

VON NEUMANN-STEINHAUS DECOMPOSITION OF PARALLELEPIPEDS

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ABSTRACT. A problem of Steinhaus was to partition a finite interval I of the real line into countably infinitely many pairwise disjoint sets that are congruent in the sense that each set is a translate of a fixed set A . This paper describes von Neumann’s solution of the Steinhaus problem, and an extension to parallelepipeds in \mathbf{R}^n .

1. A PROBLEM OF HUGO STEINHAUS

In a beautiful essay, “A walk through Johnny von Neumann’s garden,” Freeman Dyson [1] describes some of von Neumann’s most important contributions to mathematics, physics, and computer science. He writes:

In another corner of the garden, there is a little flower all by itself, a short paper . . . [that] solves a problem raised by the Polish mathematician Hugo Steinhaus. . . . Johnny solved the Steinhaus problem quickly and never returned to it. The theorem that he proved is counterintuitive, and the proof is astonishing.

Steinhaus’s problem (first stated in 1921 in a paper of Stefan Mazurkiewicz [3]) was to partition a finite interval of the real line into countably infinitely many pairwise disjoint sets that are congruent in the sense that each set is a translate of a fixed set A .

In 1928, von Neumann [4] solved Steinhaus’s problem for open, for closed, and for half-open, half closed intervals. For every finite interval I , von Neumann proved the existence of a set A and a sequence of real numbers $(b_n)_{n=1}^\infty$ such that the congruent sets

$$\{A + b_n : n = 1, 2, 3, \dots\}$$

are pairwise disjoint and

$$I = \bigcup_{n=1}^{\infty} (A + b_n).$$

The set A is necessarily non-measurable.

This note describes von Neumann’s almost forgotten “little flower.” It gives von Neumann’s proof for intervals in \mathbf{R} (perhaps, the first exposition in English) and extends the result to parallelepipeds in \mathbf{R}^n .

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2. ADDITIVE NUMBER THEORY

Steinhaus's problem is easily and naturally restated in the language of additive number theory. Let Ω be a semigroup, written additively, and let A , B , and C be subsets of Ω . We define the *sumset*

$$A + B = \{a + b : a \in A \text{ and } b \in B\}.$$

The set C is the *sum* of the sets A and B if

$$A + B = C.$$

We have the additive identity:

$$C = A + B = \bigcup_{a \in A} (a + B) = \bigcup_{b \in B} (A + b).$$

The translates $a + B$ are congruent for all $a \in A$ and the translates $A + b$ are congruent for all $b \in B$. It follows that the set C is the union of countably infinitely many congruent sets of the form $A + b$ if and only if C is also the union of any number of congruent countably infinite sets of the form $a + B$.

The set C is the *direct sum* of the sets A and B , denoted

$$A \oplus B = C$$

if $A + B = C$ and if every element of C has a *unique* representation in the form $a + b$ with $a \in A$ and $b \in B$. Thus, $A \oplus B = C$ means that if $(a, b) \in A \times B$ and $(a', b') \in A \times B$ satisfy

$$a + b = a' + b'$$

then $(a, b) = (a', b')$. We have $A \oplus B = C$ if and only if the sets $\{A + b : b \in B\}$ are pairwise disjoint if and only if the sets $\{a + B : a \in A\}$ are pairwise disjoint.

This remark gives an equivalent form of Steinhaus's problem. The original problem: Prove that every interval is the union of a countably infinite number of pairwise disjoint uncountable congruent sets. The equivalent problem: Prove that every interval is the union of uncountably many pairwise disjoint sets that are congruent to a countably infinite set.

The following elementary results are used in Section 7.

Lemma 1. *Let $\Omega_1, \dots, \Omega_n$ be semigroups, written additively, and let $\Omega_1 \times \dots \times \Omega_n$ be the semigroup of n -tuples with addition defined coordinate-wise. Let A_i, B_i , and C_i be nonempty subsets of Ω_i for $i \in \{1, 2, \dots, n\}$ and let $A_1 \times \dots \times A_n, B_1 \times \dots \times B_n$, and $C_1 \times \dots \times C_n$ be the corresponding subsets of $\Omega_1 \times \dots \times \Omega_n$. Then*

$$A_i \oplus B_i = C_i$$

for all $i \in \{1, \dots, n\}$ if and only if

$$(A_1 \times \dots \times A_n) \oplus (B_1 \times \dots \times B_n) = C_1 \times \dots \times C_n.$$

Lemma 2. *Let Ω and Ω' be semigroups, written additively, and let $T : \Omega \rightarrow \Omega'$ be a one-to-one semigroup homomorphism. Let A, B , and C be nonempty subsets of Ω . Define the subsets $A' = T(A)$, $B' = T(B)$, and $C' = T(C)$ of Ω' . Then $A \oplus B = C$ if and only if $A' \oplus B' = C'$.*

3. CONSTRUCTION OF THE SET B AND THE GROUP G

Von Neumann proved the following theorem.

Theorem 1. *Every finite interval I is the union of countably infinitely many pairwise disjoint congruent sets. Equivalently, for every finite interval I there is an uncountable set A and a countably infinite set B such that $A \oplus B = I$.*

Let Ω be a σ -algebra of subsets of \mathbf{R} that contains the intervals of \mathbf{R} . A *normalized measure* on Ω is an extended real-valued countably additive nonnegative function μ on Ω such that the measure of an interval is the length of the interval. The measure μ is translation-invariant if $\mu(A + b) = \mu(A)$ for all $A \in \Omega$ and $b \in \mathbf{R}$. In 1905 Giuseppe Vitali [6] (cf. von Neumann [5, Chapter VII]) proved the existence of subsets of the additive abelian group \mathbf{R} that are not measurable with respect to any normalized translation-invariant measure on Ω . Vitali's proof uses the axiom of choice to obtain a set of coset representatives of the quotient group \mathbf{R}/\mathbf{Q} in the unit interval $[0, 1]$. Note that \mathbf{Q} is a countably infinite and dense subgroup of \mathbf{R} .

Adapting Vitali's idea, von Neumann applied the axiom of choice to the intersections of an interval I with the cosets of a different countably infinite and dense subgroup G of \mathbf{R} . The key to the proof is the construction, for every $\varepsilon > 0$, of a countably infinite set B of \mathbf{Q} -independent real numbers with exactly two accumulation points. The subgroup G is the group generated by B , that is, the set of all integral linear combinations of elements of B .

Lemma 3. *Let β_0 and β_1 be real numbers with $\beta_0 < \beta_1$ and let*

$$0 < \varepsilon < \frac{\beta_1 - \beta_0}{2}.$$

There exists a countably infinite set of real numbers $B = \{b_0, b_1, b_2, b_3, \dots\}$ such that

- (1) *the set B is linearly independent over \mathbf{Q} ,*
- (2) *for all $n = 1, 2, 3, \dots$,*

$$\beta_0 < b_0 < b_{2n} < \beta_0 + \varepsilon$$

and

$$\beta_1 - \varepsilon < b_{2n+1} < b_1 < \beta_1$$

- (3)

$$\lim_{n \rightarrow \infty} b_{2n} = b_0 \quad \text{and} \quad \lim_{n \rightarrow \infty} b_{2n+1} = b_1$$

- (4) *for all $\delta > 0$, the set*

$$\{b_n \in B : b_0 + \delta < b_n < b_1 - \delta\}$$

is finite,

- (5) *the additive abelian subgroup G of \mathbf{R} that is generated by B is countably infinite and dense in \mathbf{R} .*

Proof. Let $\{c_0, c_1, c_2, \dots\}$ be any countably infinite set of real numbers that are linearly independent over \mathbf{Q} . For example, if $\theta = \pi$ or e or any transcendental real number, then the set $\{1, \theta, \theta^2, \theta^3, \dots\}$ is linearly independent over \mathbf{Q} . If $(q_n)_{n=0}^\infty$ is any sequence of nonzero rational numbers, then the set $\{q_0 c_0, q_1 c_1, q_2 c_2, \dots\}$ is also linearly independent over \mathbf{Q} .

Choose nonzero rational numbers q_0 and q_1 so that the numbers

$$b_0 = q_0 c_0 \quad \text{and} \quad b_1 = q_1 c_1$$

satisfy the inequalities

$$\beta_0 < b_0 < \beta_0 + \varepsilon \quad \text{and} \quad \beta_1 - \varepsilon < b_1 < \beta_1.$$

Choose $\varepsilon' > 0$ such that

$$b_0 < b_0 + \varepsilon' < \beta_0 + \varepsilon \quad \text{and} \quad \beta_1 - \varepsilon < b_1 - \varepsilon' < b_1.$$

For every positive integer n , choose nonzero rational numbers q_{2n} and q_{2n+1} such that the numbers

$$b_{2n} = q_{2n} c_{2n} \quad \text{and} \quad b_{2n+1} = q_{2n+1} c_{2n+1}$$

satisfy the inequalities

$$b_0 < b_{2n} < b_0 + \frac{\varepsilon'}{n} \quad \text{and} \quad b_1 - \frac{\varepsilon'}{n} < b_{2n+1} < b_1.$$

The sequence $(b_n)_{n=0}^{\infty}$ satisfies properties (1), (2), and (3). Properties (2) and (3) imply property (4).

The subgroup G of \mathbf{R} generated by B is the set of all integral linear combinations of elements of the countably infinite set B and so G is countably infinite. The number $\theta = b_1/b_0$ is irrational because b_0 and b_1 are \mathbf{Q} -independent. By Kronecker's theorem (Hardy and Wright [2, Theorem 439]), the set of numbers $\{n_0 + n_1\theta : n_0, n_1 \in \mathbf{Z}\}$ is dense in \mathbf{R} and so the set of numbers

$$\{b_0(n_0 + n_1\theta) : n_0, n_1 \in \mathbf{Z}\} = \{n_0 b_0 + n_1 b_1 : n_0, n_1 \in \mathbf{Z}\}$$

is also dense in \mathbf{R} . The group G contains this set and so G is dense in \mathbf{R} . This proves (5). \square

The following result, implicit in [4], is used in the proof of Theorem 1.

Lemma 4. *Let $\varepsilon > 0$ and let $\beta_0 = -2\varepsilon$ and $\beta_1 = 2\varepsilon$. Let*

$$B = B(\varepsilon) = \{b_0, b_1, b_2, b_3, \dots\}$$

be the set constructed in Lemma 3, and let $G = G(\varepsilon)$ be the additive subgroup of \mathbf{R} generated by B . Let J be a finite interval of length $\text{length}(J) > 8\varepsilon$. No finite union of pairwise disjoint translates of B contains the set $J \cap G$.

Proof. We have $b_0 = \min(B)$ and $b_1 = \max(B)$, and

$$2\varepsilon < b_1 - b_0 < 4\varepsilon.$$

By Lemma 3, the set B has the property that, for all $\delta > 0$, the set

$$B \cap ([b_0, b_0 + \delta) \cup (b_1 - \delta, b_1])$$

is a cofinite subset of B , and so, for all $a \in \mathbf{R}$ and $\delta > 0$, the set

$$(a + B) \cap ([a + b_0, a + b_0 + \delta) \cup (a + b_1 - \delta, a + b_1])$$

is a cofinite subset of the translate $a + B$. Moreover,

$$\{a + b_0, a + b_1\} \subseteq a + B \subseteq [a + b_0, a + b_1].$$

Let a_1, \dots, a_k be real numbers such that the k translates $a_1 + B, \dots, a_k + B$ are pairwise disjoint. We shall prove that

$$X = \bigcup_{i=1}^k (a_i + B)$$

does not contain $J \cap G$. The endpoints of these translates, that is, the $2k$ numbers $a_i + b_j$ for $i \in \{1, \dots, k\}$ and $j \in \{0, 1\}$, are distinct. Choose δ such that

$$0 < 3\delta < \min \{|(a_i + b_j) - (a_{i'} + b_{j'})| : (i, j) \neq (i', j')\}.$$

This inequality implies that the $2k$ intervals $[a_i + b_0, a_i + b_0 + \delta)$ and $(a_i + b_1 - \delta, a_i + b_1]$ for $i \in \{1, \dots, k\}$ are also pairwise disjoint. Moreover, consecutive intervals are separated by intervals of length at least δ .

We have

$$0 < 3\delta < b_1 - b_0 < \beta_1 - \beta_0 = 4\varepsilon < \text{length}(J)$$

and so the interval J contains a subinterval J^* of length at least δ that is disjoint from the set

$$X^* = \bigcup_{i=1}^k [a_i + b_0, a_i + b_0 + \delta) \cup \bigcup_{i=1}^k (a_i + b_1 - \delta, a_i + b_1].$$

If X contains $J \cap G$, then $X \setminus X^*$ contains $J^* \cap G$. Because G is dense in \mathbf{R} , the interval J^* contains infinitely many points of G , that is, $J^* \cap G$ is infinite. However, the set X^* contains all but finitely many elements of X , that is, $X \setminus X^*$ is finite. The finite set $X \setminus X^*$ does not contain the infinite set $J^* \cap G$, and so X does not contain $J \cap G$. This completes the proof. \square

4. BEGINNING THE PROOF OF VON NEUMANN'S THEOREM

Let $\varepsilon > 0$ and let $\beta_0 = -2\varepsilon$ and $\beta_1 = 2\varepsilon$. Apply Lemma 3 to obtain the set

$$B = B(\varepsilon) = \{b_0, b_1, b_2, \dots\} \subseteq (-2\varepsilon, 2\varepsilon)$$

with accumulation points $b_0 \in (-2\varepsilon, -\varepsilon)$ and $b_1 \in (\varepsilon, 2\varepsilon)$. Let $G = G(\varepsilon)$ be the countably infinite and dense subgroup of \mathbf{R} generated by B . Apply the axiom of choice to obtain an uncountable set V of coset representatives of the quotient group \mathbf{R}/G . Then

$$\mathbf{R} = \bigcup_{v \in V} (v + G).$$

For all $v \in V$, the coset $v + G$ in the quotient group \mathbf{R}/G is also countably infinite and dense in \mathbf{R} .

Let I be an open, closed, or half-open, half-closed interval with

$$0 < 8\varepsilon < \text{length}(I).$$

For all $v \in V$ the translate

$$I_v = I - v = \{x - v : x \in I\}$$

is an interval in \mathbf{R} with $\text{length}(I_v) = \text{length}(I)$. We have $I = v + I_v$ and

$$I \cap (v + G) = (v + I_v) \cap (v + G) = v + (I_v \cap G).$$

Because the cosets $v+G \in \mathbf{R}/G$ partition \mathbf{R} , the sets $\{I \cap (v+G) : v \in V\}$ partition I and

$$I = \bigcup_{v \in V} (I \cap (v+G)) = \bigcup_{v \in V} (v + (I_v \cap G)).$$

Suppose that for all $v \in V$ there is a set A_v such that

$$I_v \cap G = A_v \oplus B.$$

Let

$$A = \bigcup_{v \in V} (v + A_v).$$

We obtain

$$\begin{aligned} I &= \bigcup_{v \in V} (v + (I_v \cap G)) \\ &= \bigcup_{v \in V} (v + (A_v \oplus B)) \\ &= \bigcup_{v \in V} ((v + A_v) \oplus B) \\ &= \left(\bigcup_{v \in V} (v + A_v) \right) \oplus B \\ &= A \oplus B. \end{aligned}$$

The sumset $A \oplus B = I$ is a direct sum because $(v + A_v) \oplus B = I \cap (v + G)$ and the sets $\{I \cap (v + G) : v \in V\}$ are pairwise disjoint. Thus, to solve Steinhaus's problem, it suffices to prove the following result.

Theorem 2. *Let J be a finite interval. Let*

$$0 < 8\varepsilon < \text{length}(J)$$

and let $B = B(\varepsilon)$ be the set and $G = G(\varepsilon)$ the group constructed in Lemma 3 from $\beta_0 = -2\varepsilon$ and $\beta_1 = 2\varepsilon$. Let $C = J \cap G$. There is a set A such that

$$A \oplus B = C.$$

5. A CORE LEMMA

The proof of von Neumann's theorem is inductive and based on the following lemma.

Lemma 5. *Let c_0, c_1 , and ε be real numbers with*

$$0 < \varepsilon < \frac{c_1 - c_0}{8}.$$

Let $B = B(\varepsilon)$ be the set obtained from Lemma 3 with $\beta_0 = -2\varepsilon$ and $\beta_1 = 2\varepsilon$, and let $G = G(\varepsilon)$ be the additive abelian group generated by B . Let J' be the open interval (c_0, c_1) . Let

$$x \in J' \cap G$$

and let a_1, \dots, a_k be real numbers such that

$$x \notin \bigcup_{i=1}^k (a_i + B).$$

There exists a real number a such that

$$(1) \quad x \in a + B \subseteq J' \cap G$$

and

$$(2) \quad (a + B) \cap \bigcup_{i=1}^k (a_i + B) = \emptyset.$$

Proof. We shall prove that property (1) holds for infinitely many numbers a and that property (2) excludes only finitely many numbers a for which property (1) holds.

Let $B = \{b_n : n = 0, 1, 2, \dots\}$, where $b_0 = \min(B)$ and $b_1 = \max(B)$ are the accumulation points of B , and, for all positive integers n , there are the inequalities

$$-2\varepsilon < b_0 < b_{2n} < -\varepsilon$$

and

$$\varepsilon < b_{2n+1} < b_1 < 2\varepsilon.$$

It follows that $2\varepsilon < b_1 - b_0 < 4\varepsilon$.

We have $x \in a + B$ if and only if $x = a + b_j$ for some $b_j \in B$ if and only if $a = x - b_j$ for some $b_j \in B$. Because $x \in G$ and the set B generates G , it follows that, for all $b_j \in B$, if $a = x - b_j$, then

$$a + B = \{a + b_n : b_n \in B\} = \{x - b_j + b_n : b_n \in B\} \subseteq G.$$

We shall prove that $a + B \subseteq J'$ for infinitely many a , or, equivalently, that

$$\{x - b_j + b_n : b_n \in B\} \subseteq J'$$

for infinitely many $b_j \in B$. For all $b_n \in B$ we have

$$x - b_j + b_0 \leq x - b_j + b_n \leq x - b_j + b_1.$$

Therefore, $x - b_j + B \subseteq (c_0, c_1)$ if

$$c_0 < x - b_j + b_0 \quad \text{and} \quad x - b_j + b_1 < c_1$$

or, equivalently, if

$$x - c_1 + b_1 < b_j < x - c_0 + b_0.$$

Thus, it suffices to prove that the open interval

$$J'' = (x - c_1 + b_1, x - c_0 + b_0)$$

contains infinitely many numbers $b_j \in B$. We shall prove that J'' contains either b_0 or b_1 . Because b_0 and b_1 are accumulation points of B , it follows that the open interval J'' contains infinitely many numbers $b_j \in B$.

The interval J'' has length

$$\begin{aligned} \text{length}(J'') &= (x - c_0 + b_0) - (x - c_1 + b_1) \\ &= (c_1 - c_0) - (b_1 - b_0) > 8\varepsilon - 4\varepsilon \\ &= 4\varepsilon > b_1 - b_0. \end{aligned}$$

Because $x \in J' = (c_0, c_1)$, we have

$$x - c_1 + b_1 < b_1 \quad \text{and} \quad b_0 < x - c_0 + b_0.$$

There are two cases. In the first case, $x - c_1 + b_1 < b_0$ and

$$x - c_1 + b_1 < b_0 < x - c_0 + b_0$$

and so $b_0 \in J''$.

In the second case, $x - c_1 + b_1 \geq b_0$. If $b_1 \geq x - c_0 + b_0$, then $J'' \subseteq (b_0, b_1)$ and $b_1 - b_0 < \text{length}(J'') \leq b_1 - b_0$, which is absurd. Therefore,

$$x - c_1 + b_1 < b_1 < x - c_0 + b_0$$

and $b_1 \in J''$. This proves that $x \in a + B \subseteq J' \cap G$ for infinitely many $a \in \mathbf{R}$.

Next we prove that there are only finitely many a such that

$$x \in a + B \quad \text{and} \quad (a + B) \cap \bigcup_{i=1}^k (a_i + B) \neq \emptyset.$$

For all $i \in \{1, \dots, k\}$, the set $a_i + B$ is a translate of B that does not contain x . We shall prove that there are at most two translates of B that both contain x and intersect $a_i + B$.

Let $i \in \{1, 2, \dots, k\}$. For $h \in \{1, 2, 3\}$, let v_h be a real number such that

$$x \in v_h + B \quad \text{and} \quad (v_h + B) \cap (a_i + B) \neq \emptyset.$$

There exist elements $b_{p_h} \in B$ such that

$$v_h = x - b_{p_h}$$

and

$$(x - b_{p_h} + B) \cap (a_i + B) \neq \emptyset.$$

It follows that there are elements b_{q_h} and b_{r_h} in B such that

$$x - b_{p_h} + b_{q_h} = a_i + b_{r_h}$$

and so

$$x - a_i = b_{p_h} - b_{q_h} + b_{r_h}$$

for $h \in \{1, 2, 3\}$. We obtain

$$(3) \quad b_{p_1} + b_{q_2} + b_{r_1} = b_{p_2} + b_{q_1} + b_{r_2}$$

and

$$(4) \quad b_{p_1} + b_{q_3} + b_{r_1} = b_{p_3} + b_{q_1} + b_{r_3}.$$

Because the set B is \mathbf{Q} -independent, relation (3) implies

$$b_{p_2} = b_{p_1} \quad \text{or} \quad b_{p_2} = b_{q_2} \quad \text{or} \quad b_{p_2} = b_{r_1}$$

and relation (4) implies

$$b_{p_3} = b_{p_1} \quad \text{or} \quad b_{p_3} = b_{q_3} \quad \text{or} \quad b_{p_3} = b_{r_1}.$$

Suppose that $b_{p_2} \neq b_{p_1}$. If $b_{p_2} = b_{q_2}$, then

$$x - a_i = b_{r_2}$$

and $x = a_i + b_{r_2} \in a_i + B$, which is absurd. Therefore, $b_{p_2} \neq b_{p_1}$ implies $b_{p_2} = b_{r_1}$.

Suppose that $b_{p_3} \neq b_{p_1}$. If $b_{p_3} = b_{q_3}$, then

$$x - a_i = b_{r_3}$$

and $x = a_i + b_{r_3} \in a_i + B$, which is absurd. Therefore, $b_{p_3} \neq b_{p_1}$ implies $b_{p_3} = b_{r_1}$.

It follows that either $b_{p_1} = b_{p_2}$ or $b_{p_1} = b_{p_3}$ or $b_{p_2} = b_{r_1} = b_{p_3}$, and so at most two translates of B contain x and also intersect $a_i + B$ for some $i \in \{1, \dots, k\}$.

Therefore, at most $2k$ translates of B contain x and intersect $\bigcup_{i=1}^k A_i$. Because there are infinitely many translates of B that contain x and are contained in $J' \cap G$, the pigeon hole principle completes the proof of the Lemma. \square

6. FINISHING THE PROOF OF VON NEUMANN'S THEOREM

Let J be a finite interval J with endpoints $\inf(J) = c_0$ and $\sup(J) = c_1$. Let

$$0 < 8\varepsilon < \text{length}(J) = c_1 - c_0.$$

Let $B = B(\varepsilon) = \{b_0, b_1, b_2, \dots\}$ be the set and $G = G(\varepsilon)$ the group constructed in Lemma 3 with $\beta_0 = -2\varepsilon$ and $\beta_1 = 2\varepsilon$. The numbers $b_0 = \min(B)$ and $b_1 = \max(B)$ satisfy

$$2\varepsilon < b_1 - b_0 < 4\varepsilon.$$

Let

$$C = J \cap G.$$

Because G is dense in J , we have $\inf(C) = c_0$ and $\sup(C) = c_1$. The set C does not necessarily contain the numbers c_0 and c_1 .

If $c_0 \in C$, then

$$c_0 \in (c_0 - b_0) + B = \{c_0 - b_0 + b_n : b_n \in B\} \subseteq G.$$

The inequality

$$c_0 \leq c_0 - b_0 + b_n \leq c_0 + (b_1 - b_0) < c_0 + 4\varepsilon < c_1$$

implies $c_0 - b_0 + b_n \in J$ for all $b_n \in B$ and so $(c_0 - b_0) + B \subseteq J$. Therefore,

$$(c_0 - b_0) + B \subseteq J \cap G = C.$$

Similarly, if $c_1 \in C$, then

$$(c_1 - b_1) + B \subseteq J \cap G = C.$$

Suppose that $c_0 \in C$ and $c_1 \in C$. If

$$y \in ((c_0 - b_0) + B) \cap ((c_1 - b_1) + B)$$

then there exist $b_p, b_q \in B$ such that

$$y = c_0 - b_0 + b_p = c_1 - b_1 + b_q$$

and so

$$8\varepsilon < c_1 - c_0 = (b_p - b_0) + (b_1 - b_q) < 4\varepsilon + 4\varepsilon = 8\varepsilon$$

which is absurd. Therefore, if $c_0 \in C$ and $c_1 \in C$, then the translates $(c_0 - b_0) + B$ and $(c_1 - b_1) + B$ are disjoint.

We shall now prove Theorem 2. Because G is dense in J and countably infinite, the subset C of G is countably infinite. Let

$$C = \{x_n : n = 0, 1, 2, \dots\}$$

be an enumeration of the elements of C that satisfies the following initial conditions:

- (i) If $c_0 \in C$ and $c_1 \in C$, then $x_0 = c_0$ and $x_1 = c_1$. Let $a_0 = x_0 - b_0$ and $a_1 = x_1 - b_1$. We have $x_0 \in a_0 + B$ and $x_1 \in a_1 + B$. The translates $a_0 + B$ and $a_1 + B$ are disjoint and contained in C .
- (ii) If $c_0 \in C$ but $c_1 \notin C$, then $x_0 = c_0$. Let $a_0 = x_0 - b_0$. We have $x_0 \in a_0 + B \subseteq C$.
- (iii) If $c_0 \notin C$ but $c_1 \in C$, then $x_0 = c_1$. Let $a_0 = x_0 - b_1$. We have $x_0 \in a_0 + B \subseteq C$.
- (iv) If $c_0 \notin C$ and $c_1 \notin C$, then let x_0 be any element of C and let $a_0 + B$ be any translate of B such that $x_0 \in a_0 + B \subseteq C$. Lemma 5 implies the existence of such translates.

We use induction to construct an infinite sequence $(a_j + B)_{j=0}^{\infty}$ of pairwise disjoint translates of B such that $A \oplus B = C$, where $A = \{a_j : j = 0, 1, 2, \dots\}$.

Suppose that $n \geq 0$ and that $\{a_j + B : j = 0, 1, \dots, n\}$ are pairwise disjoint translates of B such that

$$\{x_0, x_1, \dots, x_n\} \subseteq \bigcup_{j=0}^n (a_j + B) \subseteq C.$$

By Lemma 4, the set $\bigcup_{j=0}^n (a_j + B)$ is a proper subset of C . Let ℓ be the smallest positive integer such that $x_\ell \notin \bigcup_{j=0}^n (a_j + B) \subseteq C$. Note that $\ell \geq n + 1$. By the core Lemma 5, there is a translate $a_{n+1} + B \subseteq C$ such that

$$x_\ell \in a_{n+1} + B$$

$$(a_{n+1} + B) \cap \left(\bigcup_{j=0}^n (a_j + B) \right) = \emptyset$$

and

$$\{x_0, x_1, \dots, x_n, x_{n+1}, \dots, x_\ell\} \subseteq \bigcup_{j=0}^{n+1} (a_j + B) \subseteq C.$$

Continuing inductively, we obtain a countably infinite set $A = \{a_0, a_1, a_2, \dots\}$ such that

$$A + B = \bigcup_{j=0}^{\infty} (a_j + B) = C.$$

This completes the proof of von Neumann's theorem.

7. EXTENSION TO PARALLELEPIPEDS

A parallelepiped in \mathbf{R}^n is a set of the form

$$P = \left\{ \sum_{i=1}^n t_i \mathbf{v}_i : t_i \in I_i \text{ for all } i \in \{1, \dots, n\} \right\}$$

where $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a linearly independent set of vectors and $\{I_1, I_2, \dots, I_n\}$ is a set of intervals that may be open, closed, or half open, half closed. We shall extend von Neumann's theorem to parallelepipeds.

Theorem 3. *Every parallelepiped in \mathbf{R}^n is the union of countably infinitely many pairwise disjoint congruent sets.*

Proof. Let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ be the standard basis in \mathbf{R}^n . Let $\{I_1, I_2, \dots, I_n\}$ be a set of intervals that may be open, closed, or half open, half closed. Consider the rectangular parallelepiped

$$P_0 = I_1 \times I_2 \times \dots \times I_n = \left\{ \sum_{i=1}^n t_i \mathbf{e}_i : t_i \in I_i \text{ for all } i \in \{1, \dots, n\} \right\}$$

and the parallelepiped

$$P = I_1 \times I_2 \times \dots \times I_n = \left\{ \sum_{i=1}^n t_i \mathbf{v}_i : t_i \in I_i \text{ for all } i \in \{1, \dots, n\} \right\}$$

where $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a linearly independent set of vectors. By Theorem 1, there are infinite sets A_1, A_2, \dots, A_n and countably infinite sets B_1, B_2, \dots, B_n such that

$$I_i = A_i \oplus B_i$$

for all $i \in \{1, \dots, n\}$. Let

$$A = A_1 \times \dots \times A_n \subseteq \mathbf{R}^n$$

and

$$B = B_1 \times \dots \times B_n \subseteq \mathbf{R}^n$$

The set B is countably infinite. It follows from Lemma 1 that

$$P_0 = \prod_{i=1}^n I_i = \prod_{i=1}^n (A_i \oplus B_i) = (A_1 \times \dots \times A_n) \oplus (B_1 \times \dots \times B_n) = A \oplus B$$

and so P_0 is the union of countably infinitely many pairwise disjoint congruent sets.

Define the linear isomorphism $T : \mathbf{R}^n \rightarrow \mathbf{R}^n$ by $T(\mathbf{e}_i) = \mathbf{v}_i$ for all $i \in \{1, 2, \dots, n\}$. Then $T(P_0) = P$. By Lemma 2, the parallelepiped P is the union of countably infinitely many pairwise disjoint congruent sets. This completes the proof. \square

It is not clear if decomposition into countably infinitely many pairwise disjoint congruent sets is possible for other geometrical objects, such as, for example, triangles in the plane.

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