$ with bound $M_f = 1$ and so, the hypothesis (H_1) is satisfied. Next, let $x_1,x_2,y_1,y_2 \in \mathbb{R}$ be such that $x_1 \geq y_1$ and $x_2 \geq y_2$. Then, we have

$$\begin{aligned} 0 &\leq f_1(t,x_1,x_2) - f_1(t,y_1,y_2) \\ &\leq \frac{1}{4}\left[\frac{x_1}{1+x_1} - \frac{y_1}{1+y_1}\right] + \frac{1}{2}\left[\frac{x_2}{1+x_2} - \frac{y_2}{1+y_2}\right] \\ &\leq \frac{1}{4} \cdot (x_1 - y_1) + \frac{1}{4} \cdot (x_2 - y_2), \end{aligned}$$

for all $t \in [0,1]$. So the hypothesis (H$_2$) holds with $\ell_1 = \frac{1}{2} = \ell_2$. Moreover, $f_1(t, \alpha_0, \alpha_0) = f_1\left(t, \frac{1}{4}, \frac{1}{4}\right) = \frac{1}{10} \geq 0$ and $\alpha_1 = 1 > 0$ for each $t \in [0,1]$, and so the hypothesis (H$_4$) holds. If we take $r = 4$, then we have

$$|\alpha_1|\,a + M_f a = 1 + \frac{11}{14} \leq \Big(1 - \frac{1}{2}\Big) \cdot 4 = \big[1 - (\ell_1 + \ell_2)a^2\big]r,$$

and so, the condition (3.6) is satisfied. Thus, all the conditions of Theorem 3.2 are satisfied. Hence, the HDE (5.2) has a unique local solution x^* in the closed ball $B_4[\frac{1}{4}]$ of $C(J,\mathbb{R})$. This further in view of Remark 3.2 implies that the HDE (5.2) has a unique local positive solution x^* and and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by

$$\begin{aligned} x_0(t) &= \frac{1}{4}, \; t \in [0,1], \\ x_{n+1}(t) &= \frac{1}{4} + t + \int_0^t (t-s)\, f_1\big(t, x_n(s), M_{x_n}(s)\big)\, ds, \; t \in [0,1], \end{aligned}$$

is monotone nondecreasing and converges to x^*. Moreover, the unique local solution x^* is Ulam-Hyers stable on $[0,1]$ in view of Definition 4.1. Consequently the HDE (5.2) is a locally Ulam-Hyers stable on the interval $[0,1]$.

Remark 5.1. The approximation results of this paper may be extended to nonlinear IVPs of higher order ordinary differential equations

$$\left.\begin{aligned} x^{(n)}(t) &= f\big(t, x(t), M_x(t)\big), \; t \in J, \\ x^{(i)}(t_0) &= \alpha_{(i)}, \; i = 0,1,2,\ldots,n-1, \end{aligned}\right\} \tag{5.3}$$

by using similar arguments with appropriate modifications.

6. Concluding Remark

Finally, while concluding this paper, we remark that unlike the Schauder fixed point theorem we do not require any convexity argument in the proof of main existence theorem, Theorem 3.1. Similarly, we do not require the usual Lipschitz condition in the proof of uniqueness theorem, Theorem 3.2, but a weaker form of one sided or partial Lipschitz condition is enough to serve the purpose. However, in both the cases we are able to achieve the existence of local solution by monotonic convergence of the successive approximations. Moreover, in order to illustrate the underlined ideas and the procedure of finding the approximate solution, in this paper a simple form of a differential equation with maxima (1.1) is considered for the study, however other complex nonlinear IVPs of HDEs with maxima for integer or fractional orders derivatives may also be considered and the present study can also be extended to such sophisticated nonlinear differential equations with appropriate modifications. These and other such problems form the further research scope in the subject of nonlinear differential and integral equations with maxima for applications. Some of the results in this direction will be reported elsewhere.

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Declarations

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References

[1] D.D. Bainov, S. Hristova, *Differential equations with maxima*, Chapman & Hall/CRC Pure and Applied Mathematics, 2011.

[2] E.A. Coddington, *An Introduction to Ordinary Differential Equations*, Dover Publications Inc. New York, 1989.

[3] B. C. Dhage, Hybrid fixed point theory in partially ordered normed linear spaces and applications to fractional integral equations, *Differ. Equ. Appl.*, **5** (2013), 155-184.

[4] B.C. Dhage, A coupled hybrid fixed point theorem for sum of two mixed monotone coupled operators in a partially ordered Banach space with applications, *Tamkang J. Math.*, **50**(1) (2019), 1-36.

[5] B.C. Dhage, Coupled and mixed coupled hybrid fixed point principles in a partially ordered Banach algebra and PBVPs of nonlinear coupled quadratic differential equations, *Differ. Equ. Appl.*, **11**(1) (2019), 1-85.

[6] B.C. Dhage, A Schauder type hybrid fixed point theorem in a partially ordered metric space with applications to nonlinear functional integral equations, *Jñānābha*, **52** (2) (2022), 168-181.

[7] B.C. Dhage, S.B. Dhage, Approximating solutions of nonlinear first order ordinary differential equations, *GJMS Special issue for Recent Advances in Mathematical Sciences and Applications-13, GJMS*, **2** (2) (2013), 25-35.

[8] B.C. Dhage, S.B. Dhage, Approximating solutions of nonlinear PBVPs of second-order differential equations via hybrid fixed point theory, *Electronic Journal of Differential Equations,***20**, (2015), 1-10.

[9] B.C. Dhage, S.B. Dhage, Dhage iteration method for approximating positive solutions of nonlinear first order ordinary quadratic differential equations with maxima, *Nonlinear Analysis Forum* **21** (2), (2016), 167-100.

[10] B.C. Dhage, S.B. Dhage, Dhage iteration method for generalized quadratic fractional integral equations with maxima, *Dynamic Systems and Applications*, **26** (1), (2017), 435-452.

[11] B.C. Dhage, S.B. Dhage, Dhage iteration method for initial value problems of nonlinear second order hybrid functional differential equations, *Electronic Journal of Mathematical Analysis and Applications*, **6** (1), (2018), 79-93.

[12] B.C. Dhage, S.B. Dhage, Dhage iteration method for approximating the positive solutions of IVPs for nonlinear first order quadratic neutral functional differential equations with delay and maxima, *International Journal of Applied Mathematics* **31** (1), (2018), 1-21.

[13] B.C. Dhage, J.B. Dhage, S.B. Dhage, Approximating existence and uniqueness of solution to a nonlinear IVP of first order ordinary iterative differential equations, *Nonlinear Studies*, **29** (1) (2022), 303-314.

[14] A. Granas, J.Dugundji, *Fixed Point Theory*, Springer 2003.

[15] J. Huang, S. Jung, Y. Li, On Hyers-Ulam stability of nonlinear differential equations, *Bull. Korean Math. Soc.*, **52** (2) (2015), 685-697.

[16] V. Lakshmikantham and S. Leela, *Differential and Integral Inequalities*, Academic Press, New York, 1969.

[17] R. Lyons, A.S. Vatsala, R.A. Chiquet, Picard's itertive method for Caputo fractional differential equatins with numerical results, *Mathematics*, (2017), 5, 65.

[18] D. Otrocol, I.A. Rus, *Functional-differential equations with maxima of mixed type argument*, Fixed Point Theory, **9**(2008), no. 1, pp. 207–220.

[19] H. Temimi, A.R. Ansari, A computational iteraion method for solving nonlinear ordinary differential equations, *LMS. J. Comput. Mathematics*, **18**(1) (2015), 770-753.

[20] A.K. Tripathy, *Hyers-Ulam stability of ordinary differential equations*, Chapman and Hall / CRC, London, NY, 2021.

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