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SOME OPEN-NEWTON-COTES TYPE INEQUALITIES FOR CONVEX FUNCTIONS IN FRACTIONAL CALCULUS

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ABSTRACT. The main goal of this paper is to establish some error bounds for Open-Newton-Cotes formula with $n = 1$ for differentiable convex functions in fractional calculus. For this, first we prove an integral identity having Riemann-Liouville fractional integrals and ordinary derivative. Then, using this identity we establish some error bounds for Open-Newton-Cotes formula with $n = 1$ for differentiable convex functions in fractional calculus. It is worth to mention that these error bounds are very important in error analysis because with the help of them error bounds can be found for particular function. We also give some applications for special means. Finally, we add an examples and show the validity of inequalities with a graph for different values of fractional parameter α.

1. Introduction

The area of mathematics known as mathematical analysis covers the theory of measure, limits, differentiation, integration, and convex functions. Convex functions are fundamental as positive or increasing functions, and they have emerged as a key topic in the field of mathematical analysis research.

Inequalities are at the core of mathematical analysis, and they have developed into a crucial tool in that process up until the early 20th century, when we started to view them as a separate field of modern mathematics. The pioneering work in this field was the book "Inequalities" [1] by Hardy, Littlewood, and Pólya. Other books (see, e.g., [2], [3]) are of great value in this field as well.

In recent years, many researchers have developed numerical integration formulas and found their error bounds using different techniques. To determine the error bounds of numerical integration formulas, mathematical inequalities are used, and the authors used various functions such as convex functions, bounded functions, Lipschitzian functions, and so on. For example, some error bounds for trapezoidal and midpoint formulas of numerical integration using the convex functions were found in [4, 5]. A number of papers have been published on the error bounds of Simpson's formula using the convex functions in different calculi and some of these bounds can be found in [6, 7, 8, 9, 10, 11, 12, 13]. Some error bounds for Newton's formula in numerical integration have also been established by using the convex functions in different calculi and these bounds can be found in [14, 15, 16, 17, 18]. In open Newton-cotes formulas, Milne's formula is very important and its error bounds for four times twice differentiable functions were found in [19]. In [20], the authors used general form of the convexity and established some new Maclaurin's formula type inequalities and discussed their applications.

In this paper, we will use the well-known Riemann-Liouville fractional integrals (RLFIs) that are given below:

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Definition 1.1. [20, 40] Let $f \in L_1[a,b]$. The (RLFIs) $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ with $a \geq 0$ and order $\alpha > 0$ are given as:

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt,\ x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt, x < b,$$

respectively, where Γ is the well-known Gamma function.

However, because of their significance, researchers have used fractional calculus to create a variety of fractional integral inequalities that are useful in approximation theory. The bounds of mathematical integration formulas can be determined using inequalities such as Hermite-Hadamard, Simpson's, midpoint, Ostrowski's, and trapezoidal inequalities. In [21], the Hermite-Hadamard type inequality and the bounds for trapezoidal formula were established. Differentiable convexity was used in Set [22] to establish fractional Ostrowski's type inequalities. Through the use of Riemann-Liouville fractional integrals (RLFIs), İşcan and Wu [23] established certain bounds for numerical integration as well as an inequality of the Hermite-Hadamard type for reciprocal convex functions. Sarikaya and Yildirim established the midpoint bounds and a new version of the fractional inequality of the Hermite-Hadamard type in [24]. Sarikaya et al. [25] used the general convexity and RLFIs to get the bounds for Simpson's $1/3$ formula. In [26], the authors used the RLFIs to discover some new boundaries for Simpson's $1/3$ formula. The s-convexity was utilized by the authors of [27] to analyse different Simpson's $1/3$ formula bounds. Generalized RLFIs were introduced as a new class of fractional integrals in 2020 by Sarikaya and Ertugral [28], they also established Hermite-Hadamard type inequalities related to the newly defined class of integrals. The ability to be transformed into the classical integral, RLFIs, k-RLFIs, Hadamard fractional integrals, etc. is the main benefit of the newly defined class of fractional integral operators. Zhao et al. used generalized RLFIs and reciprocal convex functions in [29] to get some bounds for a trapezoidal formula. Using the generalized RLFIs, Budak et al. [30] found certain approximations for Simpson's $1/3$ formula for differentiable convex functions.

Recently, Sitthiwirattham et al. [31] found some bounds for Simpson's $3/8$ formula using the RLFIs. For further inequalities that can be addressed using fractional integrals, see [32, 33, 34, 35, 36, 37, 38, 39] and the references therein.

Inspired by the ongoing studies, we establish some new error bounds for one of the Open-Newton-Cotes formulas with in the setting of fractional calculus. We use RLFIs and establish the bounds for differentiable convex functions and give some examples to show the validation of these new bounds. These error bounds or inequalities are very important in error analysis because with the help of them one can find the error bounds of Open-Newton-Cotes formula for $n = 1$ (see, [40, p. 200]).

2. Main Results

In this part, we give some inequalities related to Open-Newton-Cotes formulas for differentiable convex functions in the setting of fractional calculus.

Lemma 2.1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function over (a,b). If $f' \in L_1[a,b]$, then the following equality holds:*

$$\begin{aligned}
&\frac{1}{2}\left[f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right]-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right] \\
&= \frac{(b-a)}{2}\left[\int_0^{\frac{1}{3}} t^{\alpha}\left[f'(tb+(1-t)a)-f'(ta+(1-t)b)\right]dt\right. \\
&\quad +\int_{\frac{1}{3}}^{\frac{2}{3}}\left(t^{\alpha}-\frac{1}{2}\right)\left[f'(tb+(1-t)a)-f'(ta+(1-t)b)\right]dt \\
&\quad \left.+\int_{\frac{2}{3}}^{1}\left(t^{\alpha}-1\right)\left[f'(tb+(1-t)a)-f'(ta+(1-t)b)\right]dt\right].
\end{aligned} \tag{2.1}$$

Proof. The right-side of (2.1) gives

$$\begin{aligned}
&\frac{(b-a)}{2}\left[\int_0^{\frac{1}{3}} t^{\alpha}\left[f'(tb+(1-t)a)-f'(ta+(1-t)b)\right]dt\right. \\
&\quad +\int_{\frac{1}{3}}^{\frac{2}{3}}\left(t^{\alpha}-\frac{1}{2}\right)\left[f'(tb+(1-t)a)-f'(ta+(1-t)b)\right]dt \\
&\quad \left.+\int_{\frac{2}{3}}^{1}\left(t^{\alpha}-1\right)\left[f'(tb+(1-t)a)-f'(ta+(1-t)b)\right]dt\right] \\
&= \frac{(b-a)}{2}\left[I_1-I_2+I_3-I_4+I_5-I_6\right].
\end{aligned} \tag{2.2}$$

From integration by parts, we have

$$\begin{aligned}
I_1 &= \int_0^{\frac{1}{3}} t^{\alpha} f'(tb+(1-t)a)\,dt \\
&= \frac{1}{b-a}\left[t^{\alpha} f(tb+(1-t)a)\Big|_0^{\frac{1}{3}}-\alpha\int_0^{\frac{1}{3}} t^{\alpha-1} f(tb+(1-t)a)\,dt\right] \\
&= \frac{1}{b-a}\left[\left(\frac{1}{3}\right)^{\alpha} f\left(\frac{2a+b}{3}\right)-\alpha\int_0^{\frac{1}{3}} t^{\alpha-1} f(tb+(1-t)a)\,dt\right],
\end{aligned}$$

$$\begin{aligned}
I_3 &= \int_{\frac{1}{3}}^{\frac{2}{3}}\left(t^{\alpha}-\frac{1}{2}\right) f'(tb+(1-t)a)\,dt \\
&= \frac{1}{b-a}\left[\left(\left(\frac{2}{3}\right)^{\alpha}-\frac{1}{2}\right) f\left(\frac{a+2b}{3}\right)-\left(\left(\frac{1}{3}\right)^{\alpha}-\frac{1}{2}\right) f\left(\frac{2a+b}{3}\right)\right. \\
&\quad \left.-\alpha\int_{\frac{1}{3}}^{\frac{2}{3}} t^{\alpha-1} f(tb+(1-t)a)\,dt\right]
\end{aligned}$$

and

$$\begin{aligned} I_5 &= \int_{\frac{2}{3}}^{1} \left(t^{\alpha}-1\right) f'\left(tb+(1-t)a\right)dt \\ &= \frac{1}{b-a}\left[\left(1-\left(\frac{2}{3}\right)^{\alpha}\right) f\left(\frac{a+2b}{3}\right) - \alpha \int_{\frac{2}{3}}^{1} t^{\alpha-1} f\left(tb+(1-t)a\right)dt\right]. \end{aligned}$$

Thus from RLFIs, we have

$$\frac{(b-a)}{2}\left[I_1+I_3+I_5\right] = \frac{1}{4}\left[f\left(\frac{a+2b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right] - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} J_{b-}^{\alpha} f(a). \tag{2.3}$$

Similarly, we have

$$\begin{aligned} & -\frac{(b-a)}{2}\left[I_2+I_4+I_6\right] \\ &= \frac{1}{4}\left[f\left(\frac{a+2b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right] - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} J_{a+}^{\alpha} f(b). \end{aligned} \tag{2.4}$$

Hence, we get the required identity by plugging (2.3) and (2.4) in (2.2). □

Theorem 2.2. *Let f satisfies assumptions of Lemma 2.1. If $|f'|$ is convex function, then the following inequality holds:*

$$\begin{aligned} & \left|\frac{1}{2}\left[f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right] - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b)+J_{b-}^{\alpha} f(a)\right]\right| \\ \leq & \frac{(b-a)}{2}\left[\frac{1+2^{\alpha+1}+(\alpha-2)3^{\alpha}}{3^{\alpha+1}(\alpha+1)} + \frac{2^{\alpha+1}-1}{3^{\alpha+1}(\alpha+1)} - \frac{1}{6} + A_1(\alpha)\right]\left[\left|f'(a)\right|+\left|f'(b)\right|\right], \end{aligned} \tag{2.5}$$

where

$$A_1(\alpha) = \begin{cases} \frac{2^{\alpha+1}-1}{3^{\alpha+1}(\alpha+1)} - \frac{1}{6}, & 0<\alpha\leq \frac{\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{1}{3}\right)} \\ \left(\frac{1}{2}\right)^{\frac{1}{\alpha}} + \frac{2^{\alpha+1}+1}{3^{\alpha+1}(\alpha+1)} - 2\frac{\left(\frac{1}{2}\right)^{\frac{\alpha+1}{\alpha}}}{\alpha+1} - \frac{1}{2}, & \frac{\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{1}{3}\right)} < \alpha \leq \frac{\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{2}{3}\right)} \\ \frac{1}{6} - \frac{2^{\alpha+1}-1}{3^{\alpha+1}(\alpha+1)}, & \alpha > \frac{\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{2}{3}\right)}. \end{cases}$$

Proof. By taking modulus in (2.1) and using convexity of $|f'|$, we have

$$\begin{aligned} & \left\|\frac{1}{2}\left[f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right] - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b)+J_{b-}^{\alpha} f(a)\right]\right\| \\ = & \frac{(b-a)}{2}\left[\int_{0}^{\frac{1}{3}} t^{\alpha}\left[\left|f'(tb+(1-t)a)\right|+\left|f'(ta+(1-t)b)\right|\right]dt\right. \\ & + \int_{\frac{1}{3}}^{\frac{2}{3}} \left|t^{\alpha}-\frac{1}{2}\right| \left[\left|f'(tb+(1-t)a)\right|+\left|f'(ta+(1-t)b)\right|\right]dt \\ & \left. + \int_{\frac{2}{3}}^{1} \left(1-t^{\alpha}\right)\left[\left|f'(tb+(1-t)a)\right|+\left|f'(ta+(1-t)b)\right|\right]dt\right] \end{aligned}$$

$$\leq \frac{(b-a)}{2}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]\left[\int_0^{\frac{1}{3}} t^\alpha dt+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|t^\alpha-\frac{1}{2}\right|dt+\int_{\frac{2}{3}}^{1}(1-t^\alpha)\,dt\right]$$

$$= \frac{(b-a)}{2}\left[\frac{1}{3^{\alpha+1}(\alpha+1)}+A_1(\alpha)+\frac{2^{\alpha+1}+(\alpha-2)3^\alpha}{3^{\alpha+1}(\alpha+1)}\right]\left[\left|f'(a)\right|+\left|f'(b)\right|\right].$$

Thus, the proof is completed. □

Remark 2.3. When we set $\alpha=1$, then we have the following inequality:

$$\left|\frac{1}{2}\left[f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{5(b-a)}{72}\left[\left|f'(a)\right|+\left|f'(b)\right|\right].$$

Theorem 2.4. *If all conditions of Lemma 2.1 hold and $|f'|^q$, $q>1$ is convex function, then the following inequality holds:*

$$\left|\frac{1}{2}\left[f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right]-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]\right| \tag{2.6}$$

$$\leq (b-a)\left[\left(\frac{1}{3^{\alpha p+1}(\alpha p+1)}\right)^{\frac{1}{p}}\left(\left(\frac{|f'(b)|^q+5|f'(a)|^q}{18}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+5|f'(b)|^q}{18}\right)^{\frac{1}{q}}\right)\right.$$
$$\left.+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|t^\alpha-\frac{1}{2}\right|^p dt\right)^{\frac{1}{p}}\left(\frac{|f'(b)|^q+|f'(a)|^q}{6}\right)^{\frac{1}{q}}\right],$$

where $p+q=pq$.

Proof. Taking modulus of inequality (2.1) and using Hölder inequality, we have

$$\left|\frac{1}{2}\left[f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right]-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]\right|$$

$$\leq \frac{(b-a)}{2}\left[\left(\int_0^{\frac{1}{3}} t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\left(\int_0^{\frac{1}{3}}\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right.\right.$$
$$\left.+\left(\int_0^{\frac{1}{3}}\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right)$$
$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|t^\alpha-\frac{1}{2}\right|^p dt\right)^{\frac{1}{p}}\left(\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right)$$

$$+\left(\int_{\frac{2}{3}}^{1}(1-t)^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_{\frac{2}{3}}^{1}\left|f'(tb+(1-t)a)\right|^{q}dt\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_{\frac{2}{3}}^{1}\left|f'(ta+(1-t)b)\right|^{q}dt\right)^{\frac{1}{q}}\right].$$

We have the following relation by using the convexity of $|f'|^q$, $q>1$

$$\left|\frac{1}{2}\left[f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right]-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\right|$$
$$\leq \frac{(b-a)}{2}\left[\left(\int_0^{\frac{1}{3}}t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\left(\int_0^{\frac{1}{3}}\left[t\left|f'(b)\right|^q+(1-t)\left|f'(a)\right|^q\right]dt\right)^{\frac{1}{q}}\right.\right.$$
$$\left.+\left(\int_0^{\frac{1}{3}}\left[t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right)$$
$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|t^{\alpha}-\frac{1}{2}\right|^p dt\right)^{\frac{1}{p}}\left(\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left[t\left|f'(b)\right|^q+(1-t)\left|f'(a)\right|^q\right]dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left[t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right)$$
$$+\left(\int_{\frac{2}{3}}^{1}(1-t^{\alpha})^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{2}{3}}^{1}\left[t\left|f'(b)\right|^q+(1-t)\left|f'(a)\right|^q\right]dt\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_{\frac{2}{3}}^{1}\left[t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right]$$
$$=\frac{(b-a)}{2}\left[\left(\frac{1}{3^{\alpha p+1}(\alpha p+1)}\right)^{\frac{1}{p}}\right.$$
$$\times\left(\left(\frac{\left|f'(b)\right|^q+5\left|f'(a)\right|^q}{18}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+5\left|f'(b)\right|^q}{18}\right)^{\frac{1}{q}}\right)$$
$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|t^{\alpha}-\frac{1}{2}\right|^p dt\right)^{\frac{1}{p}}\left(\left(\frac{\left|f'(b)\right|^q+\left|f'(a)\right|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{6}\right)^{\frac{1}{q}}\right)$$
$$\left.+\left(\frac{1}{3^{\alpha p+1}(\alpha p+1)}\right)^{\frac{1}{p}}\left(\left(\frac{5\left|f'(b)\right|^q+\left|f'(a)\right|^q}{18}\right)^{\frac{1}{q}}+\left(\frac{5\left|f'(a)\right|^q+\left|f'(b)\right|^q}{18}\right)^{\frac{1}{q}}\right)\right].$$

Thus, the proof is completed. □

Remark 2.5. When we set $\alpha = 1$ in Theorem 2.4, we have the following inequality:

$$\begin{aligned} &\left|\frac{1}{2}\left[f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ \leq\ & (b-a)\left[\left(\frac{1}{3^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\left(\frac{|f'(b)|^q+5|f'(a)|^q}{18}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+5|f'(b)|^q}{18}\right)^{\frac{1}{q}}\right)\right. \\ &\left.+\left(\frac{1}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{|f'(b)|^q+|f'(a)|^q}{6}\right)^{\frac{1}{q}}\right]. \end{aligned}$$

3. Examples

In this section, we give some mathematical examples and their graphs to show the validity of new inequalities.

Example 3.1. Let $f:[1,2]\to\mathbb{R}$ be a function such that $f(x)=x^2$ and $f'(x)=2x$ is a convex function, then from Theorem 2.2

$$\begin{aligned} LHS &= \left|\frac{1}{2}\left[f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right]-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]\right| \\ &= \left|\frac{41}{18}-\frac{(10+\alpha(13+5\alpha))\Gamma(1+\alpha)}{2\Gamma(3+\alpha)}\right| \end{aligned}$$

and

$$RHS = 3\left[\frac{1+2^{\alpha+1}+(\alpha-2)3^\alpha}{3^{\alpha+1}(\alpha+1)}+A_1(\alpha)\right].$$

Since $A_1(\alpha)$ has three cases, therefore for the first case when $0<\alpha\leq\frac{\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{1}{3}\right)}$ the figure 1 show that the $LHS<RHS$. For the second case of $A_1(\alpha)$ when $\frac{\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{1}{3}\right)}<\alpha\leq\frac{\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{2}{3}\right)}$ and $\alpha>\frac{\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{2}{3}\right)}$, the figure 2 and 3 show that $LHS<RHS$, respectively.

Example 3.2. Let f be as in Example 3.1 with $p=q=2$. Then $|f'(x)|^q=4x^2$ and $|f'|^q$ is a convex function on $[1,2]$. There fore we can apply Theorem 2.4 to the this defined function f. The left hand side of the inequality (2.6) is

$$\begin{aligned} LHS &= \left|\frac{1}{2}\left[f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right]-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]\right| \\ &= \left|\frac{41}{18}-\frac{(10+\alpha(13+5\alpha))\Gamma(1+\alpha)}{2\Gamma(3+\alpha)}\right|. \end{aligned}$$

Since

$$\begin{aligned} \int_{\frac{1}{3}}^{\frac{2}{3}}\left|t^\alpha-\frac{1}{2}\right|^p dt &= \int_{\frac{1}{3}}^{\frac{2}{3}}\left|t^\alpha-\frac{1}{2}\right|^2 dt \\ &= \frac{2^{2\alpha+1}-1}{(2\alpha+1)3^{2\alpha+1}}-\frac{2^{\alpha+1}-1}{(\alpha+1)3^{\alpha+1}}+\frac{1}{12}, \end{aligned}$$

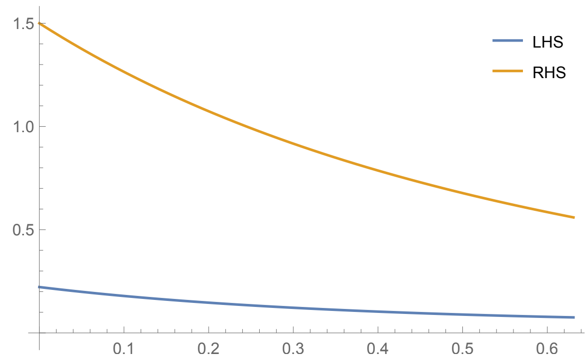

FIGURE 1. An example to the inequality (2.5) depending on $0 < \alpha \leq \frac{\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{1}{3}\right)}$, computed and plotted with Mathematica.

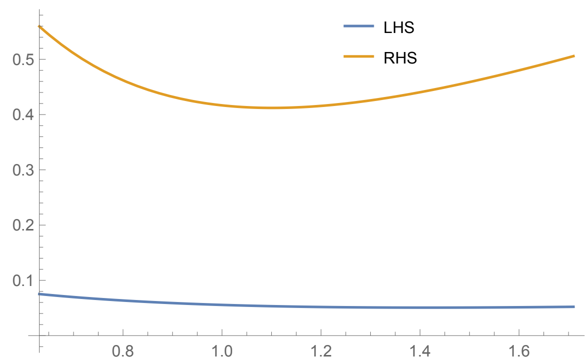

FIGURE 2. An example to the inequality (2.5) depending on $\frac{\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{1}{3}\right)} < \alpha \leq \frac{\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{2}{3}\right)}$, computed and plotted with Mathematica.

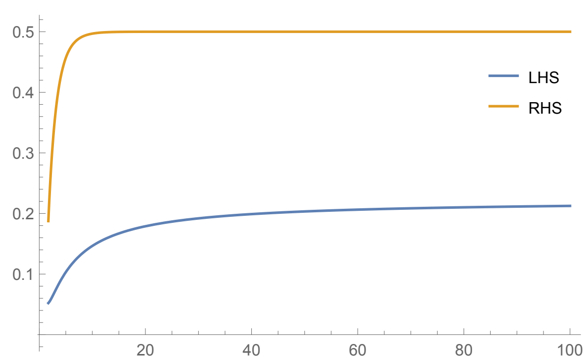

FIGURE 3. An example to the inequality (2.5) depending on $\alpha > \frac{\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{2}{3}\right)}$, computed and plotted with Mathematica.

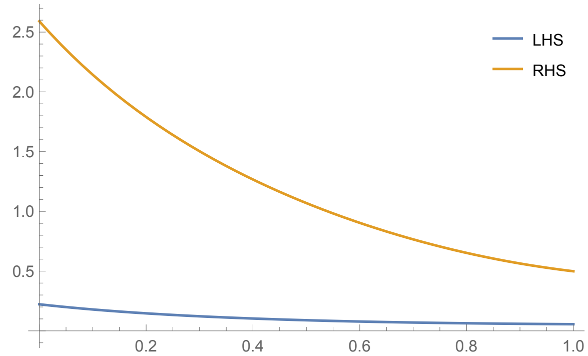

FIGURE 4. An example to the inequality (2.6) depending on $\alpha > 0$, computed and plotted with Mathematica.

the right hand side of the inequality (2.6) reduces to

$$RHS \quad : \quad = \left(\frac{1}{3^{2\alpha+1}(2\alpha+1)} \right)^{\frac{1}{2}} \left(\sqrt{2} + \frac{\sqrt{42}}{3} \right) + \left(\frac{2^{2\alpha+1}-1}{(2\alpha+1)3^{2\alpha+1}} - \frac{2^{\alpha+1}-1}{(\alpha+1)3^{\alpha+1}} + \frac{1}{12} \right)^{\frac{1}{2}} \sqrt{\frac{10}{3}}.$$

From the figure 4, it is clear that $LHS < RHS$.

4. Applications to Special Means

For arbitrary real numbers $y, y_1, y_2, ..., y_n, w$ we have:

The Arithmetic mean:

$$\mathscr{A}(y_1, y_2, ..., y_n) = \frac{y_1 + y_2 + ... + y_n}{n}.$$

Tne Geometric mean

$$\mathscr{G}(w, y) = \sqrt{wy}, \ \ y, w > 0.$$

The Harmonic mean

$$\mathscr{H}(w, y) = \frac{2}{\frac{1}{w} + \frac{1}{y}}, \ \ y, w > 0.$$

The p-Logarithmic mean

$$\mathscr{L}_p(w, y) = \left(\frac{y^{p+1} - w^{p+1}}{(y-w)(p+1)} \right)^{\frac{1}{p}}, y, w > 0, y \neq w \text{ and } p \in \mathbb{R} \setminus \{0, -1\}.$$

The identical mean

$$\mathscr{I}(a, b) = \begin{cases} a, & a = b \\ \frac{1}{e}\left(\frac{b^b}{a^a} \right)^{\frac{1}{b-a}}, & a \neq b. \end{cases}$$

Proposition 4.1. For $a,b>0$ and $n\in\mathbb{N}$, we have

$$\left|\mathscr{A}\left(\mathscr{A}(a,a,b),\mathscr{A}(a,b,b)\right)-\mathscr{L}_n(a,b)\right|\leq\frac{5n(b-a)}{36}\mathscr{A}\left(a^{n-1},b^{n-1}\right).$$

Proof. Applying Theorem 2.2 with $f(x)=x^n$ and $\alpha=1$, we get the required result. □

Proposition 4.2. For $a,b>0$, we have

$$\left|\ln\left[\frac{\mathscr{I}(a,b)}{\mathscr{G}\left(\mathscr{A}(a,a,b),\mathscr{A}(a,b,b)\right)}\right]\right|\leq\frac{5(b-a)}{36}\mathscr{H}^{-1}(a,b).$$

Proof. Applying Theorem 2.2 with $f(x)=-\ln x$ and $\alpha=1$, we get the desired result. □

5. Conclusion

In this work, we have proved some error bounds for one of the Open Newton-Cotes formulas for differentiable convex functions in fractional calculus. We gave some examples and their graphs to show the validity of newly established inequalities for different values of α. Moreover we presented some applications yo special means. It is an interesting and new problem that the upcoming researchers can obtain similar inequalities for other fractional integrals and for coordinated convex functions.

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