Oscillation of a Class of First Order 2-dim Functional Difference Systems

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Abstract

In this work, we discuss the oscillatory behaviour of all vector solutions of 2-dimensional nonlinear neutral delay difference systems of the form:

$$\Delta \begin{bmatrix} r(k)+q(k)r(k-l) \\ s(k)+q(k)s(k-l) \end{bmatrix} = \begin{bmatrix} a_{11}(k) & a_{12}(k) \\ a_{21}(k) & a_{22}(k) \end{bmatrix} \begin{bmatrix} \nu_1(r(k-\alpha_1)) \\ \nu_2(s(k-\alpha_2)) \end{bmatrix}, k \geq \rho,$$

where $\rho = \max\{l, \alpha_1, \alpha_2\}$, $l > 0, \alpha_1 \geq 0, \alpha_2 \geq 0$ are integers, $a_{11}(k), a_{12}(k), a_{21}(k), a_{22}(k)$, $q(k)$ are real sequences and $\nu_1, \nu_2 \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ are non-decreasing functions with $u\nu_1(u) > 0, u \neq 0, v\nu_2(v) > 0, v \neq 0$. Owing to our discussion, the citing results are verified by the illustrative examples.
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1 Introduction

Neutral differential/difference equation(NDE) is a tool for mathematical model arising in the lossless transmission lines model in high speed computers (see for e.g. [14]). Also, we find some special cases of NDE comprising of linear and nonlinear type in [14]. It is interesting to see that NDE is a medium to describe the model [6] of an insect population with long larval and short adult phases such as the periodical cicada. Seldom, we can extend the

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model [6] into two insect populations when they are interdependent with their life span in the mathematical form of system of NDEs. We refer to the reader the work [6] in which the authors Bocharov and Hadeler have discussed two age-structured population model using system of NDEs. Keeping in view of the dynamic behaviour of such systems, we consider the 2-dimensional first order nonlinear neutral delay difference systems of the form:

$$(FDS1) \qquad \Delta \begin{bmatrix} r(k)+q(k)r(k-l) \\ s(k)+q(k)s(k-l) \end{bmatrix} = \begin{bmatrix} a_{11}(k) & a_{12}(k) \\ a_{21}(k) & a_{22}(k) \end{bmatrix} \begin{bmatrix} \nu_1(r(k-\alpha_1)) \\ \nu_2(s(k-\alpha_2)) \end{bmatrix}, k \geq \rho,$$

where $\rho = \max\{l, \alpha_1, \alpha_2\}$, $l > 0, \alpha_1 \geq 0, \alpha_2 \geq 0$ are integers, $a_{11}(k), a_{12}(k), a_{21}(k), a_{22}(k)$, $q(k)$ are real sequences and $\nu_1, \nu_2 \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ are non-decreasing functions with $u\nu_1(u) > 0$ for $u \neq 0, v\nu_2(v) > 0$ for $v \neq 0$. The objective of our work is to discuss the oscillatory behaviour of all vector solutions $R(k) = [r(k), s(k)]^T$ of $(FDS1)$. The motivation of the present work has come from the work [24] in which the author Tripathy has presented the oscillation criteria for 2-dim linear neutral delay difference systems of the form:

$$(FDS2) \qquad \Delta \begin{bmatrix} r(k)-q(k)r(k-l) \\ s(k)-q(k)s(k-l) \end{bmatrix} = \begin{bmatrix} a_{11}(k) & a_{12}(k) \\ a_{21}(k) & a_{22}(k) \end{bmatrix} \begin{bmatrix} r(k-\alpha_1) \\ s(k-\alpha_2) \end{bmatrix}, k \geq \rho.$$

The system $(FDS2)$ ensures the necessary and sufficient conditions under which all bounded vector solutions of $(FDS2)$ either oscillates or converges to zero as $k \to \infty$. In the literature, we find some works [7], [13], [15], [16], [17], [20] on non-neutral systems, but not like the works [18], [22], [23] which are in closed forms and some neutral systems [4], [5], [7], [8], [9], [10], [11], [21] but not like the works [24], [25], [26] which are in closed forms.

In [25], Tripathy and Das have studied $(FDS1)$ which doesn't meet the requirement for *an all solution oscillatory problem*. Subject to the constant coefficient method may be an alternative, the authors have undertaken the problem with the autonomous delay system

$$\Delta \begin{bmatrix} r(k)-qr(k-l) \\ s(k)-qs(k-l) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} r(k-\alpha_1) \\ s(k-\alpha_2) \end{bmatrix}, k \geq \rho,$$

where $a_{11}, a_{12}, a_{21}, a_{22}, q \in \mathbb{R}$, $l > 1$ and $\alpha_1, \alpha_2 \in \mathbb{N}$. After all, they have gone through an application of the constant coefficient results to the non-linear neutral difference systems

$$\Delta \begin{bmatrix} r(k)-q(k)h_1(r(k-l)) \\ s(k)-q(k)h_2(s(k-l)) \end{bmatrix} + \begin{bmatrix} a_{11}(k) & a_{12}(k) \\ a_{21}(k) & a_{22}(k) \end{bmatrix} \begin{bmatrix} \nu_1(r(k-\alpha_1)) \\ \nu_2(s(k-\alpha_2)) \end{bmatrix} = 0, k \geq \rho$$

by means of linearized oscillation technique for their problem, where

$$\lim_{|s|\to\infty} \frac{h_1(s)}{s} = 1, \quad \lim_{|s|\to\infty} \frac{h_2(s)}{s} = 1.$$

However, the work is restricted to the range $1 < q(k) < \infty$ only. Therefore, summarising the above fact, we emphasize on the problem for the system $(FDS1)$. Introduction to difference equations and system of difference equations, we refer to the monographs by Agarwal et al. [1, 2, 3] and by Elyadi [12].

Definition 1.1. *By a solution of* $(FDS1)$ *we mean a vector* $R(k) = [r(k), s(k)]^T$ *which satisfies* $(FDS1)$ *for* $k \in \mathbb{N}(-\rho) = \{-\rho, -\rho+1, ...0, 1, 2, ...\}$. *We say that the solution* $R(k)$ *oscillates componentwise or simply oscillates or strongly oscillates, if each component oscillates. Otherwise, the solution* $R(k)$ *is called non-oscillatory. Therefore, a solution of* $(FDS1)$ *is non-oscillatory, if it has a component which is eventually positive or eventually negative and strongly non-oscillatory if both components of* $R(k)$ *are non-oscillatory. A vector solution* $R(k)$ *of* $(FDS1)$ *has the property oscillates, if each component of* $R(k)$ *is having the property.*

2 Unbounded Oscillation Criteria

In this section, the sufficient conditions for oscillation of all unbounded vector solutions $R(k) = [r(k), s(k)]^T$ of the system $(FDS1)$ are established.

Theorem 2.1. *Suppose that* $q(k) \geq 0, a_{11}(k) < 0, a_{12}(k) > 0, a_{21}(k) > 0$ *and* $a_{22}(k) < 0$ *for large* k. *Let* $\nu_1, \nu_2 \in BC(\mathbb{R}, \mathbb{R})$ *be such that*
(C_1) $\quad u\nu_1(u) > 0 \text{ and } u\nu_2(u) > 0 \text{ for } u \neq 0.$
If
(C_2) $\quad \sum_{k=0}^{\infty} a_{12}(k) < \infty, \ \ \sum_{k=0}^{\infty} a_{21}(k) < \infty,$
then every unbounded vector solution of $(FDS1)$ *strongly oscillates.*

Proof. If possible, let $R(k) = [r(k), s(k)]^T$ be a strongly nonoscillatory unbounded vector solution of $(FDS1)$ for any large $k \geq k_0 > 0$. Without any loss of generality, we consider the following four cases:
Case − 1 : $r(k) > 0, r(k-l) > 0, r(k-\alpha_1) > 0$ and $s(k) > 0, s(k-l) > 0, s(k-\alpha_2) > 0$ for $k \geq k_1$.
Case − 2 : $r(k) < 0, r(k-l) < 0, r(k-\alpha_1) < 0$ and $s(k) < 0, s(k-l) < 0, s(k-\alpha_2) < 0$ for $k \geq k_1$.
Case − 3 : $r(k) > 0, r(k-l) > 0, r(k-\alpha_1) > 0$ and $s(k) < 0, s(k-l) < 0, s(k-\alpha_2) < 0$ for $k \geq k_1$.
Case − 4 : $r(k) < 0, r(k-l) < 0, r(k-\alpha_1) < 0$ and $s(k) > 0, s(k-l) > 0, s(k-\alpha_2) > 0$ for $k \geq k_1$.
For the system (FDS_1), we define

$$\beta_1(k) = \sum_{j=k}^{\infty} a_{12}(j)\nu_2(s(j-\alpha_2)), \quad \beta_2(k) = \sum_{j=k}^{\infty} a_{21}(j)\nu_1(r(j-\alpha_1));$$
$$h_1(k) = r(k) + q(k)r(k-l), \quad h_2(k) = s(k) + q(k)s(k-l).$$

Therefore, it follows that

$$\Delta[h_1(k) + \beta_1(k)] = a_{11}(k)\nu_1(r(k-\alpha_1)) \leq 0, \tag{2.1}$$
$$\Delta[h_2(k) + \beta_2(k)] = a_{22}(k)\nu_2(r(k-\alpha_2)) \leq 0 \tag{2.2}$$

for $k \geq k_1 > k_0$ due to **Case − 1**. So, we can find $k_2 > k_1$ such that $[h_1(k) + \beta_1(k)]$ and $[h_2(k) + \beta_2(k)]$ are monotonic for $k \geq k_2$. Indeed, $h_1(k) > 0, h_2(k) > 0$ and $\lim_{k\to\infty} \beta_1(k) < \infty$, $\lim_{k\to\infty} \beta_2(k) < \infty$ due to (C_1) and (C_2) implies that $\lim_{k\to\infty} h_1(k)$ and $\lim_{k\to\infty} h_2(k)$ are exist, that is, $h_1(k)$ and $h_2(k)$ are bounded. This leads a contradiction to the fact that $h_1(k) \geq r(k)$ and $h_2(k) \geq s(k)$. The argument for **Case − 2** is similar to that of **Case − 1**.

In **Case − 3**, (2.1) and (2.2) can be viewed as

$$\Delta[h_1(k) + \beta_1(k)] = a_{11}(k)\nu_1(r(k - \alpha_1)) \leq 0, \tag{2.3}$$
$$\Delta[h_2(k) + \beta_2(k)] = a_{22}(k)\nu_2(s(k - \alpha_2)) \geq 0 \tag{2.4}$$

for which $[h_1(k) + \beta_1(k)]$ and $[h_2(k) + \beta_2(k)]$ are monotonic. If $[h_1(k) + \beta_1(k)] > 0$, then $\lim_{k\to\infty}[h_1(k) + \beta_1(k)]$ exist and it is all about the **Case − 1**. If $[h_1(k) + \beta_1(k)] < 0$, then $\beta_1(k)$ is bounded and hence $h_1(k)$ is bounded, a contradiction. If we put $-s(k) = t(k)$ in (2.4), we find

$$\Delta[t(k) + q(k)t(k - l) - \sum_{j=k}^{\infty} a_{21}(j)\nu_1(r(j - \alpha_1))] = a_{22}(k)\nu_2(t(k - \alpha_2))$$

which is similar to (2.3) and hence the argument follows. **Case − 4** is similar to **Case − 3**. This complete the proof of the theorem. □

Theorem 2.2. *Let $-1 < q(k) \leq 0$ for large k. If all conditions of Theorem 2.1 hold, then the conclusion of the theorem remains intact.*

Proof. On the contrary, we proceed as in Theorem 2.1 comprising of four cases. For **Case − 1**, we can find $k_2 > k_1 + \rho$ such that $[h_1(k)+\beta_1(k)]$ and $[h_2(k)+\beta_2(k)]$ are monotonic for $k \geq k_2$. If $[h_1(k) + \beta_1(k)] > 0$, then $\lim_{k\to\infty}[h_1(k) + \beta_1(k)]$ exists and hence $\lim_{k\to\infty} h_1(k)$ exists. Upon the choice of the sign of $h_1(k)$, we need $h_1(k) > -\beta_1(k)$ for $k \geq k_2$ in which $h_1(k) > 0$ for $k \geq k_2$ leads to the fact that $r(k) \geq r(k - l)$ and

$$h_1(k) = r(k) + q(k)r(k - l) \geq r(k - l) + q(k)r(k - l) = r(k - l)(1 + q(k))$$

tends to ∞ as $k \to \infty$, a contradiction. Ultimately, $r(k) \leq r(k - l)$ for $k \geq k_2$. But, again this is also not possible due to

$$r(k) \leq -q(k)r(k - l) < r(k - l) < r(k - 2l) < r(k - 3l) < \cdots < r(k_2) < \infty,$$

that is, $r(k)$ is bounded. The similar argument can be made for $h_2(k)$. If $[h_1(k) + \beta_1(k)] < 0$ for $k \geq k_2$, then $h_1(k) < -\beta_1(k) \leq 0$ implies that $r(k) \leq r(k_2) < \infty$ by the above argument, a contradiction. Proof of the rest cases are analogous and hence the details are omitted. □

Theorem 2.3. *Let $-\infty < q(k) < -1$ for large k. Assume that $\nu_1, \nu_2 \in BC(\mathbb{R}, \mathbb{R})$ and $(C_1), (C_2)$ hold. If*

(C_3) $\quad \sum_{k=0}^{\infty} a_{11}(k) = -\infty, \;\; \sum_{k=0}^{\infty} a_{22}(k) = -\infty,$

then every unbounded vector solution of (FDS_1) strongly oscillates.

Proof. We proceed as in Theorem 2.2 and if $h_1(k) > 0$ for $k > k_2$, then it follows that $r(k) > -q(k)r(k-l) \geq r(k-2l) \geq r(k-3l) \geq \cdots \geq r(k_2)$, that is, $\liminf_{k\to\infty} r(k) > 0$. Hence, we can find a $k_3 > k_2$ and $\tau > 0$ such that $r(k-\alpha_1) \geq \tau$ for $k \geq k_3$. Summing (2.3) from k_3 to ∞, we obtain a contradiction to (C_3) due to

$$\sum_{k=k_3}^{\infty} a_{11}(k)\nu_1(r(k-\alpha_1)) = \sum_{k=k_3}^{\infty} \Delta[h_1(k)+\beta_1(k)],$$

that is,

$$\sum_{k=k_3}^{\infty} a_{11}(k)\nu_1(\tau) > -[h_1(k_3)+\beta_1(k_3)].$$

Similar observation can be made for (2.4). If $h_1(k) < 0$, then owing to the existence of $\lim_{k\to\infty} h_1(k)$, we meant a choice upon $r(k) \geq r(k-l)$ and $r(k) \leq r(k-l)$. As soon as $r(k) \leq r(k-l)$ for $k > k_2$, then $r(k)$ is bounded for $k > k_2$, which is absurd. When $r(k) \geq r(k-l)$ for $k \geq k_2$, then we proceed as above to obtain a contradiction to (C_3). Analogous argument also holds for $h_2(k)$. The rest of the proof is similar and hence the details are omitted. □

3 Some Further Oscillatory Results

In this section, we discuss the oscillation criteria for any vector solution of the system $(FDS1)$.

Theorem 3.1. *Let all conditions of Theorem 2.1 be hold. Assume that*
(C_4) *there exist a subsequence* $\{k_j^*\} \subset \{k\}$ *such that*
$\sum_{j=1}^{\infty} a_{11}(k_j^*) = -\infty = \sum_{l=1}^{\infty} a_{22}(k_l^*)$.
Then every vector solution of $(FDS1)$ *strongly oscillates.*

Proof. Let $R(k) = [r(k), s(k)]^T$ be a strongly nonoscillatory vector solution of $(FDS1)$ for any large $k \geq k_0 > 0$. Setting (2.1) and (2.2) as in Theorem 2.1, we have so called four cases.

Case-1: Let $k_2 > k_1 + \rho$. It follows that $[h_1(k)+\beta_1(k)]$ and $[h_2(k)+\beta_2(k)]$ are monotonic for $k \geq k_2$. If $[h_1(k)+\beta_1(k)] > 0$, then $\lim_{k\to\infty}[h_1(k)+\beta_1(k)]$ exists and therefore, $\lim_{k\to\infty} h_1(k)$ exists. Denote $\beta_1^*(k) = [h_1(k)+\beta_1(k)]$ and $\beta_2^*(k) = [h_2(k)+\beta_2(k)]$. Now, (2.1) becomes

$$\Delta\beta_1^*(k) = a_{11}(k)\nu_1(r(k-\alpha_1)). \tag{3.1}$$

Since $h_1(k) = r(k) + q(k)r(k-l) \geq r(k)$, then we can find $L > 0$ and $k_3 > k_2$ such that $0 \leq r(k) \leq L$ for $k \geq k_3$. Hence, there exist a subsequence $\{k_j^*\} \subset \{k\}$ such that $\liminf_{j\to\infty} r(k_j^* - \alpha_1) \geq L_1$. Rewriting (3.1) for k_j^*, it follows that

$$\Delta\beta_1^*(k_j^*) \leq a_{11}(k_j^*)\nu_1(L_1), \quad k_j^* \geq k_3.$$

Consequently,

$$\nu_1(L_1)\sum_{j=1}^{\infty} a_{11}(k_j^*) \geq \sum_{j=1}^{\infty} \Delta\beta_1^*(k_j^*) > -\infty$$

gives a contradiction to (C_4). The above argument is ultimate for $h_2(k)$. The rest cases follow from Theorem 2.1. This completes the proof of the theorem. □

Theorem 3.2. *Let $-1 < q(k) \leq 0$ for any large k. Assume that all conditions of Theorem 3.1 hold. Then the conclusion of the theorem remains intact.*

Proof. On the contrary, we proceed as in Theorem 3.1 to have four possible cases. For **Case-1**, we can find $k_2 > k_1 + \rho$ such that $[h_1(k) + \beta_1(k)]$ and $[h_2(k) + \beta_2(k)]$ are monotonic for $k \geq k_2$. If $[h_1(k) + \beta_1(k)] > 0$, then $\lim_{k\to\infty}[h_1(k) + \beta_1(k)]$ exist. If $h_1(k) > 0$, then we claim that $r(k)$ is bounded. If not, then $r(k)$ is unbounded. So, we can compare $r(k)$ and $r(k-l)$. Upon the choice of $r(k)$ and $r(k-l)$ for $k \geq k_2$, we obtain the respective contradictions as in Theorem 2.2. So, our claim holds and similar argument can be done, if $h_2(k) > 0$. Next, we come to the case while $h_1(k) < 0(h_2(k) < 0)$, that is, $r(k) < -q(k)r(k-l) < r(k-l) < r(k-2l) < \cdots < r(k_2) < \infty$ and the fact is that $r(k)$ is bounded. Proceeding as in Theorem 3.1, we get a contradiction to (C_5). Returning to the case $[h_1(k) + \beta_1(k)] < 0$ for $k \geq k_2$, we see that $h_1(k) < -\beta_1(k) < 0$ implies that $r(k) < r(k_2) < \infty$, that is, $r(k)$ is bounded. So, we can apply the above argument to go against (C_4). The proof for the rest cases are analogous to Theorem 3.1 and hence the details are omitted. □

Theorem 3.3. *Let $-\infty < b < q(k) \leq -1$ for large k. If all conditions of Theorem 3.2 hold, then the conclusion of the theorem remains intact.*

Proof. On the contrary, the proof follows from the proof of Theorem 3.2. Only, we show that $r(k)$ is bounded in each case. When $h_1(k) > 0$ for $k \geq k_2$, $r(k) > -q(k)r(k-l)$ which is equivalent to say that

$$r(k) > q(k)r(k-l) > r(k-2l) > r(k-3l) > \cdots > r(k_2).$$

Now, we claim that $r(k)$ is bounded. If not, then there exists a subsequence $\{k_j^*\} \subset \{k\}$ such that $r(k_j^* - \alpha_1) > M$. Hence, (3.1) becomes

$$\Delta\beta_1^*(k_j^*) = a_{11}(k_j^*)\nu_1(r(k_j^* - \alpha_1)).$$

Taking summation from $j = 1$ to ∞, we get

$$\sum_{j=1}^{\infty} a_{11}(k_j^*) > \frac{\sum_{j=1}^{\infty} \Delta\beta_1^*(k_j^*)}{\nu_1(M)} > -\infty,$$

a contradiction to (C_4). Therefore, $h_1(k) < 0$ for $k \geq k_2$. It follows that

$$h_1(k) = r(k) + q(k)r(k-l) \geq q(k)r(k-l) \geq br(k-l)$$

implies that $r(k) \geq \frac{1}{b}h_1(k+l)$ and hence $\liminf_{k\to\infty} r(k) > 0$. Proceeding as in Theorem 3.1, we obtain a contradiction to (C_4). This completes the proof of the theorem. □

4 Discussion and Examples

During our discussion, it is observed that unbounded vector solutions of system $(FDS1)$ are strongly oscillatory with the hypotheses (C_2) and (C_3). As soon as we switch on to any vector solution(bounded/unbounded), then we consider (C_4) instead of (C_3). Since (C_3) doesn't imply (C_4) always, then the results of the Section 2 and Section 3 are entrusting the study. Here, we have not seen the existence of nonoscillatory vector solution of $(FDS1)$. However, we take into account the work [25] for the existence results. In the following we state the results without proof:

Theorem 4.1. *Suppose that* $q(k) \geq 0,\ a_{11}(k) < 0, a_{12}(k) > 0, a_{21}(k) > 0,$ *and* $a_{22}(k) < 0$ *for large* k *and let* $\nu_1, \nu_2 \in \mathcal{C}(\mathbb{R}, \mathbb{R})$. *Assume that* ν_1 *and* ν_2 *are Lipschitzian in the intervals of the form* $[\delta_1, \delta_2]$, $-\infty < \delta_1 < \delta_2 < \infty$. *If* (C_1), (C_2) *and*
(C_5) $\quad \sum_{k=0}^{\infty} a_{11}(k) > -\infty,\ \ \sum_{k=0}^{\infty} a_{22}(k) > -\infty$
hold, then $(FDS1)$ *admits a bounded strongly nonoscillatory vector solution.*

Theorem 4.2. *Let* $-1 < q(k) \leq 0$ *for large* k. *If all conditions of Theorem 4.1 hold, then* $(FDS1)$ *admits a bounded strongly nonoscillatory vector solution.*

Theorem 4.3. *Let* $-\infty < q(k) < -1$ *for large* k. *If all conditions of Theorem 4.1 hold, then* $(FDS1)$ *admits a bounded strongly nonoscillatory vector solution.*

Remark 4.4. *In the light of the preceding work, it would be interesting to study the following nonlinear nonautonomous neutral difference systems of the form:*

$$\Delta \begin{bmatrix} r(k) + q(k)r(k-l) \\ s(k) + q(k)s(k-l) \end{bmatrix} = \begin{bmatrix} a_{11}(k) & a_{12}(k) \\ a_{21}(k) & a_{22}(k) \end{bmatrix} \begin{bmatrix} \nu_1(r(k-\alpha_1)) \\ \nu_2(s(k-\alpha_2)) \end{bmatrix} + \begin{bmatrix} f_1(k) \\ f_2(k) \end{bmatrix}, k \geq \rho.$$

Remark 4.5. *In our discussion, the prototype of* ν_1 *and* ν_2 *could be of the type*

$$\begin{bmatrix} \nu_1(u) \\ \nu_2(v) \end{bmatrix} = \begin{bmatrix} \frac{u\ sgn\ u}{\sigma^2+u^2} \\ \frac{v\ sgn\ v}{\eta^2+v^2} \end{bmatrix};\ \ \sigma, \eta \in \mathbb{R} \setminus \{0\}.$$

We conclude this section with the following illustrative examples:

Example 4.6. *Consider the system* $(FDS1)$ *in which* $q(k) = e^{-k}$, $l = 2$, $\alpha_1 = 2$, $\alpha_2 = 4$,

$$a_{11}(k) = -[1 + B(k)] \left[\frac{1}{B(k)} + \frac{e^{-(k+1)}}{B(k)} + e^{-k} \right], a_{12}(k) = \frac{1}{2}[1 + 4B(k)]e^{-(k+1)},$$

$$a_{21}(k) = 2[1 + B(k)]e^{-(k+1)}, a_{22}(k) = -[1 + 4B(k)] \left[\frac{1}{B(k)} + \frac{e^{-(k+1)}}{B(k)} + e^{-k} \right],$$

$$\begin{bmatrix} \nu_1(r(k-\alpha_1)) \\ \nu_2(s(k-\alpha_2)) \end{bmatrix} = \begin{bmatrix} \frac{r(k-2)}{1+r^2(k-2)} \\ \frac{s(k-4)}{1+s^2(k-4)} \end{bmatrix},$$

where

$$B(k) = \begin{cases} 0, \ if\ k\ is\ even, \\ 1, \ if\ k\ is\ odd \end{cases} \quad for\ k \geq 0.$$

Indeed, we can put $B(k)+B(k-1)=1$ for $k \geq 1$. Clearly,

$$\sum_{k=0}^{\infty} a_{12}(k) = \frac{1}{2}\sum_{k=0}^{\infty}[1+4B(k)]e^{-(k+1)} = \frac{1}{2}\sum_{k=0}^{\infty} e^{-(k+1)} + 2\sum_{k=0}^{\infty} B(k)e^{-(k+1)}$$
$$< \frac{1}{2}\sum_{k=0}^{\infty} e^{-(k+1)} + 2\sum_{k=0}^{\infty} e^{-(k+1)} = \frac{5}{2}\sum_{k=0}^{\infty} e^{-(k+1)} < \infty$$

and

$$\sum_{k=0}^{\infty} a_{21}(k) = \sum_{k=0}^{\infty} 2[1+B(k)]e^{-(k+1)} = 2\sum_{k=0}^{\infty} e^{-(k+1)} + 2\sum_{k=0}^{\infty} B(k)e^{-(k+1)}$$
$$< 2\sum_{k=0}^{\infty} e^{-(k+1)} + 2\sum_{k=0}^{\infty} e^{-(k+1)} = 4\sum_{k=0}^{\infty} e^{-(k+1)} < \infty.$$

If we choose the subsequences $a_{11}(2k+1)$ and $a_{22}(2k+1)$, then we find

$$\sum_{k=0}^{\infty} a_{11}(2k+1) = -\sum_{k=0}^{\infty}[1+B(2k+1)]\left[\frac{1}{B(2k+1)} + \frac{e^{-(2k+2)}}{B(2k+1)} + e^{-(2k+1)}\right]$$
$$= -2\sum_{k=0}^{\infty}(1+e^{-2k-2}+e^{-2k-1}) = -\infty,$$

$$\sum_{k=0}^{\infty} a_{22}(2k+1) = -\sum_{k=0}^{\infty}[1+4B(2k+1)]\left[\frac{1}{B(2k+1)} + \frac{e^{-(k+1)}}{B(2k+1)} + e^{-k}\right]$$
$$= -5\sum_{k=0}^{\infty}(1+e^{-(2k+2)}+e^{-(2k+1)}) = -\infty.$$

So, all conditions of Theorem 3.1 are satisfied and hence all vector solutions of $(FDS1)$ are oscillatory. In particular, $[r(k), s(k)]^T = [B(k)(-1)^k, 2B(k)(-1)^k]^T$ is one of such solution of the given system. We notice that

$$\Delta\begin{bmatrix} r(k)+e^{-k}r(k-2) \\ s(k)+e^{-k}s(k-2) \end{bmatrix} = \begin{bmatrix} -(1+e^{-(k+1)})(-1)^k + (\frac{1}{e}-1)(-1)^k e^{-k}B(k) \\ -2(1+e^{-(k+1)})(-1)^k + 2(\frac{1}{e}-1)(-1)^k e^{-k}B(k) \end{bmatrix}$$

and

$$\begin{bmatrix} -[1+B(k)]\left[\frac{1}{B(k)} + \frac{e^{-(k+1)}}{B(k)} + e^{-k}\right] & \frac{1}{2}[1+4B(k)]e^{-(k+1)} \\ 2[1+B(k)]e^{-(k+1)} & -[1+4B(k)]\left[\frac{1}{B(k)} + \frac{e^{-(k+1)}}{B(k)} + e^{-k}\right] \end{bmatrix}\begin{bmatrix} \frac{r^2(k-2)}{1+r^2(k-2)} \\ \frac{s^2(k-4)}{1+s^2(k-4)} \end{bmatrix}$$
$$= \begin{bmatrix} -(1+e^{-(k+1)})(-1)^k + (\frac{1}{e}-1)(-1)^k e^{-k}B(k) \\ -2(1+e^{-(k+1)})(-1)^k + 2(\frac{1}{e}-1)(-1)^k e^{-k}B(k) \end{bmatrix}.$$

Example 4.7. *Consider the system* $(FDS1)$ *in which* $q(k) = e^{-k}$, $l = 2$, $\alpha_1 = \alpha_2 = 2$,

$$a_{11}(k) = -\frac{1+(k-2)^2B(k)}{(k-2)B(k)}\{k+1+e^{-k-1}(k-1)+2e^{-k}(k-2)B(k) - B(k) - e^{-k-1}(k-1)B(k)\},$$

$$a_{12}(k) = \frac{1}{2}\left[1+4(k-2)^2B(k)\right]e^{-k}, a_{21}(k) = 2\left[1+(k-2)^2B(k)\right]e^{-k},$$

$$a_{22}(k) = -\frac{1+4(k-2)^2B(k)}{2(k-2)B(k)}\{2k+2+2e^{-k-1}(k-1)+4e^{-k}(k-2)B(k) - 2B(k) - 2e^{-k-1}(k-1)B(k)\},$$

$$\begin{bmatrix} \nu_1(r(k-\alpha_1)) \\ \nu_2(s(k-\alpha_2)) \end{bmatrix} = \begin{bmatrix} \frac{r(k-2)}{1+r^2(k-2)} \\ \frac{s(k-4)}{1+s^2(k-4)} \end{bmatrix},$$

and $B(k)$ *is same as defined in Example 4.6. Indeed,*

$$\sum_{k=2}^{\infty} a_{11}(k) = -\sum_{k=2}^{\infty} \frac{1+(k-2)^2B(k)}{(k-2)B(k)}\{k+1+e^{-k-1}(k-1)+2e^{-k}(k-2)B(k) - B(k) - e^{-k-1}(k-1)B(k)\} = -\infty$$

and

$$\sum_{k=2}^{\infty} a_{22}(k) = -\sum_{k=2}^{\infty} \frac{1+4(k-2)^2B(k)}{2(k-2)B(k)}\{2k+2+2e^{-k-1}(k-1)+4e^{-k}(k-2)B(k) - 2B(k) - 2e^{-k-1}(k-1)B(k)\} = -\infty.$$

To verify the condition (C_2), *we have*

$$\begin{aligned}\sum_{k=2}^{\infty} a_{12}(k) &= \sum_{k=2}^{\infty} \frac{1}{2}\left[1+4(k-2)^2B(k)\right]e^{-k} \\ &< \frac{1}{2}\sum_{k=0}^{\infty}\left[1+4k^2\right]e^{-k} = \frac{1}{2}\sum_{k=0}^{\infty} e^{-k} + 2\sum_{k=0}^{\infty} k^2e^{-k} \\ &= \frac{e}{2(e-1)} + \frac{2e}{1-e}\sum_{k=0}^{\infty} k^2\Delta(e^{-k}).\end{aligned}$$

Using summation by parts, we see that

$$\frac{e}{1-e}\sum_{k=0}^{\infty} k^2\Delta(e^{-k}) = \frac{1}{e-1}\sum_{k=0}^{\infty}(2k+1)(e^{-k}).$$

Further application of summation by parts, we obtain

$$\frac{1}{e-1}\sum_{k=0}^{\infty}(2k+1)(e^{-k}) = -\frac{e}{(e-1)^2}\sum_{k=0}^{\infty}(2k+1)\Delta(e^{-k})$$
$$= \frac{2}{(e-1)^2}\sum_{k=0}^{\infty}e^{-k} = \frac{2e}{(e-1)^3}.$$

Consequently,

$$\sum_{k=2}^{\infty}a_{12}(k) < \frac{e}{2(e-1)} + \frac{2e}{(e-1)^3}.$$

Similarly,

$$\sum_{k=0}^{\infty}a_{21}(k) = \sum_{k=0}^{\infty}2\left[1+(k-2)^2B(k)\right]e^{-k} < \infty.$$

Hence, all conditions of Theorem 2.1 are satisfied and hence all unbounded vector solutions of $(FDS1)$ *are oscillatory. In particular,* $[r(k),s(k)]^T = [kB(k)(-1)^k, 2kB(k)(-1)^k]^T$ *is one of such unbounded vector solution of the given system.*

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