Long-time behavior of wave equations with nonlocal nonlinear damping and nonlinear colored noise on $\mathbb{R}^n$

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Abstract

In this paper, we investigate the long-time behavior of wave equations with nonlocal nonlinear damping and nonlinear colored noise defined on the whole space $\mathbb{R}^n$. We first establish a continuous cocycle for the equations. And then the dissipative property of solutions is obtained by utilizing the barrier method to overcome the difficulty brought by the nonlocal nonlinear damping. Finally, we obtain the existence and uniqueness of pullback random attractors. The asymptotic compactness of the cocycle associated with the problem is derived by the aid of energy equation and uniform tail-estimates to overcome the obstacle caused by the lack of compact Sobolev embeddings on unbounded domains.

Keywords: Wave equation, Nonlocal nonlinear damping, Colored noise, Random attractors

1 Introduction

We consider the asymptotic behavior of solutions of the following nonlocal nonlinear damping wave equations driven by nonlinear colored noise on the entire space $\mathbb{R}^n$:

$$\begin{cases} u_{tt} - \triangle u + \sigma(\|\nabla u\|^2)g(u_t) + \nu u + f(u) = h(t,x) + R(t,x,u)\zeta_\delta(\theta_t\omega), \\ u(\tau,x) = u_0(x), \quad u_t(\tau,x) = u_{1,0}(x), \quad t > \tau, \ \tau \in \mathbb{R}, \end{cases} \tag{1.1}$$

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where $x \in \mathbb{R}^n$, ν is a positive constant, $h \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$, ζ_δ is the colored noise with correlation time $\delta > 0$. The damping coefficient $\sigma(\cdot) \in C^1(\mathbb{R}^+)$ and $\sigma(s) > 0$ for all $s \in \mathbb{R}^+$, f, g and R are nonlinear functions on $\mathbb{R}$ which satisfy certain conditions. Here, $\|\cdot\|$ stands for the usual L^2-norm.

For the deterministic case (i.e.,$R \equiv 0$), the asymptotic behaviour of solutions for wave equations with various different nonlocal damping were studied intensively, such as nonlocal weak damping $\sigma(\|\nabla u\|^2)u_t$, nonlocal fractional damping $\sigma(\|\nabla u\|^2)(-\triangle)^\theta u_t (0 < \theta \leq 1)$ and nonlocal nonlinear damping $\sigma(\|\nabla u\|^2)g(u_t)$(see, e.g., [3,8,22] and the references therein).

The main object of this paper is to analyze the asymptotic behavior of solutions for (1.1) under the influence of the nonlocal nonlinear damping

$$\sigma(\|\nabla u\|^2)g(u_t). \tag{1.2}$$

The damping (1.2) given by the product of two nonlinearities was first used by Silva and Narciso in [12], in which they discussed the well-posedness and long-time dynamics for the following extensible beam model

$$u_{tt} + \triangle^2 u - k\phi(\|\nabla u\|^2)\triangle u + \sigma(\|\nabla u\|^2)g(u_t) + f(u) = h, \tag{1.3}$$

where $k\phi(s) \geq -\alpha_0, \sigma(s) > 0, c_{g'}|s|^\gamma \leq |g'(s)| \leq c_{g'}(1 + |s|^\gamma)$ and $|f'(s)| \leq c_{f'}(1 + |s|^\rho)$. Later, Narciso [11] investigated the well-posedness as well as the asymptotic behavior of solutions for a quasi-linear Kirchhoff wave model with nonlocal nonlinear damping

$$u_{tt} - \phi(\|\nabla u\|^2)\triangle u + \sigma(\|\nabla u\|^2)g(u_t) + f(u) = h. \tag{1.4}$$

Recently, Zhou and Sun [34] showed the well-posedness and long-time dynamics of the wave equation (1.4) when $\phi(\|\nabla u\|^2) \equiv 1$ and the growth exponent p of the nonlinearity $f(u)$ satisfies $2 \leq p \leq p*$ with $p^* = \frac{6\gamma}{\gamma+1}(\geq 3)$.

The dynamics of stochastic wave equations driven by additive or linear multiplicative white noise have been studied in [14, 17, 18, 25, 27] to the case of bounded domains, and in [19, 29, 31] to the case of unbounded domains. As far as some researchers are aware, we can only define a random dynamical system for the stochastic equation when the diffusion term is a linear function, and that is why we are currently unable to prove the existence of pathwise random attractors for stochastic wave equations with nonlinear white noise.

The colored noise is used to approximate the Wiener process in order to overcome the difficulty caused by the nowhere differentiability of the sample paths. In many complex systems, it is more reasonable to consider the random influences modeled by colored noise rather than white noise since the stochastic fluctuations are actually correlated, see, e.g., [1,4,5,10,13,15,24]. Recently, the long term dynamics of PDEs with colored noise has been extensively investigated in [7,9,16,20,21,23,26,28,32,33]. For instance, the existence of random attractors of wave equations with colored noise on unbounded domains was proved by Wang in [23]. In the present paper, we focus on the random attractors of (1.1) with nonlocal nonlinear damping and nonlinear colored noise, because not only there has no any results to (1.1), but also these problems have more challenges and more interesting when the equation include nonlocal nonlinear damping and random term.

In order to obtain the existence of pullback random attractors for the continuous cocycle associated with (1.1), as we all know, the key step is to establish the pullback asymptotic compactness of the cocycle in $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. The main difficulties come from the following aspects:

(i) The Sobolev embeddings are no longer compact in unbounded domains. This is essentially distinct from the case of bounded domains and is a major obstacle for proving the asymptotic compactness of the solution operator.

(ii) Due to the influence of the nonlocal nonlinear damping, the energy is not decreasing along trajectories and the typical method based on the construction of a suitable Gronwall's inequality to prove dissipativity fails in our case.

To overcome these difficulties, we first utilize the barrier method to overcome the difficulty brought by the nonlinear term $\sigma(\|\nabla u\|^2)g(u_t)$ and then prove the dissipation as in [3, 22, 34]. Hereafter, we use the idea of energy equation along with the uniform tail-estimates of solutions to establish the desired pullback asymptotic compactness in $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, see Lemma 4.4 for more details.

This paper is organized as follows. In the next section, we recall some basic concepts and results on colored noise as well as pullback random attractors for continuous cocycles. Section 3 is devoted to the existence of continuous cocycle of (1.1) in $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. In the last section, we prove the existence and uniqueness of pullback random attractors.

Hereafter, the inner product and norm of $L^2(\mathbb{R}^n)$ will be denoted by $(\cdot, \cdot)$ and $\| \cdot \|$, respectively. The letter c and $c_i (i = 1, 2, \cdots)$ are generic positive constants, which may be different from line to line.

2 Preliminaries

In this section, we recall some basic concepts and results on colored noise as well as pullback random attractors for continuous cocycles, see [2, 4, 15, 31].

2.1 Pullback random attractors for continuous cocycles

Let $(X, \|\cdot\|_X)$ be a complete separable metric space with Borel σ-algebra $\mathcal{B}(X)$. We first recall the definition of Hausdorff semi-distance of two non-empty sets A, B:

$$dist_X(A, B) = \sup_{x\in A} \inf_{y\in B} d_X(x, y).$$

Definition 2.1 *Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ be a metric dynamical system with probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and measure-preserving group $\theta : \mathbb{R} \times \Omega \to \Omega$ of translations on Ω. A mapping $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ is called a continuous cocycle on X over $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ if for all $\tau \in \mathbb{R}, \omega \in \Omega$ and $t, s \in \mathbb{R}^+$, the following conditions are satisfied:*

(i) $\Phi(\cdot, \tau, \cdot, \cdot) : \mathbb{R}^+ \times \Omega \times X \to X$ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable;

(ii) $\Phi(0, \tau, \omega, \cdot)$ is the identity on X;

(iii) $\Phi(s + t, \tau, \omega, \cdot) = \Phi(t, \tau + s, \theta_s\omega, \cdot)\Phi(s, \tau, \omega, \cdot)$;

(iv) $\Phi(t, \tau, \omega, \cdot) : X \to X$ is continuous.

Let $\mathcal{D}$ be a collection of some families of nonempty subsets of X.

Definition 2.2 *A family $\mathcal{K} = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ is called a $\mathcal{D}-$pullback absorbing set for Φ if for all $D \in \mathcal{D}, \tau \in \mathbb{R}$ and $\omega \in \Omega$, there exists $T = T(D, \tau, \omega) > 0$ such that, for all $t \geqslant T$,*

$$\Phi(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)) \subseteq K(\tau, \omega).$$

Definition 2.3 *The continuous cocycle Φ is called $\mathcal{D}-$pullback asymptotically compact in X if for all $D \in \mathcal{D}, \tau \in \mathbb{R}$ and $\omega \in \Omega$, the sequence $x_n \in \Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, D(\tau - t_n, \theta_{-t_n}\omega))$ has a convergent subsequence in X as $t_n \to +\infty$.*

Definition 2.4 *A family $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ is called a $\mathcal{D}-$pullback random attractor of Φ in X if for every $\tau \in \mathbb{R}, \omega \in \Omega$, the following conditions are satisfied:*

(i) $\mathcal{A}(\tau, \cdot) : \Omega \to X$ is measurable with respect to $\mathcal{F}$, and $\mathcal{A}(\tau, \omega)$ is compact in X;

(ii) $\mathcal{A}$ is invariant: $\Phi(t, \tau, \omega, \mathcal{A}(\tau, \omega)) = \mathcal{A}(t + \tau, \theta_t\omega), \forall t \in \mathbb{R}^+$;

(iii) $\mathcal{A}$ attracts every member of $\mathcal{D}$ in X: for every $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$,

$$\lim_{t\to+\infty} dist_X(\Phi(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)), \mathcal{A}(\tau, \omega)) = 0,$$

where $dist_X(\cdot, \cdot)$ is the Hausdorff semi-distance in X.

We have the following result for continuous cocycles on X.

Theorem 2.5 *([30]). Suppose X is a separable Banach space. Let $\mathcal{D}$ be an inclusion-closed collection of some families of nonempty subsets of X, and Φ be a continuous cocycle on X over $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$. Furthermore, we assume*

(i) Φ has a closed measurable $\mathcal{D}-$pullback absorbing set $\mathcal{K} = \{K(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ in $\mathcal{D}$;

(ii) Φ is $\mathcal{D}-$pullback asymptotically compact in X.

Then Φ has a unique $\mathcal{D}-$pullback random attractor $\mathcal{A}$ in $\mathcal{D}$ which is given by

$$\mathcal{A}(\tau,\omega) = \bigcap_{t_0>0} \overline{\bigcup_{t\geqslant t_0} \Phi(t, \tau - t, \theta_{-t}\omega, K(\tau - t, \theta_{-t}\omega))}^X .$$

2.2 Colored noise

To describe the colored noise, we introduce a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$ equipped with the compact-open topology, $\mathcal{F}$ is the Borel σ-algebra of Ω, $\mathbb{P}$ is the Wiener measure. The classical transformation $\{\theta_t\}_{t\in\mathbb{R}}$ on Ω is given by $\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t)$ for all $(\omega, t) \in \Omega \times \mathbb{R}$. Let W be a two-sided real-valued Wiener process on $(\Omega, \mathcal{F}, \mathbb{P})$, for each $\delta > 0$, we define

$$\zeta_\delta : \Omega \to \mathbb{R} \ \ by \ \ \zeta_\delta(\omega) = \frac{1}{\delta}\int_{-\infty}^{0} e^{\frac{s}{\delta}} dW(s).$$

Then the process $\zeta_\delta(\theta_t\omega)$ is called a real-valued colored noise (also known as an Ornstein-Uhlenbeck process) which is the unique stationary solution of the one-dimensional stochastic differential equation

$$d\zeta_\delta + \frac{1}{\delta}\zeta_\delta dt = \frac{1}{\delta}dW.$$

Note that there exists a subset of full probability measure (still denoted by Ω) such that for all $\omega \in \Omega$, $\zeta_\delta(\theta_t\omega)$ is continuous in $t \in \mathbb{R}$ and $\lim_{t\to\pm\infty} \frac{\zeta_\delta(\theta_t\omega)}{t} = 0$.

3 Continuous cocycle

In this section, we establish the existence of continuous cocycle of (1.1) in $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. Throughout this paper, we make the following assumptions on the nonlinear functions in (1.1). Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfy, for all $s, s_1, s_2 \in \mathbb{R}$,

$$\liminf_{|s|\to\infty} sf(s) > 0, \tag{3.1}$$

$$f(0) = 0, \ |f(s_1) - f(s_2)| \leq \alpha_1(|s_1|^{p-1} + |s_2|^{p-1})|s_1 - s_2|, \tag{3.2}$$

$$F(s) \geq \alpha_2 |s|^{p+1}, \tag{3.3}$$

where $p \geq 1$ for $n = 1, 2$ and $1 \leq p < \frac{n}{n-2}$ for $n \geq 3$, α_1 and α_2 are positive constants, and $F(r) = \int_0^r f(s)ds$ for all $r \in \mathbb{R}$.

Let $R : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be continuous such that for all $t, s, s_1, s_2 \in \mathbb{R}$ and $x \in \mathbb{R}^n$,

$$|R(t, x, s)| \leq \beta_1(t, x)|s|^q + \beta_2(t, x), \tag{3.4}$$

$$|R(t, x, s_1) - R(t, x, s_2)| \leq \beta_3(t, x)(|s_1|^{q-1} + |s_2|^{q-1} + \beta_4(t, x))|s_1 - s_2|, \tag{3.5}$$

where $1 \leq q < \frac{p+1}{2}$, $\beta_1 \in L_{loc}^{\frac{2p+2}{p+1-2q}}(\mathbb{R}, L^{\frac{2p+2}{p+1-2q}}(\mathbb{R}^n))$, $\beta_2 \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$, and $\beta_3, \beta_4 \in L^\infty(\mathbb{R}, L^\infty(\mathbb{R}^n))$.

The nonlinear damping $g(\cdot) \in C^1(\mathbb{R})$ is a monotone increasing function, $g(0) = 0$, and there exist two positive constants κ_1 and κ_2 such that

$$\kappa_1 |s|^{\gamma-1} \leq g'(s) \leq \kappa_2(1 + |s|^{\gamma-1}), \quad \forall s \in \mathbb{R}, \tag{3.6}$$

where $\gamma \geq 1$ for $n = 1, 2$ and $1 \leq \gamma \leq \frac{n+2}{n-2}$ for $n \geq 3$.

To define a continuous non-autonomous cocycle for the nonlocal nonlinear damping wave equations (1.1) driven by nonlinear colored noise, we first give the definition of weak solutions to problem (1.1). Given $\tau \in \mathbb{R}, \omega \in \Omega, u_0 \in H^1(\mathbb{R}^n)$ and $u_{1,0} \in L^2(\mathbb{R}^n)$. A mapping $u(\cdot, \tau, \omega, u_0, u_{1,0}) : [\tau, +\infty) \to H^1(\mathbb{R}^n)$ is called a (weak) solution of (1.1) if for $u(\tau, \tau, \omega, u_0, u_{1,0}) = u_0$, $u_t(\tau, \tau, \omega, u_0, u_{1,0}) = u_{1,0}$,

$$u(\cdot, \tau, \omega, u_0, u_{1,0}) \in L^\infty(\tau, \tau + T; H^1(\mathbb{R}^n)) \cap C([\tau, \tau + T], L^2(\mathbb{R}^n)), \tag{3.7}$$

$$u_t(\cdot, \tau, \omega, u_0, u_{1,0}) \in L^\infty(\tau, \tau + T; L^2(\mathbb{R}^n)) \cap C([\tau, \tau + T], H^{-1}(\mathbb{R}^n)), \tag{3.8}$$

and u satisfies that for all $T > 0$ and $\psi \in C_0^\infty((\tau, \tau + T) \times \mathbb{R}^n)$,

$$-\int_\tau^{\tau+T} (u_t, \psi_t)dt + \int_\tau^{\tau+T} (\nabla u, \nabla \psi)dt + \int_\tau^{\tau+T} (\sigma(\|\nabla u\|^2)g(u_t), \psi)dt$$

$$+\nu \int_\tau^{\tau+T} (u, \psi)dt + \int_\tau^{\tau+T} \int_{\mathbb{R}^n} f(u(t, x))\psi(t, x)dxdt$$

$$= \int_\tau^{\tau+T} (h(t), \psi)dt + \int_\tau^{\tau+T} \int_{\mathbb{R}^n} R(t, x, u(t, x))\zeta_\delta(\theta_t \omega)\psi(t, x)dxdt, \tag{3.9}$$

for every $\psi \in C_0^\infty((\tau, \tau + T) \times \mathbb{R}^n)$ and for almost all $t \in [\tau, \tau + T]$.

If, in addition, $u(t, \tau, \cdot, u_0, u_{1,0}) : \Omega \to H^1(\mathbb{R}^n)$ is $(\mathcal{F}, \mathcal{B}(H^1(\mathbb{R}^n)))$-measurable, and $u_t(t, \tau, \cdot, u_0, u_{1,0}) : \Omega \to L^2(\mathbb{R}^n)$ is $(\mathcal{F}, \mathcal{B}(L^2(\mathbb{R}^n)))$-measurable, then u is called a measurable solution.

Since the nonlocal nonlinear damping wave equations (1.1) can be viewed as a deterministic equation parametrized by $\omega \in \Omega$, then by the Galerkin method as in [6], we prove that for every $\tau \in \mathbb{R}, \omega \in \Omega$, and $(u_0, u_{1,0}) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, problem (1.1) under the assumptions (3.1)-(3.6) admits a unique weak solution $u(t, \tau, \omega, u_0, u_{1,0})$ for all $t \geq \tau$ in the sense of (3.7)-(3.8) such that u is continuously depends on $(u_0, u_{1,0}) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, and u is $(\mathcal{F}, \mathcal{B}(H^1(\mathbb{R}^n)) \times \mathcal{B}(L^2(\mathbb{R}^n)))$-measurable in $\omega \in \Omega$. Furthermore, the solution u of (1.1) satisfies the energy equation

$$\frac{d}{dt}\left(\|u_t\|^2 + \nu\|u\|^2 + \|\nabla u\|^2 + 2\int_{\mathbb{R}^n} F(u(t,x))dx\right) + 2\sigma(\|\nabla u\|^2)\int_{\mathbb{R}^n} g(u_t(t,x))u_t(t,x)dx$$

$$= 2(h(t), u_t(t)) + 2\zeta_\delta(\theta_t\omega)\int_{\mathbb{R}^n} R(t,x,u(t,x))u_t(t,x)dx, \tag{3.10}$$

and

$$\frac{d}{dt}(u(t), u_t(t)) + \nu\|u\|^2 + \|\nabla u\|^2 + \sigma(\|\nabla u\|^2)\int_{\mathbb{R}^n} g(u_t(t,x))u(t,x)dx + \int_{\mathbb{R}^n} f(u(t,x))u(t,x)dx$$

$$= \|u_t\|^2 + (h(t), u(t)) + \zeta_\delta(\theta_t\omega)\int_{\mathbb{R}^n} R(t,x,u(t,x))u(t,x)dx, \tag{3.11}$$

for almost all $t > \tau$.

The rest of this paper is devoted to the existence of random atractors of (1.1). To that end, we need to define a continuous cocycle in terms of the solution operator of (1.1). Given $t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega$ and $(u_0, u_{1,0}) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, define a mapping $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ given by

$$\Phi(t, \tau, \omega, (u_0, u_{1,0})) = (u(t+\tau, \tau, \theta_{-\tau}\omega, u_0), u_t(t+\tau, \tau, \theta_{-\tau}\omega, u_{1,0})), \tag{3.12}$$

where u is the unique solution of (1.1) and u is $(\mathcal{F}, \mathcal{B}(H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)))$-measurable in $\omega \in \Omega$. Therefore, Φ is a continuous cocycle on $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ over $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ in the sense of [30].

4 Pullback random atractors

In this section, we first construct a pullback random absorbing set, and then prove the pullback asymptotic compactness of solutions. Ultimately, we achieve the existence of pullback random attractors for (1.1).

Recall that a family $D=\{D(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}$ of bounded nonempty subsets in $H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ is tempered if for every $C>0,\tau\in\mathbb{R}$ and $\omega\in\Omega$,

$$\lim_{t\to+\infty} e^{-Ct}\|D(\tau-t,\theta_{-t}\omega)\|^2_{H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}=0, \tag{4.1}$$

where $\|D\|_{H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}=\sup_{\varrho\in D}\|\varrho\|_{H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}$. Let $\mathcal{D}$ be the collection of all tempered families of bounded nonempty subsets of $H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$, i.e.

$$\mathcal{D}=\{D=\{D(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}:D\ satisfies\ (4.1)\}. \tag{4.2}$$

From now on, we also assume f satisfies: for all $s\in\mathbb{R}$ and $\varrho\in(0,1]$,

$$f(s)s-\varrho F(s)\geq 0. \tag{4.3}$$

Since $\kappa_1,\kappa_2,l,\nu,\sigma_0$ and σ_1 are positive constants, $\gamma\geq 1$, and $\varrho\in(0,1]$, we find that there exists a sufficiently small positive number ε such that

$$0<\varepsilon<\min\left\{1,\nu,\frac{4-2\varrho}{\varrho},\frac{\sigma_0\kappa_1}{\gamma(2\sigma_1\kappa_2)^{\frac{\gamma+1}{\gamma}}}\right\},\frac{\sigma_0\kappa_1}{2\gamma}-\frac{\varepsilon\sigma_1^2\kappa_2^2}{\nu}>0,l\sigma_o-2\varepsilon-\frac{1}{4}\varepsilon\varrho-\frac{\varepsilon^2\varrho}{8\nu}>0. \tag{4.4}$$

We further assume that

$$\int_{-\infty}^{\tau} e^{\frac{1}{4}\varepsilon\varrho s}\|h(s)\|^2ds<\infty,\quad\forall\ \tau\in\mathbb{R}, \tag{4.5}$$

and for any $C>0$,

$$\lim_{t\to+\infty}e^{-Ct}\int_{-\infty}^{0}e^{\frac{1}{4}\varepsilon\varrho s}\|h(s-t)\|^2ds=0. \tag{4.6}$$

Note that condition (4.6) implies that the non-autonomous term $h(t)$ is tempered in $L^2(\mathbb{R}^n)$ as $t\to-\infty$. To derive the uniform estimates of solutions for large time, we now assume that the functions β_1 and β_2 in (3.4) satisfy:

$$\beta_1\in L^\infty(\mathbb{R},L^{\frac{2p+2}{p+1-2q}}(\mathbb{R}^n)),\beta_2\in L^\infty(\mathbb{R},L^2(\mathbb{R}^n)). \tag{4.7}$$

4.1 Construction of pullback random absorbing sets

We first derive uniform estimates of the solutions for large time.

Lemma 4.1 *Suppose* (3.1)-(3.6) *and* (4.5) *hold. Then for any* $\tau\in\mathbb{R},\omega\in\Omega$ *and* $D\in\mathcal{D}$, *there exists* $T=T(\tau,\omega,D)>0$ *such that for all* $t\geq T$ *and all* $r\in[-t,0]$, *the solution* u *of* (1.1) *satisfies,*

$$\|u_t(\tau+r,\tau-t,\theta_{-\tau}\omega,u_{1,0})\|^2+\|u(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0)\|^2_{H^1(\mathbb{R}^n)}$$
$$+\int_{\mathbb{R}^n}F(u(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0))dx$$
$$+\int_{\tau-t}^{\tau+r}e^{\frac14\varepsilon\varrho(s-\tau-r)}\left(\|u_t(s,\tau-t,\theta_{-\tau}\omega,u_{1,0})\|^2+\|u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2_{H^1(\mathbb{R}^n)}\right)ds$$
$$\le L_1e^{-\frac14\varepsilon\varrho r}\int_{-\infty}^0 e^{\frac14\varepsilon\gamma s}(1+\|h(s+t)\|^2+|\zeta_\delta(\theta_s\omega)|^{\frac{2(p+1)}{p+1-2q}})ds,$$

where $(u_0,u_{1,0})\in D(\tau-t,\theta_{-t}\omega)$ and L_1 is a positive number independent of τ,ω and D.

Proof. By (3.10) and (3.11) we get

$$\frac{d}{dt}V_\varepsilon(t)+\varepsilon\|\nabla u\|^2+2\sigma(\|\nabla u\|^2)\int_{\mathbb{R}^n}g(u_t(t,x))u_t(t,x)dx$$
$$-\varepsilon\|u_t\|^2+\varepsilon\nu\|u\|^2+\varepsilon\int_{\mathbb{R}^n}f(u(t,x))u(t,x)dx$$
$$=(h(t)+R(t,\cdot,u(t))\zeta_\delta(\theta_t\omega),\varepsilon u+2u_t)-\varepsilon\sigma(\|\nabla u\|^2)\int_{\mathbb{R}^n}g(u_t(t,x))u(t,x)dx, \tag{4.8}$$

where

$$V_\varepsilon(t)=\|u_t\|^2+\nu\|u\|^2+\|\nabla u\|^2+2\int_{\mathbb{R}^n}F(u(t,x))dx+\varepsilon(u,u_t).$$

We now estimate the right-hand side of (4.8). By Young's inequality, we have

$$|(R(t,\cdot,u(t))\zeta_\delta(\theta_t\omega),\varepsilon u+2u_t)|$$
$$\le\varepsilon\|R(t,\cdot,u(t))\zeta_\delta(\theta_t\omega)\|\ \|u(t)\|+2\|R(t,\cdot,u(t))\zeta_\delta(\theta_t\omega)\|\ \|u_t(t)\|$$
$$\le\frac18\varepsilon\nu\|u(t)\|^2+\frac12\varepsilon\|u_t(t)\|^2+c_1\|R(t,\cdot,u(t))\zeta_\delta(\theta_t\omega)\|^2, \tag{4.9}$$

where $c_1=\frac{2\varepsilon}{\nu}+\frac{2}{\varepsilon}$. By means of (3.3), (3.4), (4.7) and the Hölder inequality, we get

$$\|R(t,\cdot,u(t))\zeta_\delta(\theta_t\omega)\|^2\le2\int_{\mathbb{R}^n}|\zeta_\delta(\theta_t\omega)|^2\left(|\beta_1(t,x)|^2|u(t,x)|^{2q}+|\beta_2(t,x)|^2\right)dx$$
$$\le\frac12\varepsilon\varrho c_1^{-1}\alpha_2\int_{\mathbb{R}^n}|u(t,x)|^{p+1}dx+c_2\int_{\mathbb{R}^n}\left(|\zeta_\delta(\theta_t\omega)|^2\beta_1(t,x)|^2\right)^{\frac{p+1}{p+1-2q}}dx+2|\zeta_\delta(\theta_t\omega)|^2\|\beta_2(t)\|^2$$
$$\le\frac12\varepsilon\varrho c_1^{-1}\int_{\mathbb{R}^n}F(u(t,x))dx+c_3|\zeta_\delta(\theta_t\omega)|^{\frac{2(p+1)}{p+1-2q}}+c_3|\zeta_\delta(\theta_t\omega)|^2, \tag{4.10}$$

where $c_3 > 0$ depends on $\varepsilon, \nu, \gamma, \alpha_2, \beta_1$ and β_2. Inserting (4.10) to (4.9) yields

$$|(R(t,\cdot,u(t))\zeta_\delta(\theta_t\omega), \varepsilon u + 2u_t)| \le \frac{1}{8}\varepsilon\nu\|u(t)\|^2 + \frac{1}{2}\varepsilon\|u_t(t)\|^2$$

$$+\frac{1}{2}\varepsilon\varrho\int_{\mathbb{R}^n} F(u(t,x))dx + c_1c_3|\zeta_\delta(\theta_t\omega)|^{\frac{2(p+1)}{p+1-2q}} + c_1c_3|\zeta_\delta(\theta_t\omega)|^2. \tag{4.11}$$

Using Young's inequality, we have

$$|(h(t), \varepsilon u + 2u_t)| \le \frac{1}{8}\varepsilon\nu\|u(t)\|^2 + \frac{1}{2}\varepsilon\|u_t(t)\|^2 + c_1\|h(t)\|^2. \tag{4.12}$$

Due to (4.3), there holds

$$\varepsilon\int_{\mathbb{R}^n} f(u(t,x))u(t,x)dx \ge \varepsilon\varrho\int_{\mathbb{R}^n} F(u(t,x))dx. \tag{4.13}$$

According to (3.7) and the continuity of σ, there exists a constant $\widehat{C} > 0$ such that

$$\sigma(\|\nabla u\|^2) \le \max_{0\le\lambda\le\widehat{C}} \sigma(\lambda) = \sigma_1, \tag{4.14}$$

where $\sigma_1 = \sigma_1(\|(u_0, u_{1,0})\|_{H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}) > 0$. Using the Hölder inequality and embedding $H^1(\mathbb{R}^n) \hookrightarrow L^{\gamma+1}(\mathbb{R}^n)$, we get

$$-\varepsilon\sigma(\|\nabla u\|^2)\int_{\mathbb{R}^n} g(u_t(t,x))u(t,x)dx$$

$$\le \varepsilon\sigma_1\kappa_2\int_{\mathbb{R}^n}|u_t(t,x)||u(t,x)|dx + \varepsilon\sigma_1\kappa_2\int_{\mathbb{R}^n}|u_t(t,x)|^\gamma|u(t,x)|dx$$

$$\le \varepsilon\sigma_1\kappa_2\|u_t\|\|u\| + \varepsilon\sigma_1\kappa_2\left(\int_{\mathbb{R}^n}|u_t(t,x)|^{\gamma+1}dx\right)^{\frac{\gamma}{\gamma+1}}\left(\int_{\mathbb{R}^n}|u(t,x)|^{\gamma+1}dx\right)^{\frac{1}{\gamma+1}}$$

$$\le \varepsilon\sigma_1\kappa_2\|u_t\|\|u\| + \varepsilon\sigma_1\kappa_2\|u_t\|^\gamma_{L^{\gamma+1}}\|u\|_{L^{\gamma+1}}$$

$$\le \varepsilon\sigma_1\kappa_2\|u_t\|\|u\| + \varepsilon d_1\left(\|u_t\|^\gamma_{L^{\gamma+1}}\|\nabla u\|^{\frac{\gamma-1}{\gamma+1}}\right)^{\frac{\gamma+1}{\gamma}} + \frac{1}{2}\varepsilon\|\nabla u\|^{\frac{2}{\gamma+1}(\gamma+1)}$$

$$\le \frac{\varepsilon\sigma_1^2\kappa_2^2}{\nu}\|u_t\|^{\gamma+1}_{L^{\gamma+1}} + \frac{1}{4}\varepsilon\nu\|u\|^2 + \varepsilon d_1\|u_t\|^{\gamma+1}_{L^{\gamma+1}}\|\nabla u\|^{\frac{\gamma-1}{\gamma}} + \frac{1}{2}\varepsilon\|\nabla u\|^2 + \varepsilon c_4, \tag{4.15}$$

where we have used the fact that

$$\frac{\varepsilon\sigma_1^2\kappa_2^2}{\nu}\|u_t\|^2 \le \frac{\varepsilon\sigma_1^2\kappa_2^2}{\nu}\|u_t\|^{\gamma+1}_{L^{\gamma+1}} + \varepsilon c_4,$$

and $d_1 = 2^{\frac{1}{\gamma}}(\sigma_1\kappa_2)^{\frac{\gamma+1}{\gamma}}, c_4 > 0$ depends on γ. Inserting (4.11)-(4.13) and (4.15) into (4.8) yield

$$\frac{d}{dt}V_\varepsilon(t) - 2\varepsilon\|u_t\|^2 + \frac{1}{2}\varepsilon\nu\|u\|^2 + \frac{1}{2}\varepsilon\|\nabla u\|^2$$

$$+\frac{1}{2}\varepsilon\varrho\int_{\mathbb{R}^n}F(u(t,x))dx+2\sigma(\|\nabla u\|^2)\int_{\mathbb{R}^n}g(u_t(t,x))u_t(t,x)dx$$

$$\leq \varepsilon d_1\|u_t\|_{L^{\gamma+1}}^{\gamma+1}\|\nabla u\|^{\frac{\gamma-1}{\gamma}}+\frac{\varepsilon\sigma_1^2\kappa_2^2}{\nu}\|u_t\|_{L^{\gamma+1}}^{\gamma+1}$$

$$+c_1c_3|\zeta_\delta(\theta_t\omega)|^{\frac{2(p+1)}{p+1-2q}}+c_1c_3|\zeta_\delta(\theta_t\omega)|^2+c_1\|h(t)\|^2+\varepsilon c_4. \tag{4.16}$$

Thanks to (4.16), it leads to

$$\frac{d}{dt}V_\varepsilon(t)+\frac{1}{4}\varepsilon\varrho V_\varepsilon(t)+(-2\varepsilon-\frac{1}{4}\varepsilon\varrho)\|u_t\|^2+\frac{1}{4}\varepsilon\nu(2-\varrho)\|u\|^2+\frac{1}{4}\varepsilon(2-\varrho)\|\nabla u\|^2$$

$$-\frac{1}{4}\varepsilon^2\varrho(u,u_t)+2\sigma(\|\nabla u\|^2)\int_{\mathbb{R}^n}g(u_t(t,x))u_t(t,x)dx$$

$$\leq \varepsilon d_1\|u_t\|_{L^{\gamma+1}}^{\gamma+1}\|\nabla u\|^{\frac{\gamma-1}{\gamma}}+\frac{\varepsilon\sigma_1^2\kappa_2^2}{\nu}\|u_t\|_{L^{\gamma+1}}^{\gamma+1}$$

$$+c_1c_3|\zeta_\delta(\theta_t\omega)|^{\frac{2(p+1)}{p+1-2q}}+c_1c_3|\zeta_\delta(\theta_t\omega)|^2+c_1\|h(t)\|^2+\varepsilon c_4. \tag{4.17}$$

Note that

$$\frac{1}{4}\varepsilon^2\varrho|(u,u_t)|\leq\frac{1}{8}\varepsilon^2\nu\varrho\|u\|^2+\frac{\varepsilon^2\varrho}{8\nu}\|u_t\|^2. \tag{4.18}$$

Then it follows from (4.17)-(4.18) that

$$\frac{d}{dt}V_\varepsilon(t)+\frac{1}{4}\varepsilon\varrho V_\varepsilon(t)+\frac{1}{8}\varepsilon\nu(4-2\varrho-\varepsilon\varrho)\|u\|^2+\frac{1}{4}\varepsilon(2-\varrho)\|\nabla u\|^2$$

$$+2\sigma(\|\nabla u\|^2)\int_{\mathbb{R}^n}g(u_t(t,x))u_t(t,x)dx$$

$$\leq(2\varepsilon+\frac{1}{4}\varepsilon\varrho+\frac{\varepsilon^2\varrho}{8\nu})\|u_t\|^2+\varepsilon d_1\|u_t\|_{L^{\gamma+1}}^{\gamma+1}\|\nabla u\|^{\frac{\gamma-1}{\gamma}}+\frac{\varepsilon\sigma_1^2\kappa_2^2}{\nu}\|u_t\|_{L^{\gamma+1}}^{\gamma+1}$$

$$+c_1c_3|\zeta_\delta(\theta_t\omega)|^{\frac{2(p+1)}{p+1-2q}}+c_1c_3|\zeta_\delta(\theta_t\omega)|^2+c_1\|h(t)\|^2+\varepsilon c_4. \tag{4.19}$$

Since σ is a strictly positive continuous function on $\mathbb{R}^+$, then there exists a constant $\sigma_0>0$ such that

$$\sigma(\|\nabla u\|^2)\geq\sigma_0>0, \tag{4.20}$$

where $\sigma_0=\sigma_0(\|(u_0,u_{1,0})\|_{H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)})>0$. In addition, by (3.6), we have

$$2\sigma(\|\nabla u\|^2)\int_{\mathbb{R}^n}g(u_t(t,x))u_t(t,x)dx\geq\sigma_0\int_{\mathbb{R}^n}g(u_t(t,x))u_t(t,x)dx+\frac{\sigma_0\kappa_1}{\gamma}\|u_t\|_{L^{\gamma+1}}^{\gamma+1}. \tag{4.21}$$

According to (3.6), there exist $l>0$ and $L>0$ such that $g'(s)\geq l$ when $|s|>L$. Then we get

$$\int_{\mathbb{R}^n} g(u_t(t,x))u_t(t,x)dx \geq l\int_{\mathbb{O}} |u_t(t,x)|^2dx,$$

where $\mathbb{O}=\{x\in\mathbb{R}^n : |x|>L\}$. Thus,

$$\sigma_0\int_{\mathbb{R}^n} g(u_t(t,x))u_t(t,x)dx-(2\varepsilon+\frac{1}{4}\varepsilon\varrho+\frac{\varepsilon^2\varrho}{8\nu})\|u_t\|^2$$

$$\geq (l\sigma_0-2\varepsilon-\frac{1}{4}\varepsilon\varrho-\frac{\varepsilon^2\varrho}{8\nu})\int_{\mathbb{O}}|u_t(t,x)|^2dx-c_5, \tag{4.22}$$

where $c_5>0$ depends on ε,ν,ϱ. By condition (4.4), there exists a sufficiently small positive number ε such that

$$l\sigma_0-2\varepsilon-\frac{1}{4}\varepsilon\varrho-\frac{\varepsilon^2\varrho}{8\nu}>0, \frac{\sigma_0\kappa_1}{\gamma}-\frac{\varepsilon\sigma_1^2\kappa_2^2}{\nu}>\frac{\sigma_0\kappa_1}{2\gamma}\triangleq d_2. \tag{4.23}$$

It follows from (4.19)-(4.23) that

$$\frac{d}{dt}V_\varepsilon(t)+\frac{1}{4}\varepsilon\varrho V_\varepsilon(t)+\frac{1}{8}\varepsilon\nu(4-2\varrho-\varepsilon\varrho)\|u\|^2+\frac{1}{4}\varepsilon(2-\varrho)\|\nabla u\|^2$$

$$\leq \|u_t\|_{L^{\gamma+1}}^{\gamma+1}\left(\varepsilon d_1(V_\varepsilon(t))^{\frac{\gamma-1}{2\gamma}}-d_2\right)+c_6\left(1+\|h(t)\|^2+|\zeta_\delta(\theta_t\omega)|^{\frac{2(p+1)}{p+1-2q}}\right), \tag{4.24}$$

where $c_6>0$ depends on c_1,c_3,c_4 and c_5. When $\gamma=1$, from condition (4.4), we can rewrite (4.24) as

$$\frac{d}{dt}V_\varepsilon(t)+\frac{1}{4}\varepsilon\varrho V_\varepsilon(t)+\frac{1}{8}\varepsilon\nu(4-2\varrho-\varepsilon\varrho)\|u\|^2+\frac{1}{4}\varepsilon(2-\varrho)\|\nabla u\|^2+(d_2-\varepsilon d_1)\|u_t\|^2$$

$$\leq c_6\left(1+\|h(t)\|^2+|\zeta_\delta(\theta_t\omega)|^{\frac{2(p+1)}{p+1-2q}}\right). \tag{4.25}$$

Solving (4.25) on $[\tau-t,\tau+r]$ with $t>0$ and all $r\in[-t,0]$, after replacing ω by $\theta_{-\tau}\omega$, we get

$$\|u_t(\tau+r,\tau-t,\theta_{-\tau}\omega,u_{1,0})\|^2+\nu\|u(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+\|\nabla u(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0)\|^2$$

$$+2\int_{\mathbb{R}^n}F(u(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0))dx+\varepsilon(u(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0),u_t(\tau+r,\tau-t,\theta_{-\tau}\omega,u_{1,0}))$$

$$+(d_2-\varepsilon d_1)\int_{\tau-t}^{\tau+r}e^{\frac{1}{4}\varepsilon\varrho(s-\tau-r)}\|u_t(s,\tau-t,\theta_{-\tau}\omega,u_{1,0})\|^2ds$$

$$+\frac{1}{8}\varepsilon\nu(4-2\varrho-\varepsilon\varrho)\int_{\tau-t}^{\tau+r}e^{\frac{1}{4}\varepsilon\varrho(s-\tau-r)}\|u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2ds$$

$$+\frac{1}{4}\varepsilon(2-\varrho)\int_{\tau-t}^{\tau+r}e^{\frac{1}{4}\varepsilon\varrho(s-\tau-r)}\|\nabla u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2ds$$

$$\leq e^{-\frac{1}{4}\varepsilon\varrho(t+r)}\left(\|u_{1,0}\|^2+\nu\|u_0\|^2+\|\nabla u_0\|^2+2\int_{\mathbb{R}^n}F(u_0(t,x))dx+\varepsilon(u_0,u_{1,0})\right)$$

$$+c_6e^{-\frac{1}{4}\varepsilon\varrho r}\int_{-t}^{r}e^{\frac{1}{4}\varepsilon\varrho s}\left(1+\|h(s+\tau)\|^2+|\zeta_\delta(\theta_s\omega)|^{\frac{2(p+1)}{p+1-2q}}\right)ds. \tag{4.26}$$

For the first term on the right-hand side of (4.26), by (3.2) we have

$$e^{-\frac{1}{4}\varepsilon\varrho(t+r)}\left(\|u_{1,0}\|^2+\nu\|u_0\|^2+\|\nabla u_0\|^2+2\int_{\mathbb{R}^n}F(u_0(t,x))dx+\varepsilon(u_0,u_{1,0})\right)$$

$$\leq c_7e^{-\frac{1}{4}\varepsilon\varrho(t+r)}\left(1+\|u_{1,0}\|^2+\|u_0\|^2_{H^1(\mathbb{R}^n)}+\|u_0\|^{p+1}_{H^1(\mathbb{R}^n)}\right)$$

$$\leq c_8e^{-\frac{1}{4}\varepsilon\varrho(t+r)}\left(1+\|D(\tau-t,\theta_{-t}\omega)\|^{p+1}\right)\to 0,\quad as\ t\to\infty. \tag{4.27}$$

Combining with (4.5) and (4.26)-(4.27) we infer that there exists $T=T(\tau,\omega,D)>0$ such that for all $t\geq T$ and $r\in[-t,0]$,

$$\|u_t(\tau+r,\tau-t,\theta_{-\tau}\omega,u_{1,0})\|^2+\nu\|u(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+\|\nabla u(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0)\|^2$$

$$+2\int_{\mathbb{R}^n}F(u(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0))dx+\varepsilon(u(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0),u_t(\tau+r,\tau-t,\theta_{-\tau}\omega,u_{1,0}))$$

$$+(d_2-\varepsilon d_1)\int_{\tau-t}^{\tau+r}e^{\frac{1}{4}\varepsilon\varrho(s-\tau-r)}\|u_t(s,\tau-t,\theta_{-\tau}\omega,u_{1,0})\|^2ds$$

$$+\frac{1}{8}\varepsilon\nu(4-2\varrho-\varepsilon\varrho)\int_{\tau-t}^{\tau+r}e^{\frac{1}{4}\varepsilon\varrho(s-\tau-r)}\|u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2ds$$

$$+\frac{1}{4}\varepsilon(2-\varrho)\int_{\tau-t}^{\tau+r}e^{\frac{1}{4}\varepsilon\varrho(s-\tau-r)}\|\nabla u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2ds$$

$$\leq c_9e^{-\frac{1}{4}\varepsilon\varrho r}\int_{-\infty}^{0}e^{\frac{1}{4}\varepsilon\varrho s}\left(1+\|h(s+\tau)\|^2+|\zeta_\delta(\theta_s\omega)|^{\frac{2(p+1)}{p+1-2q}}\right)ds. \tag{4.28}$$

By (4.4) and Young's inequality, we have

$$|\varepsilon(u(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0),u_t(\tau+r,\tau-t,\theta_{-\tau}\omega,u_{1,0}))|$$

$$\leq\frac{1}{2}\varepsilon\|u(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+\frac{1}{2}\varepsilon\|u_t(\tau+r,\tau-t,\theta_{-\tau}\omega,u_{1,0})\|^2$$

$$\leq \frac{1}{2}\nu\|u(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+\frac{1}{2}\|u_t(\tau+r,\tau-t,\theta_{-\tau}\omega,u_{1,0})\|^2. \tag{4.29}$$

Inserting (4.29) into (4.28), we can conclude that for all $t \geq T$ and all $r \in [-t, 0]$,

$$\begin{aligned}
&\frac{1}{2}\|u_t(\tau+r,\tau-t,\theta_{-\tau}\omega,u_{1,0})\|^2+\frac{1}{2}\nu\|u(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0)\|^2\\
&+\|\nabla u(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+2\int_{\mathbb{R}^n}F(u(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0))dx\\
&+(d_2-\varepsilon d_1)\int_{\tau-t}^{\tau+r}e^{\frac{1}{4}\varepsilon\varrho(s-\tau-r)}\|u_t(s,\tau-t,\theta_{-\tau}\omega,u_{1,0})\|_{L^{\gamma+1}}^{\gamma+1}ds\\
&+\frac{1}{8}\varepsilon\nu(4-2\varrho-\varepsilon\varrho)\int_{\tau-t}^{\tau+r}e^{\frac{1}{4}\varepsilon\varrho(s-\tau-r)}\|u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2ds\\
&+\frac{1}{4}\varepsilon(2-\varrho)\int_{\tau-t}^{\tau+r}e^{\frac{1}{4}\varepsilon\varrho(s-\tau-r)}\|\nabla u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2ds\\
&\leq c_9e^{-\frac{1}{4}\varepsilon\varrho r}\int_{-\infty}^{0}e^{\frac{1}{4}\varepsilon\varrho s}\left(1+\|h(s+\tau)\|^2+|\zeta_\delta(\theta_s\omega)|^{\frac{2(p+1)}{p+1-2q}}\right)ds,
\end{aligned} \tag{4.30}$$

which yields the desired uniform estimates when $\gamma = 1$.

Next, we discuss the case of $\gamma > 1$. We infer from (4.24) the following crucial result: For any $s \geq \tau$ such that

$$V_\varepsilon(s) \leq \left(\frac{d_2}{\varepsilon d_1}\right)^{\frac{2\gamma}{\gamma-1}} - c_{10} \equiv \varphi(\varepsilon), \tag{4.31}$$

then

$$\varepsilon d_1(V_\varepsilon(t))^{\frac{\gamma-1}{2\gamma}} - d_2 \leq 0, \quad \forall\, t \geq s \geq \tau, \tag{4.32}$$

i.e.,

$$V_\varepsilon(t) \leq \left(\frac{d_2}{\varepsilon d_1}\right)^{\frac{2\gamma}{\gamma-1}}, \quad \forall\, t \geq s \geq \tau. \tag{4.33}$$

Indeed, we can infer from (4.31) that

$$V_\varepsilon(s) \leq \varphi(\varepsilon) \leq \left(\frac{d_2}{\varepsilon d_1}\right)^{\frac{2\gamma}{\gamma-1}}, \quad \forall\, s \geq \tau,$$

which implies

$$\varepsilon d_1(V_\varepsilon(s))^{\frac{\gamma-1}{2\gamma}} - d_2 \leq 0, \quad \forall\, s \geq \tau.$$

Thus exploiting the continuity of $V_\varepsilon(t)$, we have (4.32) is valid for t from some interval $[s, s+T)$. If $T < +\infty$, there exists $T^* > 0$ such that

$$\varepsilon d_1 (V_\varepsilon(t))^{\frac{\gamma-1}{2\gamma}} - d_2 < 0, \quad \forall t \in [s, s+T^*) \tag{4.34}$$

and

$$\varepsilon d_1 (V_\varepsilon(s+T^*))^{\frac{\gamma-1}{2\gamma}} - d_2 = 0. \tag{4.35}$$

From (4.24) and (4.34), we deduce

$$\frac{d}{dt}V_\varepsilon(t) + \frac{1}{4}\varepsilon\varrho V_\varepsilon(t) \le c_6 \left(1 + \|h(t)\|^2 + |\zeta_\delta(\theta_t\omega)|^{\frac{2(p+1)}{p+1-2q}}\right), \quad \forall t \in [s, s+T^*]. \tag{4.36}$$

Integrating (4.36) from s to t yields

$$V_\varepsilon(t) \le e^{-\frac{1}{4}\varepsilon\varrho(t-s)}V_\varepsilon(s) + c_6 \int_s^t e^{-\frac{1}{4}\varepsilon\varrho(t-r)} \left(1 + \|h(r)\|^2 + |\zeta_\delta(\theta_r\omega)|^{\frac{2(p+1)}{p+1-2q}}\right) dr. \tag{4.37}$$

By (4.5), we have

$$V_\varepsilon(t) \le e^{-\frac{1}{4}\varepsilon\varrho(t-s)}V_\varepsilon(s) + c_{10}, \quad \forall t \in [s, s+T^*]. \tag{4.38}$$

When $t = s + T^*$, we infer from (4.31) and (4.38) that

$$V_\varepsilon(s+T^*) < V_\varepsilon(s) + c_{10} \le \left(\frac{d_2}{\varepsilon d_1}\right)^{\frac{2\gamma}{\gamma-1}}.$$

This contradicts to the second relation in (4.35). Hence $T = +\infty$ and (4.32) holds. So we infer from (4.24) that

$$\frac{d}{dt}V_\varepsilon(t) + \frac{1}{4}\varepsilon\varrho V_\varepsilon(t) + \frac{1}{8}\varepsilon\nu(4 - 2\varrho - \varepsilon\varrho)\|u\|^2 + \frac{1}{4}\varepsilon(2-\varrho)\|\nabla u\|^2$$

$$\le c_6 \left(1 + \|h(t)\|^2 + |\zeta_\delta(\theta_t\omega)|^{\frac{2(p+1)}{p+1-2q}}\right). \tag{4.39}$$

Similar to the arguments in (4.25)-(4.30), we get the desired estimates when $\gamma > 1$, which completes the proof together with (4.30). $\square$

As a consequence of Lemma 4.1, we conclude that the cocycle Φ has a $\mathcal{D}-$pullback random absorbing set.

Lemma 4.2 *Let* (3.1)-(3.6), (4.3) *and* (4.5)-(4.6) *hold. Then the cocycle* Φ *possesses a closed measurable* $\mathcal{D}-$*pullback absorbing set* $\mathcal{K} = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$*, which is given by*

$$K(\tau, \omega) = \{(u_0, u_{1,0}) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) : \|u_0\|^2_{H^1(\mathbb{R}^n)} + \|u_{1,0}\|^2 \le R(\tau, \omega)\}, \qquad (4.40)$$

where

$$R(\tau, \omega) = L_1 \int_{-\infty}^{0} e^{\frac{1}{4}\varepsilon\varrho s}(1 + \|h(s+t)\|^2 + |\zeta_\delta(\theta_s\omega)|^{\frac{2(p+1)}{p+1-2q}})ds,$$

and L_1 *is the same number as in Lemma 4.1.*

Proof. Together with (3.12) and Lemma 4.1 with $r = 0$, we find that for every $\tau \in \mathbb{R}, \omega \in \Omega$ and $D \in \mathcal{D}$, there exists $T = T(\tau, \omega, D) > 0$ such that for all $t \ge T$,

$$\Phi(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)) \subseteq K(\tau, \omega). \qquad (4.41)$$

On the other hand, by (4.5)-(4.6), one can verify that for every $c > 0, \tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{t\to+\infty} e^{-ct}\|K(\tau - t, \theta_{-t}\omega)\|_{H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} = 0. \qquad (4.42)$$

By virtue of (4.41)-(4.42) we know that $K \in \mathcal{D}$ is a closed measurable $\mathcal{D}-$pullback absorbing set of Φ. $\square$

The following uniform estimates on the tails of solutions will be crucial for proving the $\mathcal{D}-$pullback asymptotic compactness of Φ. To this end, we choose a smooth function $\rho : \mathbb{R}^n \to \mathbb{R}$ such that $0 \le \rho(x) \le 1$ for any $x \in \mathbb{R}^n$, and

$$\rho(x) = 0 \quad for\ |x| \le \frac{1}{2}; \quad and \quad \rho(x) = 1 \quad for\ |x| \ge 1. \qquad (4.43)$$

For every $m \in \mathbb{N}$, let

$$\rho_m(x) = \rho(\frac{x}{m}), \quad x \in \mathbb{R}^n.$$

Then there exists a positive number c_0 independent of m such that $|\nabla\rho_m(x)| \le \frac{1}{m}c_0$ for all $x \in \mathbb{R}^n$ and $m \in \mathbb{N}$.

Lemma 4.3 *Let* (3.1)-(3.6), (4.3) *and* (4.5)-(4.6) *hold. Then for every* $\eta > 0, \tau \in \mathbb{R}, \omega \in \Omega$ *and* $D \in \mathcal{D}$*, there exists* $T_0 = T_0(\varsigma, \tau, \omega, D) > 0$ *and* $m_0 = m_0(\varsigma, \tau, \omega) \ge 1$ *such that for all* $t \ge T_0, r \in [-t, 0]$ *and* $m \ge m_0$*, the solution* u *of* (1.1) *satisfies*

$$\int_{|x|\ge m} (|u_t(\tau + r, \tau - t, \theta_{-\tau}\omega, u_{1,0})|^2 + |u(\tau + r, \tau - t, \theta_{-\tau}\omega, u_0)|^2 dx$$

$$+ \int_{|x|\ge m} |\nabla u(\tau + r, \tau - t, \theta_{-\tau}\omega, u_0)|^2) dx < \eta e^{-\frac{1}{4}\varepsilon\varrho r},$$

for all $(u_0, u_{1,0}) \in D(\tau - t, \theta_{-t}\omega)$.

Proof. Given $m \in \mathbb{N}$, let ρ_m be the smooth function as defined by (4.43). Similar to the energy equation (3.10), we find

$$\frac{d}{dt}\int_{\mathbb{R}^n}\rho_m(x)\left(|u_t(t,x)|^2+\nu|u(t,x)|^2+|\nabla u(t,x)|^2+2F(u(t,x))\right)dx$$

$$+2\sigma(\|\nabla u\|^2)\int_{\mathbb{R}^n}\rho_m(x)g(u_t(t,x))u_t(t,x)dx$$

$$=-2\int_{\mathbb{R}^n}u_t(t,x)\nabla u(t,x)\cdot\nabla\rho_m(x)dx+2\int_{\mathbb{R}^n}\rho_m(x)h(t,x)u_t(t,x)dx$$

$$+2\zeta_\delta(\theta_t\omega)\int_{\mathbb{R}^n}\rho_m(x)R(t,x,u(t,x))u_t(t,x)dx. \tag{4.44}$$

Similar to (3.11), we have

$$\frac{d}{dt}\int_{\mathbb{R}^n}\rho_m(x)u(t,x)u_t(t,x)dx+\nu\int_{\mathbb{R}^n}\rho_m(x)|u(t,x)|^2dx+\int_{\mathbb{R}^n}\rho_m(x)|\nabla u(t,x)|^2dx$$

$$+\sigma(\|\nabla u\|^2)\int_{\mathbb{R}^n}\rho_m(x)g(u_t(t,x))u(t,x)dx+\int_{\mathbb{R}^n}\rho_m(x)f(u(t,x))u(t,x)dx$$

$$=\int_{\mathbb{R}^n}\rho_m(x)|u_t(t,x)|^2dx-\int_{\mathbb{R}^n}u(t,x)\nabla u(t,x)\cdot\nabla\rho_m(x)dx$$

$$+\int_{\mathbb{R}^n}\rho_m(x)h(t,x)u(t,x)dx+\zeta_\delta(\theta_t\omega)\int_{\mathbb{R}^n}\rho_m(x)R(t,x,u(t,x))u(t,x)dx. \tag{4.45}$$

It follows from (4.44)-(4.45) that

$$\frac{d}{dt}\int_{\mathbb{R}^n}\rho_m(x)\left(|u_t(t,x)|^2+\nu|u(t,x)|^2+|\nabla u(t,x)|^2+2F(u(t,x))+\varepsilon u(t,x)u_t(t,x)\right)dx$$

$$-\varepsilon\int_{\mathbb{R}^n}\rho_m(x)|u_t(t,x)|^2dx+\varepsilon\nu\int_{\mathbb{R}^n}\rho_m(x)|u(t,x)|^2dx+\varepsilon\int_{\mathbb{R}^n}\rho_m(x)|\nabla u(t,x)|^2dx$$

$$+\varepsilon\int_{\mathbb{R}^n}\rho_m(x)f(u(t,x))u(t,x)dx+2\sigma(\|\nabla u\|^2)\int_{\mathbb{R}^n}\rho_m(x)g(u_t(t,x))u_t(t,x)dx$$

$$=\int_{\mathbb{R}^n}\rho_m(x)\left(h(t,x)+R(t,x,u(t,x))\zeta_\delta(\theta_t\omega)\right)\left(\varepsilon u(t,x)+2u_t(t,x)\right)dx$$

$$-\varepsilon\int_{\mathbb{R}^n}u(t,x)\nabla u(t,x)\cdot\nabla\rho_m(x)dx-2\int_{\mathbb{R}^n}u_t(t,x)\nabla u(t,x)\cdot\nabla\rho_m(x)dx$$

$$-\sigma(\|\nabla u\|^2)\int_{\mathbb{R}^n}\rho_m(x)g(u_t(t,x))u(t,x)dx. \tag{4.46}$$

By (4.3) and (4.46), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n}\rho_m(x)\left(|u_t(t,x)|^2+\nu|u(t,x)|^2+|\nabla u(t,x)|^2+2F(u(t,x))+\varepsilon u(t,x)u_t(t,x)\right)dx\\
&-\varepsilon\int_{\mathbb{R}^n}\rho_m(x)|u_t(t,x)|^2dx+\varepsilon\nu\int_{\mathbb{R}^n}\rho_m(x)|u(t,x)|^2dx+\varepsilon\int_{\mathbb{R}^n}\rho_m(x)|\nabla u(t,x)|^2dx\\
&+\varepsilon\varrho\int_{\mathbb{R}^n}\rho_m(x)F(u(t,x))dx+2\sigma(\|\nabla u\|^2)\int_{\mathbb{R}^n}\rho_m(x)g(u_t(t,x))u_t(t,x)dx\\
&\le\int_{\mathbb{R}^n}\rho_m(x)\left(h(t,x)+R(t,x,u(t,x))\zeta_\delta(\theta_t\omega)\right)\left(\varepsilon u(t,x)+2u_t(t,x)\right)dx\\
&-\varepsilon\int_{\mathbb{R}^n}u(t,x)\nabla u(t,x)\cdot\nabla\rho_m(x)dx-2\int_{\mathbb{R}^n}u_t(t,x)\nabla u(t,x)\cdot\nabla\rho_m(x)dx\\
&-\sigma(\|\nabla u\|^2)\int_{\mathbb{R}^n}\rho_m(x)g(u_t(t,x))u(t,x)dx. \qquad (4.47)
\end{aligned}
$$

Following the arguments of (4.9)-(4.12), by (3.3)-(3.4) and (4.7) one can verify that the first term on the right-hand side of (4.47) is bounded by

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^n}\rho_m(x)\left(h(t,x)+R(t,x,u(t,x))\zeta_\delta(\theta_t\omega)\right)\left(\varepsilon u(t,x)+2u_t(t,x)\right)dx\right|\\
&\le\frac{1}{4}\varepsilon\nu\int_{\mathbb{R}^n}\rho_m(x)|u(t,x)|^2dx+\varepsilon\int_{\mathbb{R}^n}\rho_m(x)|u_t(t,x)|^2dx+\frac{1}{2}\varepsilon\varrho\int_{\mathbb{R}^n}\rho_m(x)F(u(t,x))dx\\
&+c_{11}\int_{\mathbb{R}^n}\rho_m(x)\left(|h(t,x)|^2+|\zeta_\delta(\theta_t\omega)\beta_1(t,x)|^{\frac{2(p+1)}{p+1-2q}}+|\zeta_\delta(\theta_t\omega)\beta_2(t,x)|^2\right)dx, \qquad (4.48)
\end{aligned}
$$

where $c_{11}>0$ depends on ε,ν and ϱ. By the property of ρ_m, we have

$$
\begin{aligned}
&\left|-\varepsilon\int_{\mathbb{R}^n}u(t,x)\nabla u(t,x)\cdot\nabla\rho_m(x)dx-2\int_{\mathbb{R}^n}u_t(t,x)\nabla u(t,x)\cdot\nabla\rho_m(x)dx\right|\\
&\le\frac{c_{12}}{m}(\|u(t)\|+\|u_t(t)\|)\|\nabla u(t)\|, \qquad (4.49)
\end{aligned}
$$

where $c_{12}>0$ depends only on ε, but not on m. Similar to (4.15), we arrive at

$$
\begin{aligned}
&-\sigma(\|\nabla u\|^2)\int_{\mathbb{R}^n}\rho_m(x)g(u_t(t,x))u(t,x)dx\\
&\le\frac{\varepsilon\sigma_1^2\kappa_2^2}{\nu}\int_{\mathbb{R}^n}\rho_m(x)|u_t(t,x)|^{\gamma+1}dx+\frac{1}{4}\varepsilon\nu\int_{\mathbb{R}^n}\rho_m(x)|u(t,x)|^2dx+\frac{1}{2}\varepsilon\int_{\mathbb{R}^n}\rho_m(x)|\nabla u(t,x)|^2dx
\end{aligned}
$$

$$+\varepsilon d_1 \int_{\mathbb{R}^n} \rho_m(x)|u_t(t,x)|^{\gamma+1}dx \left(\int_{\mathbb{R}^n} \rho_m(x)|\nabla u_t(t,x)|^2 dx\right)^{\frac{\gamma-1}{2\gamma}} + \varepsilon c_4. \tag{4.50}$$

Inserting (4.48)-(4.50) into (4.47) yields

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n} \rho_m(x)\left(|u_t(t,x)|^2 + \nu|u(t,x)|^2 + |\nabla u(t,x)|^2 + 2F(u(t,x)) + \varepsilon u(t,x)u_t(t,x)\right)dx \\
&-2\varepsilon \int_{\mathbb{R}^n} \rho_m(x)|u_t(t,x)|^2dx + \frac{1}{2}\varepsilon\nu \int_{\mathbb{R}^n} \rho_m(x)|u(t,x)|^2dx + \frac{1}{2}\varepsilon \int_{\mathbb{R}^n} \rho_m(x)|\nabla u(t,x)|^2dx \\
&+\frac{1}{2}\varepsilon\varrho \int_{\mathbb{R}^n} \rho_m(x)F(u(t,x))dx + 2\sigma(\|\nabla u\|^2)\int_{\mathbb{R}^n} \rho_m(x)g(u_t(t,x))u_t(t,x)dx \\
&\leq \varepsilon d_1 \int_{\mathbb{R}^n} \rho_m(x)|u_t(t,x)|^{\gamma+1}dx \left(\int_{\mathbb{R}^n} \rho_m(x)|\nabla u_t(t,x)|^2 dx\right)^{\frac{\gamma-1}{2\gamma}} \\
&+c_{11}\int_{\mathbb{R}^n} \rho_m(x)\left(|h(t,x)|^2 + |\zeta_\delta(\theta_t\omega)\beta_1(t,x)|^{\frac{2(p+1)}{p+1-2q}} + |\zeta_\delta(\theta_t\omega)\beta_2(t,x)|^2\right)dx \\
&+\frac{\varepsilon\sigma_1^2\kappa_2^2}{\nu}\int_{\mathbb{R}^n} \rho_m(x)|u_t(t,x)|^{\gamma+1}dx + \frac{c_{12}}{m}(\|u(t)\| + \|u_t(t)\|)\|\nabla u(t)\| + \varepsilon c_4,
\end{aligned} \tag{4.51}$$

which can be rewritten as

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n} \rho_m(x)\left(|u_t(t,x)|^2 + \nu|u(t,x)|^2 + |\nabla u(t,x)|^2 + 2F(u(t,x)) + \varepsilon u(t,x)u_t(t,x)\right)dx \\
&+\frac{1}{4}\varepsilon\varrho\int_{\mathbb{R}^n} \rho_m(x)\left(|u_t(t,x)|^2 + \nu|u(t,x)|^2 + |\nabla u(t,x)|^2 + 2F(u(t,x)) + \varepsilon u(t,x)u_t(t,x)\right)dx \\
&+\frac{1}{4}\varepsilon\nu(2-\varrho)\int_{\mathbb{R}^n} \rho_m(x)|u(t,x)|^2dx + \frac{1}{4}\varepsilon(2-\varrho)\int_{\mathbb{R}^n} \rho_m(x)|\nabla u(t,x)|^2dx \\
&+(-2\varepsilon - \frac{1}{4}\varepsilon\varrho)\int_{\mathbb{R}^n} \rho_m(x)|u_t(t,x)|^2dx + 2\sigma(\|\nabla u\|^2)\int_{\mathbb{R}^n} \rho_m(x)g(u_t(t,x))u_t(t,x)dx \\
&-\frac{1}{4}\varepsilon^2\varrho\int_{\mathbb{R}^n} \rho_m(x)u(t,x)u_t(t,x)dx \\
&\leq \varepsilon d_1 \int_{\mathbb{R}^n} \rho_m(x)|u_t(t,x)|^{\gamma+1}dx \left(\int_{\mathbb{R}^n} \rho_m(x)|\nabla u_t(t,x)|^2 dx\right)^{\frac{\gamma-1}{2\gamma}} \\
&+c_{11}\int_{\mathbb{R}^n} \rho_m(x)\left(|h(t,x)|^2 + |\zeta_\delta(\theta_t\omega)\beta_1(t,x)|^{\frac{2(p+1)}{p+1-2q}} + |\zeta_\delta(\theta_t\omega)\beta_2(t,x)|^2\right)dx \\
&+\frac{\varepsilon\sigma_1^2\kappa_2^2}{\nu}\int_{\mathbb{R}^n} \rho_m(x)|u_t(t,x)|^{\gamma+1}dx + \frac{c_{12}}{m}(\|u(t)\| + \|u_t(t)\|)\|\nabla u(t)\| + \varepsilon c_4,
\end{aligned} \tag{4.52}$$

By Young's inequality, we have

$$\frac{1}{4}\varepsilon^2\varrho\int_{\mathbb{R}^n}\rho_m(x)u(t,x)u_t(t,x)dx$$

$$\leq \frac{1}{8}\varepsilon^2\nu\varrho\int_{\mathbb{R}^n}\rho_m(x)|u(t,x)|^2dx+\frac{\varepsilon^2\varrho}{8\nu}\int_{\mathbb{R}^n}\rho_m(x)|u_t(t,x)|^2dx. \tag{4.53}$$

Inserting (4.53) into (4.52) yields

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n}\rho_m(x)\left(|u_t(t,x)|^2+\nu|u(t,x)|^2+|\nabla u(t,x)|^2+2F(u(t,x))+\varepsilon u(t,x)u_t(t,x)\right)dx\\
&+\frac{1}{4}\varepsilon\varrho\int_{\mathbb{R}^n}\rho_m(x)\left(|u_t(t,x)|^2+\nu|u(t,x)|^2+|\nabla u(t,x)|^2+2F(u(t,x))+\varepsilon u(t,x)u_t(t,x)\right)dx\\
&+\frac{1}{8}\varepsilon\nu(4-2\varrho-\varepsilon\varrho)\int_{\mathbb{R}^n}\rho_m(x)|u(t,x)|^2dx+\frac{1}{4}\varepsilon(2-\varrho)\int_{\mathbb{R}^n}\rho_m(x)|\nabla u(t,x)|^2dx\\
&+2\sigma(\|\nabla u\|^2)\int_{\mathbb{R}^n}\rho_m(x)g(u_t(t,x))u_t(t,x)dx\\
&\leq (2\varepsilon+\frac{1}{4}\varepsilon\varrho+\frac{\varepsilon^2\varrho}{8\nu})\int_{\mathbb{R}^n}\rho_m(x)|u_t(t,x)|^2dx+\frac{\varepsilon\sigma_1^2\kappa_2^2}{\nu}\int_{\mathbb{R}^n}\rho_m(x)|u_t(t,x)|^{\gamma+1}dx\\
&+\varepsilon d_1\int_{\mathbb{R}^n}\rho_m(x)|u_t(t,x)|^{\gamma+1}dx\left(\int_{\mathbb{R}^n}\rho_m(x)|\nabla u_t(t,x)|^2dx\right)^{\frac{\gamma-1}{2\gamma}}\\
&+c_{11}\int_{\mathbb{R}^n}\rho_m(x)\left(|h(t,x)|^2+|\zeta_\delta(\theta_t\omega)\beta_1(t,x)|^{\frac{2(p+1)}{p+1-2q}}+|\zeta_\delta(\theta_t\omega)\beta_2(t,x)|^2\right)dx\\
&+\frac{c_{12}}{m}(\|u(t)\|+\|u_t(t)\|)\|\nabla u(t)\|+\varepsilon c_4,
\end{aligned}\tag{4.54}$$

Collecting (3.6) and (4.14), we deduce

$$2\sigma(\|\nabla u\|^2)\int_{\mathbb{R}^n}\rho_m(x)g(u_t(t,x))u_t(t,x)dx$$

$$\geq \sigma_o\int_{\mathbb{R}^n}\rho_m(x)g(u_t(t,x))u_t(t,x)dx+\frac{\sigma_0\kappa_1}{\gamma}\int_{\mathbb{R}^n}\rho_m(x)|u_t(t,x)|^{\gamma+1}dx. \tag{4.55}$$

By (3.6), there exist $l>0$ and $L>0$ such that $g'(s)\geq l$ when $|s|>L$. Then we get

$$\sigma_o\int_{\mathbb{R}^n}\rho_m(x)g(u_t(t,x))u_t(t,x)dx-(2\varepsilon+\frac{1}{4}\varepsilon\varrho+\frac{\varepsilon^2\varrho}{8\nu})\int_{\mathbb{R}^n}\rho_m(x)|u_t(t,x)|^2dx$$

$$\geq (l\sigma_o-2\varepsilon-\frac{1}{4}\varepsilon\varrho-\frac{\varepsilon^2\varrho}{8\nu})\int_{\mathbb{O}}\rho_m(x)|u_t(t,x)|^2dx-c_5, \tag{4.56}$$

where $c_5 > 0$ and $\mathbb{O}$ are given in Lemma 4.1. Similar to (4.24), we have

$$\frac{d}{dt}\int_{\mathbb{R}^n}\rho_m(x)\left(|u_t(t,x)|^2+\nu|u(t,x)|^2+|\nabla u(t,x)|^2+2F(u(t,x))+\varepsilon u(t,x)u_t(t,x)\right)dx$$

$$+\frac{1}{4}\varepsilon\varrho\int_{\mathbb{R}^n}\rho_m(x)\left(|u_t(t,x)|^2+\nu|u(t,x)|^2+|\nabla u(t,x)|^2+2F(u(t,x))+\varepsilon u(t,x)u_t(t,x)\right)dx$$

$$+\frac{1}{8}\varepsilon\nu(4-2\varrho-\varepsilon\varrho)\int_{\mathbb{R}^n}\rho_m(x)|u(t,x)|^2dx+\frac{1}{4}\varepsilon(2-\varrho)\int_{\mathbb{R}^n}\rho_m(x)|\nabla u(t,x)|^2dx$$

$$\leq\left(\varepsilon d_1(\int_{\mathbb{R}^n}\rho_m(x)(|u_t(t,x)|^2+\nu|u(t,x)|^2+|\nabla u(t,x)|^2+2F(u(t,x)))dx)^{\frac{\gamma-1}{2\gamma}}-d_2\right)\cdot$$

$$\int_{\mathbb{R}^n}\rho_m(x)|u_t(t,x)|^{\gamma+1}dx+\frac{c_{12}}{m}(\|u(t)\|+\|u_t(t)\|)\|\nabla u(t)\|$$

$$+c_{11}\int_{\mathbb{R}^n}\rho_m(x)\left(|h(t,x)|^2+|\zeta_\delta(\theta_t\omega)\beta_1(t,x)|^{\frac{2(p+1)}{p+1-2q}}+|\zeta_\delta(\theta_t\omega)\beta_2(t,x)|^2\right)dx. \tag{4.57}$$

Since $\gamma\geq 1$, similar to the arguments in (4.25) and (4.39), we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^n}\rho_m(x)\left(|u_t(t,x)|^2+\nu|u(t,x)|^2+|\nabla u(t,x)|^2+2F(u(t,x))+\varepsilon u(t,x)u_t(t,x)\right)dx$$

$$+\frac{1}{4}\varepsilon\varrho\int_{\mathbb{R}^n}\rho_m(x)\left(|u_t(t,x)|^2+\nu|u(t,x)|^2+|\nabla u(t,x)|^2+2F(u(t,x))+\varepsilon u(t,x)u_t(t,x)\right)dx$$

$$+c_{11}\int_{\mathbb{R}^n}\rho_m(x)\left(|h(t,x)|^2+|\zeta_\delta(\theta_t\omega)\beta_1(t,x)|^{\frac{2(p+1)}{p+1-2q}}+|\zeta_\delta(\theta_t\omega)\beta_2(t,x)|^2\right)dx$$

$$+\frac{c_{12}}{m}(\|u(t)\|+\|u_t(t)\|)\|\nabla u(t)\|. \tag{4.58}$$

Integrating (4.58) on $[\tau-t,\tau+r]$ with $t>0$ and all $r\in[-t,0]$, after replacing ω by $\theta_{-\tau}\omega$, we get

$$\int_{\mathbb{R}^n}\rho_m(x)\left(|u_t(\tau+r,\tau-t,\theta_{-\tau}\omega,u_{1,0})|^2+\nu|u(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0)|^2+|\nabla u|^2\right)dx$$

$$+2\int_{\mathbb{R}^n}\rho_m(x)F(u(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0))dx$$

$$+\varepsilon\int_{\mathbb{R}^n}\rho_m(x)u(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0)u_t(\tau+r,\tau-t,\theta_{-\tau}\omega,u_{1,0})dx$$

$$\leq e^{-\frac{1}{4}\varepsilon\varrho(t+r)}\int_{\mathbb{R}^n}\rho_m(x)\left(|u_{1,0}|^2+\nu|u_0|^2+|\nabla u_0|^2+2F(u_0(x))+\varepsilon u_0(x)u_{1,0}(x)\right)dx$$

$$+c_{11}e^{-\frac{1}{4}\varepsilon\varrho r}\int_{-t}^{r}e^{\frac{1}{4}\varepsilon\varrho s}\int_{\mathbb{R}^n}\rho_m(x)\left(|h(s+\tau,x)|^2+|\zeta_\delta(\theta_s\omega)\beta_1(s,x)|^{\frac{2(p+1)}{q+1-2q}}\right)dxds$$

$$+c_{11}e^{-\frac{1}{4}\varepsilon\varrho r}\int_{-t}^{r}e^{\frac{1}{4}\varepsilon\varrho s}\int_{\mathbb{R}^n}\rho_m(x)|\zeta_\delta(\theta_s\omega)\beta_2(s,x)|^2dxds$$

$$+\frac{2c_{12}}{m}\int_{\tau-t}^{\tau+r}e^{\frac{1}{4}\varepsilon\varrho(s-\tau-r)}\left(\|u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2_{H^1(\mathbb{R}^n)}+\|u_t(s,\tau-t,\theta_{-\tau}\omega,u_{1,0})\|^2\right)ds. \tag{4.59}$$

Analogous to (4.27), the first term on the right-hand side of (4.59) is bounded. Hence, there exists $T_1=T_1(\varsigma,\tau,\omega,D)>0$ such that for all $t\geq T_1$,

$$e^{-\frac{1}{4}\varepsilon\varrho(t+r)}\int_{\mathbb{R}^n}\rho_m(x)\left(|u_{1,0}|^2+\nu|u_0|^2+|\nabla u_0|^2+2F(u_0(x))+\varepsilon u_0(x)u_{1,0}(x)\right)dx<\eta e^{-\frac{1}{4}\varepsilon\varrho r}. \tag{4.60}$$

For the second term on the right-hand side of (4.59) we have

$$\begin{aligned}
c_{11}e^{-\frac{1}{4}\varepsilon\varrho r}&\int_{-t}^{r}e^{\frac{1}{4}\varepsilon\varrho s}\int_{\mathbb{R}^n}\rho_m(x)\left(|h(s+\tau,x)|^2+|\zeta_\delta(\theta_s\omega)\beta_1(s,x)|^{\frac{2(p+1)}{q+1-2q}}\right)dxds\\
&+c_{11}e^{-\frac{1}{4}\varepsilon\varrho r}\int_{-t}^{r}e^{\frac{1}{4}\varepsilon\varrho s}\int_{\mathbb{R}^n}\rho_m(x)|\zeta_\delta(\theta_s\omega)\beta_2(s,x)|^2dxds\\
\leq c_{11}e^{-\frac{1}{4}\varepsilon\varrho r}&\int_{-\infty}^{r}e^{\frac{1}{4}\varepsilon\varrho s}\int_{|x|\geq\frac{1}{2}m}\left(|h(s+\tau,x)|^2+|\zeta_\delta(\theta_s\omega)\beta_1(s,x)|^{\frac{2(p+1)}{q+1-2q}}\right)dxds\\
&+c_{11}e^{-\frac{1}{4}\varepsilon\varrho r}\int_{-\infty}^{r}e^{\frac{1}{4}\varepsilon\varrho s}\int_{|x|\geq\frac{1}{2}m}|\zeta_\delta(\theta_s\omega)\beta_2(s,x)|^2dxds\\
&\leq c_{11}e^{-\frac{1}{4}\varepsilon\varrho r}\int_{-\infty}^{r}e^{\frac{1}{4}\varepsilon\varrho s}\int_{|x|\geq\frac{1}{2}m}|h(s+\tau,x)|^2dxds\\
&+c_{11}e^{-\frac{1}{4}\varepsilon\varrho r}\int_{-\infty}^{r}e^{\frac{1}{4}\varepsilon\varrho s}|\zeta_\delta(\theta_s\omega)|^{\frac{2(p+1)}{q+1-2q}}ds\int_{|x|\geq\frac{1}{2}m}|\beta_1(s,x)|^{\frac{2(p+1)}{q+1-2q}}dx\\
&+c_{11}e^{-\frac{1}{4}\varepsilon\varrho r}\int_{-\infty}^{r}e^{\frac{1}{4}\varepsilon\varrho s}|\zeta_\delta(\theta_s\omega)|^2ds\int_{|x|\geq\frac{1}{2}m}|\beta_2(s,x)|^2dx.
\end{aligned} \tag{4.61}$$

Combining (4.5) with (4.7) we find that there exists $m_1=m_1(\varsigma,\tau,\omega)\geq 1$ such that for all $m\geq m_1$, the right-hand side of (4.47) is bounded by $\eta e^{-\frac{1}{4}\varepsilon\varrho r}$. Thus, for all $m\geq m_1$ and $r\in[-t,0]$,

$$\begin{aligned}
c_{11}e^{-\frac{1}{4}\varepsilon\varrho r}&\int_{-t}^{r}e^{\frac{1}{4}\varepsilon\varrho s}\int_{\mathbb{R}^n}\rho_m(x)\left(|h(s+\tau,x)|^2+|\zeta_\delta(\theta_s\omega)\beta_1(s,x)|^{\frac{2(p+1)}{q+1-2q}}\right)dxds\\
&+c_{11}e^{-\frac{1}{4}\varepsilon\varrho r}\int_{-t}^{r}e^{\frac{1}{4}\varepsilon\varrho s}\int_{\mathbb{R}^n}\rho_m(x)|\zeta_\delta(\theta_s\omega)\beta_2(s,x)|^2dxds<\eta e^{-\frac{1}{4}\varepsilon\varrho r}.
\end{aligned} \tag{4.62}$$

For the last term in (4.59), by Lemma 4.1 we see that there exists $T_2 = T_2(\varsigma, \tau, \omega, D) \geq T_1$ such that for all $t \geq T_2$,

$$\frac{2c_{12}}{m}\int_{\tau-t}^{\tau+r} e^{\frac{1}{4}\varepsilon\varrho(s-\tau-r)}\left(\|u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2_{H^1(\mathbb{R}^n)} + \|u_t(s,\tau-t,\theta_{-\tau}\omega,u_{1,0})\|^2\right)ds \leq \frac{c_{13}}{m},$$

where $c_{13} > 0$ depends only on $\varepsilon, \nu, \gamma, \tau$ and ω, but not on m, which implies that there exists $m_2 = m_2(\varsigma, \tau, \omega) \geq m_1$ such that for all $m \geq m_2$ and $t \geq T_2$,

$$\begin{aligned}&\frac{2c_{12}}{m}\int_{\tau-t}^{\tau+r} e^{\frac{1}{4}\varepsilon\varrho(s-\tau-r)}\left(\|u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2_{H^1(\mathbb{R}^n)} + \|u_t(s,\tau-t,\theta_{-\tau}\omega,u_{1,0})\|^2\right)ds \\ &< \eta e^{-\frac{1}{4}\varepsilon\varrho r}.\end{aligned} \tag{4.63}$$

It follows from (4.59)-(4.60) and (4.62)-(4.63) that for all $m \geq m_2$ and $t \geq T_2$,

$$\begin{aligned}&\int_{\mathbb{R}^n}\rho_m(x)\left(|u_t(\tau+r,\tau-t,\theta_{-\tau}\omega,u_{1,0})|^2 + \nu|u(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0)|^2 + |\nabla u|^2\right)dx \\ &+2\int_{\mathbb{R}^n}\rho_m(x)F(u(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0))dx \\ &+\varepsilon\int_{\mathbb{R}^n}\rho_m(x)u(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0)u_t(\tau+r,\tau-t,\theta_{-\tau}\omega,u_{1,0})dx \\ &< 3\eta e^{-\frac{1}{4}\varepsilon\varrho r}.\end{aligned} \tag{4.64}$$

Similar to (4.29), By (4.4) and Young's inequality, we claim that

$$\begin{aligned}&\varepsilon\int_{\mathbb{R}^n}\rho_m(x)u(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0)u_t(\tau+r,\tau-t,\theta_{-\tau}\omega,u_{1,0})dx \\ \leq& \frac{1}{2}\varepsilon\int_{\mathbb{R}^n}\rho_m(x)|u(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0)|^2dx + \frac{1}{2}\varepsilon\int_{\mathbb{R}^n}\rho_m(x)|u_t(\tau+r,\tau-t,\theta_{-\tau}\omega,u_{1,0})|^2dx \\ \leq& \frac{1}{2}\nu\int_{\mathbb{R}^n}\rho_m(x)|u(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0)|^2dx + \frac{1}{2}\int_{\mathbb{R}^n}\rho_m(x)|u_t(\tau+r,\tau-t,\theta_{-\tau}\omega,u_{1,0})|^2dx,\end{aligned}$$

which along with (3.3) and (4.64) yields that for all $m \geq m_2$ and $t \geq T_2$,

$$\begin{aligned}&\int_{|x|\geq m}(\frac{1}{2}|u_t(\tau+r,\tau-t,\theta_{-\tau}\omega,u_{1,0})|^2 + \frac{1}{2}\nu|u(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0)|^2 \\ &\qquad +|\nabla u(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0)|^2)dx \\ \leq& \int_{\mathbb{R}^n}\rho_m(x)(\frac{1}{2}|u_t(\tau+r,\tau-t,\theta_{-\tau}\omega,u_{1,0})|^2 + \frac{1}{2}\nu|u(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0)|^2 \\ &\qquad +|\nabla u(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0)|^2)dx \\ &< 3\eta e^{-\frac{1}{4}\varepsilon\varrho r},\end{aligned} \tag{4.65}$$

which completes the proof. □

4.2 Existence of random attractors

In this subsection, we above all investigate the $\mathcal{D}-$pullback asymptotic compactness of Φ in $H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ by combining the idea of energy equation with the trick of uniform tail-estimates. And then we prove the existence and uniqueness of $\mathcal{D}-$pullback random attractors of Φ.

Given $(u,v)\in H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$, let

$$E(u,v)=\|v\|^2+\nu\|u\|^2+\|\nabla u\|^2+2\int_{\mathbb{R}^n}F(u(x))dx+\varepsilon(u,v),$$

and

$$\begin{aligned}\Psi(u,v)&=(h(t)+R(t,\cdot,u(t))\zeta_\delta(\theta_t\omega),\varepsilon u+2v)-\varepsilon\sigma(\|\nabla u\|^2)\int_{\mathbb{R}^n}g(u_t(t,x))u(t,x)dx\\&\quad+\frac{1}{2}\varepsilon(2+\varrho)\|v\|^2-\frac{1}{2}\varepsilon\nu(2-\varrho)\|u\|^2-\frac{1}{2}\varepsilon(2-\varrho)\|\nabla u\|^2+\frac{1}{2}\varepsilon^2(u,v)\\&\quad-2\sigma(\|\nabla u\|^2)\int_{\mathbb{R}^n}g(u_t(t,x))u_t(t,x)dx+\varepsilon\int_{\mathbb{R}^n}(\varrho F(u(t,x))-f(u(t,x))u(t,x))dx.\end{aligned}$$

Then the energy equation (4.8) can be rewritten as

$$\frac{d}{dt}E(u(t),u_t(t))+\frac{1}{2}\varepsilon\varrho E(u(t),u_t(t))=\Psi(u(t),u_t(t)). \tag{4.66}$$

Integrating (4.66) from τ to t, we have

$$\begin{aligned}&E(u(t,\tau,\omega,u_0),u_t(t,\tau,\omega,u_{1,0}))\\&=e^{\frac{1}{2}\varepsilon\varrho(\tau-t)}E(u_0,u_{1,0})+\int_\tau^t e^{\frac{1}{2}\varepsilon\varrho(s-t)}\Psi(u(s,\tau,\omega,u_0),u_t(s,\tau,\omega,u_{1,0}))ds.\end{aligned} \tag{4.67}$$

Next, we prove the $\mathcal{D}-$pullback asymptotic compactness of Φ by the energy equation (4.67).

Lemma 4.4 *Let* (3.1)-(3.6), (4.3) *and* (4.5)-(4.6) *hold. Then the cocycle* Φ *is* $\mathcal{D}-$*pullback asymptotically compact in* $H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$; *that is, for any* $\tau\in\mathbb{R},\omega\in\Omega$ *and* $(u_0^{(n)},u_{1,0}^{(n)})\in D(\tau-t_n,\theta_{-t_n}\omega)$ *with* $D\in\mathcal{D}$, *the sequence*

$$\left\{\Phi(t_n,\tau-t_n,\theta_{-t_n}\omega,(u_0^{(n)},u_{1,0}^{(n)}))\right\}_{n=1}^{\infty}$$

has a convergent subsequence in $H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ *as* $t_n\to\infty$.

Proof. By the definition of cocycle Φ in (3.12), for all $n \in \mathbb{N}$,

$$\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, (u_0^n, u_{1,0}^n)) = (u(\tau, \tau - t_n, \theta_{-\tau}\omega, u_0^n), u_t(\tau, \tau - t_n, \theta_{-\tau}\omega, u_{1,0}^n)).$$

Therefore, we need to show the sequence $\{(u(\tau, \tau-t_n, \theta_{-\tau}\omega, u_0^n), u_t(\tau, \tau-t_n, \theta_{-\tau}\omega, u_{1,0}^n))\}_{n=1}^{\infty}$ is precompact in $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

By Lemma 4.1 with $r = 0$, we find that $\{(u(\tau, \tau-t_n, \theta_{-\tau}\omega, u_0^n), u_t(\tau, \tau-t_n, \theta_{-\tau}\omega, u_{1,0}^n))\}_{n=1}^{\infty}$ is bounded in $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and hence there exists $(\xi, \varsigma) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ such that, up to a subsequence,

$$(u(\tau, \tau - t_n, \theta_{-\tau}\omega, u_0^n), u_t(\tau, \tau - t_n, \theta_{-\tau}\omega, u_{1,0}^n)) \rightharpoonup (\xi, \varsigma) \textit{ in } H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n), \tag{4.68}$$

which implies

$$\liminf_{n\to\infty} \|(u(\tau, \tau - t_n, \theta_{-\tau}\omega, u_0^n), u_t(\tau, \tau - t_n, \theta_{-\tau}\omega, u_{1,0}^n))\|_{H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} \geq \|(\xi, \varsigma)\|_{H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}. \tag{4.69}$$

Here, it remains to show

$$\limsup_{n\to\infty} \|(u(\tau, \tau - t_n, \theta_{-\tau}\omega, u_0^n), u_t(\tau, \tau - t_n, \theta_{-\tau}\omega, u_{1,0}^n))\|_{H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} \leq \|(\xi, \varsigma)\|_{H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}, \tag{4.70}$$

which along with (4.69) shows the strong convergence of $(u(\tau, \tau - t_n, \theta_{-\tau}\omega, u_0^{(n)}), u_t(\tau, \tau - t_n, \theta_{-\tau}\omega, u_{1,0}^{(n)}))$ in $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, and hence completes the proof. Next, we prove (4.70) by the energy equation (4.67). By Lemma 4.1, there exists $N = N(\tau, \omega, D) \in \mathbb{N}$ such that for $n \geq \mathbb{N}$ and $r \in [-t_n, 0]$,

$$\begin{aligned} &\|u(\tau + r, \tau - t_n, \theta_{-\tau}\omega, u_0^n)\|^2_{H^1(\mathbb{R}^n)} + \|u_t(\tau + r, \tau - t_n, \theta_{-\tau}\omega, u_{1,0}^n)\|^2 \\ &\quad + \int_{\mathbb{R}^n} F(u(\tau + r, \tau - t_n, \theta_{-\tau}\omega, u_0^n))dx \leq c_{14} e^{-\frac{1}{4}\varepsilon\varrho r}, \end{aligned} \tag{4.71}$$

where $c_{14} = c_{14}(\tau, \omega) > 0$ is independent of n.

Since as $t_n \to \infty$, for each $m \in \mathbb{N}$, there exists $N_m = N_m(\tau, \omega, D, m) \geq \mathbb{N}$ such that $t_n \geq m$ for all $n \geq \mathbb{N}$. Thus, by (4.71) we have for $n \geq N_m$,

$$\begin{aligned} &\|u(\tau - m, \tau - t_n, \theta_{-\tau}\omega, u_0^n)\|^2_{H^1(\mathbb{R}^n)} + \|u_t(\tau - m, \tau - t_n, \theta_{-\tau}\omega, u_{1,0}^n)\|^2 \\ &\quad + \int_{\mathbb{R}^n} F(u(\tau - m, \tau - t_n, \theta_{-\tau}\omega, u_0^n))dx \leq c_{14} e^{\frac{1}{4}\varepsilon\varrho m}. \end{aligned} \tag{4.72}$$

Taking advantage with (4.72) and a diagonal process, we deduce that for each $m \in \mathbb{N}$, there exist $(\xi_m, \varsigma_m) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ such that, up to a subsequence, as $n \to \infty$,

$$(u(\tau - m, \tau - t_n, \theta_{-\tau}\omega, u_0^n), u_t(\tau - m, \tau - t_n, \theta_{-\tau}\omega, u_{1,0}^n)) \rightharpoonup (\xi_m, \varsigma_m) \ in \ H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n). \tag{4.73}$$

Thus, (4.72)-(4.73) imply that for all $m \in \mathbb{N}$,

$$\|\xi_m\|^2_{H^1(\mathbb{R}^n)} + \|\varsigma_m\|^2 \leq c_{14} e^{\frac{1}{4}\varepsilon\varrho m}. \tag{4.74}$$

Combining with (4.73) and the weak continuity of solutions of (1.1) with respect to initial data in $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, we find that for all $r \in [-m, 0]$, as $n \to \infty$,

$$u(\tau + r, \tau - m, \theta_{-\tau}\omega, u(\tau - m, \tau - t_n, \theta_{-\tau}\omega, u_0^n)) \rightharpoonup u(\tau + r, \tau - m, \theta_{-\tau}\omega, \xi_m) \ in \ H^1(\mathbb{R}^n);$$

that is, for all $r \in [-m, 0]$,

$$u(\tau + r, \tau - t_n, \theta_{-\tau}\omega, u_0^n) \rightharpoonup u(\tau + r, \tau - m, \theta_{-\tau}\omega, \xi_m) \ in \ H^1(\mathbb{R}^n). \tag{4.75}$$

Similarly, we have for $r \in [-m, 0]$, as $n \to \infty$,

$$u_t(\tau + r, \tau - t_n, \theta_{-\tau}\omega, u_{1,0}^n) \rightharpoonup u_t(\tau + r, \tau - m, \theta_{-\tau}\omega, \eta_m) \ in \ L^2(\mathbb{R}^n). \tag{4.76}$$

Combining (4.68) and (4.75)-(4.76) with $r = 0$, we have

$$\xi = u(\tau, \tau - m, \theta_{-\tau}\omega, \xi_m) \quad and \quad \varsigma = u_t(\tau, \tau - m, \theta_{-\tau}\omega, \varsigma_m). \tag{4.77}$$

It follows from (4.67) and (4.77) that

$$\begin{aligned}
&E(\xi, \varsigma) = E(u(\tau, \tau - m, \theta_{-\tau}\omega, \xi_m), u_t(\tau, \tau - m, \theta_{-\tau}\omega, \varsigma_m)) \\
&= e^{-\frac{1}{2}\varepsilon\varrho m} E(\xi_m, \varsigma_m) + \int_{\tau-m}^{\tau} e^{\frac{1}{2}\varepsilon\varrho(s-\tau)} \Psi(u(s, \tau - m, \theta_{-\tau}\omega, \xi_m), u_t(s, \tau - m, \theta_{-\tau}\omega, \varsigma_m)) ds \\
&= e^{-\frac{1}{2}\varepsilon\varrho m} E(\xi_m, \varsigma_m) + \int_{\tau-m}^{\tau} e^{\frac{1}{2}\varepsilon\varrho(s-\tau)} (h(s), \varepsilon u(s, \tau - m, \theta_{-\tau}\omega, \xi_m) + 2u_t(s, \tau - m, \theta_{-\tau}\omega, \varsigma_m)) ds \\
&\qquad + \int_{\tau-m}^{\tau} e^{\frac{1}{2}\varepsilon\varrho(s-\tau)} (R(s, \cdot, u(s, \tau - m, \theta_{-\tau}\omega, \xi_m))\zeta_\delta(\theta_{s-\tau}\omega), \\
&\qquad\qquad \varepsilon u(s, \tau - m, \theta_{-\tau}\omega, \xi_m) + 2u_t(s, \tau - m, \theta_{-\tau}\omega, \varsigma_m)) ds \\
&\quad - \varepsilon \int_{\tau-m}^{\tau} e^{\frac{1}{2}\varepsilon\varrho(s-\tau)} \sigma(\|\nabla u\|^2) \int_{\mathbb{R}^n} g(u_t(s, \tau - m, \theta_{-\tau}\omega, \varsigma_m)) u(s, \tau - m, \theta_{-\tau}\omega, \xi_m) dx ds \\
&\qquad + \frac{1}{2}\varepsilon(2 + \varrho) \int_{\tau-m}^{\tau} e^{\frac{1}{2}\varepsilon\varrho(s-\tau)} \|u_t(s, \tau - m, \theta_{-\tau}\omega, \varsigma_m)\|^2 ds
\end{aligned}$$

$$-\frac{1}{2}\varepsilon\nu(2-\varrho)\int_{\tau-m}^{\tau}e^{\frac{1}{2}\varepsilon\varrho(s-\tau)}\|u(s,\tau-m,\theta_{-\tau}\omega,\xi_m)\|^2ds$$

$$-\frac{1}{2}\varepsilon(2-\varrho)\int_{\tau-m}^{\tau}e^{\frac{1}{2}\varepsilon\varrho(s-\tau)}\|\nabla u(s,\tau-m,\theta_{-\tau}\omega,\xi_m)\|^2ds$$

$$-2\int_{\tau-m}^{\tau}e^{\frac{1}{2}\varepsilon\varrho(s-\tau)}\sigma(\|\nabla u\|^2)\int_{\mathbb{R}^n}g(u_t(s,\tau-m,\theta_{-\tau}\omega,\varsigma_m))u_t(s,\tau-m,\theta_{-\tau}\omega,\varsigma_m)dxds$$

$$+\frac{1}{2}\varepsilon^2\varrho\int_{\tau-m}^{\tau}e^{\frac{1}{2}\varepsilon\varrho(s-\tau)}(u(s,\tau-m,\theta_{-\tau}\omega,\xi_m),u_t(s,\tau-m,\theta_{-\tau}\omega,\varsigma_m))ds$$

$$+\varepsilon\int_{\tau-m}^{\tau}e^{\frac{1}{2}\varepsilon\varrho(s-\tau)}\int_{\mathbb{R}^n}(\varrho F(u(s,\tau-m,\theta_{-\tau}\omega,\xi_m))$$

$$-f(u(s,\tau-m,\theta_{-\tau}\omega,\xi_m))u(s,\tau-m,\theta_{-\tau}\omega,\xi_m))dxds. \tag{4.78}$$

Since $u(\tau,\tau-t_n,\theta_{-\tau}\omega,u_0^n)=u(\tau,\tau-m,\theta_{-\tau}\omega,u(\tau-m,\tau-t_n,\theta_{-\tau}\omega,u_0^n))$, by (4.67) we have

$$E(u(\tau,\tau-t_n,\theta_{-\tau}\omega,u_0^n),u_t(\tau,\tau-t_n,\theta_{-\tau}\omega,u_{1,0}^n))$$

$$=e^{-\frac{1}{2}\varepsilon\varrho m}E(u(\tau-m,\tau-t_n,\theta_{-\tau}\omega,u_0^n),u_t(\tau-m,\tau-t_n,\theta_{-\tau}\omega,u_{1,0}^n))$$

$$+\int_{\tau-m}^{\tau}e^{\frac{1}{2}\varepsilon\varrho(s-\tau)}\Psi(u(s,\tau-m,\theta_{-\tau}\omega,u(\tau-m,\tau-t_n,\theta_{-\tau}\omega,u_0^n)),$$

$$u_t(s,\tau-m,\theta_{-\tau}\omega,u_t(\tau-m,\tau-t_n,\theta_{-\tau}\omega,u_{1,0}^n)))ds$$

$$=e^{-\frac{1}{2}\varepsilon\varrho m}E(u(\tau-m,\tau-t_n,\theta_{-\tau}\omega,u_0^n),u_t(\tau-m,\tau-t_n,\theta_{-\tau}\omega,u_{1,0}^n))$$

$$+\int_{\tau-m}^{\tau}e^{\frac{1}{2}\varepsilon\varrho(s-\tau)}\Psi(u(s,\tau-t_n,\theta_{-\tau}\omega,u_0^n),u_t(s,\tau-t_n,\theta_{-\tau}\omega,u_{1,0}^n))ds$$

$$=e^{-\frac{1}{2}\varepsilon\varrho m}E(u(\tau-m,\tau-t_n,\theta_{-\tau}\omega,u_0^n),u_t(\tau-m,\tau-t_n,\theta_{-\tau}\omega,u_{1,0}^n))$$

$$+\int_{-m}^{0}e^{\frac{1}{2}\varepsilon\varrho s}(h(s+\tau),\varepsilon u(s+\tau,\tau-t_n,\theta_{-\tau}\omega,u_0^n)+2u_t(s+\tau,\tau-t_n,\theta_{-\tau}\omega,u_{1,0}^n))ds$$

$$+\int_{-m}^{0}e^{\frac{1}{2}\varepsilon\varrho s}(R(s+\tau,\cdot,u(s+\tau,\tau-t_n,\theta_{-\tau}\omega,u_0^n))\zeta_\delta(\theta_s\omega),$$

$$\varepsilon u(s+\tau,\tau-t_n,\theta_{-\tau}\omega,u_0^n)+2u_t(s+\tau,\tau-t_n,\theta_{-\tau}\omega,u_{1,0}^n))ds$$

$$-\varepsilon\int_{-m}^{0}e^{\frac{1}{2}\varepsilon\varrho s}\sigma(\|\nabla u\|^2)\int_{\mathbb{R}^n}g(u_t(s+\tau,\tau-t_n,\theta_{-\tau}\omega,u_{1,0}^n))u(s+\tau,\tau-t_n,\theta_{-\tau}\omega,u_0^n)dxds$$

$$+\frac{1}{2}\varepsilon(2+\varrho)\int_{-m}^{0}e^{\frac{1}{2}\varepsilon\varrho s}\|u_t(s+\tau,\tau-t_n,\theta_{-\tau}\omega,u_{1,0}^n)\|^2ds$$

$$-\frac{1}{2}\varepsilon\nu(2-\varrho)\int_{-m}^{0}e^{\frac{1}{2}\varepsilon\varrho s}\|u(s+\tau,\tau-t_n,\theta_{-\tau}\omega,u_0^n)\|^2ds$$

$$-\frac{1}{2}\varepsilon(2-\varrho)\int_{-m}^{0} e^{\frac{1}{2}\varepsilon\varrho s}\|\nabla u(s+\tau,\tau-t_n,\theta_{-\tau}\omega,u_0^n)\|^2 ds$$

$$-2\int_{-m}^{0} e^{\frac{1}{2}\varepsilon\varrho s}\sigma(\|\nabla u\|^2)\int_{\mathbb{R}^n} g(u_t(s+\tau,\tau-t_n,\theta_{-\tau}\omega,u_{1,0}^n))u_t(s+\tau,\tau-t_n,\theta_{-\tau}\omega,u_{1,0}^n)dxds$$

$$+\frac{1}{2}\varepsilon^2\varrho\int_{-m}^{0} e^{\frac{1}{2}\varepsilon\varrho s}(u(s+\tau,\tau-t_n,\theta_{-\tau}\omega,u_0^n),u_t(s+\tau,\tau-t_n,\theta_{-\tau}\omega,u_{1,0}^n))ds$$

$$+\varepsilon\int_{-m}^{0} e^{\frac{1}{2}\varepsilon\varrho s}\int_{\mathbb{R}^n}(\varrho F(u(s+\tau,\tau-t_n,\theta_{-\tau}\omega,u_0^n))$$

$$-f(u(s+\tau,\tau-t_n,\theta_{-\tau}\omega,u_0^n))u(s+\tau,\tau-t_n,\theta_{-\tau}\omega,u_0^n))dxds. \tag{4.79}$$

Next, we estimate the limit of each term on the right-hand side of (4.79) as $n\to\infty$. For the first term, by (4.72) we get for all $n\geq N_m$,

$$e^{-\frac{1}{2}\varepsilon\varrho m}E(u(\tau-m,\tau-t_n,\theta_{-\tau}\omega,u_0^n),u_t(\tau-m,\tau-t_n,\theta_{-\tau}\omega,u_{1,0}^n))\leq c_{15}e^{-\frac{1}{4}\varepsilon\varrho m}, \tag{4.80}$$

where $c_{15}=c_{15}(\tau,\omega)>0$ is independent of n and m.
By (4.71), (4.75)-(4.76) and the Lebesgue dominated convergence theorem, we get

$$\lim_{n\to\infty}\int_{-m}^{0} e^{\frac{1}{2}\varepsilon\varrho s}(h(s+\tau),\varepsilon u(s+\tau,\tau-t_n,\theta_{-\tau}\omega,u_0^n)+2u_t(s+\tau,\tau-t_n,\theta_{-\tau}\omega,u_{1,0}^n))ds$$

$$\leq\int_{-m}^{0} e^{\frac{1}{2}\varepsilon\varrho s}(h(s+\tau),\varepsilon u(s+\tau,\tau-m,\theta_{-\tau}\omega,\xi_m)+2u_t(s+\tau,\tau-m,\theta_{-\tau}\omega,\varsigma_m))ds. \tag{4.81}$$

By Lemma 4.3 we get that for every $\eta>0$, there exists $k_0=k_0(\tau,\omega,\eta,m)\in\mathbb{N}$ and $\widetilde{N}_m=\widetilde{N}_m(\tau,\omega,\eta,m)\geq N_m$ such that for all $n\geq\widetilde{N}_m$ and $r\in[-m,0]$,

$$\int_{|x|\geq k_0}|u(\tau+r,\tau-t_n,\theta_{-\tau}\omega,u_0^n)|^2dx<\eta e^{-\frac{1}{4}\varepsilon\varrho r}. \tag{4.82}$$

Combining with (4.75), (4.82) and the compactness of embedding $H^1\hookrightarrow L^2$ in bounded domains, we get that for all $r\in[-m,0]$, as $n\to\infty$,

$$u(\tau+r,\tau-t_n,\theta_{-\tau}\omega,u_0^n)\to u(\tau+r,\tau-m,\theta_{-\tau}\omega,\xi_m)\ in\ L^2(\mathbb{R}^n). \tag{4.83}$$

According to (4.71), (4.83), and the Lebesgue dominated convergence theorem as well as the interpolation inequality, we conclude

$$\lim_{n\to\infty}\int_{-m}^{0}\|u(s+\tau,\tau-t_n,\theta_{-\tau}\omega,u_0^n)-u(s+\tau,\tau-m,\theta_{-\tau}\omega,\xi_m)\|^2ds=0, \tag{4.84}$$

and

$$\lim_{n\to\infty}\int_{-m}^{0}\|u(s+\tau,\tau-t_n,\theta_{-\tau}\omega,u_0^n)-u(s+\tau,\tau-m,\theta_{-\tau}\omega,\xi_m)\|_{L^{2q}(\mathbb{R}^n)}^{2q}ds=0, \quad (4.85)$$

Combining (3.5) with (4.85), we get

$$\lim_{n\to\infty}\int_{-m}^{0}\|R(s+\tau,\cdot,u(s+\tau,\tau-t_n,\theta_{-\tau}\omega,u_0^n))-R(s+\tau,\cdot,u(s+\tau,\tau-m,\theta_{-\tau}\omega,\xi_m))\|^2ds=0. \quad (4.86)$$

It follows from (4.71), (4.75)-(4.76), (4.86) and the Lebesgue dominated convergence theorem that as $n\to\infty$,

$$\begin{aligned}&\int_{-m}^{0}e^{\frac{1}{2}\varepsilon\varrho s}(R(s+\tau,\cdot,u(s+\tau,\tau-t_n,\theta_{-\tau}\omega,u_0^n))\zeta_\delta(\theta_s\omega),\\ &\varepsilon u(s+\tau,\tau-t_n,\theta_{-\tau}\omega,u_0^n)+2u_t(s+\tau,\tau-t_n,\theta_{-\tau}\omega,u_{1,0}^n))ds\\ &\to\int_{-m}^{0}e^{\frac{1}{2}\varepsilon\varrho s}(R(s+\tau,\cdot,u(s+\tau,\tau-m,\theta_{-\tau}\omega,\xi_m))\zeta_\delta(\theta_s\omega),\\ &\varepsilon u(s+\tau,\tau-m,\theta_{-\tau}\omega,\xi_m)+2u_t(s+\tau,\tau-m,\theta_{-\tau}\omega,\varsigma_m))ds.\end{aligned} \quad (4.87)$$

Similarly, by (4.71), (4.83) and the Lebesgue dominated convergence theorem, we have

$$\lim_{n\to\infty}\int_{-m}^{0}\|u(s+\tau,\tau-t_n,\theta_{-\tau}\omega,u_0^n)-u(s+\tau,\tau-m,\theta_{-\tau}\omega,\xi_m)\|_{L^{\gamma+1}(\mathbb{R}^n)}^{\gamma+1}ds=0, \quad (4.88)$$

which follows from (4.75)-(4.76), (3.6) and the Lebesgue dominated convergence theorem that as $n\to\infty$,

$$\begin{aligned}&\int_{-m}^{0}e^{\frac{1}{2}\varepsilon\varrho s}\sigma(\|\nabla u\|^2)\int_{\mathbb{R}^n}g(u_t(s+\tau,\tau-t_n,\theta_{-\tau}\omega,u_{1,0}^n))u(s+\tau,\tau-t_n,\theta_{-\tau}\omega,u_0^n)dxds\\ &\to\int_{-m}^{0}e^{\frac{1}{2}\varepsilon\varrho s}\sigma(\|\nabla u\|^2)\int_{\mathbb{R}^n}g(u_t(s+\tau,\tau-m,\theta_{-\tau}\omega,\varsigma_m))u(s+\tau,\tau-m,\theta_{-\tau}\omega,\xi_m)dxds,\end{aligned} \quad (4.89)$$

and

$$\begin{aligned}&\int_{-m}^{0}e^{\frac{1}{2}\varepsilon\varrho s}\sigma(\|\nabla u\|^2)\int_{\mathbb{R}^n}g(u_t(s+\tau,\tau-t_n,\theta_{-\tau}\omega,u_{1,0}^n))u_t(s+\tau,\tau-t_n,\theta_{-\tau}\omega,u_{1,0}^n)dxds\\ &\to\int_{-m}^{0}e^{\frac{1}{2}\varepsilon\varrho s}\sigma(\|\nabla u\|^2)\int_{\mathbb{R}^n}g(u_t(s+\tau,\tau-m,\theta_{-\tau}\omega,\varsigma_m))u(s+\tau,\tau-m,\theta_{-\tau}\omega,\varsigma_m)dxds.\end{aligned} \quad (4.90)$$

By (4.75)-(4.76) and (4.84) we have

$$\lim_{n\to\infty}\int_{-m}^{0} e^{\frac{1}{2}\varepsilon\varrho s}\|u_t(s+\tau,\tau-t_n,\theta_{-\tau}\omega,u_{1,0}^n)\|^2 ds$$

$$=\int_{-m}^{0} e^{\frac{1}{2}\varepsilon\varrho s}\|u_t(s+\tau,\tau-m,\theta_{-\tau}\omega,\varsigma_m)\|^2 ds, \tag{4.91}$$

and

$$\lim_{n\to\infty}\int_{-m}^{0} e^{\frac{1}{2}\varepsilon\varrho s}(u(s+\tau,\tau-t_n,\theta_{-\tau}\omega,u_0^n),u_t(s+\tau,\tau-t_n,\theta_{-\tau}\omega,u_{1,0}^n))ds$$

$$=\int_{-m}^{0} e^{\frac{1}{2}\varepsilon\varrho s}(u(s+\tau,\tau-m,\theta_{-\tau}\omega,\xi_m),u_t(s+\tau,\tau-m,\theta_{-\tau}\omega,\varsigma_m))ds. \tag{4.92}$$

On the other hand, by (4.75)-(4.76), (4.84) and Fatou's lemma, we achieve

$$\liminf_{n\to\infty}\int_{-m}^{0} e^{\frac{1}{2}\varepsilon\varrho s}\|u(s+\tau,\tau-t_n,\theta_{-\tau}\omega,u_0^n)\|^2 ds$$

$$\geq\int_{-m}^{0} e^{\frac{1}{2}\varepsilon\varrho s}\|u(s+\tau,\tau-m,\theta_{-\tau}\omega,\xi_m)\|^2 ds, \tag{4.93}$$

and

$$\liminf_{n\to\infty}\int_{-m}^{0} e^{\frac{1}{2}\varepsilon\varrho s}\|\nabla u(s+\tau,\tau-t_n,\theta_{-\tau}\omega,u_0^n)\|^2 ds$$

$$\geq\int_{-m}^{0} e^{\frac{1}{2}\varepsilon\varrho s}\|\nabla u(s+\tau,\tau-m,\theta_{-\tau}\omega,\xi_m)\|^2 ds, \tag{4.94}$$

By a diagonal process, we infer from (4.84) that there exists a further subsequence (not relabeled) such that for every $m\in\mathbb{N}$,

$$u(\tau+\cdot,\tau-t_n,\theta_{-\tau}\omega,u_{\tau-t_n})\to u(\tau+\cdot,\tau-m,\theta_{-\tau}\omega,\xi_m) \ \ a.e.\ on\ (-m,0)\times\mathbb{R}. \tag{4.95}$$

Thus, by (4.3), (4.95) and Fatou's lemma, it follows that

$$\liminf_{n\to\infty}\int_{-m}^{0} e^{\frac{1}{2}\varepsilon\varrho s}\int_{\mathbb{R}^n}(f(u(s+\tau,\tau-t_n,\theta_{-\tau}\omega,u_0^n))u(s+\tau,\tau-t_n,\theta_{-\tau}\omega,u_0^n)$$

$$-\varrho F(u(s+\tau,\tau-t_n,\theta_{-\tau}\omega,u_0^n)))dxds$$

$$\geq\int_{-m}^{0} e^{\frac{1}{2}\varepsilon\varrho s}\int_{\mathbb{R}^n}(f(u(s+\tau,\tau-m,\theta_{-\tau}\omega,\xi_m))u(s+\tau,\tau-m,\theta_{-\tau}\omega,\xi_m)$$

$$-\varrho F(u(s+\tau,\tau-m,\theta_{-\tau}\omega,\xi_m)))dxds. \tag{4.96}$$

Taking the limit of (4.79) as $n\to\infty$, we infer from (4.78)-(4.81), (4.87)-(4.94) and (4.96) that

$$\begin{aligned}
&\limsup_{n\to\infty} E(u(\tau,\tau-t_n,\theta_{-\tau}\omega,u_0^n),u_t(\tau,\tau-t_n,\theta_{-\tau}\omega,u_{1,0}^n))\\
&\le E(\xi,\varsigma)+c_{15}e^{-\frac{1}{4}\varepsilon\varrho m}-e^{-\frac{1}{2}\varepsilon\varrho m}E(\xi_m,\varsigma_m)\\
&\le E(\xi,\varsigma)+c_{15}e^{-\frac{1}{4}\varepsilon\varrho m}-\varepsilon e^{-\frac{1}{2}\varepsilon\varrho m}(\xi_m,\varsigma_m).
\end{aligned} \tag{4.97}$$

Combining (4.74) with (4.97) yields

$$\limsup_{n\to\infty} E(u(\tau,\tau-t_n,\theta_{-\tau}\omega,u_0^n),u_t(\tau,\tau-t_n,\theta_{-\tau}\omega,u_{1,0}^n))\le E(\xi,\varsigma)+(\varepsilon c_{14}+c_{15})e^{-\frac{1}{4}\varepsilon\varrho m}. \tag{4.98}$$

Taking the limit of (4.98) as $m\to\infty$, we obtain

$$\limsup_{n\to\infty} E(u(\tau,\tau-t_n,\theta_{-\tau}\omega,u_0^n),u_t(\tau,\tau-t_n,\theta_{-\tau}\omega,u_{1,0}^n))\le E(\xi,\varsigma). \tag{4.99}$$

Thanks to (4.77) and (4.83), we get that as $n\to\infty$,

$$u(\tau,\tau-t_n,\theta_{-\tau}\omega,u_0^n)\to\xi\quad in\ L^2(\mathbb{R}^n). \tag{4.100}$$

Due to (4.100), there exists a further subsequence (not relabeled) such that as $n\to\infty$,

$$u(\tau,\tau-t_n,\theta_{-\tau}\omega,u_0^n)\to\xi\quad a.e.\ on\ \mathbb{R}^n. \tag{4.101}$$

Therefore, using Fatou's lemma, we infer from (3.3) and (4.101) that

$$\liminf_{n\to\infty}\int_{\mathbb{R}^n}F(u(\tau,\tau-t_n,\theta_{-\tau}\omega,u_0^n))dx\ge\int_{\mathbb{R}^n}F(\xi)dx. \tag{4.102}$$

Combining (4.75)-(4.76) with (4.100) yields

$$\lim_{n\to\infty}(u(\tau,\tau-t_n,\theta_{-\tau}\omega,u_0^n),u_t(\tau,\tau-t_n,\theta_{-\tau}\omega,u_{1,0}^n))=(\xi,\varsigma). \tag{4.103}$$

It follows from (4.99) and (4.102)-(4.103) that

$$\begin{aligned}
&\limsup_{n\to\infty}(\|u_t(\tau,\tau-t_n,\theta_{-\tau}\omega,u_{1,0}^n)\|^2+\nu\|u(\tau,\tau-t_n,\theta_{-\tau}\omega,u_0^n)\|^2\\
&\qquad+\|\nabla u(\tau,\tau-t_n,\theta_{-\tau}\omega,u_0^n)\|^2)\\
&\le\|\varsigma\|^2+\nu\|\xi\|^2+\|\nabla\xi\|^2.
\end{aligned} \tag{4.104}$$

Then (4.70) follows from (4.104) immediately. This completes the proof. □

Now, we present our main result on the existence and uniqueness of $\mathcal{D}-$pullback random attractors of Φ.

Theorem 4.5 *Let* (3.1)-(3.6), (4.3) *and* (4.5)-(4.6) *hold. Then the cocycle* Φ *has a unique* $\mathcal{D}-$*pullback random attractor in* $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

Proof. Since Φ has a closed measurable $\mathcal{D}-$pullback absorbing set by Lemma 4.2 and is $\mathcal{D}-$pullback asymptotically compact in $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ by Lemma 4.4, then the existence and uniqueness of $\mathcal{D}-$pullback random attractor of Φ follows from [30,31] immediately. $\square$

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