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NON-AUTONOMOUS EVOLUTION EQUATIONS WITH RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVE

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ABSTRACT. In this paper, we study a Cauchy problem for non-autonomous evolution equations with Riemann-Liouville fractional derivative. We propose two different concepts of fundamental solutions and classical solutions corresponding to the homogeneous problem. We also prove some existence results of non-homogeneous Cauchy problem. Our methods rely upon analytic semigroup theory, the Mittag-Leffler function and a variation of parameters formula. As an application, we apply the main results to a time dependent fractional Schrödinger type equation.

1. Introduction

The classical autonomous evolution equations have provided an important manner to solve mathematical models in science and engineering. Among various models containing many concrete nonlocal problems or memory characteristics of materials, fractional calculus has been proven as one of the most efficient analysis tools, such as in anomalous diffusion [23, 34, 39], control theory and engineering [24, 32, 33], viscoelasticity [26, 30], Hamiltonian chaos [40], biophysics [21], impulsive systems [11] and several other areas. Nevertheless, many evolution equations play an important role in mathematical research driven by the time-varying parameters, a non-autonomous evolution equation becomes the main equation for studying this issue, see e.g. [1, 17]. When considering a nonlocal problems or a memory characteristic of materials with time-varying parameters described by the fractional derivative, there are still many difficulties and challenges in obtaining the qualitative properties of these non-autonomous models. In particular, we remark that the structure of solutions for non-autonomous fractional evolution equations is not obvious, while the cases of integer orders are also natural to construct by an evolution operator, which provides great convenience in defining and establishing the properties of mild solutions or classical solutions. Moreover, when studying the solutions to non-autonomous fractional evolution equations, the useful Laplace transform for studying autonomous type equations is not as applicable. To overcome this difficulty, the technique of a variation of parameters formula will be proposed.

A common mathematical physics model $\partial_t^\alpha x = a(t)\Delta x$, $t > 0$, is represented by a fractional evolution equation with time-varying parameters, where Δ is the Laplace operator, ∂_t^α is the Caputo

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fractional derivative of order $\alpha \in (0,2)$, and $a(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function. If $a(\cdot)$ reduces to a constant function, this model of order $\alpha \in (0,1)$ can simulate anomalous diffusion phenomena, ensuring the behavior of a subdiffusion process [27]. For the case of order $\alpha \in (1,2)$, it will ensure the behavior of a superdiffusion process in anomalous diffusion phenomena [3, 12]. Several abstract theories, such as semigroup theory, cosine family theory, and resolvent family theory, will be more useful in obtaining the qualitative properties of this model, see e.g. [13, 14, 15, 36, 37, 41, 42]. Replacing $a(t)$ by $A(t)$, a more general closed linear operator, was considered in [4] for $A(t) = C(t)\Delta$ with a bounded linear operator $C(\cdot)$, in [35] for $A(t) = \mathrm{div}(B(t,\cdot)D)$ with coefficient matrix $B(t,\cdot)$ and first order partial derivative operator D, and in [20] for $A(t) = L(t)$ with a general second order uniform elliptic operator of the main part form $\sum_{i,j=1}^{d} D_i(a_{i,j}(t,\cdot)D_j)$. In particular, closer to this current work, the existence of solution for fractional Löwner-Kufarev equation

$$ {}_0^L D_t^\alpha x(t,z) = zF(t,z)x_z(t,z), \quad {}_0J_t^{1-\alpha}x(0,z) = x_0(z), \quad \alpha \in (0,1), $$

was obtained on Hilbert space with a unit disk by Bajlekova [4], where ${}_0^L D_t^\alpha$ is the Riemann-Liouville fractional derivative of order $\alpha \in (0,1)$, ${}_0J_t^{1-\alpha}$ is the Riemann-Liouville fractional integral of order $1-\alpha$, (see Definitions 7-8), and $F(t,z)$, $x_0(z)$ are analytic functions. More recently, Mahdi [25] proved maximal L^p-regularity results in the context of Hilbert spaces. He and Zhou [12] established the existence and uniqueness of solutions for a non-autonomous fractional evolution equation of order $\alpha \in (1,2)$ in a more general Banach space.We also remark that there are several excellent works on time-variable coefficient parabolic partial differential equations. Kim et al. [19] proved unique solvability for the following evolution equations with the Caputo fractional derivative

$$ -\partial_t^\alpha u + a_{ij}(t,z)D_{ij}u + b_i(t,z)D_i u + c(t,z)u = f(t,z,u), \quad t > 0,\ z \in \mathbb{R}^d. $$

Here $\alpha \in (0,2)$, the indices i, j move from 1 to d, D_i, D_{ij} are the derivatives respect to z. The coefficients $a_{ij}(t,z)$ are piecewise continuous in t and uniformly continuous in z, and the lower order coefficients b_i and c are only bounded measurable functions. Dong and Kim [8] generalized the results in [19] associated with the coefficients $a_{ij}(t,z)$ satisfying the uniform ellipticity condition and having no regularity in the time variable for the parabolic regime $\alpha \in (0,1)$, as well as the weighted mixed-norm estimates and solvability in non-divergence form in [9]. Dong and Liu [10] improved the weighted mixed-norm estimate and solvability in non-divergence form under the coefficients with locally small mean oscillations.

Inspired by the above works, we are interested in the solvability of Cauchy problem for the following non-autonomous fractional evolution equation

$$ (1) \qquad {}_s^L D_t^\alpha x(t) + A(t)x(t) = f(t), \quad {}_sJ_t^{1-\alpha}x(s) = x_s, \quad t \in (s,T], $$

where operator $A(t)$ generates an analytic semigroup $T_t(\varsigma)$, $\varsigma \geq 0$, for each $t \in [s,T]$ with $s \geq 0$, non-homogeneous function $f : (s,T] \to X$ is Hölder continuous, and X is usual Banach space. To derive the main results, we will separate the Cauchy problem into its homogeneous and non-homogeneous components. These results exhibit novelties in three distinct aspects.

(a) We discuss the structure of solutions, introducing two distinct concepts: fundamental solutions and classical solutions. Specifically, this is the first time the concept of fundamental solutions has been considered for the current problem. While these two concepts may appear different, they are, in fact, compatible. Through careful analysis, we observe that fundamental solutions exhibit a singular value at the initial time. This situation can pose significant challenges in finding a solution.

(b) In the context of first order non-autonomous evolution equations, the evolution operator $U(t,s)$ plays a crucial role in representing solutions. This operator also possesses a generalized semigroup property, namely $U(t,s) = U(t,r)U(r,s)$ for $s \leq r \leq t$. Leveraging this property, we can delve into the solvability and stability of solutions. Notably, in a specific scenario, the evolution operator $U(t,s)$ can degenerate into a one parameter semigroup $T(t)$ within the framework of an autonomous setting. However, we find that fundamental solution does not possess the generalized semigroup property of evolution operator. Fortunately, the fundamental solution can degenerate into a solution operator $T_\alpha(t)$ corresponding to the autonomous fractional evolution equations, as seen in Remark 26, notably, even this solution operator $T_\alpha(t)$ does not possess a semigroup property. Relied on time-variable operator $A(t)$, the solution exhibits several new characteristics. For instance, a solution admits uniform continuity but displays a weak singularity at the initial time. Furthermore, the representation of formal solutions bears a stronger resemblance to that of first-order evolution equations or fractional-order evolution equations.

(c) If considered Ω by a bounded open subset of $\mathbb{R}^d$ with regular boundary $\partial\Omega$, problem (1) can be regarded as an abstract version of parabolic type partial differential equation

$$ {}_s^L D_t^\alpha x(t,z) + A(t,z,D)x(t,z) = f(t,z), \quad \text{for } (t,z) \in (s,T] \times \Omega, $$

in $L^p(\Omega)$, where $A(t,z,D)$ is a linear uniform strongly elliptic operator with coefficients depending on $t \in [s,T]$ and $z \in \overline{\Omega}$, satisfying

$$ A(t,z,D) = \sum_{|\beta| \leq 2m} a_\beta(t,z) D^\beta, $$

and there is a constant $c > 0$, for every $z \in \overline{\Omega}$, $t \in [s,T]$ and $\xi \in \mathbb{R}^d$ such that

$$ (-1)^m \mathrm{Re} \sum_{|\beta|=2m} a_\beta(t,x)\xi^\beta \geq c|\xi|^{2m}, $$

where the coefficients $a_\beta(t,z)$ ($|\beta| \leq 2m$) are smooth functions of variable z in $\overline{\Omega}$ for every $t \in [s,T]$ and satisfy for some constants $C > 0$ and $0 < \vartheta \leq 1$

$$ |a_\beta(t,z) - a_\beta(\tau,z)| \leq C|t-\tau|^\vartheta, \quad z \in \overline{\Omega}, \quad \tau, t \in [s,T]. $$

Although the requirements of coefficients $a_\beta(t,z)$ are somewhat stricter than those considered in [8, 19, 20], where the authors considered piecewise continuous or merely measurable coefficients, the Hölder continuity of the coefficients $a_\beta(t,z)$ is advantageous for analyzing the existence of solutions to an abstract problem. However, the results we obtained may be more applicable to practical issues, not just in terms of the solvability of fractional parabolic problem, but also in regards to the regularity of solutions. Taking these factors into account, we shall address the solvability to problem (1) using the operator $A(t)$ that generates an analytic semigroup $T_t(\varsigma)$, for $\varsigma \geq 0$. Additionally, if the function f is Hölder continuous, the current parabolic-type partial differential equation with initial value $x_0 \in L^p(\Omega)$ is solvable, and it possesses a unique classical solution.

The remaining part of the paper is organized as follows. In Sect. 2 we give a brief overview of the preliminaries of fractional derivatives, we summarize some properties of Riemann-Liouville fractional derivative and two special functions. In Sect. 3, we first establish the existence of solutions for linear bounded operators. Subsequently, we construct a solution operator for linear unbounded operators. Furthermore, we provide a thorough analysis of pertinent properties. Leveraging the concept of fundamental solutions, we demonstrate their existence and provide a solvability result for classical solutions in the homogeneous case. In Sect. 4, we prove the existence of classical solutions for the Cauchy problem (1). In Sect. 5, an application is presented to illustrate the main results.

2. Preliminaries

The purpose of this section is to briefly introduce some notations, definitions, and preliminary facts including fractional derivative and integral, two special functions, for more details, we refer to see [18, 32, 41]. We set X by a Banach space equipped with the norm $\|\cdot\|$, $D(A) \subset X$ stands for the domain of the operator A with the graph norm $\|x\|_A = \|x\| + \|Ax\|$ for all $x \in D(A)$. Denote $\mathscr{B}(X)$ by the Banach space of all linear bounded operators from X to X equipped with norm $\|\cdot\|_{\mathscr{B}}$ of the uniform operator topology. Throughout this paper, we set $C > 0$ by some genetic constant.

Let $*$ denote the convolution for functions $f, g \in L^1(0,T;X)$ as follows

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau, \quad t > 0.$$

Let $g_\alpha(\cdot)$ be the Riemann-Liouville fractional kernel of order $\alpha \in \mathbb{R}_+$ defined by $g_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha)$, for $t > 0$, where $\Gamma(\cdot)$ is the usual Gamma function. By a simple calculation, for any $\alpha, \varsigma > 0$, it follows that $(g_\alpha * g_\varsigma)(t-s) = g_{\alpha+\varsigma}(t-s)$, for $t > s \geq 0$, i.e.,

$$\frac{1}{\Gamma(\alpha)\Gamma(\varsigma)} \int_s^t (t-\tau)^{\alpha-1}(\tau-s)^{\varsigma-1} d\tau = \frac{1}{\Gamma(\alpha+\varsigma)}(t-s)^{\alpha+\varsigma-1}, \quad t > s. \tag{2}$$

Let us recall the Mittag-Leffler function $E_{\alpha,\sigma}(\cdot)$ for $\alpha > 0$, $\sigma \in \mathbb{R}$.

Definition 1. *An entire function $E_{\alpha,\sigma}(\cdot) : \mathbb{C} \to \mathbb{C}$ is called a Mittag-Leffler function for $\alpha > 0$, $\sigma \in \mathbb{R}$, given by*

$$E_{\alpha,\sigma}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \sigma)} \quad z \in \mathbb{C}.$$

Noting that for $0 < \alpha < 2$, $\sigma \in \mathbb{R}$, $\pi\alpha/2 < \omega < \min(\pi, \alpha\pi)$, $N \geq 1$, Podlubny [32, pp.32-34] proved the asymptotic expansion of $E_{\alpha,\sigma}(z)$ as $z \to \infty$ by

$$E_{\alpha,\sigma}(z) = \begin{cases} \dfrac{1}{\alpha} z^{\frac{1-\sigma}{\alpha}} e^{z^{\frac{1}{\alpha}}} + \varepsilon_{\alpha,\sigma}(z), & \text{if } |\arg z| \leq \omega, \\ \varepsilon_{\alpha,\sigma}(z), & \text{if } \omega \leq |\arg z| \leq \pi, \end{cases}$$

where

$$\varepsilon_{\alpha,\sigma}(z) = -\sum_{k=1}^{N-1} \frac{z^{-k}}{\Gamma(\sigma - \alpha k)} + O(|z|^{-N}), \quad \text{as } z \to \infty.$$

By this asymptotic expansion, the following two properties hold, see Podlubny [32, Theorem 1.5 and Theorem 1.6].

Lemma 2. *If $0<\alpha<2$ and $\sigma\in\mathbb{R}$, $\pi\alpha/2<\omega<\min(\pi,\alpha\pi)$, then*

$$|E_{\alpha,\sigma}(z)|\le\frac{C}{1+|z|},\quad z\in\mathbb{C},\ \omega\le|\arg z|\le\pi.$$

Lemma 3. *If $0<\alpha<2$ and $\sigma\in\mathbb{R}$, $\pi\alpha/2<\omega<\min(\pi,\pi\alpha)$, then*

$$|E_{\alpha,\sigma}(z)|\le C(1+|z|)^{\frac{1-\sigma}{\alpha}}e^{\left(Re z^{\frac{1}{\alpha}}\right)}+\frac{C}{1+|z|},\quad z\in\mathbb{C},\ |\arg z|\le\omega.$$

Remark 4. *In particular, from the asymptotic expansion of $E_{\alpha,\sigma}(z)$, it follows that $E_{\alpha,\sigma}(z)=\varepsilon_{\alpha,\sigma}(z)$ for $\omega\le|\arg z|\le\pi$ as $z\to\infty$, taking $N=2$ in $\varepsilon_{\alpha,\sigma}(\cdot)$, it yields*

$$\varepsilon_{\alpha,\sigma}(z)=-\frac{z^{-1}}{\Gamma(\sigma-\alpha)}+O(|z|^{-2}),\quad \text{as } z\to\infty.$$

It is notice that if $\sigma-\alpha=-n$, $(n=0,1,2,\cdots)$, and taking into account the well-known property of the Gamma function

$$\frac{1}{\Gamma(-n)}=0,\quad n=0,1,2,\cdots,$$

then there holds

$$|E_{\alpha,\sigma}(z)|\le\frac{C}{1+|z|^2},\quad z\in\mathbb{C},\ \omega\le|\arg z|\le\pi.$$

See [22, Remark 2.2] *for example.*

An entire function closely associated with Mittag-Leffler function is the Wright type function.

Definition 5. *An entire function $\zeta_\alpha(\cdot):\mathbb{C}\to\mathbb{C}$ is called a Wright type function for $\alpha\in(0,1)$, given by*

$$\zeta_\alpha(z)=\sum_{n=0}^{\infty}\frac{(-z)^n}{n!\Gamma(1-\alpha(n+1))},\quad z\in\mathbb{C}.$$

From the definition of Wright type function, the following properties hold.

Lemma 6. [41] *For any $\alpha\in(0,1)$, there hold*

(i): $\zeta_\alpha(\upsilon)\ge0$, *where* $\upsilon\in[0,\infty)$;

(ii): $\int_0^\infty\zeta_\alpha(\upsilon)e^{-z\upsilon}d\upsilon=E_{\alpha,1}(-z)$, $z\in\mathbb{C}$;

(iii): $\int_0^\infty\alpha\upsilon\zeta_\alpha(\upsilon)e^{-z\upsilon}d\upsilon=E_{\alpha,\alpha}(-z)$, $z\in\mathbb{C}$;

(iv): *for $-1<\delta<\infty$, it yields*

$$\int_0^\infty\upsilon^\delta\zeta_\alpha(\upsilon)d\upsilon=\frac{\Gamma(1+\delta)}{\Gamma(1+\alpha\delta)}.$$

We mention that the following two identities from Lemma 6 (iv) will be useful throughout this paper.

$$\int_0^\infty\zeta_\alpha(\upsilon)d\upsilon=1,\quad\int_0^\infty\alpha\upsilon\zeta_\alpha(\upsilon)d\upsilon=\frac{1}{\Gamma(\alpha)}.\tag{3}$$

Definition 7. *Let $\alpha > 0$. For any $0 \le s < t \le T$, a function $x \in L^1(s,T;X)$ denotes the Riemann-Liouville fractional integral of order α by*

$$ {}_sJ_t^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_s^t (t-\tau)^{\alpha-1} x(\tau) d\tau, \quad t > s. $$

Definition 8. *Let $\alpha > 0$. For any $0 \le s < t \le T$, a function $x \in L^1(s,T;X)$ with ${}_sJ_t^{n-\alpha} x \in W^{n,1}(s,T;X)$ is called the Riemann-Liouville fractional derivative of order α by*

$$ {}_s^LD_t^{\alpha} x(t) = \frac{d^n}{dt^n} {}_sJ_t^{n-\alpha} x(t), \quad t > s, $$

where $n = [\alpha]+1$, $[\alpha]$ means the integer part of α.

It is obvious that ${}_sJ_t^{\alpha} x(t) = x(t)$ as $\alpha \to 0$ as well as ${}_s^LD_t^{\alpha} x(t) = x'(t)$ as $\alpha \to 1$. We recall that the definition of Caputo's fractional derivative of order $\alpha \in (0,2)$, that is, for a function $x \in C^n([s,T],X)$ and ${}_sJ_t^{n-\alpha} x \in W^{n,1}(s,T;X)$, $n = [\alpha]+1$, where $W^{n,1}(s,T;X)$ stands for the space of functions x such that $x^{(n)} \in L^1(s,T;X)$, the derivative is given by $\partial_t^{\alpha} x(t) = {}_s^LD_t^{\alpha}\Big(x(t) - \sum_{k=0}^{n-1} \frac{x^k(s)}{k!}(t-s)^k\Big)$, in which $\partial_t^{\alpha} x$ and ${}_s^LD_t^{\alpha} x$ are equivalent if $x^{(k)}(s) = 0$ for $k = 0, \cdots, [\alpha]$. Using the properties of fractional calculus and taking the Mittag-Leffler functions of item by item integration for $\alpha \in (0,1)$, $\omega \in \mathbb{C}$, we have

$$ \begin{aligned} {}_sJ_t^{1-\alpha}((t-s)^{\alpha-1} E_{\alpha,\alpha}(\omega(t-s)^{\alpha})) &= E_{\alpha,1}(\omega(t-s)^{\alpha}), \ \ t > s, \\ {}_s^LD_t^{\alpha}((t-s)^{\alpha-1} E_{\alpha,\alpha}(\omega(t-s)^{\alpha})) &= \omega(t-s)^{\alpha-1} E_{\alpha,\alpha}(\omega(t-s)^{\alpha}), \ \ t > s. \end{aligned} \tag{4} $$

3. A linear problem

In this section, we discuss some properties for a linear problem of (1). Next, we get an existence result of problem (1) when operator $A(\cdot) \in \mathscr{B}(X)$. In particular, there is no assumption on the density of the domain of $A(t)$ for all $t \in [0,T]$.

Lemma 9. *The Cauchy problem (1) is equivalent to the integral equation*

$$ x(t) = g_{\alpha}(t-s)x_s + \int_s^t g_{\alpha}(t-\tau)(-A(\tau)x(\tau) + f(\tau)) d\tau, \tag{5} $$

for $0 \le s < t \le T$ provided that (5) exists.

Proof. Similarly to [41, Theorem 4.1], the proof is easy to check, so we omit it here. □

Definition 10. *An X-valued function x is called a classical solution of problem (1) if x is continuous on $(s,T]$ with ${}_s^LD_t^{\alpha} x \in C((s,T],X)$, and it belongs to $D(A(t))$ for every $t \in [s,T]$ satisfying problem (1).*

3.1. *Linear bounded operator.*

Based on Lemma 9, we have an immediate result associated with the linear bounded operator $A(\cdot)$. For this purpose, we introduce a Banach space $C_{\alpha}([s,T],X)$ that is the weighted continuous function space defined by

$$ C_{\alpha}([s,T],X) := \Big\{ x \in C((s,T],X) : \lim_{t \to s+} (t-s)^{1-\alpha} \|x(t)\| \text{ exists and is finite} \Big\}, $$

equipped with the norm $\|x\|_{\alpha} = \sup_{t \in [s,T]} (t-s)^{1-\alpha} \|x(t)\|$.

Theorem 11. *Let X be a Banach space and let $A(t)$ be a bounded linear operator on X for every $s \le t \le T$. If $f \in C([s,T],X)$ and the map $t \mapsto A(t)$ is continuous in the uniform operator topology, then for every $x_0 \in X$, problem (1) has a unique classical solution x in $C_\alpha([s,T],X)$.*

Proof. Clearly, $C_\alpha([s,T],X)$ is a subset of $C((s,T],X)$, it suffices to check the conclusion in $C_\alpha([s,T],X)$. We will use the Picard method to establish this result. Let $\rho_A = \sup\{\|A(t)\|_{\mathscr{B}},\ t \in [s,T]\}$ and define a mapping $\mathscr{T}$ on $C_\alpha([s,T],X)$ by

$$(\mathscr{T}x)(t) = g_\alpha(t-s)x_s + \int_s^t g_\alpha(t-\tau)(-A(\tau)x(\tau)+f(\tau))d\tau. \tag{6}$$

We first check that $\mathscr{T}$ maps $C_\alpha([s,T],X)$ into itself. For $x \in C_\alpha([s,T],X)$, one has

$$\|(\mathscr{T}x)(t)\| \le g_\alpha(t-s)\|x_s\| + \int_s^t g_\alpha(t-\tau)(\rho_A(\tau-s)^{\alpha-1}\|x\|_\alpha + \|f(\tau)\|)d\tau. \tag{7}$$

Using (2), we deduce that

$$\|\mathscr{T}x\|_\alpha \le \frac{1}{\Gamma(\alpha)}\|x_s\| + \frac{\rho_A\Gamma(\alpha)}{\Gamma(2\alpha)}(T-s)^\alpha\|x\|_\alpha + \frac{(T-s)}{\Gamma(\alpha+1)}\|f\|_\infty,$$

where $\|f\|_\infty = \sup\{\|f(t)\|,\ t \in [s,T]\}$ is the norm of $C([s,T];X)$, and then $\|\mathscr{T}x\|_\alpha < \infty$. Moreover, we know that $g_\alpha(t-s)x_s$ belongs to $x \in C_\alpha([s,T],X)$, by the boundedness of operator $A(\cdot)$, it is easy to verify that $\mathscr{T}x \in C_\alpha([s,T],X)$. Additionally, by induction from (2) and (6) we have

$$\|(\mathscr{T}^n x)(t) - (\mathscr{T}^n y)(t)\| \le \frac{\rho_A^n\Gamma(\alpha)}{\Gamma((n+1)\alpha)}(t-s)^{(n+1)\alpha-1}\|x-y\|_\alpha,$$

therefore,

$$\|\mathscr{T}^n x - \mathscr{T}^n y\|_\alpha \le \frac{\rho_A^n\Gamma(\alpha)}{\Gamma((n+1)\alpha)}(T-s)^{n\alpha}\|x-y\|_\alpha.$$

Since there exists a positive integer $\hat{n}$ enough large such that

$$\frac{\rho_A^{\hat{n}}\Gamma(\alpha)}{\Gamma((\hat{n}+1)\alpha)}(t-s)^{\hat{n}\alpha} \le \frac{\rho_A^{\hat{n}}\Gamma(\alpha)}{\Gamma((\hat{n}+1)\alpha)}(T-s)^{\hat{n}\alpha} < 1,$$

by a well known generalization of the Banach contraction principle, $\mathscr{T}$ has a unique fixed point x^* in $C_\alpha([s,T],X)$ for which

$$x^*(t) = g_\alpha(t-s)x_s + \int_s^t g_\alpha(t-\tau)(-A(\tau)x^*(\tau)+f(\tau))d\tau. \tag{8}$$

Since x^* is continuous, from Lemma 9, ${}_sJ_t^{1-\alpha}x^*$ exists and we obtain that ${}_sJ_t^{1-\alpha}x^*$ is absolutely continuous. Thus ${}_sJ_t^{1-\alpha}x^*$ is differentiable, and taking its derivative yields

$${}_s^LD_t^\alpha x^*(t) + A(t)x^*(t) = f(t).$$

Moreover, ${}_sJ_t^{1-\alpha}x^*(t) = x_s$ as $t \to s$, and it is easy to derive that ${}_s^LD_t^\alpha x^* \in C((s,T],X)$ and $x^* \in D(A(t))$ for all $t \in [s,T]$ for every $x_s \in X$ by the same arguments. Hence, x^* is a classical solution of the Cauchy problem (1). Since every solution of (1) is also a solution of (8), the solution of (1) is unique. Thus, we get that $x^* \in C_\alpha([s,T],X)$ is a unique classical solution to problem (1). The proof is completed. □

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3.2. *Linear unbounded operators.* In this subsection, we shall discuss the case of linear unbounded operator, where $-A(t)$ is an infinitesimal generator of analytic semigroup $T_t(\varsigma)$ on the Banach space X for all $\varsigma \geq 0$ and every $t \in [0,T]$. To achieve to our goals, we will need the following assumptions, see e.g. [1]:

(P1): the domain $D(A(t)) = \mathscr{D}(\mathscr{A})$ of $A(t)$, $0 \leq t \leq T$ is dense in X, and it is independent of t;

(P2): for $t \in [0,T]$, the resolvent $R(\lambda;A(t))$ of $A(t)$ exists for all λ with $Re\lambda \leq 0$ and there is a constant $C > 0$ such that

$$\|R(\lambda;A(t))\|_{\mathscr{B}} \leq \frac{C}{|\lambda|+1}, \quad \text{for } Re\lambda \leq 0,\ t \in [0,T]; \tag{9}$$

(P3): (Acquistapace-Terremi's condition) there exist two constants $L > 0$ and $0 < \vartheta \leq 1$ such that

$$\|(A(t)-A(\tau))A(r)^{-1}\|_{\mathscr{B}} \leq L|t-\tau|^{\vartheta}, \quad \text{for } \tau,t,r \in [0,T]. \tag{10}$$

Note that, $(P2)$ and the density of $\mathscr{D}(\mathscr{A})$ in X imply that for every $t \in [0,T]$, $-A(t)$ is the infinitesimal generator of a uniformly bounded analytic semigroup $T_t(\varsigma)$, $\varsigma \geq 0$, satisfying

$$\|T_t(\varsigma)\|_{\mathscr{B}} \leq C, \quad \text{for } \varsigma \geq 0, \quad \text{and} \quad \|A(t)T_t(\varsigma)\|_{\mathscr{B}} \leq C\varsigma^{-1}, \quad \text{for } \varsigma > 0. \tag{11}$$

From $(P2)$, it also yields that there exists an angle $\theta \in (0,\pi/2)$ such that

$$\rho(A(t)) \supset \Sigma = \{\lambda \in \mathbb{C}\setminus\{0\}:\ \theta \leq |\arg\lambda| \leq \pi\} \cup \{0\},$$

and (9) holds as

$$\|R(\lambda;A(t))\|_{\mathscr{B}} \leq \frac{\widetilde{C}}{|\lambda|+1}, \quad \text{for } \lambda \in \Sigma,\ t \in [0,T],$$

possibly with a different constant $\widetilde{C} > 0$.

Without losing generality, from the results of [31] the representation of semigroup in a Dunford integral form is given by

$$T_t(\varsigma) = \frac{1}{2\pi i}\int_{\mathscr{C}} e^{-\varsigma z}R(z;A(t))dz,$$

where $\mathscr{C}$ is a smooth path in Σ connecting $+\infty e^{-i\theta}$ to $+\infty e^{i\theta}$ for some $\theta \in (0,\pi/2)$.

For $0 \leq t \leq T$, we introduce two operators

$$\phi_t(\varsigma) = \int_0^\infty \zeta_\alpha(\upsilon)T_t(\varsigma^\alpha\upsilon)d\upsilon,\ \varsigma \geq 0; \quad \psi_t(\varsigma) = \varsigma^{\alpha-1}\int_0^\infty \alpha\upsilon\zeta_\alpha(\upsilon)T_t(\varsigma^\alpha\upsilon)d\upsilon,\ \varsigma > 0.$$

The two operators we introduced here will indeed be instrumental in constructing fundamental solutions. We will directly discuss some properties of $\phi_t(\varsigma)$ and $\psi_t(\varsigma)$ that are frequently used throughout this paper.

Lemma 12. *Let $(P1)$-$(P2)$ be satisfied, for each $t \in [0,T]$, operator families $\{\phi_t(\varsigma)\}_{\varsigma\geq 0}$, $\{\psi_t(\varsigma)\}_{\varsigma>0}$ are linear and bounded, i.e.,*

$$\|\phi_t(\varsigma)\|_{\mathscr{B}} \leq C, \text{ for } \varsigma \geq 0, \quad \text{and} \quad \|\psi_t(\varsigma)\|_{\mathscr{B}} \leq Cg_\alpha(\varsigma), \text{ for } \varsigma > 0.$$

Proof. Since $-A(t)$ is the infinitesimal generator of an analytic semigroup $T_t(\varsigma)$, $\varsigma \geq 0$, the operator families $\{\phi_t(\varsigma)\}_{t\in[0,T],\varsigma\geq 0}$, $\{\psi_t(\varsigma)\}_{t\in[0,T],\varsigma>0}$ are linear clearly. Moreover, for any $x \in X$, from (3) and (11), we have

$$\|\phi_t(\varsigma)\|_{\mathscr{B}} \leq \int_0^\infty \zeta_\alpha(\upsilon)\|T_t(\varsigma^\alpha\upsilon)\|_{\mathscr{B}} d\upsilon \leq C\int_0^\infty \zeta_\alpha(\upsilon)d\upsilon,\ \varsigma \geq 0.$$

From the same arguments, we get

$$\|\psi_t(\varsigma)\|_{\mathscr{B}} \leq \varsigma^{\alpha-1}\int_0^\infty \alpha\upsilon\zeta_\alpha(\upsilon)\|T_t(\varsigma^\alpha\upsilon)\|_{\mathscr{B}} d\upsilon \leq C\varsigma^{\alpha-1}\int_0^\infty \alpha\upsilon\zeta_\alpha(\upsilon)d\upsilon,\ \varsigma > 0.$$

Thus, the desired results are satisfied. □

Lemma 13. *Let* $(P1)$*-*$(P2)$ *be satisfied, for each* $t \in [0,T]$*, there hold on* $\mathscr{B}(X)$

$$\begin{aligned}\phi_t(\varsigma) &= \frac{1}{2\pi i}\int_{\mathscr{C}} E_{\alpha,1}(-z\varsigma^\alpha)R(z;A(t))dz, \quad \varsigma \geq 0,\\ \psi_t(\varsigma) &= \varsigma^{\alpha-1}\frac{1}{2\pi i}\int_{\mathscr{C}} E_{\alpha,\alpha}(-z\varsigma^\alpha)R(z;A(t))dz, \quad \varsigma > 0.\end{aligned}$$

Proof. From the Dunford integral of $T_t(\cdot)$, it follows that

$$\phi_t(\varsigma) = \int_0^\infty \zeta_\alpha(\upsilon)T_t(\varsigma^\alpha\upsilon)d\upsilon = \frac{1}{2\pi i}\int_{\mathscr{C}}\int_0^\infty \zeta_\alpha(\upsilon)e^{-z\varsigma^\alpha\upsilon}R(z;A(t))d\upsilon dz.$$

It is clear from Lemma 12, by virtue of Lemma 6 (ii), the Fibini theorem shows that

$$\frac{1}{2\pi i}\int_{\mathscr{C}}\int_0^\infty \zeta_\alpha(\upsilon)e^{-z\varsigma^\alpha\upsilon}R(z;A(t))d\upsilon dz = \frac{1}{2\pi i}\int_{\mathscr{C}} E_\alpha(-z\varsigma^\alpha)R(z;A(t))dz.$$

Therefore, the first identity is proved. On the other hand, for $\varsigma > 0$, a similar way is employed to check that

$$\begin{aligned}\psi_t(\varsigma) &= \varsigma^{\alpha-1}\frac{1}{2\pi i}\int_{\mathscr{C}}\int_0^\infty \alpha\upsilon\zeta_\alpha(\upsilon)e^{-z\varsigma^\alpha\upsilon}R(z;A(t))d\upsilon dz\\ &= \varsigma^{\alpha-1}\frac{1}{2\pi i}\int_{\mathscr{C}} E_{\alpha,\alpha}(-z\varsigma^\alpha)R(z;A(t))dz.\end{aligned}$$

Hence, the another identity is obtained. The proof is completed. □

Remark 14. *Let* $X = L^p[a,b]$*,* $1 \leq p \leq \infty$*, for some constant* $k > 1$*, let* $A(t)$ *be a multiplication operator defined by*

$$A(t)x(s) = -|s-t|^{-k}x(s).$$

From [17], the assumptions (P1)-(P3) hold, and therefore $-A(t)$ *is an infinitesimal generator of analytic semigroup* $T_t(\varsigma) = e^{-\varsigma A(t)}$ $(\varsigma \geq 0, t \in [a,b])$ *of bounded linear operators on X. Hence, from the properties of Wright-type function, operators* $\phi_t(\cdot)$*,* $\psi_t(\cdot)$ *can be expressed as the corresponding Mittag-Leffler functions by Lemma 6 (ii)-(iii), i.e.,*

$$\begin{aligned}\phi_t(\varsigma) &= \int_0^\infty \zeta_\alpha(\upsilon)e^{-\varsigma^\alpha\upsilon A(t)}d\upsilon =: E_{\alpha,1}(-\varsigma^\alpha A(t)),\\ \psi_t(\varsigma) &= \varsigma^{\alpha-1}\int_0^\infty \alpha\upsilon\zeta_\alpha(\upsilon)e^{-\varsigma^\alpha\upsilon A(t)}d\upsilon =: \varsigma^{\alpha-1}E_{\alpha,\alpha}(-\varsigma^\alpha A(t)).\end{aligned}$$

Lemma 15. *Let $(P1)$-$(P2)$ be satisfied, for each $t \in [0,T]$, there hold on $\mathscr{B}(X)$*

$$\phi_t(r-s) = {}_sJ_r^{1-\alpha}\psi_t(r-s), \quad r \in [s,T]; \quad \frac{d}{dr}\phi_t(r) = -A(t)\psi_t(r),\ r \in (0,T].$$

Proof. Lemma 13 implies that ${}_sJ_r^{1-\alpha}\psi_t(r-s)$ is bounded on $\mathscr{B}(X)$, by Fubini's theorem, it is equal to

$$\frac{1}{\Gamma(1-\alpha)}\frac{1}{2\pi i}\int_{\mathscr{C}}\int_s^r (r-\tau)^{-\alpha}(\tau-s)^{\alpha-1}E_{\alpha,\alpha}(-z(\tau-s)^{\alpha})R(z;A(t))d\tau dz.$$

By virtue of identity in Lemma 4, one has

$${}_sJ_r^{1-\alpha}(r-s)^{\alpha-1}E_{\alpha,\alpha}(-z(r-s)^{\alpha}) = E_{\alpha,1}(-z(r-s)^{\alpha}),$$

which implies

$${}_sJ_r^{1-\alpha}\psi_t(r-s) = \frac{1}{2\pi i}\int_{\mathscr{C}} E_{\alpha}(-z(r-s)^{\alpha})R(z;A(t))dz = \phi_t(r-s).$$

On the other hand, by the differentiability of analytic semigroup, namely $\frac{d}{dr}T_t(r) = -A(t)T_t(r)$ for $r > 0$, we have

$$\begin{aligned}\frac{d}{dr}\phi_t(r) &= \int_0^{\infty}\zeta_{\alpha}(\upsilon)\frac{d}{dr}T_t(r^{\alpha}\upsilon)d\upsilon\\ &= -r^{\alpha-1}\int_0^{\infty}\alpha\upsilon\zeta_{\alpha}(\upsilon)A(t)T_t(r^{\alpha}\upsilon)d\upsilon = -A(t)\psi_t(r).\end{aligned}$$

The proof is completed. □

Lemma 16. *Let $(P1)$-$(P3)$ be satisfied, the following statements are true*

(i) *for $\eta \in (0,T]$, $\varsigma, t_1, t_2 \in [0,T]$, there holds*

$$\|(A(t_1)-A(t_2))\psi_{\varsigma}(\eta)\|_{\mathscr{B}} \le \frac{C}{\eta}|t_1-t_2|^{\vartheta};$$

(ii) *for $\eta \in (0,T]$, $t, \varsigma_1, \varsigma_2 \in [0,T]$, there holds*

$$\|A(t)(\psi_{\varsigma_1}(\eta)-\psi_{\varsigma_2}(\eta))\|_{\mathscr{B}} \le \frac{C}{\eta}|\varsigma_1-\varsigma_2|^{\vartheta};$$

(iii) *for $0 < \eta_1 < \eta_2 \le T$, $\varsigma \in [0,T]$, there holds*

$$\|A(\varsigma)(\psi_{\varsigma}(\eta_2)-\psi_{\varsigma}(\eta_1))\|_{\mathscr{B}} \le \frac{C}{\eta_1\eta_2}|\eta_2-\eta_1|;$$

(iv) *for $0 < \eta_1 < \eta_2 \le T$, there holds*

$$\|\psi_{\varsigma}(\eta_2)-\psi_{\varsigma}(\eta_1)\|_{\mathscr{B}} \le C|\eta_2^{\alpha-1}-\eta_1^{\alpha-1}|.$$

Moreover, $A(t)\psi_{\varsigma}(\eta) \in \mathscr{B}(X)$ for $\eta \in (0,T]$, $t, \varsigma \in [0,T]$ and $\|A(t)\psi_{\varsigma}(\eta)\|_{\mathscr{B}} \le C\eta^{-1}$ for $\eta \in (0,T]$.

Proof. From (3), (10) and (11), we have

$$\begin{aligned}\|(A(t_1)-A(t_2))\psi_\varsigma(\eta)\|_{\mathscr{B}} \leq&\|(A(t_1)-A(t_2))A(\varsigma)^{-1}\|_{\mathscr{B}}\|A(\varsigma)\psi_\varsigma(\eta)\|_{\mathscr{B}}\\ \leq& L|t_1-t_2|^{\vartheta}\eta^{\alpha-1}\int_0^\infty \alpha\upsilon\zeta_\alpha(\upsilon)\|A(\varsigma)T_\varsigma(\eta^\alpha\upsilon)\|_{\mathscr{B}}d\upsilon\\ \leq& C|t_1-t_2|^{\vartheta}\eta^{-1},\end{aligned}$$

which proves the first inequality.

Note that from $(P2)$ it follows that $\|A(t)R(\lambda;A(t))\|_{\mathscr{B}} \leq C+1$ for $0\leq t\leq T$. Since $0\in\rho(A(\varsigma))$ for $\varsigma\in[0,T]$, from (10) and the trigonometric inequality, it yields

$$\|A(t)A(\varsigma)^{-1}\|_{\mathscr{B}} \leq \|(A(t)-A(\varsigma))A(\varsigma)^{-1}\|_{\mathscr{B}}+1\leq LT^{\vartheta}+1, \quad \text{for } 0\leq t\leq T,$$

therefore, we have

$$\|A(t)(R(\lambda;A(\varsigma_1))-R(\lambda;A(\varsigma_2)))\|_{\mathscr{B}}\leq \widetilde{L}|\varsigma_1-\varsigma_2|^{\vartheta}, \tag{12}$$

where $\widetilde{L}=(LT^{\vartheta}+1)(C+1)^2L>0$. From Lemma 13 we get the following representation

$$\psi_{\varsigma_1}(\eta)-\psi_{\varsigma_2}(\eta)=\eta^{\alpha-1}\frac{1}{2\pi i}\int_{\mathscr{C}}E_{\alpha,\alpha}(-z\eta^\alpha)(R(z;A(\varsigma_1))-R(z;A(\varsigma_2)))dz.$$

Noting that $\mathscr{C}$ is a smooth path in Σ running from $+\infty e^{-i\theta}$ to $+\infty e^{i\theta}$ for some $\theta\in(0,\pi/2)$, and the following relation also holds

$$-z\eta^\alpha\in\{z\in\mathbb{C}:\ \omega\leq|\arg z|\leq\pi\}\cup\{0\},$$

substituting $\omega\in(\alpha\pi/2,\pi/2)$ in Lemma 2. From the analyticity of Mittag-Leffler function and the resolvent in $\mathscr{C}$, by the Cauchy integral theorem, Let $\delta=\eta^{-\alpha}>0$, we may shift the path of integration $\mathscr{C}$ to $\mathscr{C}'=\mathscr{C}_1\cup\mathscr{C}_2\cup\mathscr{C}_3$ by

$$\mathscr{C}_1=\{re^{i\theta},\ r\geq\delta\},\ \ \mathscr{C}_2=\{\delta e^{i\varphi},\ \theta\leq|\varphi|\leq\pi\},\ \ \mathscr{C}_3=\{re^{-i\theta},\ r\geq\delta\}. \tag{13}$$

Firstly, in view of (12) we get

$$\begin{aligned}&\left\|\frac{1}{2\pi i}\int_{\mathscr{C}_1\cup\mathscr{C}_3}E_{\alpha,\alpha}(-z\eta^\alpha)A(t)(R(z;A(\varsigma_1))-R(z;A(\varsigma_2)))dz\right\|_{\mathscr{B}}\\ \leq& C|\varsigma_1-\varsigma_2|^{\vartheta}\int_{\mathscr{C}_1\cup\mathscr{C}_3}|E_{\alpha,\alpha}(-z\eta^\alpha)||dz|.\end{aligned}$$

This means from Remark 4 that

$$\int_{\mathscr{C}_1\cup\mathscr{C}_3}|E_{\alpha,\alpha}(-z\eta^\alpha)||dz|\leq\int_{\mathscr{C}_1\cup\mathscr{C}_3}\frac{C}{1+|z\eta^\alpha|^2}|dz|.$$

Note that

$$\begin{aligned}\int_{\mathscr{C}_1\cup\mathscr{C}_3}\frac{1}{1+|z\eta^\alpha|^2}|dz|\leq&\int_\delta^\infty\frac{1}{1+|re^{i\theta}\eta^\alpha|^2}dr+\int_\delta^\infty\frac{1}{1+|re^{-i\theta}\eta^\alpha|^2}dr\\ =&\int_\delta^\infty\frac{2}{1+r^2\eta^{2\alpha}}dr.\end{aligned}$$

By a simple calculation, due to the integral $\int_0^\infty \frac{1}{1+z^2}dz = \pi/2$, it yields

$$\int_{\mathscr{C}_1\cup\mathscr{C}_3} |E_{\alpha,\alpha}(-z\eta^\alpha)|\,|dz| \le C\eta^{-\alpha}\int_{\delta\eta^\alpha}^\infty \frac{1}{1+r^2}dr \le C\eta^{-\alpha}.$$

Secondly, we check the integral in $\mathscr{C}_2$. Since $e^{Re(-\delta e^{i\varphi}\eta^\alpha)^{\frac{1}{\alpha}}} = e^{\cos((\pi-\varphi)/\alpha)\delta^{\frac{1}{\alpha}}\eta}$, by virtue of Lemma 2 and Lemma 3, we see that

$$|E_{\alpha,\alpha}(-z\eta^\alpha)| \le C(1+|\delta\eta^\alpha|)^{\frac{1-\alpha}{\alpha}}e^{\cos((\pi-\varphi)/\alpha)\delta^{\frac{1}{\alpha}}\eta} + \frac{C}{1+|\delta\eta^\alpha|} \le C2^{\frac{1-\alpha}{\alpha}}e + \frac{C}{2}, \tag{14}$$

for $z=\delta e^{i\varphi}$ and $\theta \le |\varphi| \le \pi$ with $\delta = \eta^{-\alpha} > 0$. Hence, it follows from (12) and (14) that

$$\begin{aligned}
&\left\|\frac{1}{2\pi i}\int_{\mathscr{C}_2} E_{\alpha,\alpha}(-z\eta^\alpha)A(t)(R(z;A(\varsigma_1)) - R(z;A(\varsigma_2)))dz\right\|_{\mathscr{B}} \\
\le&\widetilde{L}|\varsigma_1-\varsigma_2|^\vartheta \int_{\mathscr{C}_2} |E_{\alpha,\alpha}(-z\eta^\alpha)|\,|dz| \\
\le&C|\varsigma_1-\varsigma_2|^\vartheta \left(\int_\theta^\pi + \int_{-\pi}^{-\theta}\right)\delta d\varphi \\
\le&C|\varsigma_1-\varsigma_2|^\vartheta \eta^{-\alpha}.
\end{aligned}$$

We thus imply that

$$\left\|\frac{1}{2\pi i}\int_{\mathscr{C}} E_{\alpha,\alpha}(-z\eta^\alpha)A(t)(R(z;A(\varsigma_1)) - R(z;A(\varsigma_2)))dz\right\|_{\mathscr{B}} \le C|\varsigma_1-\varsigma_2|^\vartheta\eta^{-\alpha}.$$

Consequently, we have

$$\|A(t)\psi_{\varsigma_1}(\eta) - A(t)\psi_{\varsigma_2}(\eta)\|_{\mathscr{B}} \le C|\varsigma_1-\varsigma_2|^\vartheta\eta^{-1}.$$

Let us check (iii). By Lemma 13, for $0<\eta_1\le\eta_2\le T$, we have

$$\psi_\varsigma(\eta_2) - \psi_\varsigma(\eta_1) = \frac{1}{2\pi i}\int_{\mathscr{C}} (\eta_2^{\alpha-1}E_{\alpha,\alpha}(-z\eta^\alpha) - \eta_1^{\alpha-1}E_{\alpha,\alpha}(-z\eta_1^\alpha))R(z;A(\varsigma))dz.$$

Using the identity $\frac{d}{d\eta}(\eta^{\alpha-1}E_{\alpha,\alpha}(-z\eta^\alpha)) = \eta^{\alpha-2}E_{\alpha,\alpha-1}(-z\eta^\alpha)$, it yields

$$A(\varsigma)\psi_\varsigma(\eta_2) - A(\varsigma)\psi_\varsigma(\eta_1) = \frac{1}{2\pi i}\int_{\mathscr{C}}\int_{\eta_1}^{\eta_2} \eta^{\alpha-2}E_{\alpha,\alpha-1}(-z\eta^\alpha)A(\varsigma)R(z;A(\varsigma))d\eta dz.$$

Hence, in view of the analyticity of Mittag-Liffler function, we imply that

$$\left\|\int_{\mathscr{C}} E_{\alpha,\alpha-1}(-z\eta^\alpha)A(\varsigma)R(z;A(\varsigma))dz\right\|_{\mathscr{B}} \le C\int_{\mathscr{C}'} |E_{\alpha,\alpha-1}(-z\eta^\alpha)||dz|.$$

By the similar proof that of (ii), one can check that

$$\int_{\mathscr{C}'} |E_{\alpha,\alpha-1}(-z\eta^\alpha)||dz| = \int_{\mathscr{C}_1\cup\mathscr{C}_2\cup\mathscr{C}_3} |E_{\alpha,\alpha-1}(-z\eta^\alpha)||dz| \le C\eta^{-\alpha},$$

which implies that

$$\left\|\int_{\mathscr{C}} E_{\alpha,\alpha-1}(-z\eta^{\alpha})A(\varsigma)R(z;A(\varsigma))dz\right\|_{\mathscr{B}} \le C\eta^{-\alpha}.$$

Therefore, we have

$$\|A(\varsigma)\psi_{\varsigma}(\eta_2)-A(\varsigma)\psi_{\varsigma}(\eta_1)\|_{\mathscr{B}} \le C\int_{\eta_1}^{\eta_2}\eta^{-2}d\eta = \frac{C}{\eta_1\eta_2}|\eta_2-\eta_1|.$$

Let us check (iv). From (iii), it follows that

$$\psi_{\varsigma}(\eta_2)-\psi_{\varsigma}(\eta_1) = \frac{1}{2\pi i}\int_{\mathscr{C}}\int_{\eta_1}^{\eta_2}\eta^{\alpha-2}E_{\alpha,\alpha-1}(-z\eta^{\alpha})R(z;A(\varsigma))d\eta dz.$$

Let $\eta>0$ and $\delta=\eta^{-\alpha}>0$ be fixed, by the same way in (14), we get $|E_{\alpha,\alpha-1}(-z\eta^{\alpha})|\le C$, for $z\in\mathscr{C}_2$. In view of the Cauchy integral theorem and the analyticity of Mittag-Liffler function, similarly to the proof of (ii), by Lemma 2, we have

$$\int_{\mathscr{C}'}|E_{\alpha,\alpha-1}(-z\eta^{\alpha})|/|z|\,|dz| \le \int_{\delta}^{\infty}\frac{2C}{r(1+r\eta^{\alpha})}dr + C\left(\int_{\theta}^{\pi}+\int_{-\pi}^{-\theta}\right)d\varphi.$$

Due to the integral $\int_1^{\infty}\frac{1}{z(1+z)}dz=\ln 2$, it yields

$$\int_{\mathscr{C}'}|E_{\alpha,\alpha-1}(-z\eta^{\alpha})|/|z|\,|dz| \le 2C\ln 2+2C(\pi-\theta).$$

This deduces that

$$\|\psi_{\varsigma}(\eta_2)-\psi_{\varsigma}(\eta_1)\|_{\mathscr{B}} \le C\int_{\eta_1}^{\eta_2}\eta^{\alpha-2}d\eta = C|\eta_2^{\alpha-1}-\eta_1^{\alpha-1}|.$$

Since $T_{\varsigma}(\eta)$ is an analytic semigroup for $\eta\ge 0$, it follows that $A(t)T_{\varsigma}(\eta)$ also is a bounded linear operator for $t,\varsigma\in[0,T]$, $\eta\in(0,T]$. From (11) and Lemma 6 (iv), $A(t)\psi_{\varsigma}(\eta)\in\mathscr{B}(X)$ is obvious for $\eta\in(0,T]$, $t,\varsigma\in[0,T]$. The proof is completed. □

Lemma 17. *Let* $(P1)$-$(P3)$ *be satisfied, for each* $t\in[0,T]$*, there holds on* $\mathscr{B}(X)$

$$-\int_{\varsigma}^{t}A(t)\psi_t(t-\tau)d\tau = \phi_t(t-\varsigma)-I.$$

Proof. For every $\varepsilon>0$, since $T_t(\eta)$ is an analytic semigroup on X, we see that $T_t(0)=I$. Hence, by (3), we get that $\phi_t(0)=I$. By Lemma 15, integrating on term $\frac{d}{d\tau}\phi_t(\tau)$ from 0 to $t-\varsigma$ we have

$$\int_0^{t-\varsigma}\partial_{\tau}\phi_t(\tau)d\tau = \phi_t(t-\varsigma)-\phi_t(0) = \phi_t(t-\varsigma)-I.$$

On the other hand, by Lemma 15 again we have

$$\int_0^{t-\varsigma}\partial_{\tau}\phi_t(\tau)d\tau = -\int_0^{t-\varsigma}A(t)\psi_t(\tau)d\tau,$$

which means that

$$-\int_0^{t-\varsigma}A(t)\psi_t(\tau)d\tau = \phi_t(t-\varsigma)-I.$$

We thus get the desired result. □

Lemma 18. *Assume that*

(P4): *A function $\xi(\cdot,s)\in L^1(s,T;\mathscr{B}(X))\cap C((s,T];\mathscr{B}(X))$ such that the integral*

$$H_\xi(t,s):=\int_s^t \frac{\|\xi(t,s)-\xi(\tau,s)\|_{\mathscr{B}}}{t-\tau}d\tau,$$

exists on $\mathbb{R}_+$ for $t\in[s+\varepsilon,T]$ with any $\varepsilon>0$.

Then, there holds

$$ {}_s^LD_t^\alpha\int_s^t \psi_\tau(t-\tau)\xi(\tau,s)d\tau=\xi(t,s)-\int_s^t A(\tau)\psi_\tau(t-\tau)\xi(\tau,s)d\tau,\quad t\in(s,T].$$

Proof. Let

$$w_\xi(t,s):=\int_s^t \psi_\tau(t-\tau)\xi(\tau,s)d\tau,\ \ t>s.$$

Then w_ξ is well-defined on $(s,T]$. In fact, from the assumption of ξ and by Lemma 12, we have

$$\begin{aligned}\|w_\xi(t,s)\|_{\mathscr{B}}\le&\int_s^t\|\psi_\tau(t-\tau)(\xi(\tau,s)-\xi(t,s))\|_{\mathscr{B}}d\tau+\int_s^t\|\psi_\tau(t-\tau)\xi(t,s)\|_{\mathscr{B}}d\tau\\ \le&C\int_s^t g_\alpha(t-\tau)\,\|\xi(\tau,s)-\xi(t,s)\|_{\mathscr{B}}d\tau+C\int_s^t g_\alpha(t-\tau)\,\|\xi(t,s)\|_{\mathscr{B}}d\tau\\ \le&\left(CH_\xi(t,s)+C\|\xi(t,s)\|_{\mathscr{B}}\right)(t-s)^\alpha,\end{aligned}$$

which belongs to $L^1(s,T;\mathbb{R}_+)$. Hence, the integral ${}_sJ_t^{1-\alpha}w_\xi$ exists in view of the definition of Riemann-Liouville fractional integral for almost every $t\in[s,T]$, (see e.g. [7, Theorem 2.1]), and ${}_sJ_t^{1-\alpha}w_\xi$ itself is also an element in $L^1(s,T;\mathbb{R}_+)$. Then, from Fubini's theorem, we have

$$\begin{aligned}\int_s^t g_{1-\alpha}(t-\tau)w_\xi(\tau,s)d\tau=&\int_s^t\int_s^\tau g_{1-\alpha}(t-\tau)\psi_\upsilon(\tau-\upsilon)\xi(\upsilon,s)d\upsilon d\tau\\ =&\int_s^t\int_\upsilon^t g_{1-\alpha}(t-\tau)\psi_\upsilon(\tau-\upsilon)\xi(\upsilon,s)d\tau d\upsilon.\end{aligned}$$

Lemma 15 implies

$${}_sJ_t^{1-\alpha}w_\xi(t,s)=\int_s^t\phi_\upsilon(t-\upsilon)\xi(\upsilon,s)d\upsilon,$$

which means that

$${}_s^LD_t^\alpha\int_s^t\psi_\tau(t-\tau)\xi(\tau,s)d\tau=\frac{d}{dt}\int_s^t\phi_\upsilon(t-\upsilon)\xi(\upsilon,s)d\upsilon.$$

It suffices to show that

$$v_\xi(t)=\int_s^t\phi_\upsilon(t-\upsilon)\xi(\upsilon,s)d\upsilon,\quad 0\le s<t\le T,$$

is differentiable in t. Let $h>0$, by a direct calculation we have

$$\frac{v_\xi(t+h)-v_\xi(t)}{h}=\frac{1}{h}\int_t^{t+h}\phi_\upsilon(t+h-\upsilon)\xi(\upsilon,s)d\upsilon+\frac{1}{h}\int_s^t(\phi_\upsilon(t+h-\upsilon)-\phi_\upsilon(t-\upsilon))\xi(\upsilon,s)d\upsilon.$$

From $\phi_t(0)=I$ and $\phi_\upsilon(t+h-\upsilon)\xi(\upsilon,s)\in L^1(s,T;\mathscr{B}(X))$, from [6, Proposition 1.4.29] we have

$$\frac{1}{h}\int_t^{t+h}\phi_\upsilon(t+h-\upsilon)\xi(\upsilon,s)d\upsilon \to \phi_t(0)\xi(t,s)=\xi(t,s), \quad \text{as } h\to 0,$$

in $L^1(s,T;\mathscr{B}(X))$. Note that by using Lemma 15 again,

$$\frac{1}{h}(\phi_\upsilon(t+h-\upsilon)-\phi_\upsilon(t-\upsilon)) \to -A(\upsilon)\psi_\upsilon(t-\upsilon), \quad \text{as } h\to 0.$$

Therefore, by virtue of Lebesgue dominated convergence theorem, it remains to check that the integrand term $A(\cdot)\psi_\cdot(t-\cdot)\xi(\cdot,s)$ in $L^1(s,t;\mathscr{B}(X))$ for $t\in(s,T]$. We first have

$$\begin{aligned}-\int_s^t A(\upsilon)\psi_\upsilon(t-\upsilon)\xi(\upsilon,s)d\upsilon &= \int_s^t (A(t)\psi_t(t-\upsilon)-A(\upsilon)\psi_\upsilon(t-\upsilon))\xi(\upsilon,s)d\upsilon \\ &\quad -\int_s^t A(t)\psi_t(t-\upsilon)\xi(\upsilon,s)d\upsilon.\end{aligned} \tag{15}$$

The Lemma 16 (i)-(ii) imply that

$$\|A(t)\psi_t(t-\upsilon)-A(\upsilon)\psi_\upsilon(t-\upsilon)\|_{\mathscr{B}} \le C(t-\upsilon)^{\vartheta-1},$$

therefore the integral

$$\int_s^t (A(t)\psi_t(t-\upsilon)-A(\upsilon)\psi_\upsilon(t-\upsilon))\xi(\upsilon,s)d\upsilon, \quad \text{exists on } [s,t].$$

In addition, by virtue of Lemma 17 we have

$$\begin{aligned}\int_s^t A(t)\psi_t(t-\upsilon)\xi(\upsilon,s)d\upsilon &= \int_s^t A(t)\psi_t(t-\upsilon)(\xi(\upsilon,s)-\xi(t,s))d\upsilon + \int_s^t A(t)\psi_t(t-\upsilon)\xi(t,s)d\upsilon \\ &= \int_s^t A(t)\psi_t(t-\upsilon)(\xi(\upsilon,s)-\xi(t,s))d\upsilon + (I-\phi_t(t-s))\xi(t,s).\end{aligned}$$

By the inequality $\|A(\upsilon)\psi_\upsilon(t-\upsilon)\|_{\mathscr{B}} \le C(t-\upsilon)^{-1}$, it follows from the assumption of ξ that

$$\int_s^t A(t)\psi_t(t-\upsilon)(\xi(\upsilon,s)-\xi(t,s))d\upsilon, \quad \text{exists on } [s+\varepsilon,t],$$

for any $\varepsilon>0$. Moreover, $(I-\phi_t(t-s))\xi(t,s)\in L^1(s,T;\mathscr{B}(X))\cap C((s,T];\mathscr{B}(X))$ due to $\xi(t,s)\in L^1(s,T;\mathscr{B}(X))\cap C((s,T];\mathscr{B}(X))$. As a consequence, we get

$${}_s^LD_t^\alpha \int_s^t \psi_\tau(t-\tau)\xi(\tau,s)d\tau = \frac{d}{dt}v_\xi(t) = \xi(t,s) - \int_s^t A(\upsilon)\psi_\upsilon(t-\upsilon)\xi(\upsilon,s)d\upsilon.$$

The proof is completed. $\square$

Remark 19. *It is notice that Lemma 18 also holds for $\xi\in L^1(s,T;X)\cap C((s,T];X)$, specially if ξ is Hölder continuous function with type (γ,K), i.e., there exists constants $\gamma\in(0,1)$ and $K>0$ such that*

$$\|\xi(t,\varsigma)-\xi(\tau,\varsigma)\| \le K|t-\tau|^\gamma, \quad t,\tau\in[s,T],$$

then, (P4) is satisfied immediately. If $x\in\mathscr{D}(\mathscr{A})$, the conclusion in Lemma 18 just needs $\xi(\cdot,s)x\in L^1(s,T;X)\cap C((s,T];X)$ without condition (P4). Additionally, suppose that $A(\cdot)$ is a linear bounded operator on $\mathscr{B}(X)$ associated with $\xi(\cdot,s)\in L^1(s,T;\mathscr{B}(X))\cap C((s,T];\mathscr{B}(X))$, the conclusion of Lemma 18 holds without the condition (P4).

Remark 20. *The Hölder continuity of ξ is a quite useful assumption for studying the solvability of problem (1). It also satisfies assumption (P4). However, we have identified several special cases where the Hölder continuity does not hold assumption (P4) is still met. Specifically, operators such as $R_1(t,s)$ and $R(t,s)$, which are defined further below, can be employed to obtain the fundamental solutions of the problem (1) even without Hölder continuity.*

In the sequel, we start with a formal computation that will lead us to construct the fundamental solution. For $0 \leq s < t \leq T$, assume that operator $R(t,s)$ satisfies $(P4)$, set

$$U_\alpha(t,s) = \psi_s(t-s) + W(t,s), \tag{16}$$

where

$$W(t,s) = \int_s^t \psi_\tau(t-\tau)R(\tau,s)d\tau.$$

From Lemma 13 and Lemma 15, we first have

$${}^L_sD^\alpha_t \psi_s(t-s) = \frac{d}{dt} {}_sJ^{1-\alpha}_t \psi_s(t-s) = \frac{d}{dt}\int_0^\infty \zeta_\alpha(\upsilon)T_s((t-s)^\alpha \upsilon)d\upsilon = -A(s)\psi_s(t-s).$$

On the other hand, in view of Lemma 18, we have

$${}^L_sD^\alpha_t W(t,s) = {}^L_sD^\alpha_t \int_s^t \psi_\tau(t-\tau)R(\tau,s)d\tau = R(t,s) - \int_s^t A(\tau)\psi_\tau(t-\tau)R(\tau,s)d\tau.$$

From the following identity $A(t)U_\alpha(t,s) = A(t)\psi_s(t-s) + A(t)W(t,s)$, one finds that

$$\begin{aligned} {}^L_sD^\alpha_t U_\alpha(t,s) + A(t)U_\alpha(t,s) = & -A(s)\psi_s(t-s) + R(t,s) - \int_s^t A(\tau)\psi_\tau(t-\tau)R(\tau,s)d\tau \\ & + A(t)\psi_s(t-s) + A(t)W(t,s). \end{aligned}$$

Let $R_1(t,s) = -(A(t)-A(s))\psi_s(t-s)$. We thus obtain

$${}^L_sD^\alpha_t U_\alpha(t,s) + A(t)U_\alpha(t,s) = -R_1(t,s) + R(t,s) - \int_s^t R_1(t,\tau)R(\tau,s)d\tau.$$

According to above arguments, if $U_\alpha(t,s)$ is a solution of

$${}^L_sD^\alpha_t U_\alpha(t,s) + A(t)U_\alpha(t,s) = 0,$$

then the integral equation

$$R_1(t,s) + \int_s^t R_1(t,\tau)R(\tau,s)d\tau = R(t,s) \tag{17}$$

must be satisfied.

Remark 21. *We obverse that the identity (17) formally corresponds to that in* [31, pp150.(6.6)], *where the author considered the first order non-autonomous evolution equation. However, since the operator $\psi_s(t-s)$ in the fractional setting differs from $e^{-(t-s)A(t)}$ in the first order evolution operator, there are significant differences in their properties. Inspired by the technique of* [31], *we construct a solution for $R(t,s)$ as defined in (17). At the same time, it also proposes a reasonable explanation for constructing a fundamental solution $U_\alpha(t,s)$.*

For this purpose, we next establish some useful properties.

Lemma 22. *Let $(P1)$-$(P3)$ be satisfied, then the operator $R_1(t,s) = -(A(t)-A(s))\psi_s(t-s)$ is linear bounded on $0 \le s < t \le T$ in the following sense,*

$$\|R_1(t,s)\|_{\mathscr{B}} \le C(t-s)^{\vartheta-1}, \quad \text{for } 0 \le s < t \le T.$$

Moreover, operator $R_1(t,s)$ is continuous in the uniform operator topology on $0 \le s \le t-\varepsilon \le T$ for every $\varepsilon > 0$, i.e., for every $0 < \beta < \vartheta < 1$, there holds

$$\|R_1(t,s) - R_1(\sigma,s)\|_{\mathscr{B}} \le C(t-\sigma)^{\beta}(\sigma-s)^{\vartheta-\beta-1},$$

for all $0 \le s < \sigma \le t \le T$.

Proof. The linearity is obvious. From (3) and (11), it follows that

$$\|A(s)\psi_s(t-s)\|_{\mathscr{B}} \le (t-s)^{\alpha-1}\int_0^\infty \alpha\upsilon\zeta_\alpha(\upsilon)\|A(s)T_s((t-s)^\alpha\upsilon)\|_{\mathscr{B}}d\upsilon \le C(t-s)^{-1},$$

and therefore from (11) and

$$\|(A(t)-A(s))A(s)^{-1}\|_{\mathscr{B}}\|A(s)\psi_s(t-s)\|_{\mathscr{B}} \le C(t-s)^{\vartheta-1},$$

we obtain the desired inequality

$$\|R_1(t,s)\|_{\mathscr{B}} \le \|(A(t)-A(s))\psi_s(t-s)\|_{\mathscr{B}} \le C(t-s)^{\vartheta-1}, \quad \text{for } 0 \le s < t \le T.$$

Clearly, the $R_1(t,s) - R_1(\sigma,s)$ is equal to

$$-(A(t)-A(\sigma))\psi_s(t-s) - (A(\sigma)-A(s))(\psi_s(t-s)-\psi_s(\sigma-s)).$$

From (i) in Lemma 16 one has

$$\|(A(t)-A(\sigma))\psi_s(t-s)\|_{\mathscr{B}} \le C(t-\sigma)^{\vartheta}(\sigma-s)^{-1}.$$

By using $(P3)$ and (iii) in Lemma 16, we obverse that

$$\|(A(\sigma)-A(s))(\psi_s(t-s)-\psi_s(\sigma-s))\|_{\mathscr{B}} \le C(\sigma-s)^{\vartheta-2}(t-\sigma).$$

Additionally, by using (11), we also get that

$$\|(A(\sigma)-A(s))(\psi_s(t-s)-\psi_s(\sigma-s))\|_{\mathscr{B}} \le 2LC(\sigma-s)^{\vartheta-1}.$$

Therefore, in view of the interpolation of $\vartheta \in (0,1]$, we have

$$\begin{aligned}\|(A(\sigma)-A(s))(\psi_s(t-s)-\psi_s(\sigma-s))\|_{\mathscr{B}} &\le \left(C(\sigma-s)^{\vartheta-2}(t-\sigma)\right)^{\vartheta}\left(2LC(\sigma-s)^{\vartheta-1}\right)^{1-\vartheta} \\ &\le C(\sigma-s)^{-1}(t-\sigma)^{\vartheta},\end{aligned}$$

which implies that $\|R_1(t,s)-R_1(\sigma,s)\|_{\mathscr{B}} \le C(t-\sigma)^{\vartheta}(\sigma-s)^{-1}$.

On the other hand, we have

$$\begin{aligned}\|R_1(t,s)-R_1(\sigma,s)\|_{\mathscr{B}} \le& \|R_1(t,s)\|_{\mathscr{B}} + \|R_1(\sigma,s)\|_{\mathscr{B}} \\ \le& C\left((t-s)^{\vartheta-1}+(\sigma-s)^{\vartheta-1}\right) \le 2C(\sigma-s)^{\vartheta-1},\end{aligned}$$

for $t > \sigma$. Interpolating the two estimates for $\|R_1(t,s) - R_1(\sigma,s)\|_{\mathscr{B}}$ we find

$$\begin{aligned}\|R_1(t,s) - R_1(\sigma,s)\|_{\mathscr{B}} \leq& C\left((t-\sigma)^{\vartheta}(\sigma-s)^{-1}\right)^{\beta/\vartheta}\left((\sigma-s)^{\vartheta-1}\right)^{1-\beta/\vartheta}\\ \leq& C(t-\sigma)^{\beta}(\sigma-s)^{\theta-\beta-1}.\end{aligned}$$

The proof is completed. □

We begin with solving the integral equation (17) for $R(t,s)$. We know that $R_1(t,s)$ satisfies Lemma 22, then (17) can be solved by successive approximation as follows:

For $m \geq 1$ and $0 \leq s < t \leq T$, we define inductively

$$R_{m+1}(t,s) = \int_s^t R_1(t,\tau)R_m(\tau,s)d\tau, \tag{18}$$

and let

$$R(t,s) = \sum_{m=1}^{\infty} R_m(t,s). \tag{19}$$

Lemma 23. *Let $(P1)$-$(P3)$ be satisfied, then operator $R(t,s)$ satisfies*

$$\|R(t,s)\|_{\mathscr{B}} \leq C(t-s)^{\vartheta-1}, \quad t \in (s,T].$$

It also is continuous in the uniform operator topology on $\mathscr{B}(X)$ for all $0 \leq s < \sigma < t \leq T$, i.e., for every $0 < \beta < \vartheta \leq 1$, there holds

$$\|R(t,s) - R(\sigma,s)\|_{\mathscr{B}} \leq C(t-\sigma)^{\beta}(\sigma-s)^{\vartheta-\beta-1}.$$

Moreover, $R(t,s)$ is a unique solution of the integral equation (17).

Proof. By induction, we know that $R_m(t,s)$ is continuous in the uniform operator topology for $0 \leq s < t \leq T$ and

$$\|R_m(t,s)\|_{\mathscr{B}} \leq \frac{(C\Gamma(\vartheta))^m}{\Gamma(m\vartheta)}(t-s)^{m\vartheta-1}. \tag{20}$$

We note that the integral defining $R_{m+1}(t,s)$ is an improper integral whose existence is an immediate consequence of (20). The continuity of $R_{m+1}(t,s)$ also follows easily from the continuity of $R_m(t,s)$, $R_1(t,s)$ and (20).

The estimate (20) implies that the series (19) converges in the uniform operator topology for $0 \leq s \leq t-\varepsilon \leq T$ and every $\varepsilon > 0$. Moreover, using the definition of Mittag-Leffler functions and Lemma 3, we first have $E_{\vartheta,\vartheta}(C\Gamma(\vartheta)(t-s)^{\vartheta}) \leq C$ for $t \in [s,T]$ and then

$$\begin{aligned}\|R(t,s)\|_{\mathscr{B}} \leq \sum_{m=1}^{\infty}\|R_m(t,s)\|_{\mathscr{B}} \leq \sum_{m=1}^{\infty}\frac{(C\Gamma(\vartheta))^m}{\Gamma(m\vartheta)}(t-s)^{m\vartheta-1} =& C\Gamma(\vartheta)E_{\vartheta,\vartheta}(C\Gamma(\vartheta)(t-s)^{\vartheta})\\ \leq& C(t-s)^{\vartheta-1}.\end{aligned}$$

As a consequence, from (18), for $0 \leq s < t \leq T$ it follows that

$$R(t,s) = \sum_{m=1}^{\infty} R_m(t,s) = R_1(t,s) + \sum_{m=1}^{\infty}\int_s^t R_1(t,\tau)R_m(\tau,s)d\tau. \tag{21}$$

The continuity of $R_m(t,s)$, $m \geq 1$, Lemma 22 and (20) imply that one can interchange the summation and integral in (21), and then $R(t,s)$ is a solution of the integral equation (17).

Now, let $R(t,s)$ be a solution of (17), the $R(t,s)-R(\sigma,s)$ is equal to

$$R_1(t,s)-R_1(\sigma,s)+\int_\sigma^t R_1(t,\tau)R(\tau,s)d\tau+\int_s^\sigma (R_1(t,\tau)-R_1(\sigma,\tau))R(\tau,s)d\tau.$$

By virtue of Lemma 22, for $0<\beta<\vartheta\leq 1$, we get

$$\begin{aligned}\left\|\int_\sigma^t R_1(t,\tau)R(\tau,s)d\tau\right\|_{\mathscr{B}} \leq & C\int_\sigma^t (t-\tau)^{\vartheta-1}(\tau-s)^{\vartheta-1}d\tau \\ \leq & C(\sigma-s)^{\vartheta-\beta-1}(t-\sigma)^{\beta}\int_\sigma^t (t-\tau)^{\vartheta-\beta-1}(\tau-s)^{\beta}d\tau \\ \leq & C(\sigma-s)^{\vartheta-\beta-1}(t-\sigma)^{\beta},\end{aligned}$$

for $0\leq s<\sigma<t\leq T$. (2) and Lemma 22 also imply that

$$\begin{aligned}\left\|\int_s^\sigma (R_1(t,\tau)-R_1(\sigma,\tau))R(\tau,s)d\tau\right\|_{\mathscr{B}} \leq & C\int_s^\sigma (t-\sigma)^{\beta}(\sigma-\tau)^{\vartheta-\beta-1}(\tau-s)^{\vartheta-1}d\tau \\ \leq & C(t-\sigma)^{\beta}(\sigma-s)^{\vartheta-\beta-1}.\end{aligned}$$

Together with these estimates, we get the desired inequality

$$\begin{aligned}\|R(t,s)-R(\sigma,s)\|_{\mathscr{B}} \leq & \|R_1(t,s)-R_1(\sigma,s)\|_{\mathscr{B}}+\left\|\int_\sigma^t R_1(t,\tau)R(\tau,s)d\tau\right\|_{\mathscr{B}} \\ & +\left\|\int_s^\sigma (R_1(t,\tau)-R_1(\sigma,\tau))R(\tau,s)d\tau\right\|_{\mathscr{B}} \\ \leq & C(t-\sigma)^{\beta}(\sigma-s)^{\vartheta-\beta-1}.\end{aligned}$$

Let us finally verify the uniqueness of solution. Let $\widetilde{R}(t,s)\in\mathscr{B}(X)$ for $t>s$ be any other solution satisfying (17) that may have a weak singularity at $t=s$. Thus, we easily verify that $\bar{R}(t,s)=R(t,s)-\widetilde{R}(t,s)$ satisfies

$$\|\bar{R}(t,s)\|_{\mathscr{B}}\leq\int_s^t \|R_1(t,\tau)\|_{\mathscr{B}}\|\bar{R}(\tau,s)\|_{\mathscr{B}}d\tau\leq C\int_s^t (t-\tau)^{\vartheta-1}\|\bar{R}(\tau,s)\|_{\mathscr{B}}d\tau,$$

in which $\|\bar{R}(t,s)\|_{\mathscr{B}}$ equals to zero by a generalization Gronwall inequality [38, Theorem 1.28]. Consequently, $\bar{R}(t,s)$ is identically zero, i.e., $R(t,s)\equiv\widetilde{R}(t,s)$. The proof is completed. □

Lemma 24. *Let (P1)-(P3) be satisfied, then operator $U_\alpha(t,s)$ is linear bounded on $\mathscr{B}(X)$ for $t\in(s,T]$, i.e.,*

$$\|U_\alpha(t,s)\|_{\mathscr{B}}\leq C(t-s)^{\alpha-1},\quad t\in(s,T].$$

Moreover, it is continuous in the uniform operator topology on $\mathscr{B}(X)$ satisfying

$$\|U_\alpha(t,s)-U_\alpha(\sigma,s)\|_{\mathscr{B}}\leq C(t-\sigma)^{\alpha_0}(\sigma-s)^{\vartheta_0-1},$$

where $\alpha_0=\min\{1-\alpha,\alpha\}$, for all $\vartheta_0=\min\{2\alpha-1,\vartheta\}$, $0\leq s<\sigma<t\leq T$.

Proof. Clearly, $U_\alpha(t,s)$ is a linear operator. Also, Lemma 22 and Lemma 23 show that $U_\alpha(t,s)$ is well-defined. From Lemma 12 and Lemma 23 it follows that for $t\in(s,T]$

$$\begin{aligned}\|U_\alpha(t,s)\|_{\mathscr{B}} \le&\|\psi_s(t-s)\|_{\mathscr{B}}+\int_s^t\|\psi_\tau(t-\tau)\|_{\mathscr{B}}\|R(\tau,s)\|_{\mathscr{B}}d\tau\\ \le&C(t-s)^{\alpha-1}+C\int_s^t(t-\tau)^{\alpha-1}(\tau-s)^{\vartheta-1}d\tau\\ \le&\left(C+C(T-s)^{\vartheta}\right)(t-s)^{\alpha-1}\\ \le&C(t-s)^{\alpha-1}.\end{aligned}$$

We next check that $\psi_s(t-s)$ is uniformly continuous for $t\in(s,T]$. In fact, let $0\le s<\sigma<t\le T$, from Lemma 16 (iv) and Lemma 23, we get that

$$\begin{aligned}\|U_\alpha(t,s)-U_\alpha(\sigma,s)\|_{\mathscr{B}} \le&\|\psi_s(t-s)-\psi_s(\sigma-s)\|_{\mathscr{B}}+\int_\sigma^t\|\psi_\tau(t-\tau)R(\tau,s)\|_{\mathscr{B}}d\tau\\ &+\int_s^\sigma\|(\psi_\tau(\sigma-\tau)-\psi_\tau(t-\tau))R(\tau,s)\|_{\mathscr{B}}d\tau\\ \le&C|(t-s)^{\alpha-1}-(\sigma-s)^{\alpha-1}|+C\int_\sigma^t(t-\tau)^{\alpha-1}(\tau-s)^{\vartheta-1}d\tau\\ &+C\int_s^\sigma|(t-\tau)^{\alpha-1}-(\sigma-\tau)^{\alpha-1}|(\tau-s)^{\vartheta-1}d\tau.\end{aligned}$$

Noting that $(t-\tau)^{\alpha-1}<(\sigma-\tau)^{\alpha-1}$, we have

$$\begin{aligned}&\int_s^\sigma|(t-\tau)^{\alpha-1}-(\sigma-\tau)^{\alpha-1}|(\tau-s)^{\vartheta-1}d\tau\\ =&\int_s^\sigma(\sigma-\tau)^{\alpha-1}(\tau-s)^{\vartheta-1}d\tau-\int_s^t(t-\tau)^{\alpha-1}(\tau-s)^{\vartheta-1}d\tau+\int_\sigma^t(t-\tau)^{\alpha-1}(\tau-s)^{\vartheta-1}d\tau\\ =&B(\alpha,\vartheta)((\sigma-s)^{\alpha+\vartheta-1}-(t-s)^{\alpha+\vartheta-1})+\int_\sigma^t(t-\tau)^{\alpha-1}(\tau-s)^{\vartheta-1}d\tau,\end{aligned}$$

where $B(\cdot,\cdot)$ is the Beta function. Hence, from the inequality

$$\xi_1^a-\xi_2^a\le(\xi_1-\xi_2)^a,\quad 0\le\xi_2\le\xi_1<\infty,\quad 0\le a\le 1,$$

for $(t-s)^{\alpha-1}<(\sigma-s)^{\alpha-1}$, we have

$$\begin{aligned}\|U_\alpha(t,s)-U_\alpha(\sigma,s)\|_{\mathscr{B}} \le&C((\sigma-s)^{\alpha-1}-(t-s)^{\alpha-1})+2C(t-\sigma)^{\alpha}(\sigma-s)^{\vartheta-1}\\ &+C((\sigma-s)^{\alpha+\vartheta-1}-(t-s)^{\alpha+\vartheta-1})\\ \le&C(t-\sigma)^{1-\alpha}(\sigma-s)^{2\alpha-2}+C(t-\sigma)^{\alpha}(\sigma-s)^{\vartheta-1},\end{aligned}$$

we thus get the desired conclusion. The proof is completed. □

Remark 25. *The operator $U_\alpha(t,s)\neq U_\alpha(t,r)U_\alpha(r,s)$, the reason is that the Mittag-Leffler function $E_{\alpha,\alpha}(z)$ does not enjoy the semigroup property by Lemma 13, that is $E_{\alpha,\alpha}(t+s)\neq E_{\alpha,\alpha}(t)E_{\alpha,\alpha}(s)$ for $t,s\in\mathbb{R}_+$, $\alpha\in(0,1)$, and thus $\psi_s(t-s)=\psi_r(t-r)\psi_s(r-s)$ is invalid, this means that $U_\alpha(t,s)$ has not "good" properties to compared with that of the classical evolution operator.*

Remark 26. *It is worth noticing that if $A(t)$ degenerates to $A : D(A) \subset X \to X$ (independent of t), the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$, then $U_\alpha(t,s)$ degenerates to $T_\alpha(t-s)$, which is a resolvent operator (see e.g. [41]) given by*

$$T_\alpha(t) = t^{\alpha-1}\int_0^\infty \alpha\upsilon M_\alpha(\upsilon)T(t^\alpha\upsilon)d\upsilon, \quad t>0.$$

Additionally, we also see that $T_\alpha(t) = t^{\alpha-1}E_{\alpha,\alpha}(-At^\alpha)$ (see e.g. [5]), where $T_\alpha(t+s) \neq T_\alpha(t)T_\alpha(s)$ for $\alpha\in(0,1)$ and it does not possess the semigroup property, this situation is compatible for $\psi_s(t-s)\neq \psi_r(t-r)\psi_s(r-s)$.

Lemma 27. *Let $(P1)$-$(P3)$ hold and let*

$$\eta_g(t,s) := \int_s^t U_\alpha(t,\tau)g(\tau,s)d\tau, \quad 0\leq s<t\leq T,$$

where $g(\cdot,s)$ satisfies the assumption (P4) with values in X, then $\eta_g \in \mathscr{D}(\mathscr{A})$ for $t\in(s,T]$ and it is a solution to

$${}_s^LD_t^\alpha \eta_g(t,s) + A(t)\eta_g(t,s) = g(t,s).$$

Proof. From the assumption of function g, by virtue of Lemma 23, it yields

$${}_s^LD_t^\alpha \int_s^t \psi_\tau(t-\tau)g(\tau,s)d\tau = g(t,s) - \int_s^t A(\tau)\psi_\tau(t-\tau)g(\tau,s)d\tau.$$

Let $G(t,s) = \int_s^t R(t,\tau)g(\tau,s)d\tau$, by Lemma 23, it is easy to check that function $G(t,s)$ satisfies the assumption (P4) in Lemma 18. Hence, by Fubini's theorem we get

$$\begin{aligned}{}_s^LD_t^\alpha \int_s^t W(t,\tau)g(\tau,s)d\tau &= {}_s^LD_t^\alpha \int_s^t\int_\tau^t \psi_\upsilon(t-\upsilon)R(\upsilon,\tau)g(\tau,s)d\upsilon d\tau\\ &= {}_s^LD_t^\alpha \int_s^t \psi_\tau(t-\tau)G(\tau,s)d\tau\\ &= G(t,s) - \int_s^t A(\tau)\psi_\tau(t-\tau)G(\tau,s)d\tau,\end{aligned}$$

it means that

$$(22)\quad {}_s^LD_t^\alpha \eta_g(t,s) = g(t,s) - \int_s^t A(\tau)\psi_\tau(t-\tau)g(\tau,s)d\tau + G(t,s) - \int_s^t A(\tau)\psi_\tau(t-\tau)G(\tau,s)d\tau.$$

Therefore, it suffices to verify that $\eta_g(t,s) \in \mathscr{D}(\mathscr{A})$ on $[s+\varepsilon, T]$ with any $\varepsilon > 0$. In fact, by Lemma 16 and Lemma 17 it follows that

$$\begin{aligned}
&\left\|\int_s^t A(t)\psi_\tau(t-\tau)g(\tau,s)d\tau\right\| \\
\leq &\left\|\int_s^t A(t)\psi_t(t-\tau)g(t,s)d\tau\right\| + \left\|\int_s^t A(t)\psi_\tau(t-\tau)(g(\tau,s)-g(t,s))d\tau\right\| \\
&+\left\|\int_s^t A(t)(\psi_\tau(t-\tau)-\psi_t(t-\tau))g(t,s)d\tau\right\| \\
\leq &(C+1)\|g(t,s)\| + C\int_s^t \frac{\|g(\tau,s)-g(t,s)\|}{t-\tau}d\tau + C\int_s^t (t-\tau)^{\vartheta-1}\|g(t,s)\|d\tau \\
\leq &CH_g(t,s) + C\|g(t,s)\|,
\end{aligned}$$

which means that $\int_s^t \psi_\tau(t-\tau)g(\tau,s)d\tau \in \mathscr{D}(\mathscr{A})$ on $[s+\varepsilon, T]$. Similarly, we derive that

$$\int_s^t W(t,\tau)g(\tau,s)d\tau \in \mathscr{D}(\mathscr{A}).$$

Hence, $\eta_g \in \mathscr{D}(\mathscr{A})$. It yields

$$\begin{aligned}
A(t)\eta_g(t,s) =&A(t)\int_s^t \psi_\tau(t-\tau)g(\tau,s)d\tau + A(t)\int_s^t W(t,\tau)g(\tau,s)d\tau \\
=&A(t)\int_s^t \psi_\tau(t-\tau)g(\tau,s)d\tau + A(t)\int_s^t \psi_\tau(t-\tau)G(\tau,s)d\tau.
\end{aligned} \tag{23}$$

Consequently, combined (22) and (23), we have

$$\begin{aligned}
{}_s^LD_t^\alpha \eta_g(t,s) + A(t)\eta_g(t,s) =&g(t,s) - \int_s^t A(\tau)\psi_\tau(t-\tau)g(\tau,s)d\tau \\
&+G(t,s) - \int_s^t A(\tau)\psi_\tau(t-\tau)G(\tau,s)d\tau \\
&+A(t)\int_s^t \psi_\tau(t-\tau)g(\tau,s)d\tau + A(t)\int_s^t \psi_\tau(t-\tau)G(\tau,s)d\tau \\
=&g(t,s) - \int_s^t R_1(t,\tau)g(\tau,s)d\tau + G(t,s) - \int_s^t R_1(t,\tau)G(\tau,s)d\tau.
\end{aligned}$$

Additionally, it yields

$$\int_s^t R_1(t,\tau)G(\tau,s)d\tau = \int_s^t\int_s^\tau R_1(t,\tau)R(\tau,\nu)g(\nu,s)d\nu d\tau = \int_s^t\int_\nu^t R_1(t,\tau)R(\tau,\nu)g(\nu,s)d\tau d\nu.$$

Identity (17) shows that

$$\int_s^t R_1(t,\tau)g(\tau,s)d\tau + G(t,s) - \int_s^t R_1(t,\tau)G(\tau,s)d\tau = 0.$$

Consequently, g is a solution to the desired equation. The proof is completed.

□

Lemma 28. *Let $R(t,s)$ be given in (19). There holds ${}_s^LD_t^\alpha w_R(\cdot,s)x \in C((s,T],X)$ for any $x \in X$.*

Proof. By Lemma 18, Remark 19, $\|A(t)\psi_t(\eta)\|_{\mathscr{B}} \le C\eta^{-1}$ for $\eta > 0$, $t \in [0,T]$ and the continuity of R, it just needs to prove $\int_s^t A(\tau)\psi_\tau(t-\tau)R(\tau,s)x d\tau \in C([s+\varepsilon,T],X)$ with any $\varepsilon > 0$ and any $x \in X$. Let $\{t_n\}_{n\ge1}^\infty$ such that $s+\varepsilon \le t < t_n \le T$ and $t_n \to t$ as $n \to \infty$, we need to verify that

$$\int_s^{t_n} A(\tau)\psi_\tau(t_n-\tau)R(\tau,s)xd\tau - \int_s^t A(\tau)\psi_\tau(t-\tau)R(\tau,s)xd\tau \to 0, \quad \text{as } t_n \to t.$$

By Lemma 18, we know that

$$\begin{aligned}\left\|\int_t^{t_n} A(\tau)\psi_\tau(t_n-\tau)R(\tau,s)xd\tau\right\| \le C(t_n-t)^\vartheta + &\left\|\int_t^{t_n} A(t_n)\psi_{t_n}(t_n-\upsilon)(R(\upsilon,s)-R(t_n,s))xd\upsilon\right\| \\ &+ \|(I-\phi_{t_n}(t_n-t))R(t_n,s)x\| \\ \le& C(t_n-t)^\vartheta\|x\| + C\int_t^{t_n}(t_n-\upsilon)^{\beta-1}(\upsilon-s)^{\vartheta-\beta-1}d\upsilon\|x\| \\ &+ C\|(I-\phi_{t_n}(t_n-t))x\|(t_n-s)^{\vartheta-1} \\ \le& C(t_n-t)^\vartheta\|x\| + C(t-s)^{\vartheta-\beta-1}(t_n-t)^\beta\|x\| \\ &+ C\|(I-\phi_{t_n}(t_n-t))x\|(t_n-s)^{\vartheta-1} \to 0, \quad \text{as } n\to\infty,\end{aligned}$$

where we have used Lemma 23 and $\phi_t(0) = I$. Moveover, we note that

$$\|A(\tau)(\psi_\tau(t_n-\tau)-\psi_\tau(t-\tau))(R(\tau,s)-R(t,s))x\| \le C(t-\tau)^{\beta-1}(\tau-s)^{\vartheta-\beta-1}\|x\|,$$

is L^1 integrable in a.e. $[s,t]$, and then from the continuity of $A(\tau)\psi_\tau(t-\tau)$ for $t \in (\tau,T]$, we also have $A(\tau)\psi_\tau(t_n-\tau)(R(\tau,s)-R(t,s))x \to A(\tau)\psi_\tau(t-\tau)(R(\tau,s)-R(t,s))x$ as $n\to\infty$ for a.e. $\tau\in[s,t]$. By Lebesgue dominated convergence theorem, we get

$$\int_s^t A(\tau)(\psi_\tau(t_n-\tau)-\psi_\tau(t-\tau))(R(\tau,s)-R(t,s))xd\tau \to 0, \quad \text{as } n\to\infty. \tag{24}$$

On the other hand, we have

$$\begin{aligned}\int_s^t A(\tau)(\psi_\tau(t_n-\tau)-\psi_\tau(t-\tau))R(t,s)xd\tau =& \int_s^t (A(t_n)\psi_{t_n}(t_n-\tau)-A(t)\psi_t(t-\tau))R(t,s)xd\tau \\ &+ \int_s^t (K(t_n,\tau)-K(t,\tau))R(t,s)xd\tau =: J_1+J_2,\end{aligned}$$

where $K(t,\tau) = A(\tau)\psi_\tau(t-\tau) - A(t)\psi_t(t-\tau)$. From Lemma 17, we first have

$$\begin{aligned}J_1 =& \int_t^{t_n} A(t_n)\psi_{t_n}(t_n-\tau)R(t,s)xd\tau d\tau + \int_0^{t_n} A(t_n)\psi_{t_n}(t_n-\tau)R(t,s)xd\tau \\ &- \int_0^t A(t)\psi_t(t-\tau)R(t,s)xd\tau \\ =& (\phi_t(t-s)-\phi_{t_n}(t_n-t)-\phi_{t_n}(t_n-s)-I)R(t,s)x,\end{aligned}$$

which tends to zero as $n\to\infty$. In view of $K(t_n,\tau)x \to K(t,\tau)x$ as $n\to\infty$ for a.e. $[s,t]$ and from Lemma 18 we get

$$\|K(t_n,\tau)x - K(t,\tau)x\| \le C((t_n-\tau)^{\vartheta-1}+(t-\tau)^{\vartheta-1})\|x\| \le C(t-\tau)^{\vartheta-1},$$

is L^1 integrable for a.e. $[s,t]$. Therefore, by Lebesgue dominated convergence theorem, for any $x \in X$ we get

$$\int_s^t (K(t_n,\tau) - K(t,\tau))x d\tau \to 0, \quad \text{as } n \to \infty,$$

which means $J_2 \to 0$ as $n \to \infty$, and then

$$\int_s^t A(\tau)(\psi_\tau(t_n - \tau) - \psi_\tau(t - \tau))R(t,s)x d\tau \to 0, \quad \text{as } n \to \infty. \tag{25}$$

Together (24) and (25), we obtain the desired result. The proof is completed. □

In particular, we get the following conclusion.

Corollary 29. *Let $(P1)$-$(P3)$ hold. If ξ is Hölder continuous with type (γ, K). There holds ${}_s^L D_t^\alpha w_\xi(\cdot, s)x \in C((s,T],X)$ for any $x \in X$.*

In the sequel, we introduce a concept of fundamental solution.

Definition 30. *An operator-valued function $U_\alpha(t,s) \in C((s,T],\mathscr{B}(X))$ is called a fundamental solution of equation*

$${}_s^L D_t^\alpha x(t) + A(t)x(t) = 0, \quad 0 \le s < t \le T, \tag{26}$$

if

(i) *the derivative ${}_s^L D_t^\alpha U_\alpha(t,s)$ exists in $\mathscr{B}(X)$, and is also strongly continuous on $(s,T]$.*
(ii) *the range of $U_\alpha(t,s)$ is included in $\mathscr{D}(\mathscr{A})$ for $0 \le s < t \le T$.*
(iii) *for any $y \in \mathscr{D}(\mathscr{A})$, there hold*

$${}_s^L D_t^\alpha U_\alpha(t,s)y + A(t)U_\alpha(t,s)y = 0, \quad 0 \le s < t \le T, \tag{27}$$

$${}_s J_t^{1-\alpha} U_\alpha(t,s)y \to y, \quad \text{as } t \to s. \tag{28}$$

Theorem 31. *Let $(P1)$-$(P3)$ be satisfied, then there exists a fundamental solution $U_\alpha(t,s)$ of equation (26). Moreover, for any $x_s \in \mathscr{D}(\mathscr{A})$, the problem*

$${}_s^L D_t^\alpha x(t) + A(t)x(t) = 0, \quad {}_s J_t^{1-\alpha} x(s) = x_s, \tag{29}$$

has a unique classical solution $x(t) = U_\alpha(t,s)x_s$, $t \in (s,T]$.

Proof. By virtue of Lemma 23, we know that $R(t,s) : (s,T] \mapsto \mathscr{B}(X)$ satisfies the assumption of $(P4)$. Hence, we have

$${}_s^L D_t^\alpha \int_s^t \psi_\tau(t-\tau)R(\tau,s)d\tau = R(t,s) - \int_s^t A(\tau)\psi_\tau(t-\tau)R(\tau,s)d\tau.$$

In particular, for $t \in (s,T]$, we have

$$\left\| {}_s^L D_t^\alpha \int_s^t \psi_\tau(t-\tau)R(\tau,s)d\tau \right\|_{\mathscr{B}} \le C(t-s)^{\vartheta-1}.$$

In fact, from Lemma 23, it suffices to verify the estimate of integrand term.

$$\begin{aligned}
&\int_s^t A(\tau)\psi_\tau(t-\tau)R(\tau,s)d\tau \\
&=\int_s^t (A(\tau)-A(t))\psi_\tau(t-\tau)R(\tau,s)d\tau+\int_s^t A(t)(\psi_\tau(t-\tau)-\psi_t(t-\tau))R(\tau,s)d\tau \\
&\quad+\int_s^t A(t)\psi_t(t-\tau)(R(\tau,s)-R(t,s))d\tau+\int_s^t A(t)\psi_t(t-\tau)R(t,s)d\tau \\
&=:I_1+I_2+I_3+I_4.
\end{aligned}$$

By virtue of (i), (ii) in Lemma 16 and Lemma 23, we have

$$\|I_i\|_{\mathscr{B}} \le C\int_s^t (t-\tau)^{\vartheta-1}(\tau-s)^{\vartheta-1}d\tau \le C(t-s)^{2\vartheta-1},$$

for $i=1,2$. Lemma 23 implies that for every $0<\beta<\vartheta\le 1$ we have

$$\|I_3\|_{\mathscr{B}} \le C\int_s^t (t-\tau)^{\beta-1}(\tau-s)^{\vartheta-\beta-1}d\tau \le C(t-s)^{\vartheta-1}.$$

Lemma 17 shows that $\|I_4\|_{\mathscr{B}}\le C(t-s)^{\vartheta-1}$. Hence, ${}^L_sD^\alpha_t U_\alpha(t,s)$ exists in $\mathscr{B}(X)$ for $t\in(s,T]$ as follows

$${}^L_sD^\alpha_t U_\alpha(t,s) = -A(s)\psi_s(t-s)+R(t,s)-\int_s^t A(\tau)\psi_\tau(t-\tau)R(\tau,s)d\tau.$$

By virtue of (iii) in Lemma 16, Lemma 23 and Lemma 28, we deduce that ${}^L_sD^\alpha_t U_\alpha(t,s)$ is strongly continuous on $[s+\varepsilon,T]$ with any $\varepsilon>0$. Thus, the requirement (i) in the sense of Definition 30 is satisfied.

For any $y\in\mathscr{D}(\mathscr{A})$, it follows that $A(t)U_\alpha(t,s)y=U_\alpha(t,s)A(t)y$ commuted with the property $A(t)T_t(\varsigma)y=T_t(\varsigma)A(t)y$. Since

$$A(t)U_\alpha(t,s)y=A(t)\psi_s(t-s)y+A(t)\int_s^t \psi_\tau(t-\tau)R(\tau,s)yd\tau,$$

by repeating the above proof process, we get the requirement (ii) that the range of $U_\alpha(t,s)$ is included in $\mathscr{D}(\mathscr{A})$ for $t\in(s,T]$.

From the construction of a fundamental solution at the beginning of the this subsection, it is readily seen that (27) holds. Next, for any $y\in\mathscr{D}(\mathscr{A})$, it suffices to prove ${}_sJ^{1-\alpha}_t U_\alpha(t,s)y\to y$ as $t\to s$. In fact, Lemma 13 and Lemma 15 show that

$$\begin{aligned}
{}_sJ^{1-\alpha}_t U_\alpha(t,s)y &= {}_sJ^{1-\alpha}_t\psi_s(t-s)y+{}_sJ^{1-\alpha}_t W(t,s)y \\
&=\phi_s(t-s)y+\frac{1}{\Gamma(1-\alpha)}\int_s^t\int_\sigma^t (t-\tau)^{-\alpha}\psi_\sigma(\tau-\sigma)R(\sigma,s)yd\tau d\sigma \\
&=\phi_s(t-s)y+\int_s^t \phi_\sigma(t-\sigma)R(\sigma,s)yd\tau.
\end{aligned}$$

Furthermore, from Lemma 12 and Lemma 23, it follows that

$$\left\|\int_s^t \phi_\sigma(t-\sigma)R(\sigma,s)yd\tau\right\| \le C\int_s^t \|R(\sigma,s)y\|d\tau \le C(t-s)^\vartheta\|y\|.$$

This shows that

$$ {}_sJ_t^{1-\alpha}U_\alpha(t,s)y \to y, \quad \text{as } t \to s. $$

Thus, there exists a fundamental solution in the sense of Definition 30.

Clearly, the solution $x(t) = U_\alpha(t,s)x_s$ is a classical solution in the sense of Definition 10 due to $U_\alpha(t,s)$ is a fundamental solution and is uniformly continuous in Lemma 24. So it remains to prove the uniqueness of classical solution. Clearly, the above proofs imply that $A(t)U_\alpha(t,s) \in \mathscr{B}(X)$ for every $t \in (s,T]$. This means that $x \in \mathscr{D}(\mathscr{A})$. We now introduce, for every $\lambda > 0$, the Yosida approximation of $A(t)$ by $A_\lambda(t) = \lambda A(t)(\lambda I + A(t))^{-1}$. Obviously, $\lim_{\lambda\to\infty} A_\lambda(t)x(t) = A(t)x(t)$ for $x \in \mathscr{D}(\mathscr{A})$ by the density of $\mathscr{D}(\mathscr{A})$ in X, it also has $\|R(z,A_\lambda(t))\|_{\mathscr{B}} \le C'/(1+|z|)$, $z \in \Sigma$, and

$$ \|(A_\lambda(t) - A_\lambda(s))(A_\lambda(r))^{-1}\| \le L'|t-s|^\vartheta, $$

where $C', L' > 0$ are determined by C and L which are defined as in (9) and (10), respectively. Therefore, one finds that the map $t \mapsto A_\lambda(t)$ is continuous in the uniform operator topology in view of $(P3)$ and $\|A_\lambda(t)\| \le C'\lambda$ from $\Sigma \subset \rho(A(t))$. Now, let us consider the approximation problem

$$ {}_s^LD_t^\alpha x_\lambda(t) + A_\lambda(t)x_\lambda(t) = 0, \quad {}_sJ_t^{1-\alpha}x_\lambda(s) = x_s. \tag{30} $$

It is readily seen that (30) has a solution x_λ which is given by $x_\lambda(t) = U_{\lambda,\alpha}(t,s)x_s$, where

$$ U_{\lambda,\alpha}(t,s) = \psi_{\lambda,s}(t-s) + W_\lambda(t,s), $$

is a fundamental solution of the first equation in problem (30) and $T_{\lambda,s}(t) = \dfrac{1}{2\pi i}\displaystyle\int_\Gamma e^{-tz}R(z;A_\lambda(s))dz$,

$$ \psi_{\lambda,s}(t) = t^{\alpha-1}\int_0^\infty \alpha\upsilon\zeta_\alpha(\upsilon)T_{\lambda,s}(t^\alpha\upsilon)d\upsilon, \quad W_\lambda(t,s) = \int_s^t \psi_{\lambda,\tau}(t-\tau)R(\tau,s)d\tau. $$

By repeating the proofs process as in corresponding lemmas, the operators $\psi_{\lambda,s}(t)$, $W_\lambda(t,s)$ and $U_{\lambda,\alpha}(t,s)$ enjoy the properties that of $\psi_s(t)$, $W(t,s)$ and $U_\alpha(t,s)$, respectively.

Now, let $w_\lambda(t) = x(t) - x_\lambda(t)$, then it satisfies

$$ {}_s^LD_t^\alpha w_\lambda(t) + A_\lambda(t)w_\lambda(t) = \xi_\lambda(t), \quad {}_sJ_t^{1-\alpha}w_\lambda(s) = 0. \tag{31} $$

where $\xi_\lambda(t) = (A_\lambda(t) - A(t))x(t)$ and $\xi_\lambda \in X$. Condition $(P3)$ shows that $\xi_\lambda(t)$ is continuous on $(s,T]$ and it is an L^1-integral function, indeed, for $x_s \in \mathscr{D}(\mathscr{A})$, from Lemma 24 we have

$$ \|\xi_\lambda(t)\| = \|A(t)(\lambda R(\lambda,A(t)) - I)U_\alpha(t,s)x_s\| \le C(t-s)^{\alpha-1}\|x_s\|_{\mathscr{D}(\mathscr{A})}, $$

which belongs to $L^1(s,T;\mathbb{R}^+)$.

Furthermore, if $\xi_\lambda(t) = 0$ for every $\lambda > 0$, then by using (5) and the similar proof of Theorem 11, it follows that Cauchy problem (31) has a unique solution as follows

$$ w_\lambda(t) = \frac{1}{\Gamma(\alpha)}\int_s^t (t-\tau)^{\alpha-1}A_\lambda(\tau)w_\lambda(\tau)d\tau. \tag{32} $$

The generalized Gronwall inequality (see e.g. [41]) shows $w_\lambda(t) = 0$. Consequently, we have $x(t) = x_\lambda(t)$ for all $\lambda > 0$ and the uniqueness follows.

If $\xi_\lambda(t) \neq 0$ for every $\lambda > 0$, from the boundedness of $\xi_\lambda(t)$ and $A_\lambda(t)$, by using the similar proof of Theorem 11, we know that w_λ is the unique solution of Cauchy problem (31) satisfying

$$w_\lambda(t) = \int_s^t U_{\lambda,\alpha}(t,\tau)\xi_\lambda(\tau)d\tau. \tag{33}$$

Admitting (33) satisfies problem (31) for the moment. Now, let $R_{\lambda,1}(t,s) = (A_\lambda(t) - A_\lambda(s))\psi_{\lambda,s}(t-s)$, and $R_\lambda(t,s)$ is the unique solution of variation of parameters formula

$$R_{\lambda,1}(t,s) + \int_s^t R_{\lambda,1}(t,\tau)R_\lambda(\tau,s)d\tau = R_\lambda(t,s),$$

where we can apply the same way as in constructing $R(t,s)$ to build $R_\lambda(t,s)$ and it enjoys the similar properties of $R(t,s)$ as follows

$$R_{\lambda,m+1}(t,s) = \int_s^t R_1(t,\tau)R_{\lambda,m}(\tau,s)d\tau,\ m \geq 1, \quad \text{and} \quad R_\lambda(t,s) = \sum_{m=1}^{\infty} R_{\lambda,m}(t,s).$$

By the formal of fundamental solution $U_{\lambda,\alpha}(t,s)$ and the boundedness of operator $A_\lambda(t)$ for every $\lambda > 0$, Lemma 18, Remark 19 and Lemma 27 show that $w_\lambda(t)$ is the unique solution to ${}_s^L D_t^\alpha w_\lambda(t) + A_\lambda(t)w_\lambda(t) = \xi_\lambda(t)$. Also ${}_sJ_t^{1-\alpha}w_\lambda(s) = 0$ is easy to check.

Let us end this proof, from $R(z;A_\lambda(\cdot)) \to R(z;A(\cdot))$ as $\lambda \to \infty$ in $\mathscr{B}(X)$, we get that $\psi_{\lambda,s}(t-s) \to \psi_s(t-s)$ in $\mathscr{B}(X)$, and $W_\lambda(t,s)x \to W(t,s)x$ as $\lambda \to \infty$ for $x \in \mathscr{D}(\mathscr{A})$. Consequently, $U_{\lambda,\alpha}(t,s)x \to U_\alpha(t,s)x$, $\xi_\lambda(t) \to 0$, as $\lambda \to \infty$, for $s+\varepsilon \leq t \leq T$ with every $\varepsilon > 0$, we have $\lim_{\lambda\to\infty} w_\lambda(t) = 0$ and then $\lim_{\lambda\to\infty} x_\lambda(t) = x(t)$ for $t \in (s,T]$. So $x(t)$ is unique. Lemma 16 shows that $A(\cdot)\psi_s(s) \in C((s,T];\mathscr{B}(X))$, Lemma 24 shows that $U_\alpha(t,s)x_s$ is strongly continuous for all $t \in (s,T]$, and then $A(t)U_\alpha(t,s)x_s$ is also strongly continuous for $t \in [s+\varepsilon,T]$, $s \in [0,T]$ for every $\varepsilon > 0$, which implies that ${}_s^L D_t^\alpha x$ belongs to $C((s,T],X)$. Thus, x is a classical solution. The proof is completed. □

4. Classical solutions to problem (1)

In this section, the classical solution of problem (1) is obtained under the properties of fundamental solution and the Hölder continuity assumption of f.

Theorem 32. *Let (P1)-(P3) be satisfied. Assume that f is Hölder continuous with type (ζ,K). Then the problem (1) has, for every $x_s \in \mathscr{D}(\mathscr{A})$, a unique classical solution given by*

$$x(t) = U_\alpha(t,s)x_s + \int_s^t U_\alpha(t,\tau)f(\tau)d\tau.$$

Proof. It follows from the pervious arguments that $U_\alpha(t,s)x_s$ is the unique classical solution of the initial value problem ${}_s^L D_t^\alpha x(t) + A(t)x(t) = 0$, ${}_sJ_t^{1-\alpha}x(s) = x_s$. Put

$$v_f(t) = \int_s^t U_\alpha(t,\tau)f(\tau)d\tau, \quad 0 \leq s < t \leq T.$$

It remains to check that v_f is the unique solution of problem

$${}_s^L D_t^\alpha x(t) + A(t)x(t) = f(t), \quad {}_sJ_t^{1-\alpha}x(s) = 0. \tag{34}$$

Since f is Hölder continuous, from Remark 19, it yields that f satisfies (P4) with values in X, we also have

$$\|v_f(t)\| \leq \int_s^t \|U_\alpha(t,\tau)f(\tau)\| \, d\tau \leq C(t-s)^\alpha \|f\|_\infty,$$

which proves $v_f(s) = 0$ and then ${}_sJ_t^{1-\alpha}v_f(s) = 0$ is easy to verify. This derives from Lemma 27 that v_f is a solution of the problem (34), and $v_f \in \mathscr{D}(\mathscr{A})$. Moreover, the uniqueness follows the classical arguments. From Lemma 23 and the Hölder continuity of f, as the same way in Lemma 28, one can prove ${}_s^LD_t^\alpha v_f \in C((s,T],X)$. Hence, by Definition 10, v_f is a unique classical solution to problem (34) as well as x is a unique solution to problem (1). The proof is completed. □

Remark 33. *Let us mention that if $A(t)$ degenerates to linear unbounded operator A, we see that the operator $U_\alpha(t,0)$ can be regarded as $T_\alpha(t)$ in Remark 26, and there is an analogous form of classical solution in the setting of autonomous fractional evolution equations given by*

$$x(t) = t^{\alpha-1}T_\alpha(t)x_0 + \int_0^t (t-\tau)^{\alpha-1}T_\alpha(t-\tau)f(\tau)d\tau, \quad t \in (0,T].$$

For more details, see [36, 41].

5. An application

In this section, we apply the abstract theory developed in this work to a classic parabolic type equation, as an application, we concern a time dependent fractional Schrödinger type equation

$${}_0^LD_t^\alpha x(t,z) - \Delta x(t,z) + m(t,z)x(t,z) = f(t,z), \;\; t>0, \; z \in \mathbb{R}^d, \quad {}_0J_t^{1-\alpha}x(0,z) = 0,$$

where $d \geq 1$, the potential m is not bounded, see [29] associated with the case of $\alpha = 1$. We assume that there is a non-negative potential $W \in L^1_{loc}(\mathbb{R}^d, dz)$ such that m satisfies the following properties (where c_1, c_2 are positive constants)

$$c_1W(z) \leq m(t,z) \leq c_2W(z), \quad a.e.\ z \in \mathbb{R}^d, \;\text{ and all } t \in [0,T],$$
$$|m(t,z) - m(\tau,z)| \leq c_2W(z)|t-\tau|^\vartheta, \quad a.e.\ z \in \mathbb{R}^d, \;\text{ and all } t,\tau \in [0,T], \; \vartheta \in (0,1].$$

We now define a sesquilinear form defined on $V \times V$ for every fixed t in $[0,T]$ by

$$a(t,u,v) = \sum_{k=0}^d \int_{\mathbb{R}^d} \partial_k u \cdot \partial_k v dz + \int_{\mathbb{R}^d} m(t,z)u \cdot v dz,$$

$$D(a(t,\cdot,\cdot)) = \left\{ u \in H^1(\mathbb{R}^d), \; \int_{\mathbb{R}^d} m(t,z)|u|^2 dz < \infty \right\}.$$

Under these arguments, the following spaces are equivalent

$$D(a(t,\cdot,\cdot)) = V =: \left\{ u \in H^1(\mathbb{R}^d), \; \int_{\mathbb{R}^d} W(z)|u(z)|^2 dz < \infty \right\}, \quad \text{for all } t \in [0,T].$$

The space V endowed with the norm

$$\|u\|_V := \left[\int_{\mathbb{R}^d} |\nabla u|^2 dz + \int_{\mathbb{R}^d} |u|^2 dz + \int_{\mathbb{R}^d} W|u|^2 dz \right]^{1/2}$$

is a Hilbert space and $V \hookrightarrow L^2(\mathbb{R}^d)$ (see e.g. [29]). Additionally, $a(t,\cdot,\cdot)$ is densely defined and satisfies the continuity condition, i.e., there exists a non-negative constant M (independent of t) such that

$$|a(t,u,v)| \le M\|u\|_V\|v\|_V, \quad \text{for all } u,v \in V, \quad \text{and all } t \in [0,T].$$

Moreover the coercive condition holds, i.e., there exist a $\delta > 0$ and a real number γ such that

$$Rea(t,u,u) \ge \delta\|u\|_V^2 - \gamma\|u\|_{L^2(\mathbb{R}^d)}^2, \quad \text{for all } u \in V, \quad \text{and all } t \in [0,T].$$

Fix $v \in V$ and consider the functional

$$\phi(t,v) = a(t,u,v), \quad v \in V,\ t \in [0,T].$$

By the continuity condition of $a(t,\cdot,\cdot)$, let us define $\mathscr{A}(t)u = \phi(t,\cdot)$, it is easy to check that $\mathscr{A}(t)$ is a continuous operator from V into V' by the following space form

$$D(\mathscr{A}(t)) := \left\{u \in V,\ \exists g \in V',\ \text{s.t. } a(t,u,v) = \langle g,v\rangle,\ \forall v \in V\right\},\ \mathscr{A}(t)u = g,$$

where $\langle\cdot,\cdot\rangle$ the dualization between V and the dual space V' (i.e. $\langle u,v\rangle$ denotes the value of u at v for $v \in V$ and $u \in V'$). Therefore, it follows that $A(t) := -\Delta + m(t,\cdot)$ is indeed $\mathscr{A}(t)$ in V'. This means that

$$D(A(t)) = \left\{u \in D(\mathscr{A}(t)) :\ \mathscr{A}(t)u \in V'\right\}, \text{ and } A(t)u = \mathscr{A}(t)u \text{ for } u \in D(A(t)).$$

Hence, the time-varying parameter fractional Schrödinger type equation can be abstracted as the equation (1). On the other hand, one can see that there is a uniform constant $M > 0$ such that the sesquilinear form satisfies

$$|a(t,u,v) - a(\tau,u,v)| \le M|t-\tau|^{\vartheta}\|u\|_V\|v\|_V, \quad \text{for all } t,\tau \in [0,T],\ u,v \in V.$$

Furthermore, we find that the operator $-A(t)$ generates an analytic semigroup in V'. Hence $(P1)$ and $(P2)$ are valid. By [38, Theorem 1.24], the domain of $\mathscr{A}(t)$ coincides with V and so it is independent of t, then $\mathscr{A}(t)$ can commute in $A(t)$. In addition, by the definition of $\mathscr{A}(\cdot)$, we have for $u \in V$, $v \in V$

$$\begin{aligned}\langle(\mathscr{A}(t) - \mathscr{A}(\tau))\mathscr{A}(\tau)^{-1}u,v\rangle =& \langle\mathscr{A}(t)\mathscr{A}(\tau)^{-1}u,v\rangle - \langle\mathscr{A}(\tau)\mathscr{A}(\tau)^{-1}u,v\rangle \\ =& a(t,\mathscr{A}(\tau)^{-1}u,v) - a(\tau,\mathscr{A}(\tau)^{-1}u,v).\end{aligned}$$

Therefore, the Hölder condition on the form implies that

$$\begin{aligned}|\langle(\mathscr{A}(t) - \mathscr{A}(\tau))\mathscr{A}(\tau)^{-1}u,v\rangle| \le& M|t-\tau|^{\vartheta}\|\mathscr{A}(\tau)^{-1}u\|_V\|v\|_V \\ \le& C|t-\tau|^{\vartheta}\|u\|_V\|v\|_V.\end{aligned}$$

Hence, $(\mathscr{A}(t) - \mathscr{A}(\tau))\mathscr{A}(\tau)^{-1}u \in V'$ for $u \in V$. By the density of V in V', it follows that

$$\|(A(t) - A(\tau))A(\tau)^{-1}\|_{\mathscr{B}(V')} \le C|t-\tau|^{\vartheta}.$$

We conclude that $(P3)$ is also satisfied. Assume that f is Hölder continuous with type (ζ, K), by Theorem 32, there exists a unique classical solution. In particular, if $f \equiv 0$, there exists a fundamental solution of present problem by Theorem 30.

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