Existence of solutions for generalized 2D fractional integral equations via Petryshyn's fixed point theorem

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Abstract

In this article, our purpose is to establish the existence results of the solutions for fractional Volterra-type integral equations of two variables. We use the method of measure of non-compactness and Petryshyn's fixed point theorem to obtain these results. Our results contain many previously obtained existence results with more relaxed conditions. Finally, we also give an example to validate our obtained results.

Keywords. Volterra integral equations, Measures of noncompactness(MNC), Fixed-point theorems, Fractional integral equation (FIE).
MSC2020: 45D05, 47H10.

1 Introduction

There are many physical and engineering phenomena that are suitably modeled into differential and integral equations of fractional order [1, 3, 6, 7, 8, 17, 18, 23, 24, 26]. Due to this fact, the problem related to finding the existence of a solution to these equations has great importance. Recent area of research involves the study of integral equations with fractional integrals. Fractional integrals are a generalization of ordinary integrals, and they have been shown to have many applications in physics, engineering, and other fields [23, 30]. Recent research has focused on the existence and uniqueness of solutions to integral equations with fractional integrals, as well as the regularity and stability of these solutions [14, 21, 22, 32]. For example, some researchers have studied the existence and uniqueness of solutions to fractional integral equations with nonlocal boundary conditions, while

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others have investigated the existence and uniqueness of solutions to systems of fractional integral equations. In recent studies, it has been observed that the method of measure of non-compactness is a very powerful tool to handle these types of problems with fixed point theorems [2, 4, 5, 11, 12, 13, 15, 16, 19, 31, 32].

In this study, we establish a existence theorem for the solution of 2D FIE, which is expressed in terms of condensing operators in $[0, a] \times [0, b]$. We consider the following nonlinear 2D FIE:

$$z(h,\tau) = q\left(h, \tau, z(h,\tau), \int_0^h \int_0^\tau f(h,\tau,\zeta,\rho,z(\zeta,\rho))d\rho d\zeta, \int_0^h \int_0^\tau \frac{g(h,\tau,\zeta,\rho,z(\zeta,\rho))}{(h-\zeta)^{1-s_1}(\tau-\rho)^{1-s_2}} d\rho d\zeta)\right), \tag{1}$$

where $(h,\tau) \in I = [0,a] \times [0,b], 0 < s_1, s_2 \leq 1$.
Das et al. [9] studied the existence of solutions for 2D equation

$$z(h,\tau) = B(h,\tau) + q\left(h,, z(h,\tau), \int_0^h \int_0^\tau f(h,\tau,\zeta,\rho,z(\zeta,\rho))d\rho d\zeta\right) \tag{2}$$

for $(h,\tau) \in [0,1] \times [0,1]$.

Further, famous 2D integral equations of Fredholm and Hammerstein type [28] have the form

$$z(h,\tau) = B(h,\tau) + \int_0^1 \int_0^1 f(h,\tau,\zeta,\rho,z(\zeta,\rho))d\rho d\zeta, \tag{3}$$

and

$$z(h,\tau) = B(h,\tau) + \int_0^h \int_0^\tau p_1(h,\tau,\zeta,\rho)p_2(\zeta,\rho,z(\zeta,\rho))d\rho d\zeta. \tag{4}$$

The main purpose of this article is to obtain the results regarding the existence of solution of Eq. (1). To serve this purpose, we use a generalization of Darbo's fixed theorem [2] namely Petryshyn's fixed point theorem [29] with the method of measure of non-compactness. However, many authors studied various types of fractional integral equations with the help of Darbo's fixed point theorem, one can see [8, 9, 11, 14]. Instead of Darbo's fixed point theorem, hear we use Petryshyn's fixed point.

One advantage of Petrashyn's fixed point theorem is its simplicity and ease of application. It only requires the mapping to be a condensing with a constant less than 1, which is a relatively easy condition to check in practice. This makes it a useful tool for proving the existence and uniqueness of fixed points in a variety of settings, particularly in the context of numerical analysis and optimization.

Petrashyn's fixed point theorem is a special case of Darbo's fixed point theorem, and applies specifically to condensing mappings. Darbo's fixed point theorem is a more general result that applies to a wider class of mappings, but requires a slightly stronger condition on the continuity of the mapping.

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By using Petryshyn's fixed point theorem, it is not necessary to verify that the involved operator maps a closed convex subset onto itself. This feature is also an advantage that mentioned compared to other similar methods such as the use of Darbo's and Schauder fixed point theorems.

The paper is classified as five sections including the introduction. In Section 2, we introduce some preliminaries and describes the idea of MNC. Section 3 is devoted to state and prove existence theorem for equations involving condensing operators by the Petryshyn's fixed point theorem. In Section 4, we present a example that verify the application of this kind of nonlinear fractional integral equations. Finally Section 5, concludes the paper.

2 Preliminaries

In entire article, we use

- E : Real Banach space;
- $B(z,r)$: Open ball having z as a center with radius r;
- $co\bar{Z}$: Closed convex hull of a set Z.

Definition 2.1. [25] Let $Z \subset E$ and

$$\gamma(Z) = \inf\left\{\sigma > 0 : Z = \bigcup_{i=1}^{m} Z_i \text{ with } \operatorname{diam} Z_i \leq \sigma,\ i = 1, 2, ..., n\right\}.$$

Hence, $\gamma(Z)$ is called the Kuratowski MNC.

Definition 2.2. [2] The Hausdorff MNC

$$\varphi(Z) = \inf\left\{\sigma > 0 : \text{there exists a finite } \sigma\text{-net for } Z \text{ in } E\ \right\}, \tag{5}$$

here, a finite δ-net for Z in E means that a set $\{z_1, z_2, ..., z_n\} \subset E$ such that $B_\sigma(E, z_1), B_\sigma(E, z_2)$, ..., $B_\sigma(E, z_n)$ over Z. These MNC are connected in the following way

$$\varphi(Z) \leq \gamma(Z) \leq 2\varphi(Z),$$

for any bounded set $Z \subset E$.

Theorem 2.1. *[29] Let $Z, \tilde{Z} \subset E$ and $\lambda \in \mathbb{R}$. Then*

(i) $\varphi(Z) = 0$ *if and only if Z is pre-compact;*

(ii) $Z \subseteq \tilde{Z} \implies \varphi(Z) \leq \varphi(\tilde{Z})$;

(iii) $\varphi(\bar{co}Z) = \varphi(Z)$;

(iv) $\varphi(Z \cup \tilde{Z}) = \max\{\varphi(Z), \varphi(\tilde{Z})\}$;

(v) $\varphi(\theta Z) = |\lambda|\varphi(Z)$;

(vi) $\varphi(Z + \tilde{Z}) \leq \varphi(Z) + \varphi(\tilde{Z})$.

Here, we consider the Banach space $C(I, \mathbb{R})$ with the usual norm

$$\|z\| = \sup\{|z(h, \tau)| : h \in [0, a], \tau \in [0, b]\}.$$

The modulus of continuity of $z \in I$ is defined as

$$\omega(z, \sigma) = \sup\{|z(h, \tau) - z(\hat{h}, \hat{\tau})| : h, \hat{h} \in [0, a], \tau, \hat{\tau} \in [0, b], |h - \hat{h}|, |\tau - \hat{\tau}| \leq \sigma\}.$$

Further,

$$\omega(Z, \sigma) = \sup\{\omega(z, \sigma) : z \in Z\},$$

$$\omega_0(Z) = \lim_{\sigma \to 0} \omega(Z, \sigma).$$

In [2] we found that $\omega_0(Z)$ is a regular MNC in $C(I)$.

Definition 2.3. [27] Let $H : E \to E$ be a continuous mapping of E which fulfill the following condition if for all $Z \subset E$ with Z bounded, $H(Z)$ is bounded and $\gamma(HZ) \leq k\gamma(Z), k \in (0, 1)$. If

$$\gamma(HZ) < \gamma(Z), \text{for all } \gamma(Z) > 0,$$

then H is called condensing or densifying mapping.

Theorem 2.2. *[29] Assume that $H : B_r \to E$ is a condensing mapping which fulfill the boundary condition*

$$H(z) = kz, \text{ for some } z \in \partial B_r \text{ then } k \leq 1.$$

Then the set of fixed points of H in B_r is nonempty.

3 Main results

In this part, we study the existence of the FIE (1) with the following assumptions

(1) $F \in C(I \times \mathbb{R} \times \mathbb{R}, \times\mathbb{R}, \mathbb{R})$, $f, g \in C(\hat{I} \times \mathbb{R}, \mathbb{R})$, where

$$\hat{I} = \{(h, \tau, \zeta, \rho) \in I^2 : 0 \leq \zeta \leq h \leq a, 0 \leq \rho \leq \tau \leq b\}.$$

(2) There exist non-negative constants c_1, c_2, c_3 and $c_1 < 1$ such that

$$|q(h, \tau, z, v, w) - q(h, \tau, \hat{z}, \hat{v}, \hat{w})| \leq c_1|z - \hat{z}| + c_2|v - \hat{v}| + c_3|w - \hat{w}|;$$

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(3) There exists a $r > 0$ such that F satisfies the following bounded condition

$$\sup\{|q(h,\tau,z,v,w)| : (h,\tau)\in I, z\in[-r,r], v\in[-abN, abN], w\in[-a^{s_1}b^{s_2}L, a^{s_1}b^{s_2}L]\}\leq r,$$

where $L = \sup\{|g(h,\tau,\zeta,\rho,z)| :$ for all $(h,\tau,\zeta,\rho)\in\hat{I}$ and $z\in[-r,r]\}$,
$N = \sup\{|f(h,\tau,\zeta,\rho,z)| :$ for all $(h,\tau,\zeta,\rho)\in\hat{I}$ and $z\in[-r,r]\}$.

Theorem 3.1. *Under the assumptions* $(1)-(3)$, *with* $c_1 < 1$, *for all* $z\in I$. *Then the Eq. (1) has at least one solution in* E.

Proof. Define $H : B_r \to E$ in the following way

$$(Hz)(h,\tau) = q\left(h,\tau,z(h,\tau),\int_0^h\int_0^\tau f(h,\tau,\zeta,\rho,z(\zeta,\rho))d\rho d\zeta,\int_0^h\int_0^\tau \frac{g(h,\tau,\zeta,\rho,z((\zeta,\rho)))}{(h-\zeta)^{1-s_1}(\tau-\rho)^{1-s_2}}d\rho d\zeta\right).$$

Now, we show that H is continuous on the ball B_τ. For $\sigma > 0$ and $z,x\in B_\tau$ with $\|z-x\| < \sigma$

$$\begin{aligned}
&|(Hz)(h,\tau)-(Hx)(h,\tau)|\\
&=\Bigg|q\left(h,\tau,z(h,\tau),\int_0^h\int_0^\tau f(h,\tau,\zeta,\rho,z(\zeta,\rho))d\rho d\zeta,\int_0^h\int_0^\tau \frac{g(h,\tau,\zeta,\rho,z(\zeta,\rho))}{(h-\zeta)^{1-s_1}(\tau-\rho)^{1-s_2}}d\rho d\zeta\right)\\
&-q\left(h,\tau,x(h,\tau),\int_0^h\int_0^\tau f(h,\tau,\zeta,\rho,x(\zeta,\rho))d\rho d\zeta,\int_0^h\int_0^\tau \frac{g(h,\tau,\zeta,\rho,x(\zeta,\rho))}{(h-\zeta)^{1-s_1}(\tau-\rho)^{1-s_2}}d\rho d\zeta\right)\Bigg|\\
&\leq c_1|z(h,\tau)-x(h,\tau)| + c_2\int_0^h\int_0^\tau |f(h,\tau,\zeta,\rho,z(\zeta,\rho))-f(h,\tau,\zeta,\rho,x(\zeta,\rho))|d\rho d\zeta\\
&+c_3\int_0^h\int_0^\tau \frac{|g(h,\tau,\zeta,\rho,z(\zeta,\rho))-g(h,\tau,\zeta,\rho,x(\zeta,\rho))|}{(h-\zeta)^{1-s_1}(\tau-\rho)^{1-s_2}}d\rho d\zeta\\
&\leq c_1\|z-x\| + c_2 ab\,\omega(f,\sigma) + c_3 a^{s_1}b^{s_2}\omega(g,\sigma),
\end{aligned}$$

where, for $\sigma > 0$, we denote

$$\omega(f,\sigma) = \sup\{|f(h,\tau,\zeta,\rho,z)-f(h,\tau,\zeta,\rho,x)| : (h,\tau,\zeta,\rho)\in\hat{I}, z,x\in[-r,r], \|z-x\|\leq\sigma\}.$$

$$\omega(g,\sigma) = \sup\{|g(h,\tau,\zeta,\rho,z)-g(h,\tau,\zeta,\rho,x)| : (h,\tau,\zeta,\rho)\in\hat{I}, z,x\in[-r,r], \|z-x\|\leq\sigma\}.$$

By uniform continuity of $f(h,\tau,\zeta,\rho,z)$ and $g(h,\tau,\zeta,\rho,z)$ on the set $\hat{I}\times[-r,r]$, respectively, we infer that $\omega(f,\sigma)\to 0$ and $\omega(g,\sigma)\to 0$ as $\sigma\to 0$. Hence, from above estimate H is continuous on B_r.

Now, we prove that H fulfills the condensing mapping. Select a fixed $\sigma > 0$ and $z\in Z$. For $(h_1,\tau_1),(h_2,\tau_2)\in I$ with $h_1-h_2\leq\sigma, \tau_1-\tau_2\leq\sigma$, we get

$$\begin{aligned}
&|(Hz)(h_2,\tau_2)-(Hz)(h_1,\tau_1)|\\
=&\Bigg|q\left(h_2,\tau_2,z(h_2,\tau_2),\int_0^{h_2}\int_0^{\tau_2} f(h_2,\tau_2,\zeta,\rho,z(\zeta,\rho))d\rho d\zeta,\int_0^{h_2}\int_0^{\tau_2}\frac{g(h_2,\tau_2,\zeta,\rho,z(\zeta,\rho))}{(h_2-\zeta)^{1-s_1}(\tau_2-\rho)^{1-s_2}}d\rho d\zeta\right)
\end{aligned}$$

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\begin{aligned}
& -q\left(h_1,\tau_1,z(h_1,\tau_1),\int_0^{h_1}\int_0^{\tau_1} f(h_1,\tau_1,\zeta,\rho,z(\zeta,\rho))d\rho d\zeta,\int_0^{h_1}\int_0^{\tau_1}\frac{g(h_1,\tau_1,\zeta,\rho,z(\zeta,\rho))}{(h_1-\zeta)^{1-s_1}(\tau_1-\rho)^{1-s_2}}d\rho d\zeta\right)\Bigg| \\
\leq\ & \Bigg|q\left(h_2,\tau_2,z(h_2,\tau_2),\int_0^{h_2}\int_0^{\tau_2} f(h_2,\tau_2,\zeta,\rho,z(\zeta,\rho))d\rho d\zeta,\int_0^{h_2}\int_0^{\tau_2}\frac{g(h_2,\tau_2,\zeta,\rho,z(\zeta,\rho))}{(h_2-\zeta)^{1-s_1}(\tau_2-\rho)^{1-s_2}}d\rho d\zeta\right) \\
& -q\left(h_2,\tau_2,z(h_2,\tau_2),\int_0^{h_2}\int_0^{\tau_2} f(h_2,\tau_2,\zeta,\rho,z(\zeta,\rho))d\rho d\zeta,\int_0^{h_1}\int_0^{\tau_1}\frac{g(h_1,\tau_1,\zeta,\rho,z(\zeta,\rho))}{(h_1-\zeta)^{1-s_1}(\tau_1-\rho)^{1-s_2}}d\rho d\zeta\right)\Bigg| \\
& +\Bigg|q\left(h_2,\tau_2,z(h_2,\tau_2),\int_0^{h_2}\int_0^{\tau_2} f(h_2,\tau_2,\zeta,\rho,z(\zeta,\rho))d\rho d\zeta,\int_0^{h_1}\int_0^{\tau_1}\frac{g(h_1,\tau_1,\zeta,\rho,z(\zeta,\rho))}{(h_1-\zeta)^{1-s_1}(\tau_1-\rho)^{1-s_2}}d\rho d\zeta\right) \\
& -q\left(h_2,\tau_2,z(h_2,\tau_2),\int_0^{h_1}\int_0^{\tau_1} f(h_1,\tau_1,\zeta,\rho,z(\zeta,\rho))d\rho d\zeta,\int_0^{h_1}\int_0^{\tau_1}\frac{g(h_1,\tau_1,\zeta,\rho,z(\zeta,\rho))}{(h_1-\zeta)^{1-s_1}(\tau_1-\rho)^{1-s_2}}d\rho d\zeta\right)\Bigg| \\
& +\Bigg|q\left(h_2,\tau_2,z(h_2,\tau_2),\int_0^{h_2}\int_0^{\tau_2} f(h_2,\tau_2,\zeta,\rho,z(\zeta,\rho))d\rho d\zeta,\int_0^{h_1}\int_0^{\tau_1}\frac{g(h_1,\tau_1,\zeta,\rho,z(\zeta,\rho))}{(h_1-\zeta)^{1-s_1}(\tau_1-\rho)^{1-s_2}}d\rho d\zeta\right) \\
& -q\left(h_2,\tau_2,z(h_1,\tau_1),\int_0^{h_2}\int_0^{\tau_2} f(h_2,\tau_2,\zeta,\rho,z(\zeta,\rho))d\rho d\zeta,\int_0^{h_1}\int_0^{\tau_1}\frac{g(h_1,\tau_1,\zeta,\rho,z(\zeta,\rho))}{(h_1-\zeta)^{1-s_1}(\tau_1-\rho)^{1-s_2}}d\rho d\zeta\right)\Bigg| \\
& +\Bigg|q\left(h_2,\tau_2,z(h_2,\tau_2),\int_0^{h_2}\int_0^{\tau_2} f(h_2,\tau_2,\zeta,\rho,z(\zeta,\rho))d\rho d\zeta,\int_0^{h_1}\int_0^{\tau_1}\frac{g(h_1,\tau_1,\zeta,\rho,z(\zeta,\rho))}{(h_1-\zeta)^{1-s_1}(\tau_1-\rho)^{1-s_2}}d\rho d\zeta\right) \\
& -q\left(h_1,\tau_1,z(h_2,\tau_2),\int_0^{h_2}\int_0^{\tau_2} f(h_2,\tau_2,\zeta,\rho,z(\zeta,\rho))d\rho d\zeta,\int_0^{h_1}\int_0^{\tau_1}\frac{g(h_1,\tau_1,\zeta,\rho,z(\zeta,\rho))}{(h_1-\zeta)^{1-s_1}(\tau_1-\rho)^{1-s_2}}d\rho d\zeta\right)\Bigg| \\
\leq\ & c_3\left|\int_0^{h_2}\int_0^{\tau_2}\frac{g(h_2,\tau_2,\zeta,\rho,z(\zeta,\rho)))}{(h_2-\zeta)^{1-s_1}(\tau_2-\rho)^{1-s_2}}d\rho d\zeta-\int_0^{h_1}\int_0^{\tau_1}\frac{g(h_1,\tau_1,\zeta,\rho,z(\zeta,\rho)))}{(h_1-\zeta)^{1-s_1}(\tau_1-\rho)^{1-s_2}}d\rho d\zeta\right| \\
& +c_2\left|\int_0^{h_2}\int_0^{\tau_2} f(h_2,\tau_2,\zeta,\rho,z(\zeta,\rho))d\rho d\zeta-\int_0^{h_1}\int_0^{\tau_1} f(h_1,\tau_1,\zeta,\rho,z(\zeta,\rho))d\rho d\zeta\right| \\
& +c_1|z(h_2,\tau_2)-z(h_1,\tau_1)|+\omega_1(q,\sigma) \\
\leq\ & c_3\Bigg|\int_0^{h_1}\int_0^{\tau_1}\left[\frac{g(h_2,\tau_2,\zeta,\rho,z(\zeta,\rho))}{(h_2-\zeta)^{1-s_1}(\tau_2-\rho)^{1-s_2}}-\frac{g(h_1,\tau_1,\zeta,\rho,z(\zeta,\rho))}{(h_1-\zeta)^{1-s_1}(\tau_1-\rho)^{1-s_2}}\right]d\rho d\zeta \\
& +\int_{h_1}^{h_2}\int_{\tau_1}^{\tau_2}\frac{g(h_2,\tau_2,\zeta,\rho,z(\zeta,\rho))}{(h_2-\zeta)^{1-s_1}(\tau_2-\rho)^{1-s_2}}d\rho d\zeta+\int_0^{h_1}\int_{\tau_1}^{\tau_2}\frac{g(h_2,\tau_2,\zeta,\rho,z(\zeta,\rho))}{(h_2-\zeta)^{1-s_1}(\tau_2-\rho)^{1-s_2}}d\rho d\zeta \\
& +\int_{h_1}^{h_2}\int_0^{\tau_1}\frac{g(h_2,\tau_2,\zeta,\rho,z(\zeta,\rho)))}{(h_2-\zeta)^{1-s_1}(\tau_2-\rho)^{1-s_2}}d\rho d\zeta\Bigg| \\
& +c_2\Bigg|\int_0^{h_1}\int_0^{\tau_1}[f(h_2,\tau_2,\zeta,\rho,z(\zeta,\rho))-f(h_1,\tau_1,\zeta,\rho,z(\zeta,\rho))]d\rho d\zeta \\
& +\int_{h_1}^{h_2}\int_0^{\tau_1} f(h_2,\tau_2,\zeta,\rho,z(\zeta,\rho))d\rho d\zeta+\int_0^{h_1}\int_{\tau_1}^{\tau_2} f(h_2,\tau_2,\zeta,\rho,z(\zeta,\rho))d\rho d\zeta
\end{aligned}
$$

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\begin{aligned}
& +\int_{h_1}^{h_2}\int_{\tau_1}^{\tau_2} f(h_2,\tau_2,\zeta,\rho,z(\zeta,\rho))d\rho d\zeta\Bigg| + c_1|z(h_2,\tau_2)-z(h_1,\tau_1)| + \omega_1(q,\sigma) \\
\leq\; & c_3\int_0^{h_1}\int_0^{\tau_1} \frac{|g(h_2,\tau_2,\zeta,\rho,z(\zeta,\rho))-g(h_1,\tau_1,\zeta,\rho,z(\zeta,\rho))|}{(h_2-\zeta)^{1-s_1}(\tau_2-\rho)^{1-s_2}}d\rho d\zeta \\
& +c_3\int_0^{h_1}\int_0^{\tau_1}|g(h_1,\tau_1,\zeta,\rho,z(\zeta,\rho))|\Bigg|\frac{1}{(h_2-\zeta)^{1-s_1}(\tau_2-\rho)^{1-s_2}} \\
& -\frac{1}{(h_1-\zeta)^{1-s_1}(\tau_1-\rho)^{1-s_2}}\Bigg|d\rho d\zeta + c_3\int_{h_1}^{h_2}\int_{\tau_1}^{\tau_2}\frac{|g(h_2,\tau_2,\zeta,\rho,z(\zeta,\rho))|}{(h_2-\zeta)^{1-s_1}(\tau_2-\rho)^{1-s_2}}d\rho d\zeta \\
& +c_3\int_0^{h_1}\int_{\tau_1}^{\tau_2}\frac{|g(h_2,\tau_2,\zeta,\rho,z(\zeta,\rho))|}{(h_2-\zeta)^{1-s_1}(\tau_2-\rho)^{1-s_2}}d\rho d\zeta + c_3\int_{h_1}^{h_2}\int_0^{\tau_1}\frac{|g(h_2,\tau_2,\zeta,\rho,z(\zeta,\rho))|}{(h_2-\zeta)^{1-s_1}(\tau_2-\rho)^{1-s_2}}d\rho d\zeta \\
& +c_2\int_0^{h_1}\int_0^{\tau_1}|f(h_2,\tau_2,\zeta,\rho,z(\zeta,\rho))-f(h_1,\tau_1,\zeta,\rho,z(\zeta,\rho))|d\rho d\zeta \\
& +c_2\int_{h_1}^{h_2}\int_0^{\tau_1}|f(h_2,\tau_2,\zeta,\rho,z(\zeta,\rho))|d\rho d\zeta + c_2\int_0^{h_1}\int_{\tau_1}^{\tau_2}|f(h_2,\tau_2,\zeta,\rho,z(\zeta,\rho))|d\rho d\zeta \\
& +c_2\int_{h_1}^{h_2}\int_{\tau_1}^{\tau_2}|f(h_2,\tau_2,\zeta,\rho,z(\zeta,\rho))|d\rho d\zeta + c_1|z(h_2,\tau_2)-z(h_1,\tau_1)| + \omega_1(q,\sigma) \\
\leq\; & c_3L\omega_1(g,\sigma)\Big[\frac{(h_2-h_1)^{s_1}}{s_1}-\frac{h_2^{s_1}}{s_1}\Big]\Big[\frac{(\tau_2-\tau_1)^{s_2}}{s_2}-\frac{\tau_2^{s_2}}{s_2}\Big] + c_3L\Big[\frac{h_1^{s_1}\tau_1^{s_2}}{s_1s_2}-\Big(\frac{(h_2-h_1)^{s_1}-h_2^{s_1}}{s_1}\Big) \\
& \times\Big(\frac{(\tau_2-\tau_1)^{s_2}-\tau_2^{s_2}}{s_2}\Big)\Big] + c_3L\Big[\frac{(h_2-h_1)^{s_1}(\tau_2-\tau_1)^{s_2}}{s_1s_2} \\
& +\frac{(\tau_2-\tau_1)^{s_2}[h_2^{s_1}-(h_2-h_1)^{s_1}]}{s_1s_2}+\frac{(h_2-h_1)^{s_1}[\tau_2^{s_2}-(\tau_2-\tau_1)^{s_2}]}{s_1s_2}\Big] \\
& +c_1\|z(h_2,\tau_2)-z(h_1,\tau_1)\| + +c_2ab\omega_1(f,\sigma)+\sigma c_2bN+\sigma c_3aN+\sigma^2c_2N+\omega_1(q,\sigma) \\
\leq\; & c_3L\omega_1(g,\sigma)h_1^{s_1}\tau_1^{s_2}+c_3L[h_1^{s_1}\tau_1^{s_2}+(h_2-h_1)^{s_1}(\tau_2^{s_2}-(\tau_2-\tau_1)^{s_2})+h_2^{s_1}((\tau_2-\tau_1)^{s_2})-h_2^{s_1}\tau_2s_2)] \\
& +c_3L[h_2^{s_1}(s_2-s_1)^{s_2}+(h_2-h_1)^{s_1}(\tau_2^{s_2}-(\tau_2-\tau_1)^{s_2})] \\
& +c_1\|z(h_2,\tau_2)-z(h_1,\tau_1)\| + +c_2ab\omega_1(f,\sigma)+\sigma c_2bN+\sigma c_3aN+\sigma^2c_2N+\omega_1(q,\sigma) \\
\leq\; & c_3L\omega_1(g,\sigma)h_1^{s_1}\tau_1^{s_2}+c_3L[(h_1-h_2)^{s_1}\tau_2^{s_2}+\tau_1^{s_2}(h_2-h_1)^{s_1}+h_2^{s_1}(\tau_2-\tau_1)^{s_2}] \\
& +c_3L[h_2^{s_1}(\tau_2-\tau_1)^{s_2}+\tau_1^{s_2}(h_2-h_1)^{s_1}] \\
& +c_1\|z(h_2,\tau_2)-z(h_1,\tau_1)\| + +c_2ab\omega_1(f,\sigma)+\sigma c_2bN+\sigma c_3aN+\sigma^2c_2N+\omega_1(q,\sigma) \\
\leq\; & c_3L\omega_1(g,\sigma)a^{s_1}b^{s_2}+c_3L[\sigma^{s_1}d^{s_2}+b^{s_2}\sigma^{s_1}+a^{s_1}\sigma^{s_2}]+c_3L[a^{s_1}\sigma^{s_2}+b^{s_2}\sigma^{s_1}] \\
& +c_1\|z(h_2,\tau_2)-z(h_1,\tau_1)\| + +c_2ab\omega_1(f,\sigma)+\sigma c_2bN+\sigma c_3aN+\sigma^2c_2N+\omega_1(q,\sigma).
\end{aligned}
$$

To simplify,

$$\omega_1(g,\sigma)=\sup\{|g(h,\tau,\zeta,\rho,z)-g(\hat{h},\hat{\tau},\zeta,\rho,z)| : |h-\hat{h}|\leq\sigma, |\tau-\hat{\tau}|\leq\sigma, (h,\tau,\zeta,\rho)\in\hat{I}, z\in[-r,r]\},$$
$$\omega_1(f,\sigma)=\sup\{|f(h,\tau,\zeta,\rho,z)-f(\hat{h},\hat{\tau},\zeta,\rho,z)| : |h-\hat{h}|\leq\sigma, |\tau-\hat{\tau}|\leq\sigma, (h,\tau,\zeta,\rho)\in\hat{I}, z\in[-r,r]\},$$

$$\omega_1(q,\sigma)=\sup\Bigg\{|q(h,\tau,z,v,w)-q(\hat{h},\hat{\tau},z,v,w)|:|h-\hat{h}|\leq\sigma,|\tau-\hat{\tau}|\leq\sigma,z\in[-r,r],$$
$$v\in[-abN,abN],w\in[-a^{s_1}b^{s_2}L,a^{s_1}b^{s_2}L]\Bigg\}.$$

From above relation,

$$\omega_1(HZ,\sigma)\leq c_3L\omega_1(g,\sigma)a^{s_1}b^{s_2}+c_3L[\sigma^{s_1}b^{s_2}+b^{s_2}\sigma^{s_1}+a^{s_1}\sigma^{s_2}]+c_3L[a^{s_1}\sigma^{s_2}+b^{s_2}\sigma^{s_1}]$$
$$+c_1\|z(h_2,\tau_2)-z(h_1,\tau_1)\|++c_2ab\omega_1(f,\sigma)+\sigma c_2bN+\sigma c_3aN+\sigma^2c_2N+\omega_1(q,\sigma).$$

Putting limit as $\sigma\to 0$,

$$\omega_1(HZ,\sigma)\leq c_1\omega_1(Z).$$

i.e.,

$$\varphi(HZ)\leq c_1\varphi(Z),$$

which explains that H is a densifying mapping. Now, let $z\in\partial B_r$ and if $Hz=kz$ then, we get $\|Hz\|=k\|z\|=kr$ and by assumption (3),

$$|Hz(h,\tau)|=\left|q\left(h,\tau,z(h,\tau),\int_0^h\int_0^\tau f(h,\tau,\zeta,\rho,z(\zeta,\rho))d\rho d\zeta,\int_0^h\int_0^\tau\frac{g(h,\tau,\zeta,\rho,z(\zeta,\rho))}{(h-\zeta)^{1-s_1}(\tau-\rho)^{1-s_2}}d\rho d\zeta\right)\right|$$
$$\leq r$$

for all $(h,\tau)\in I$, hence $\|Hz\|\leq r$ i.e., $k\leq 1$. □

Corollary 3.2. *Let*

(T_1) $q\in C(I\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, $f\in C(\hat{I}\times\mathbb{R},\mathbb{R})$, *where*

$$\hat{I}=\{(h,\tau,\zeta,\rho)\in I^2:0\leq\zeta\leq h\leq a,0\leq\rho\leq\tau\leq b\};\ \phi,\beta:I\to I.$$

(T_2) *The exist non-negative constants* $c_1,c_2,c_1<1$ *such that*

$$|q(h,\tau,z,w)-q(h,\tau,\hat{z},\hat{w})|\leq c_1|z-\hat{z}|+c_2|w-\hat{w}|.$$

(T_3) *There exists a* $r>0$ *such that* q *satisfies the following bounded condition*

$$\sup\{|q(h,\tau,z,w)|:(h,\tau)\in I,z\in[-r,r],w\in[-a^{s_1}b^{s_2}L,a^{s_1}b^{s_2}L]\}\leq r,$$

where

$$L=\sup\{|f(h,\tau,\zeta,\rho,z)|:\forall(h,\tau,\zeta,\rho)\in\hat{I}\text{ and }z\in[-r,r]\}.$$

If $F(h,\tau,z,w)=\hat{F}(h,\tau,z,w)$, then the following integral equation under $(T_1)-(T_3)$ has at least one solution in $C(I)$.

$$z(h,\tau)=q\left(h,\tau,z(h,\tau),\int_0^h\int_0^\tau\frac{f(h,\tau,\zeta,\rho,z(\zeta,\rho))}{(h-\zeta)^{1-s_1}(\tau-\rho)^{1-s_2}}d\rho d\zeta\right).\tag{6}$$

Proof. The proof is similar to the Theorem 3.1, so we omit the proof. □

Corollary 3.3. *Let*

(D_1) $q \in C(I \times \mathbb{R}, \mathbb{R})$, $f \in C(\hat{I} \times \mathbb{R}, \mathbb{R})$, *where*

$$\hat{I} = \{(h, \tau, \zeta, \rho) \in I^2 : 0 \leq \zeta \leq h \leq a, 0 \leq \rho \leq \tau \leq b\}; \ \phi, \beta : I \to I.$$

(D_2) *There exist non-negative constants* $c_1, c_2, c_1 < 1$ *such that*

$$|q(h, \tau, z) - q(h, \tau, \hat{z})| \leq c_1|z - \hat{z}|;$$

$$|p(h, \tau, z) - p(h, \tau, \hat{z})| \leq c_2|z - \hat{z}|.$$

(D_3) *There exists a* $r > 0$ *such that* q *satisfies the following bounded condition*

$$\sup \{|p(h, \tau, z) + q(h, \tau, z)| : (h, \tau) \in I, z \in [-a^{s_1}b^{s_2}L, a^{s_1}b^{s_2}L]\} \leq r,$$

where $L = \sup\{|f(h, \tau, \zeta, \rho, z)| :$ *for all* $(h, \tau, \zeta, \rho) \in \hat{I}$ *and* $z \in [-r, r]\}$.

If $q(h, \tau, z, w) = p(h, \tau, z) + \tilde{q}(h, \tau, z)$ then the following integral equation under $(D_1) - (D_3)$ has at least one solution in $C(I)$.

$$z(h, \tau) = p(h, \tau, z(h, \tau))) + F\left(h, \tau, \int_0^h \int_0^\tau \frac{f(h, \tau, \zeta, \rho, z(\zeta, \rho))}{(h - \zeta)^{1-s_1}(\tau - \rho)^{1-s_2}} d\rho d\zeta\right). \tag{7}$$

Proof. The proof is similar to the Theorem 3.1, so we leave it to the readers. □

4 Examples

Example 4.1. *Consider the following equation*

$$\begin{aligned} z(h, \tau) = &\frac{1}{5}e^{-\frac{h+2\tau}{3}} + \frac{1}{2 + \tau^2 + h}\left(\frac{|z(h, \tau)|}{1 + |z(h, \tau)|}\right) \\ &+ \frac{\sin(z(h, \tau))}{5(\sqrt{\tau + h} + 1)} \int_0^h \int_0^\tau e^{-\frac{h\tau + \rho\zeta}{2}} |z(\zeta, \rho)| d\rho d\zeta \\ &+ \frac{1}{\tau^2 + h^2 + 6} \int_0^h \int_0^\tau \frac{\rho\zeta \sin(\zeta + z(\zeta, \rho)) + 3h\tau \ln(1 + z(\zeta, \rho))}{(h - \zeta)^{\frac{1}{2}}(\tau - \rho)^{\frac{1}{3}}} d\rho d\zeta. \end{aligned} \tag{8}$$

Here, we study the solution in $C([0,1]\times[0,1])$. We have $s_1=\frac{1}{2}, s_2=\frac{2}{3}$ and

$$
\begin{aligned}
&q(h,\tau,z,v,w)=\frac{1}{5}e^{-\frac{h+2\tau}{3}}+\frac{1}{2+\tau^2+h}\left(\frac{|z|}{1+|z|}\right)+\frac{\sin(z(h,\tau))}{5(\sqrt{\tau+h}+1)}v+\frac{1}{\tau^2+h^2+6}w,\\
&v=\int_0^h\int_0^\tau f(h,\tau,\zeta,\rho,z(\zeta,\rho))d\rho d\zeta,\\
&f(h,\tau,\zeta,\rho,z(\zeta,\rho))=e^{-\frac{h\tau+\rho\zeta}{2}}|z(\zeta,\rho)|,\\
&|f(h,\tau,\zeta,\rho,z)|\leq|z|\\
&w=\int_0^h\int_0^\tau \frac{g(h,\tau,\zeta,\rho,z(\zeta,\rho))}{(h-\zeta)^{\frac{1}{2}}(\tau-\rho)^{\frac{1}{3}}}d\rho d\zeta,\\
&g(h,\tau,\zeta,\rho,z(\zeta,\rho))=\rho\zeta\sin(\zeta+z(\zeta,\rho))+3h\tau\ln\left(1+z(\zeta,\rho)\right),\\
&|g(h,\tau,\zeta,\rho,z)|\leq 1+3|z|
\end{aligned}
$$

for all $(h,\tau)\in[0,1]\times[0,1]$.
It is easy to see that (1) and (2) holds. We show that (3) also hold. Suppose that $\|z\|\leq r, r>0$, then we have

$$
\begin{aligned}
|z(h,\tau)|=&\left|\frac{1}{5}e^{-\frac{h+2\tau}{3}}+\frac{1}{2+\tau^2+h}\left(\frac{|z(h,\tau)|}{1+|z(h,\tau)|}\right)\right.\\
&+\frac{\sin(z(h,\tau))}{5(\sqrt{\tau+h}+1)}\int_0^h\int_0^\tau e^{-\frac{h\tau+\rho\zeta}{2}}|z(\zeta,\rho)|d\rho d\zeta\\
&+\frac{1}{\tau^2+h^2+6}\int_0^h\int_0^\tau \frac{\rho\zeta\sin(\zeta+z(\zeta,\rho))+3h\tau\ln\left(1+z(\zeta,\rho)\right)}{(h-\zeta)^{\frac{1}{2}}(\tau-\rho)^{\frac{1}{3}}}d\rho d\zeta|\\
\leq&\frac{7}{10}+\frac{1}{5}\delta+\frac{1}{36}(1+3\delta).
\end{aligned}
$$

Hence (3) holds, if $\frac{7}{10}+\frac{1}{5}r+\frac{1}{36}(1+3r)\leq r$. We can easily verify that $r=1.0155$ satisfies this inequality. Hence all the assumptions from (1) - (3) are satisfied. Thus, from the Theorem 3.1, we get Eq. (8) has at least one solution in $C([0,1]\times[0,1])$.

5 Conclusion

In this work, we studied the existence of solutions for Volterra fractional integral equations with the help of Petryshyn's fixed point theorem and the method of measure of non-compactness with relaxed conditions. We gave an examples to prove the validity of our results. The interested authors may consider the result of Eq. (1) in different Banach function spaces, e.g., Sobolev space, Orlicz space, Hölder space, etc. The importance of defending the existence result in the research of this is one of the advantages of researchers. So far, several approaches have been devised for this idea. This research is based on a more general form of the FIE, which involves some other relevant works as well.

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