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SINGLE AND SYSTEM OF FRACTIONAL NEUTRAL FUNCTIONAL q–DIFFERENTIAL EQUATIONS WITH APPLICATION ON PARTICLE IN THE PLANE

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ABSTRACT. The principal importance of this paper is to obtain the existence of solution of single and multi-dimensional fractional neutral functional q–differential equations with bounded delay based on operator equations by using Krasnoselskii's fixed point theorem. At the end, the examples which they contain some tables, figures and related algorithms with numerical effect, are added to show applications of the our results.

1. Introduction

The quantum calculus is an old subject which was first introduced by Jackson in [1] then developed by Al-Salam who started fitting the concept of q–fractional calculus [2]. Further, some researchers studied differential and q–differential equations with different types of fractional derivatives; see [3–13] for more details.

In 2008, Benchohra *et al.* [14] investigated various criteria for the existence and uniqueness of solutions for classes of functional fractional differential equations ($\mathbb{FFDE}$s) with infinite delay involving Riemann–Liouville fractional derivative as form:

$$\begin{cases} \mathscr{D}^{\sigma} y(t) = w(t, y_t), & \forall t \in [0, \vartheta_0],\ 0 < \sigma < 1, \\ y(t) = \dot{\theta}(t), & t \in (-\infty, 0], \end{cases}$$

where $\mathscr{D}^{\sigma}$ is the standard Riemman-Liouville fractional derivative, $w : [0, \vartheta_0] \times \mathbb{A} \to \mathbb{R}$ is a given function satisfying some assumptions that will be specified later, $\dot{\theta} \in \mathbb{A}$, $\dot{\theta}(0) = 0$ and $\mathbb{A}$ is called a phase space [14]. Then in 2010, Agarwal *et al.* [15] studied the following initial value problems of fractional neutral functional differential equations ($\mathbb{FNFDE}$s) with bounded delay:

$$\begin{cases} {}^{c}\mathscr{D}^{\sigma}\big(y(t) - f(t, y_t)\big) = w(t, y_t), & \forall t \in (\vartheta_0, \infty),\ \vartheta_0 > 0, \\ y_{\vartheta_0}(t) = \dot{\theta}, \end{cases}$$

where ${}^{c}\mathscr{D}^{\sigma}$ is the standard Caputo's fractional derivative of order $0 < \sigma < 1$, $w, f : [\vartheta_0, \infty) \times \mathscr{C}_0 \to \mathbb{R}^n$ are given functions with satisfying some assumptions, $\dot{\theta} \in \mathscr{C}_0 = C([-\delta, 0], \mathbb{R}^n)$ [15]. Later Baleanu *et al.* in [16], investigated that m-dimension of the problem and demonstarted that it is equivalent to an integral equation

$$y_i(t) = \dot{\theta}_i(0) - f_i\big(\vartheta_0, \dot{\theta}_1(t), \dot{\theta}_2(t), \ldots, \dot{\theta}_m(t)\big) + f_i\big(t, \mathbf{y}_t\big) + \mathscr{I}^{\sigma_i} w_i\big(t, \mathbf{y}_t\big),$$

for $t \in (t_0, t_0 + r]$, with conditions $y_{i_{\vartheta_0}} = \dot{\theta}_i$, for $i = 1, 2, \ldots, m$ [15].

Oriented by the above detailed literature, and recent excellent works [17–27], the purpose of this paper is to establish and develop the existence of the following single and multi-dimensional functional problems:

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(P1) Initial value problem of fractional neutral functional q–differential equation ($\mathbb{FNF}q\mathbb{DE}$)

$$ {}^{c}\mathscr{D}_q^{\sigma}\big(y(t)-f(t,y_t)\big)=w(t,y_t), \qquad \forall t\in(\vartheta_0,\infty),\ \vartheta_0>0, \tag{1.1}$$

under infinit delay $y_{\vartheta_0}(t)=\dot{\theta}\in\mathscr{C}_0$ for $t\in[-\delta,0]$, $\delta>0$, where $q\in J:=(0,1)$, ${}^{c}\mathscr{D}_q^{\sigma}$ is the fractional Caputo type q–derivative of order $\sigma\in J$, and $f,\, w:[\vartheta_0,+\infty)\times\mathscr{C}_0\to\mathbb{R}^n$ are given functional satisfying some assumptions. Let $\tau>0$. If $y\in C([\vartheta_0-\delta,\vartheta_0+\tau],\mathbb{R}^n)$, then for any $t\in\overline{J}_0=[\vartheta_0,\vartheta_0+\tau]$ define y_t by $y_t(u)=y(t+u)$ for $u\in[-\delta,0]$.

(P2) m-dimensional system of $\mathbb{FNF}q\mathbb{DE}$s

$$\begin{cases} {}^{c}\mathscr{D}_q^{\sigma_1}\big(y_1(t)-f_1(t,\mathbf{y}_t)\big)=w_1(t,\mathbf{y}_t),\\ {}^{c}\mathscr{D}_q^{\sigma_2}\big(y_2(t)-f_2(t,\mathbf{y}_t)\big)=w_2(t,\mathbf{y}_t),\\ \qquad\vdots\\ {}^{c}\mathscr{D}_q^{\sigma_m}\big(y_m(t)-f_m(t,\mathbf{y_t})\big)=w_m(t,\mathbf{y}_t), \end{cases} \tag{1.2}$$

for $t\in(\vartheta_0,\infty)$, $\vartheta_0>0$, under infinite delay

$$ y_{1_{\vartheta_0}}=\dot{\theta}_1,\quad y_{2_{\vartheta_0}}=\dot{\theta}_2,\quad \dots,\quad y_{m_{\vartheta_0}}=\dot{\theta}_m,$$

for $t\in[-\delta,0]$, $\delta>0$, where ${}^{c}\mathscr{D}_q^{\sigma_i}$ is the standard Caputo type fractional q–derivative of order $\sigma_i\in J$,

$$ f_i,w_i:\prod_i^m[\vartheta_0,\infty)\times\mathscr{C}_0\to\mathbb{R}^n,$$

are given functions satisfying some assumptions that will be specified in the main section, $\mathbf{y}_t=(y_{1_t},y_{2_t},\dots,y_{m_t})$ and $\dot{\theta}_i\in\mathscr{C}_0$ for $i=1,2,\dots,m$. If $y\in C([\vartheta_0-\delta,\vartheta_0+\tau],\mathbb{R}^n)$, then for each $t\in\overline{J}_0$ define y_t by $y_t(u)=y(t+u)$ for all $u\in[-\delta,0]$ where $\tau>0$ is a positive constant.

This paper is organized as follows: In Section 2, we state some useful definitions on the fundamental concepts of q–fractional calculus and state Krasnoselskii's fixed point theorem in operator equations. The main theorems on the existence of solutions for one and multi-dimensional bounded value problem (1.1) and (1.2) are proved. In Section 3. Section 4 is devoted to some illustrative examples that show the validity and applicability of our results. Finally in Section 5, the conclusion of the paper is drawn where we conclude some interesting observations.

2. Essential preliminaries

In fact, we consider the fractional q–calculus on the specific time scale

$$\mathbb{T}_{t_0}=\{0\}\cup\Big\{t:t=t_0q^n,\, n\in\mathbb{N},\, t_0\in\mathbb{R},\, q\in(0,1)\Big\}.$$

If there is no confusion concerning t_0, we shall denote $\mathbb{T}_{t_0}$ by $\mathbb{T}$. Let $a\in\mathbb{R}$. Define $[a]_q=\frac{1-q^a}{1-q}$ [1]. The q–factorial function $(x-y)_q^{(n)}$ with $n\in\mathbb{N}_0$ is defined by

$$(x-y)_q^{(n)}=\prod_{k=0}^{n-1}(x-yq^k),\qquad (x-y)_q^{(0)}=1,\qquad x,y\in\mathbb{R}, \tag{2.1}$$

here $\mathbb{N}_0:=\{0\}\cup\mathbb{N}$ [3]. Also, the function is defined for $\sigma\in\mathbb{R}$, by

$$(x-y)_q^{(\sigma)}=x^{\sigma}\prod_{k=0}^{\infty}\frac{x-yq^k}{x-yq^{\sigma+k}},\qquad a\neq 0. \tag{2.2}$$

The q–Gamma function is given by [1]

$$\Gamma_q(z) = (1-q)^{1-z}(1-q)_q^{(z-1)}, \qquad z \in \mathbb{R}\backslash\{\cdots,-2,-1,0\}.$$

Note that, $\Gamma_q(z+1) = [z]_q\Gamma_q(z)$ [9, Lemma 1]. For a function $w : \mathbb{T} \to \mathbb{R}$, the q–derivative of w, is

$$\mathscr{D}_q[w](x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)_q w(x) = \frac{w(x)-w(qx)}{(1-q)x}, \tag{2.3}$$

for all $t \in \mathbb{T}\setminus\{0\}$, and $\mathscr{D}_q[w](0) = \lim_{x\to 0}\mathscr{D}_q[w](x)$ [3]. Also, the higher order q–derivative of the function w is defined by

$$\mathscr{D}_q^n[w](x) = \mathscr{D}_q\left[\mathscr{D}_q^{n-1}[w]\right](x), \qquad \forall n \geq 1,$$

where $\mathscr{D}_q^0[w](x) = w(x)$ [3]. In fact

$$\mathscr{D}_q^n[w](x) = \frac{1}{x^n(1-q)^n}\sum_{k=0}^{n}\frac{(1-q^{-n})_q^{(k)}}{(1-q)_q^{(k)}}\, q^k w(xq^k), \tag{2.4}$$

for $x \in \mathbb{T}\setminus\{0\}$ [4]. The q–integral of the function w is defined by

$$\mathscr{I}_q[w](x) = \int_0^x w(s)\,\mathrm{d}_q s = x(1-q)\sum_{k=0}^{\infty} q^k w\left(xq^k\right), \tag{2.5}$$

for $0 \leq x \leq b$, provided the series is absolutely converges [3]. We can obtain the numerical results of $\mathscr{I}_q[w](x)$ when $n \to \infty$. If a in $[0,b]$, then

$$\int_a^b w(s)\,\mathrm{d}_q s = \mathscr{I}_q[w](b) - \mathscr{I}_q[w](a) = (1-q)\sum_{k=0}^{\infty} q^k\left[bw(bq^k) - aw(aq^k)\right], \tag{2.6}$$

whenever the series exists. The operator $\mathscr{I}_q^n$ is given

$$\mathscr{I}_q^0[w](x) = w(x), \qquad \mathscr{I}_q^n[w](x) = \mathscr{I}_q\left[\mathscr{I}_q^{n-1}[w]\right](x),$$

for $n \geq 1$ and $g \in C([0,b])$ [3]. It has been proved that $\mathscr{D}_q\left[\mathscr{I}_q[w]\right](x) = w(x)$ and $\mathscr{I}_q\left[\mathscr{D}_q[w]\right](x) = w(x) - w(0)$, whenever the function w is continuous at $x = 0$ [3]. The fractional Riemann–Liouville type q–integral of the function w is defined by

$$\mathscr{I}_q^{\sigma}[w](t) = \int_0^t \frac{(t-s)_q^{(\sigma-1)}}{\Gamma_q(\sigma)}\, w(s)\,\mathrm{d}_q s, \qquad \mathscr{I}_q^0[w](t) = w(t), \tag{2.7}$$

for $t \in [0,1]$ and $\sigma > 0$ [4, 5]. The Caputo fractional q–derivative of the function w is defined by

$${}^c\mathscr{D}_q^{\sigma}[w](t) = \mathscr{I}_q^{[\sigma]-\sigma}\left[\mathscr{D}_q^{[\sigma]}[w]\right](t) = \int_0^t \frac{(t-s)_q^{([\sigma]-\sigma-1)}}{\Gamma_q([\sigma]-\sigma)}\,\mathscr{D}_q^{[\sigma]}[w](s)\,\mathrm{d}_q s \tag{2.8}$$

for $t \in [0,1]$ and $\sigma > 0$ [5, 10]. It has been proved that

$$\mathscr{I}_q^{\nu}\left[\mathscr{I}_q^{\sigma}[w]\right](t) = \mathscr{I}_q^{\sigma+\nu}[w](t), \qquad {}^c\mathscr{D}_q^{\sigma}\left[\mathscr{I}_q^{\sigma}[w]\right](t) = w(t),$$

where $\sigma, \nu \geq 0$ [5]. Algorithms 1-8 in [28] show numerical technque for above calculus.

Lemma 2.1 ([6, 29])**.** *For $\sigma > 0$, the general solution of the fractional q–differential equation ${}^c\mathscr{D}^{\sigma}y(t) = 0$ is given by $y(t) = \sum_{i=0}^{n-1} e_i t^i$, where $e_i \in \mathbb{R}$ for $i = 0,1,2,\ldots,n-1$ and $n = [\sigma]+1$ here $[\sigma]$ denotes the integer part of the real number σ.*

Lemma 2.2 (Krasnoselskii's fixed point theorem)**.** *Let $\mathscr{A}$ be a Banach space, let $\mathfrak{B}$ be a bounded closed convex subset of $\mathscr{A}$ and let $\mathfrak{M}$, $\mathfrak{N}$ be maps of $\mathfrak{B}$ into $\mathscr{A}$ such that $\mathfrak{M}(y)+\mathfrak{N}(z) \in \mathfrak{B}$ for every pair $y,z \in \mathfrak{B}$. If $\mathfrak{M}$ is a contraction and $\mathfrak{N}$ is completely continuous, then the equation $\mathfrak{M}(y)+\mathfrak{N}(y) = y$ has a solution on $\mathfrak{B}$.*

3. Main results

The main results are twofold. We firstly consider the one-dimensional $\mathbb{FNF}q\mathbb{DE}$ under bounded delay (1.1). Secondly, we discuss m-dimensional of $\mathbb{FNF}q\mathbb{DE}$ (1.2).

3.1. *The* $\mathbb{FNF}q\mathbb{DE}$ (1.1). We define

$$\Upsilon_r(\varepsilon) = \left\{ y \in C([\vartheta_0 - \delta, \vartheta_0 + r], \mathbb{R}^n) : y_{\vartheta_0} = \dot{\theta}_0, \sup_{t \in \overline{J}_0} \left| y(t) - \dot{\theta}(0) \right| \leq \varepsilon \right\},$$

where r, ε are positive constants. For Banach space $\mathscr{A} = C(I)$, here $I = [a,b] \subset \mathbb{R}$, we consider the norm $\|y\| = \sup_{t \in [a,b]} |y(t)|$. Before stating and proving the main result of this section, we introduce the following hypotheses.

(HE1) $w(t,\rho)$ is measurable with respect to t on $\overline{J}_0$ and is continuous with respect to ρ on $\mathscr{C}_0$;

(HE2) There exist $0 < \sigma_1 < \sigma$ and a real-value function $m(t) \in L^{1/\sigma_1}(\overline{J}_0)$ such that $|w(t,y_t)| \leq m(t)$, for any $y \in \Upsilon_r(\varepsilon)$ and for each $t \in \overline{J}_0$;

(HE3) Furthermore, $f(t,y_t) = f_1(t,y_t) + f_2(t,y_t)$ for any $y \in \Upsilon_r(\varepsilon)$ such that f_1 is continuous and for any $y, z \in \Upsilon_r(\varepsilon)$,

$$|f_1(t,y_t) - f_1(t,z_t)| \leq \ell \|y - z\|, \qquad t \in \overline{J}_0,$$

where $0 < \ell < 1$, and f_2 is completely continuous, and for any bounded set $\mathfrak{B}$ in $\Upsilon_r(\varepsilon)$, the set

$$\left\{ t \to f_2(t,y_t) : y \in \mathfrak{B} \right\},$$

is equicontinuous in $C(\overline{J}_0, \mathbb{R}^n)$.

Lemma 3.1. *If there exist $r \in (0,\tau)$ and $\varepsilon \in (0,\infty)$ such that Hypotheses* (HE1) *and* (HE2) *are satisfied, then for $t \in (\vartheta_0, \vartheta_0 + r]$, the problem* (1.1) *is equivalent to the following equation*

$$\left\{ \begin{aligned} y(t) &= \dot{\theta}(0) - f(\vartheta_0, \dot{\theta}(t)) + f(t,y_t) + \mathscr{I}_q^{\sigma} w(t,y_t) \\ &= \dot{\theta}(0) - f(\vartheta_0, \dot{\theta}(t)) + f(t,y_t) + \int_{\vartheta_0}^{t} \frac{(t - q\xi)^{(\sigma-1)}}{\Gamma_q(\sigma)} w\left(\xi, y_\xi\right) \mathrm{d}_q\xi, \\ y_{\vartheta_0} &= \dot{\theta}, \end{aligned} \right. \tag{3.1}$$

for $t \in \overline{J}_0$.

Proof. Conditions (HE1) implies that $w(t,y_t)$ is Lebesgue measurable on $\overline{J}_0$ and

$$(t - q\xi)^{(\sigma-1)} \in L^{1/(1-\sigma_1)}([\vartheta_0, t]), \qquad \forall t \in \overline{J}_0.$$

The Hölder inequality and condition (HE2) imply $(t - q\xi)^{(\sigma-1)} w(\xi, y_\xi)$, is Lebesgue integrable with respect to $\xi \in [\vartheta_0, t]$, for each $t \in \overline{J}_0$, and $y \in \Upsilon_r(\varepsilon)$, and

$$\begin{aligned} \Gamma_q(\sigma) \mathscr{I}_q^{\sigma} w(t,y_t) &= \int_{\vartheta_0}^{t} \left| (t - q\xi)^{(\sigma-1)} w(\xi, y_\xi) \right| \mathrm{d}_q\xi \\ &\leq \Gamma_q(\sigma) \left\| (t - q\xi)^{(\sigma-1)} \right\|_{L^{1/(1-\sigma_1)}([\vartheta_0,t])} \cdot \|m\|_{L^{1/\sigma_1}(\overline{J}_0)}, \end{aligned} \tag{3.2}$$

where

$$\|m\|_{L^p(I)} = \left[\int_I |m(\xi)|^p \, \mathrm{d}\xi \right]^{1/p}, \tag{3.3}$$

for any L^p-intagrable function $\mathscr{Z}: I \to \mathbb{R}$. It is easy to see that if y is a solution of the problem (1.1), then y is a solution of the Eq. (3.1). On the other hand, if Eq. (3.1) is satisfied, then $\forall t \in (\vartheta_0, \vartheta_0 + r]$,

$$\begin{aligned}
{}^c\mathscr{D}_q^\sigma(y(t) - f(t,y_t)) &= {}^c\mathscr{D}_q^\sigma\left[\dot{\theta}(0) - f(\vartheta_0, \dot{\theta}) + \mathscr{I}_q^\sigma w(t,y_t)\right] = {}^c\mathscr{D}_q^\sigma\left(\mathscr{I}_q^\sigma w(t,y_t)\right)\\
&= {}^c\mathscr{D}_q^\sigma\left(\mathscr{I}_q^\sigma w\right)(t,y_t) - \frac{(t-q\vartheta_0)^{(-\sigma)}}{\Gamma_q(1-\sigma)}\left[\mathscr{I}_q^\sigma w(t,y_t)\right]_{t=\vartheta_0}\\
&= w(t,y_t) - \frac{(t-q\vartheta_0)^{(-\sigma)}}{\Gamma_q(1-\sigma)}\left[\mathscr{I}_q^\sigma w(t,y_t)\right]_{t=\vartheta_0}.
\end{aligned}$$

Eq. (3.2) implies that $\left[\mathscr{I}_q^\sigma w(t,y_t)\right]_{t=\vartheta_0} = 0$. Indeed,

$${}^c\mathscr{D}_q^\sigma\left(y(t) - f(t,y_t)\right) = w(t,y_t), \qquad \forall t \in (\vartheta_0, \vartheta_0 + r],$$

and this completes the proof. □

Theorem 3.2. *Assume that there esist $r \in (0,\tau)$ and $\varepsilon \in (0,\infty)$ such that hypotheses* (HE1)*,* (HE2) *and* (HE3) *are satisfied. Then the problem* (1.1) *has at least one solution on $[\vartheta_0, \vartheta_0 + \eta]$ for some positive number η.*

Proof. Eq. (3.1) is equivalent to

$$\begin{cases}
y(t) = \dot{\theta}(0) - f_1(\vartheta_0, \dot{\theta}) - f_2(\vartheta_0, \dot{\theta}) + f_1(\vartheta_0, y_t) + f_2(\vartheta_0, y_t)\\
\qquad\qquad + \displaystyle\int_{\vartheta_0}^t \frac{(t-q\xi)^{(\sigma-1)}}{\Gamma_q(\sigma)} w\left(\xi, y_\xi\right) \mathrm{d}_q\xi,\\
k_{\vartheta_0} = \dot{\theta},
\end{cases}$$

for $t \in \bar{J}_0$. Let $\check{\theta} \in \Upsilon_r(\varepsilon)$ be defined as $\check{\theta}_{\vartheta_0} = \dot{\theta}$, $\check{\theta}(\vartheta_0 + t) = \dot{\theta}(0)$ for almost all $t \in [0,r]$. If y is a solution of the initial bounded value problem (1.1), let

$$y(\vartheta_0 + t) = \check{\theta}(\vartheta_0 + t) + l(t), \qquad t \in [-\delta, r],$$

then $k_{\vartheta_0+t} = \check{\theta}_{\vartheta_0+t} + l_t$, for all $t \in [0,r]$. Thus l satisfies the following equation

$$\begin{aligned}
l(t) = &-f_1\left(\vartheta_0, \dot{\theta}\right) - f_2\left(\vartheta_0, \dot{\theta}\right) + f_1\left(\vartheta_0 + t, l_t + \check{\theta}_{\vartheta_0+t}\right)\\
&+ f_2\left(\vartheta_0 + t, l_t + \check{\theta}_{\vartheta_0+t}\right) + \mathscr{I}_q^\sigma w\left(\vartheta_0 + t, l_t + \check{\theta}_{\vartheta_0+t}\right), \qquad \forall t \in [0,r].
\end{aligned} \tag{3.4}$$

Since f_1, f_2 and y_t are continuous in t, there esists $r' > 0$, with $0 < t < r'$,

$$\begin{aligned}
\left|f_1\left(\vartheta_0 + t, l_t + \check{\theta}_{\vartheta_0+t}\right) - f_1\left(\vartheta_0, \dot{\theta}\right)\right| &< \frac{1}{3}\varepsilon,\\
\left|f_2\left(\vartheta_0 + t, l_t + \check{\theta}_{\vartheta_0+t}\right) - f_2\left(\vartheta_0, \dot{\theta}\right)\right| &< \frac{1}{3}\varepsilon.
\end{aligned} \tag{3.5}$$

Choose

$$\eta = \min\left\{r, r', \left[\frac{\varepsilon\Gamma_q(\sigma)(1+\beta)^{1-\sigma_1}}{3M}\right]^{1/(1+\beta)(1-\sigma_1)}\right\}, \tag{3.6}$$

where $\beta = \frac{\sigma-1}{1-\sigma_1} \in (-1,0)$ and $M = \|m\|_{L^{1/\sigma_1}(\bar{J}_0)}$. Define $\Xi_\eta(\varepsilon)$ as follows

$$\Xi_\eta(\varepsilon) = \left\{l \in \mathscr{C}_\eta\ :\ l_t(\xi) = 0,\ \text{for } \xi \in [-\delta, 0] \text{ and }, \|l\| \le \varepsilon\right\},$$

where $\mathscr{C}_\eta = C([-\delta,\eta], \mathbb{R}^n)$. Then $\Xi_\eta(\varepsilon)$ is a closed bounded and convex subset of $\mathscr{C}_r$. Now, we defined the oprators $\mathfrak{M}$ and $\mathfrak{N}$ on $\Xi_\eta(\varepsilon)$, by

$$\mathfrak{M}[l](t) = \begin{cases} 0, & t \in [-\delta, 0],\\ -f_1(\vartheta_0, \dot{\theta}) + f_1\left(\vartheta_0 + t, l_t + \check{\theta}_{\vartheta_0+t}\right), & t \in [0,\eta], \end{cases}$$

$$\mathfrak{N}[l](t) = \begin{cases} 0, & t \in [-\delta, 0], \\ -f_2(\vartheta_0, \theta) + f_2\left(\vartheta_0 + t, l_t + \check{\theta}_{\vartheta_0+t}\right) & \\ \quad + \mathscr{I}_q^{\sigma}[w]\left(\vartheta_0 + t, l_t + \check{\theta}_{\vartheta_0+t}\right), & t \in [0, \eta]. \end{cases}$$

So, the operator equation $l = \mathfrak{M}(l) + \mathfrak{N}(l)$ has a solution $l \in \Xi_\eta(\varepsilon)$ iff l is a solution of Eq. (3.4). Thus

$$k(\vartheta_0 + t) = l(t) + \check{\theta}(\vartheta_0 + t),$$

is a solution of Eq. (1.1) on $[0, \eta]$. Therefor, the existence of solution of the initial BVP (1.1) is equivalent that $l = \mathfrak{M}(l) + \mathfrak{N}(l)$ has a fixed point in $\Xi_\eta(\varepsilon)$. At present, we show that $\mathfrak{M}(l) + \mathfrak{N}(l)$ has a fixed point in $\Xi_\eta(\varepsilon)$. In this case, the proof is divided into three steps.

Step I. For every pair y, $z \in \Xi_\eta(\varepsilon)$, we have $\mathfrak{M}(y) + \mathfrak{N}(z) \in \Xi_\eta(\varepsilon)$. In fact, for the pair y, z,

$$\mathfrak{M}(y) + \mathfrak{N}(z) \in \mathscr{C}_\eta.$$

Also, $\mathfrak{M}(y)(t) + \mathfrak{N}(z)(t) = 0$, $t \in [-\delta, 0]$. Moreover, by Eqs. (3.5) and (3.6) and condition (HE2), we have

$$\begin{aligned} &\left|\mathfrak{M}(y)(t) + \mathfrak{N}(z)(t)\right| \\ &\quad \leq \left|-f_1(\vartheta_0, \dot{\theta}) + f_1\left(\vartheta_0 + t, y_t + \check{\theta}_{\vartheta_0+t}\right)\right| \\ &\qquad + \left|-f_2(\vartheta_0, \dot{\theta}) + f_2\left(\vartheta_0 + t, z_t + \check{\theta}_{\vartheta_0+t}\right)\right| + \mathscr{I}_q^{\sigma}|w|\left(\vartheta_0 + t, z_t + \check{\theta}_{\vartheta_0+t}\right) \\ &\quad \leq \frac{2}{3}\varepsilon + \frac{1}{\Gamma_q(\sigma)}\left[\int_0^t (t - q\xi)^{\frac{(\sigma-1)}{1-\sigma_1}} \, \mathrm{d}_q\xi\right]^{1-\sigma_1} \left[\int_{\vartheta_0}^{\vartheta_0+t} (m(\xi))^{\frac{1}{\sigma_1}} \, \mathrm{d}_q\xi\right]^{\sigma_1} \\ &\quad \leq \frac{2}{3}\varepsilon + \frac{M\eta^{(1+\beta)(1-\sigma_1)}}{\Gamma_q(\sigma)(1+\beta)^{1-\sigma_1}} \leq \varepsilon, \qquad \forall t \in [0, \eta]. \end{aligned}$$

Therefore,

$$\|\mathfrak{M}(k) + \mathfrak{N}(z)\| = \sup_{t \in [0,\eta]} |\mathfrak{M}(y) + \mathfrak{N}(z)| \leq \varepsilon,$$

which mesns that $\mathfrak{M}(y) + \mathfrak{N}(z) \in \Xi_\eta(\varepsilon)$ for any y, $z \in \Xi_\eta(\varepsilon)$.

Step II. $\mathfrak{M}$ is a contraction on $\Xi_\eta(\varepsilon)$. For any y, $z \in \Xi_\eta(\varepsilon)$, we have $y_t + \check{\theta}_{\vartheta_0+t}$, $z_t + \check{\theta}_{\vartheta_0+t} \in \Upsilon_r(\varepsilon)$. Hence, by (HE3), we get that

$$|\mathfrak{M}(y) - \mathfrak{M}(z)| = \left|f_1\left(\vartheta_0 + t, y_t + \check{\theta}\vartheta_0 + t\right) - f_1\left(\vartheta_0 + t, z_t + \check{\theta}_{\vartheta_0+t}\right)\right| \leq \ell\|y - z\|,$$

which implies that

$$\|\mathfrak{M}(y) - \mathfrak{M}(z)\| \leq \ell\|y - z\|.$$

In view of $0 < \ell < 1$, $\mathfrak{M}$ is contractionn on $\Xi_\eta(\varepsilon)$.

Step III. Now, we show that $\mathfrak{N}$ is completely continuouse operator. Let

$$\mathfrak{N}_1(z) = \begin{cases} 0, & t \in [-\delta, 0], \\ -f_1(\vartheta_0, \dot{\theta}) + f_2\left(t_0 + t, z_t + \check{\theta}_{\vartheta_0+t}\right), & t \in [0, \eta], \end{cases}$$

$$\mathfrak{N}_2(z) = \begin{cases} 0, & t \in [-\delta, 0], \\ \mathscr{I}_q^{\sigma} w\left(\vartheta_0 + t, l_t + \check{\theta}_{t_0+t}\right), & t \in [0, \eta]. \end{cases}$$

Clearly, $\mathfrak{N} = \mathfrak{N}_1 + \mathfrak{N}_2$. Since f_2 is completely continuous, $\mathfrak{N}_1$ is continuous and

$$E_1 = \left\{\mathfrak{N}_1(z) : z \in \Xi_\eta(\varepsilon)\right\},$$

is uniformly bounded. Form the condition that the set $\{t \to f_2(t, y_t) : y \in \mathfrak{B}\}$, be equicontinuous for any bounded set $\mathfrak{B} \subset \Upsilon_r(\varepsilon)$, we can conclude that $\mathfrak{N}_1$ is a completely continuous operator. On the other hand, for any $t \in [0, \eta]$, we have

$$\begin{aligned} |\mathfrak{N}_2(z)| &\leq \mathscr{I}_q^{\sigma} |w| \left(\vartheta_0 + t, z_t + \check{\theta}_{\vartheta_0+t}\right) \\ &\leq \frac{1}{\Gamma_q(\sigma)} \left[\int_0^t (t - q\xi)^{\frac{\sigma-1}{1-\sigma_1}}\right]^{1-\sigma_1} \left[\int_{\vartheta_0}^{\vartheta_0+t} (m(\xi))^{\frac{1}{\sigma_1}} \, \mathrm{d}_q\xi\right]^{\sigma_1} \\ &\leq \frac{M\eta^{(1+\beta)(1-\sigma_1)}}{\Gamma_q(\sigma)(1+\beta)^{1-\sigma_1}}. \end{aligned}$$

Hence, $E_2 = \left\{\mathfrak{N}_2(z) : z \in \Xi_\eta(\varepsilon)\right\}$ is uniformaly bounded. Now, we will prove that E_2, is equivontinuous. For each $0 \leq t_1 \leq t_2 \leq \eta$ and $z \in \Xi_\eta(\varepsilon)$, we get that

$$\begin{aligned} \left|\mathfrak{N}_2[z](t_2) - \mathfrak{N}_2[z](t_1)\right| &= \Bigg| \frac{1}{\Gamma_q(\sigma)} \int_0^{t_1} \left[(t_2 - q\xi)^{(\sigma-1)} - (t_1 - q\xi)^{(\sigma-1)}\right] \\ &\quad \times w\left(t_0 + \xi, l_\xi + \check{\theta}_{\vartheta_0+\xi}\right) \mathrm{d}_q\xi \\ &\quad + \frac{1}{\Gamma_q(\sigma)} \int_{t_1}^{t_2} (t_2 - q\xi)^{(\sigma-1)} w\left(\vartheta_0 + \xi, l_\xi + \check{\theta}_{\vartheta_0+\xi}\right) \mathrm{d}_q\xi \Bigg| \\ &\leq \frac{1}{\Gamma_q(\sigma)} \int_0^{t_1} \left[(t_2 - q\xi)^{(\sigma-1)} - (t_1 - q\xi)^{(\sigma-1)}\right] \\ &\quad \times \left|w\left(\vartheta_0 + \xi, l_\xi + \check{\theta}_{\vartheta_0+\xi}\right)\right| \mathrm{d}_q\xi \\ &\quad + \frac{1}{\Gamma_q(\sigma)} \int_{t_1}^{t_2} (t_2 - q\xi)^{(\sigma-1)} \left|w\left(\vartheta_0 + \xi, l_\xi + \check{\theta}_{\vartheta_0+\xi}\right)\right| \mathrm{d}_q\xi \\ &\leq \frac{M}{\Gamma_q(\sigma)} \left[\int_0^{t_1} \left[(t_1 - q\xi)^{(\sigma-1)} - (t_2 - q\xi)^{(\sigma-1)}\right]^{\frac{1}{1-\sigma_1}} \mathrm{d}_q\xi\right]^{1-\sigma_1} \\ &\quad + \frac{M}{\Gamma_q(\sigma)} \left[\int_{t_1}^{t_2} \left[(t_2 - q\xi)^{(\sigma-1)}\right]^{\frac{1}{1-\sigma_1}} \mathrm{d}_q\xi\right]^{1-\sigma_1} \\ &\leq \frac{M}{\Gamma_q(\sigma)} \left[\int_0^{\vartheta_0} (t_1 - q\xi)^{(\beta)} - (t_2 - q\xi)^{(\beta)} \mathrm{d}_q\xi\right]^{1-\sigma_1} \\ &\quad + \frac{M}{\Gamma_q(\sigma)} \left[\int_{t_1}^{t_2} (t_2 - q\xi)^{(\beta)} \mathrm{d}_q\xi\right]^{1-\sigma_1} \\ &\leq \frac{M}{\Gamma_q(\sigma)(1+\beta)^{1-\sigma_1}} \left[t_1^{1+\beta} - t_2^{1+\beta} + (t_2 - t_1)^{1+\beta}\right]^{1-\sigma_1} \\ &\quad + \frac{M}{\Gamma_q(\sigma)(1+\beta)^{1-\sigma_1}} (t_2 - t_1)^{(1+\beta)(1-\sigma_1)} \\ &\leq \frac{2M}{\Gamma_q(\sigma)(1+\beta)^{1-\sigma_1}} (t_2 - t_1)^{(1+\beta)(1-\sigma_1)}, \end{aligned}$$

which means that E_2 is equicontinuous. Moreover, it is clear that $\mathfrak{N}_2$ is continuous. So $\mathfrak{N}_2$ is completely continuous operator. Then $\mathfrak{N} = \mathfrak{N}_1 + \mathfrak{N}_2$ is completely continuous operator.

Therefore, Kransoselskii's fixed point theorem shows that $\mathfrak{M} + \mathfrak{N}$ has a fixed point on $\Xi_\eta(\varepsilon)$, and hence the problem (1.1) has a solution

$$y(t) = \dot{\theta}(0) + l(t - \vartheta_0), \qquad \forall t \in [\vartheta_0, \vartheta_0 + \eta].$$

This completes the proof. □

In this case where $f_1 = 0$, we get the following result.

Corollary 3.3. *Assume that there exist $r \in (0,\tau)$ and $\varepsilon \in (0,\infty)$ such that* (HE1) *hold. Then Problem* (1.1) *has at least one solution on $[\vartheta_0, \vartheta_0+\eta]$ for some positive number η, whenever f is continuous and for any $y, z \in \Upsilon_r(\varepsilon)$, $t \in \bar{J}_0$, $|f(t,y_t) - f(t,z_t)| \le \ell \|y-z\|$, where $0 < \ell < 1$.*

Corollary 3.4. *Assume that there exist $r \in (0,\tau)$ and $\varepsilon \in (0,\infty)$ such that* (HE1) *and* (HE2) *hold. Then Problem* (1.1) *has at least one solution on $[\vartheta_0, \vartheta_0+\eta]$ for same positive number η, whenever f is completely continuous and for any bounded set $\mathfrak{B} \in \Upsilon_r(\varepsilon)$, the set $\{t \to f(t,y_t) : y \in \mathfrak{B}\}$, is equicontinuous on $C(\bar{J}_0, \mathbb{R}^n)$.*

3.2. *The system of* $\mathbb{FNF}q\mathbb{DE}$ (1.2). Now, we consider the initial value problem of multi-dimensional system of $\mathbb{FNF}q\mathbb{DE}$ (1.2). Let I be an interval in $\mathbb{R}$ and $\mathscr{A}$ with the norm $\|y\| = \sup_{t\in J}|y(t)|$. Consider the product Banach space $(\mathscr{A}^m, \|\,.\,\|_*)$ with the norme

$$\|(y_1,y_2,\dots,y_m)\|_* = \max\Big\{\|y_1\|, \|y_2\|, \dots, \|y_m\|\Big\}.$$

With the same definitions as the previous subsection 3.1, we define

$$\Upsilon_r(\varepsilon) = \Big\{(y_1,y_2,\dots,y_m)\,\Big|\, y_{i_{\vartheta_0}} = \dot{\theta}_i,\ \sup_{t\in\bar{J}_0}\big|y_i(t) - \dot{\theta}_i(0)\big| \le \varepsilon,\ i = 1,2,\dots,m\Big\},$$

where $k_i \in C([\vartheta_0 - \delta, \vartheta_0 + r], \mathbb{R}^n)$. For obtaining our results, we need the following conditions:

(HS1) $w_i(t,\rho_1,\rho_2,\dots,\rho_m)$ is measurable with respect to t on $\bar{J}_0$, and is continuous with respect to ρ_i on $\mathscr{C}_0$ for each $i,j = 1,2,\dots,m$;

(HS2) There exists $0 < \sigma_{i1} < \sigma_i$ and a $m_i(t) \in L^{1/\sigma_{i1}}(\bar{J}_0)$ such that

$$|w_i(t,\mathbf{y}_t)| \le m_i(t), \qquad \forall (y_1,y_2,\dots,y_m) \in \Upsilon_r(\varepsilon), \forall t \in \bar{J}_0;$$

(HS3) Furthermore

$$f_i(t,\mathbf{y}_t) = f_{i1}(t,\mathbf{y}_t) + f_{i2}(t,\mathbf{y}_t),$$

for any $(y_1,y_2,\dots,y_m) \in \Upsilon_r(\varepsilon)$ such that f_{i1} is continuous, and

$$|f_{i1}(t,\mathbf{y}_t) - f_{i1}(t,\mathbf{z}_t)| \le \ell_i \|y-z\|_*, \qquad \forall t \in \bar{J}_0,$$

and for each $\mathbf{y} = (y_1,y_2,\dots,y_m)$, $\mathbf{z} = (z_1,z_2,\dots,z_m) \in \Upsilon_r(\varepsilon)$ where $\ell_i \in (0,1)$ is a constant for $i = 1,2,\dots,m$, and f_{i2} is completely continuous and for all bounded set $\mathfrak{B}$ belong to $\Upsilon_r(\varepsilon)$, the set

$$\Big\{t \mapsto f_{i2}(t,\mathbf{y}_t) : (y_1,y_2,\dots,y_m) \in \mathfrak{B}\Big\},$$

is equi-continuous on $\prod_{i=1}^m C(\bar{J}_0, \mathbb{R}^n)$.

Lemma 3.5. *Suppose that there exist $r \in (0,\tau)$ and $\varepsilon \in (0,\infty)$ such that* (HS1) *and* (HS2) *hold. Then the system* (1.2) *is equivalent to the equation with conditions as follows*

$$(3.7)\qquad \begin{cases} k_i(t) = \dot{\theta}_i(0) - f_i\big(\vartheta_0, \dot{\theta}_1(t), \dot{\theta}_2(t), \dots, \dot{\theta}_m(t)\big) + f_i\big(t,\mathbf{y}_t\big) + \mathscr{I}_q^{\sigma_i} w_i\big(t,\mathbf{y}_t\big) \\ \qquad = \dot{\theta}_i(0) - f_i\big(\vartheta_0, \dot{\theta}_1(t), \dot{\theta}_2(t), \dots, \dot{\theta}_m(t)\big) + f_i\big(t,\mathbf{y}_t\big) \\ \qquad\quad + \displaystyle\int_{\vartheta_0}^{t} \frac{(t-q\xi)^{(\sigma_i-1)}}{\Gamma_q(\sigma_i)} w_i\big(\xi,\mathbf{y}_\xi\big)\,\mathrm{d}_q\xi, \qquad \forall t \in (t_0, t_0+r], \\ y_{i_{\vartheta_0}} = \dot{\theta}_i, \qquad i = 1,2,\dots,m, \end{cases}$$

for almost all $t \in \bar{J}_0$.

Proof. Form condition (HS1), clearly $w_i(t,\mathbf{y}_t)$ is Lebesgue measurable on $\overline{J}_0$ and so

$$(t-q\xi)^{(\sigma_i-1)} \in L^{1/(1-\sigma_{i1})}([t_0,t]), \qquad \forall t \in \overline{J}_0.$$

By using the Hölder's inequality and the condition (HS2), we get that $(t-q\xi)^{(\sigma_i-1)} w_i(\xi,\mathbf{y}_\xi)$, is Lebesgue integrable with respect to $\xi \in [\vartheta_0,t]$, for each $t \in \overline{J}_0$, $i=1,2,\dots,m$, $(y_1,y_2,\dots,y_m) \in \Upsilon_r(\varepsilon)$, and

$$\int_{\vartheta_0}^{t} \left|(t-q\xi)^{(\sigma_i-1)} w_i(\xi,\mathbf{y}_\xi)\right| \mathrm{d}_q\xi \le \left\|(t-q\xi)^{(\sigma_i-1)}\right\|_{L^{1/(1-\sigma_{i1})}([\vartheta_0,t])} \cdot \|m_i\|_{L^{1/\sigma_{i1}}(\overline{J}_0)}.$$

By simple review, we conclude that if $\mathbf{y}=(y_1,y_2,\dots,y_m)$ is a solution of the problem (1.2), then y is a solution of the equation (3.7). Now, suppose that $\mathbf{y}=(y_1,y_2,\dots,y_m)$ is a solution of the equation (3.7) and $t \in (\vartheta_0, \vartheta_0+r]$. Then $y_{i_{\vartheta_0}} = \dot{\theta}_i$ and

$${}^{c}\mathscr{D}_q^{\sigma_i}\big(y_i(t) - y_i(t,\mathbf{y}_t)\big) = w_i\big(t,\mathbf{y}_t\big),$$

for all $t \in (\vartheta_0, \vartheta_0+r]$ and $i=1,2,\dots,m$. Thus, $y=(y_1,y_2,\dots,y_m)$ is a solution of the problem (1.2). This completes the proof. $\square$

Theorem 3.6. *Suppose that there exist $r \in (0,\tau), \varepsilon \in (0,\infty)$ such that hypotheses* (HS1)*,* (HS2) *and* (HS3) *hold. Then the problem* (1.2) *has at least one solution on $[\vartheta_0, \vartheta_0+\eta]$ with $\eta > 0$.*

Proof. Since the condition (HS3) holds, the equation (3.7) is equivalent to the equation

$$\begin{cases} k_i(t) = \dot{\theta}_i(0) - f_{i1}\big(\vartheta_0, \dot{\theta}_1, \dot{\theta}_2, \dots, \dot{\theta}_m\big) \\ \qquad\qquad - f_{i2}\big(\vartheta_0, \dot{\theta}_1, \dot{\theta}_2, \dots, \dot{\theta}_m\big) + f_{i1}(t,\mathbf{y}_t) + f_{i2}(t,\mathbf{y}_t) \\ \qquad\qquad + \displaystyle\int_{\vartheta_0}^{t} \frac{(t-q\xi)^{(\sigma_i-1)}}{\Gamma_q(\sigma_i)} w_i\big(\xi,\mathbf{y}_\xi\big)\,\mathrm{d}_q\xi, \\ y_{i_{\vartheta_0}} = \dot{\theta}_i, \qquad i=1,2,\dots,m, \end{cases}$$

for $t \in \overline{J}_0$. Let $\big(\check{\theta}_1, \check{\theta}_2, \dots, \check{\theta}_m\big) \in \Upsilon_r(\varepsilon)$ be defined by $\check{\theta}_{i_{\vartheta_0}} = \dot{\theta}_i$ and

$$\check{\theta}_i(\vartheta_0+t) = \dot{\theta}_i(0), \qquad \forall t \in [0,r],\, i=1,2,\dots,k.$$

If $y=(y_1,y_2,\dots,y_m)$ is a solution of problem (1.2) and

$$y_i(\vartheta_0+t) = \check{\theta}_i(\vartheta_0+t) + l_i(t), \qquad \forall t \in [-\delta,r],\, i=1,2,\dots,m,$$

then $y_{i_{\vartheta_0+t}} = \check{\theta}_{i_{\vartheta_0+t}} + l_{i_t}$ for $t \in [0,r]$ and $i=1,2,\dots,m$. Thus,

$$\begin{aligned} l_i(t) = {} & -f_{i1}\big(\vartheta_0, \dot{\theta}_1, \dot{\theta}_2, \dots, \dot{\theta}_m\big) - f_{i2}\big(\vartheta_0, \dot{\theta}_1, \dot{\theta}_2, \dots, \dot{\theta}_m\big) \\ & + f_{i1}\big(\vartheta_0+t, l_{1_t} + \check{\theta}_{1_{\vartheta_0+t}}, l_{2_t} + \check{\theta}_{2_{\vartheta_0+t}}, \dots, l_{m_t} + \check{\theta}_{m_{\vartheta_0+t}}\big) \\ & + f_{i2}\big(\vartheta_0+t, l_{1_t} + \check{\theta}_{1_{\vartheta_0+t}}, l_{2_t} + \check{\theta}_{2_{\vartheta_0+t}}, \dots, l_{m_t} + \check{\theta}_{m_{\vartheta_0+t}}\big) \\ & + \frac{1}{\Gamma_q(\sigma_i)} \int_0^t (t-q\xi)^{(\sigma_i-1)} f_i\big(\vartheta_0+\xi, l_{1_\xi} + \check{\theta}_{1_{\vartheta_0+\xi}}, l_{2_\xi} + \check{\theta}_{2_{\vartheta_0+\xi}}, \dots, l_{k_s} + \check{\theta}_{m_{\vartheta_0+\xi}}\big)\,\mathrm{d}_q\xi. \end{aligned} \tag{3.8}$$

Since f_{i1}, f_{i2} are continuous and k_{i_t} is continuous at t for each $i=1,2,\dots,m$, there exists $r'>0$ such that

$$\left| f_{i1}\big(\vartheta_0+t, l_{1_t} + \check{\theta}_{1_{\vartheta_0+t}}, l_{2_t} + \check{\theta}_{2_{\vartheta_0+t}}, \dots, l_{m_t} + \check{\theta}_{m_{\vartheta_0+t}}\big) - f_{i1}\big(\vartheta_0, \dot{\theta}_1, \dot{\theta}_2, \dots, \dot{\theta}_m\big)\right| < \frac{1}{3}\varepsilon,$$

$$\left| f_{i2}\big(\vartheta_0+t, l_{1_t} + \check{\theta}_{1_{\vartheta_0+t}}, l_{2_t} + \check{\theta}_{2_{\vartheta_0+t}}, \dots, l_{m_t} + \check{\theta}_{m_{\vartheta_0+t}}\big) - f_{i2}\big(\vartheta_0, \dot{\theta}_1, \dot{\theta}_2, \dots, \dot{\theta}_m\big)\right| < \frac{1}{3}\varepsilon,$$

for $0 < t < r'$ and $i = 1, 2, \dots, m$. Take

$$\eta = \min\left\{ r, r', \min_{1\le i\le m}\left\{ \left[\frac{\varepsilon\Gamma_q(\sigma_i)(1+\beta_i)^{1-\sigma_{i1}}}{3M_i}\right]^{1/(1+\beta_i)(1-\sigma_{i1})} \right\}\right\}, \tag{3.9}$$

where $\beta_i = \frac{\sigma_i - 1}{1-\sigma_{i1}} \in (-1, 0)$ and

$$M_i = \|m_i\|_{L^{1/\sigma_{i1}}(\bar{J}_0)}, \qquad \forall i = 1, 2, \dots, m.$$

Define

$$\Xi_\eta(\varepsilon) = \left\{ (l_1, l_2, \dots, l_m) \,\middle|\, l_i \in \mathscr{C}_\eta,\, l_i(\xi) = 0, \|l_i\| \le \varepsilon,\, \xi \in [-\delta, 0],\, i = 1, 2, \dots, m \right\}.$$

In fact, $\Xi_\eta(\varepsilon)$ is a closed, bounded and convex subset of $\prod_{i=1}^m \mathscr{C}_\eta$. Define the operators $\mathfrak{M}$ and $\mathfrak{N}$ on $\Xi_\eta(\varepsilon)$ by

$$\mathfrak{M}(z_1, z_2, \dots, z_m)(t) = \begin{pmatrix} \mathfrak{M}_1(z_1, z_2, \dots, z_k)(t) \\ \mathfrak{M}_2(z_1, z_2, \dots, z_k)(t) \\ \vdots \\ \mathfrak{M}_m(z_1, z_2, \dots, z_m)(t) \end{pmatrix},$$

$$\mathfrak{N}(z_1, z_2, \dots, z_m)(t) = \begin{pmatrix} \mathfrak{N}_1(z_1, z_2, \dots, z_m)(t) \\ \mathfrak{N}_2(z_1, z_2, \dots, z_m)(t) \\ \vdots \\ \mathfrak{N}_m(z_1, z_2, \dots, z_m)(t) \end{pmatrix},$$

where

$$\begin{aligned} &\mathfrak{M}_i(z_1, z_2, \dots, z_m)(t) \\ &= \begin{cases} 0, & t \in [-\delta, 0], \\ -f_{i1}\big(\vartheta_0, \dot{\theta}_1, \dot{\theta}_2, \dots, \dot{\theta}_k\big) \\ \quad + f_{i1}\big(\vartheta_0 + t, l_{1_t} + \check{\theta}_{1_{\vartheta_0+t}}, l_{2_t} + \check{\theta}_{2_{\vartheta_0+t}}, \dots, l_{m_t} + \check{\theta}_{m_{\vartheta_0+t}}\big), & t \in [0, \eta], \end{cases} \end{aligned}$$

$$\begin{aligned} &\mathfrak{N}_i(z_1, z_2, \dots, z_m)(t) \\ &= \begin{cases} 0, & t \in [-\delta, 0], \\ -f_{i2}\big(\vartheta_0, \dot{\theta}_1, \dot{\theta}_2, \dots, \dot{\theta}_k\big) \\ \quad + f_{i2}\big(\vartheta_0 + t, l_{1_t} + \check{\theta}_{1_{\vartheta_0+t}}, l_{2_t} + \check{\theta}_{2_{\vartheta_0+t}}, \dots, l_{m_t} + \check{\theta}_{m_{\vartheta_0+t}}\big) \\ \quad + \displaystyle\int_0^t \frac{(t-q\xi)^{(\sigma_i - 1)}}{\Gamma_q(\sigma_i)} w_i\Big(\vartheta_0 + \xi, k_{1_\xi} + \check{\theta}_{1_{\vartheta_0+\xi}}, \\ \quad l_{2_\xi} + \check{\theta}_{2_{\vartheta_0+\xi}}, \dots, l_{m_\xi} + \check{\theta}_{m_{\vartheta_0+\xi}}\Big)\, \mathrm{d}_q\xi, & t \in [0, \eta], \end{cases} \end{aligned}$$

for $i = 1, 2, \dots, m$. One can easily to check that the operator equation $\mathfrak{M}(y) + \mathfrak{N}(z) = l$ has a solution $z = (z_1, z_2, \dots, z_m)$ if and only if z_i is a solution for the equation (3.8) for all $i = 1, 2, \dots, m$. In this case,

$$y_i(\vartheta_0 + t) = l_i(t) + \check{\theta}_i(\vartheta_0 + t),$$

will be a solution of the system (1.2) on $[0, \eta]$. Thus, the existence of a solution of the system (1.2) is equivalent to the existence of a fixed point for the operator $\mathfrak{M} + \mathfrak{N}$ on $\Xi_\eta(\varepsilon)$. Hence, it is sufficient we show that $\mathfrak{M} + \mathfrak{N}$ has a fixed point in $\Xi_\eta(\varepsilon)$. We prove it in three steps.

Step I. For all $\mathbf{y}=(y_1,y_2,\dots,y_m)$, $\mathbf{z}=(z_1,z_2,\dots,z_m)\in\Xi_\eta(\varepsilon)$, $\mathfrak{M}(y)+\mathfrak{N}(z)\in\Xi_\eta(\varepsilon)$. Let $y,\,z\in\Xi_\eta(\varepsilon)$ be given. Then,

$$\mathfrak{M}_i(y)+\mathfrak{N}_i(z)\in\mathscr{C}_\eta,\qquad \forall i=1,2,\dots,k.$$

It is easy to check that

$$(\mathfrak{M}(y)+\mathfrak{N}(z))(t)=0,\qquad \forall t\in[-\delta,0].$$

Also, we have

$$\begin{aligned}
&\big|\mathfrak{M}_i(y)+\mathfrak{N}_i(z)\big|\\
&\qquad\le\Big|-f_{i1}\big(\vartheta_0,\dot\theta_1,\dot\theta_2,\dots,\dot\theta_m\big)\\
&\qquad\quad+f_{i1}\big(\vartheta_0+t,y_{1_t}+\check\theta_{1_{\vartheta_0+t}},y_{2_t}+\check\theta_{2_{\vartheta_0+t}},\dots,y_{m_t}+\check\theta_{m_{\vartheta_0+t}}\big)\Big|\\
&\qquad\quad+\Big|-f_{i2}\big(\vartheta_0,\dot\theta_1,\dot\theta_2,\dots,\dot\theta_m\big)\\
&\qquad\quad+f_{i2}\big(\vartheta_0+t,l_{1_t}+\check\theta_{1_{\vartheta_0+t}},l_{2_t}+\check\theta_{2_{\vartheta_0+t}},\dots,l_{m_t}+\check\theta_{m_{\vartheta_0+t}}\big)\Big|\\
&\qquad\quad+\frac{1}{\Gamma_q(\sigma_i)}\int_0^t\Big|(t-q\xi)^{(\sigma_i-1)}w_i\big(\vartheta_0+\xi,l_{1_\xi}+\check\theta_{1_{\vartheta_0+\xi}},\\
&\qquad\quad l_{2_\xi}+\check\theta_{2_{\vartheta_0+\xi}},\dots,l_{m_\xi}+\check\theta_{m_{\vartheta_0+\xi}}\big)\Big|\,\mathrm{d}_q\xi\\
&\qquad\le\frac{2}{3}\varepsilon+\frac{1}{\Gamma_q(\sigma_i)}\left[\int_0^t(t-q\xi)^{\left(\frac{\sigma_{i1}-1}{1-\sigma_{i1}}\right)}\,\mathrm{d}_q\xi\right]^{1-\sigma_{i1}}\left[\int_{\vartheta_0}^{\vartheta_0+t}(m_i(\xi))^{\frac{1}{\sigma_{i1}}}\,\mathrm{d}_q\xi\right]^{\sigma_{i1}}\\
&\qquad\le\frac{2}{3}\varepsilon+\frac{M_i\eta^{(1+\beta_i)(1-\sigma_{i1})}}{\Gamma_q(\sigma_i)(1+\beta_i)^{1-\sigma_{i1}}}\le\varepsilon,
\end{aligned}\tag{3.10}$$

Thus,

$$\|\mathfrak{M}_i(y)+\mathfrak{N}_i(z)\|=\sup_{t\in[0,\eta]}|(\mathfrak{M}_i(y))(t)-(\mathfrak{N}_i(z))(t)|\le\varepsilon,$$

for all $i=1,2,\dots,m$. Hence, $\mathfrak{M}(y)+\mathfrak{N}(z)\in\Xi_\eta(\varepsilon)$.

Step II. $\mathfrak{M}$ is a contraction on $\Xi_\eta(\varepsilon)$. Let $\mathbf{y}=(y_1,y_2,\dots,y_m)$, $\mathbf{z}=(z_1,z_2,\dots,z_m)\in\Xi_\eta(\varepsilon)$. Then,

$$\begin{aligned}
&\big(y_{1_t}+\check\theta_{1_{\vartheta_0+t}},y_{2_t}+\check\theta_{2_{\vartheta_0+t}},\dots,y_{m_t}+\check\theta_{m_{\vartheta_0+t}}\big),\\
&\big(z_{1_t}+\check\theta_{1_{\vartheta_0+t}},z_{2_t}+\check\theta_{2_{\vartheta_0+t}},\dots,z_{m_t}+\check\theta_{m_{\vartheta_0+t}}\big),
\end{aligned}$$

belong to $\Xi_\eta(\varepsilon)$ and so

$$\begin{aligned}
\big|\mathfrak{M}_i(y)-\mathfrak{M}_i(z)\big|&=\Big|f_{i1}\big(\vartheta_0+t,y_{1_t}+\check\theta_{1_{\vartheta_0+t}},y_{2_t}+\check\theta_{2_{\vartheta_0+t}},\dots,y_{m_t}+\check\theta_{m_{\vartheta_0+t}}\big)\\
&\qquad-f_{i1}\big(\vartheta_0+t,z_{1_t}+\check\theta_{1_{\vartheta_0+t}},z_{2_t}+\check\theta_{2_{\vartheta_0+t}},\dots,z_{k_t}+\check\theta_{m_{\vartheta_0+t}}\big)\Big|\\
&\le\ell_i\|y-z\|_*,\qquad \forall i=1,2,\dots,m.
\end{aligned}\tag{3.11}$$

This implies that $\|\mathfrak{M}(y)-\mathfrak{N}(z)\|_*\le\ell\|y-z\|_*$, where $\ell=\max\{\ell_1,\ell_2,\dots,\ell_m\}$. Since $0<\ell<1$, $\mathfrak{M}$ is a contraction on $\Xi_\eta(\varepsilon)$.

Step III. $\mathfrak{N}$ is a completely continuous operator. Suppose that

$$\mathfrak{N}_{i1}(z_1,z_2,\dots,z_m)(t)\tag{3.12}$$

$$=\begin{cases}0, & t\in[-\delta,0],\\ -f_{i2}\big(\vartheta_0,\dot\theta_1,\dot\theta_2,\dots,\dot\theta_m\big)\\ \quad+f_{i2}\big(\vartheta_0+t,z_{1_t}+\check\theta_{1_{\vartheta_0+t}},z_{2_t}+\check\theta_{2_{\vartheta_0+t}},\dots,z_{m_t}+\check\theta_{m_{\vartheta_0+t}}\big), & t\in[0,\eta],\end{cases}\tag{3.13}$$

and

$$\mathfrak{N}_{i2}\big(z_1,z_2,\dots,z_m\big)(t) \tag{3.14}$$

$$= \begin{cases} 0, & t\in[-\delta,0], \\ \dfrac{1}{\Gamma_q(\sigma_i)}\displaystyle\int_0^t (t-q\xi)^{\sigma_i-1} \\ \quad \times w_i\big(t_0+\xi, z_{1_\xi}+\tilde{\theta}_{1_{t_0+\xi}}, z_{2_\xi}+\tilde{\theta}_{2_{t_0+\xi}},\dots,z_{k_\xi}+\tilde{\theta}_{k_{t_0+\xi}}\big)\,\mathrm{d}_q\xi, & t\in[0,\eta], \end{cases} \tag{3.15}$$

for $i=1,2,\dots,m$. It is clear that

$$\mathfrak{N} = \begin{pmatrix} \mathfrak{N}_{11}+\mathfrak{N}_{12} \\ \mathfrak{N}_{21}+\mathfrak{N}_{22} \\ \vdots \\ \mathfrak{N}_{m1}+\mathfrak{N}_{m2} \end{pmatrix}.$$

Since f_{i2} is completely continuous for all $i=1,2,\dots,m$, $\mathfrak{N}_{i1}$ is continuous and also

$$E_{i1} = \Big\{\mathfrak{N}_{i1}(z)\,:\, z\in\Xi_\eta(\varepsilon)\Big\},$$

is uniformly bounded. From the condition (HS3), we conclude that E_{i1} is equi-continuous. On the other hand,

$$\begin{aligned} \big|\mathfrak{N}_{i2}[z](t)\big| &\le \frac{1}{\Gamma_q(\sigma_i)}\int_0^t (t-q\xi)^{(\sigma_i-1)} \\ &\qquad \times \big|w_i\big(\vartheta_0+\xi, z_{1_\xi}+\check{\theta}_{1_{\vartheta_0+\xi}}, z_{2_\xi}+\check{\theta}_{2_{\vartheta_0+\xi}},\dots,z_{k_\xi}+\check{\theta}_{k_{\vartheta_0+\xi}}\big)\big|\,\mathrm{d}_q\xi \\ &\le \frac{1}{\Gamma_q(\sigma_i)}\left[\int_0^t (t-q\xi)^{\frac{\sigma_i-1}{1-\sigma_{i1}}}\,\mathrm{d}_q\xi\right]^{1-\sigma_{i1}}\left[\int_{\vartheta_0}^{\vartheta_0+t} (m_i(\xi))^{\frac{1}{\sigma_{i1}}}\,\mathrm{d}_q\xi\right]^{\sigma_{i1}} \\ &\le \frac{M_i\eta^{(1+\beta_i)(1-\sigma_{i1})}}{\Gamma_q(\sigma_i)(1+\beta_i)^{1-\sigma_{i1}}}. \end{aligned} \tag{3.16}$$

This implies that

$$E_{i2} = \Big\{\mathfrak{N}_{i2}(z)\,:\, z\in\Xi_\eta(\varepsilon)\Big\},$$

is uniformly bounded. Now, we prove that E_{i2} is equi-continuous. Let $0\le\vartheta_1<\vartheta_2\le\eta$ and $z\in\Xi_\eta(\varepsilon)$ be given. Then, we have

$$\begin{aligned} &\big|\mathfrak{N}_{i2}[z](\vartheta_2)-\mathfrak{N}_{i2}[z](t_1)\big| \\ &\quad = \Bigg|\frac{1}{\Gamma_q(\sigma_i)}\int_0^{\vartheta_1}[(\vartheta_2-q\xi)^{(\sigma_i-1)}-(\vartheta_1-q\xi)^{(\sigma_i-1)}] \\ &\qquad \times w_i\Big(\vartheta_0+\xi, l_{1_\xi}+\check{\theta}_{1_{\vartheta_0+\xi}}, l_{2_\xi}+\check{\theta}_{2_{\vartheta_0+\xi}},\dots,l_{m_\xi}+\check{\theta}_{m_{\vartheta_0+\xi}}\Big)\,\mathrm{d}_q\xi \\ &\qquad + \frac{1}{\Gamma_q(\sigma_i)}\int_{\vartheta_1}^{\vartheta_2}(\vartheta_2-q\xi)^{(\sigma_i-1)} \\ &\qquad \times w_i\Big(\vartheta_0+\xi, l_{1_\xi}+\check{\theta}_{1_{\vartheta_0+\xi}}, l_{2_\xi}+\check{\theta}_{2_{\vartheta_0+\xi}},\dots,l_{m_\xi}+\check{\theta}_{m_{\vartheta_0+\xi}}\Big)\,\mathrm{d}_q\xi\Bigg| \\ &\quad \le \frac{1}{\Gamma_q(\sigma_i)}\int_0^{\vartheta_1}\Big[(\vartheta_1-q\xi)^{(\sigma_i-1)}-(\vartheta_2-q\xi)^{(\sigma_i-1)}\Big] \end{aligned}$$

$$
\begin{aligned}
&\times \left| w_i\Big(\vartheta_0+\xi, l_{1_\xi}+\check{\theta}_{1_{\vartheta_0+\xi}}, l_{2_\xi}+\check{\theta}_{2_{\vartheta_0+\xi}},\dots, l_{m_\xi}+\check{\theta}_{m_{\vartheta_0+\xi}}\Big)\right| \mathrm{d}_q\xi \\
&+\frac{1}{\Gamma_q(\sigma_i)}\int_{\vartheta_1}^{\vartheta_2}(\vartheta_2-q\xi)^{(\sigma_i-1)} \\
&\times \left| w_i\Big(\vartheta_0+\xi, l_{1_\xi}+\check{\theta}_{1_{\vartheta_0+\xi}}, l_{2_\xi}+\check{\theta}_{2_{\vartheta_0+\xi}},\dots, l_{m_\xi}+\check{\theta}_{m_{\vartheta_0+\xi}}\Big)\right| \mathrm{d}_q\xi \\
\le\ &\frac{M_i}{\Gamma_q(\sigma_i)}\left[\int_0^{\vartheta_1}\left[(\vartheta_1-q\xi)^{(\sigma_i-1)}-(\vartheta_2-q\xi)^{(\sigma_i-1)}\right]^{1/(1-\sigma_{i1})}\mathrm{d}_q\xi\right]^{1-\sigma_{i1}} \\
&+\frac{M_i}{\Gamma_q(\sigma_i)}\left[\int_{\vartheta_1}^{\vartheta_2}\left[(\vartheta_2-q\xi)^{(\sigma_i-1)}\right]^{\frac{1}{1-\sigma_{i1}}}\mathrm{d}_q\xi\right]^{1-\sigma_{i1}} \\
\le\ &\frac{M_i}{\Gamma_q(\sigma_i)}\left[\int_0^{\vartheta_1}[(\vartheta_1-q\xi)^{(\beta_i)}-(\vartheta_2-q\xi)^{(\beta_i)}]\,\mathrm{d}_q\xi\right]^{1-\sigma_{i1}} \\
&+\frac{M_i}{\Gamma_q(\sigma_i)}\left[\int_{\vartheta_1}^{\vartheta_2}(\vartheta_2-q\xi)^{(\beta_i)}\,\mathrm{d}_q\xi\right]^{1-\sigma_{i1}} \\
\le\ &\frac{M_i}{\Gamma_q(\sigma_i)(1+\beta_i)^{1-\sigma_{i1}}}\left[\vartheta_1^{1+\beta_i}-\vartheta_2^{1+\beta_i}+(\vartheta_2-\vartheta_1)^{1+\beta_i}\right]^{1-\sigma_{i1}} \\
&+\frac{M_i}{\Gamma_q(\sigma_i)(1+\beta_i)^{1-\sigma_{i1}}}(\vartheta_2-\vartheta_1)^{(1+\beta_i)(1-\sigma_{i1})} \\
\le\ &\frac{2M_i}{\Gamma_q(\sigma_i)(1+\beta_i)^{1-\sigma_{i1}}}(\vartheta_2-\vartheta_1)^{(1+\beta_i)(1-\sigma_{i1})},
\end{aligned}
$$

for all $i=1,2,\dots,m$. Thus, E_{i2} is equi-continuous. Moreover, $\mathfrak{N}_{i2}$ is continuous and so $\mathfrak{N}$ is a completely continuous operator. Now, by using the Kranoselskii's fixed point theorem we get $\mathfrak{M}+\mathfrak{N}$ has a fixed point on $\Xi_\eta(\varepsilon)$ and so the system (1.2) has a solution $k=(k_1,k_2,\dots,k_m)$, where $y_i(t)=\dot{\theta}_i(0)+l_i(t-\vartheta_0)$, for all $t\in[\vartheta_0,\vartheta_0+\eta]$, $i=1,2,\dots,m$. □

If we put $f_{i1}=0$ for all $i=1,2,\dots,m$, then we obtain next result.

Corollary 3.7. *Suppose that there exist $r\in(0,\tau)$ and $\varepsilon\in(0,\infty)$ such that the conditions* (HS1) *and* (HS2) *hold, f_i is continuous for all $i=1,2,\dots,m$ and*

$$|f_i(t,\mathbf{y}_t)-f_i(t,\mathbf{z}_t)|\le \ell_i\|y-z\|_*,$$

for all $\mathbf{y}=(y_1,y_2,\dots,y_m)$, $\mathbf{z}=(z_1,z_2,\dots,z_m)\in\Xi_r(\varepsilon)$ and $t\in\bar{J}_0$, where $\ell_i\in(0,1)$ is a constant for all $i=1,2,\dots,m$. Then the system (1.2) *has at least one solution on $[\vartheta_0,\vartheta_0+\eta]$ for some positive number η.*

If we put $f_{i2}=0$ for all $i=1,2,\dots,m$, then we obtain next result.

Corollary 3.8. *Suppose that there exist $r\in(0,\tau)$ and $\varepsilon\in(0,\infty)$ such that the conditions* (HS1) *and* (HS2) *hold, f_i is completely continuous for all $i=1,2,\dots,m$ and the family*

$$\Big\{t\vdash f_i(t,\mathbf{y}_t) : (y_1,y_2,\dots,y_m)\in\mathfrak{B}\Big\},$$

is equi-continuous on $\prod_{i=1}^m C(\bar{J}_0,\mathbb{R}^n)$ for all bounded set $\mathfrak{B}$ in $\Xi_r(\varepsilon)$. Then the system (1.2) *has at least one solution on $[\vartheta_0,\vartheta_0+\eta]$ for some positive number η.*

4. Applications with illustrative examples

Here, in this section, we consider an application to examine the validity of our theoretical results on the fractional-order representation of the motion of a particle along a straight line. In this case, we consider a constrained motion of a particle along a straight line restrained by two linear springs with equal spring constant (stiffness coefficient) under external force and fractional damping along the t-axis. The springs unless subjected to force are assumed to have free length (unstretched length) and resist a change in length. The springs are anchored on the t-axis at $t = -1$ and $t = 1$, and the vibration of the particle in this example is restricted to t-axis only (see Fig. 1). The motion of the system along t-axis is independent of the initial spring tension. The vibration of the system is represented by a system of equations with the first equation having similar form of simple harmonic oscillator which cannot produce instability.

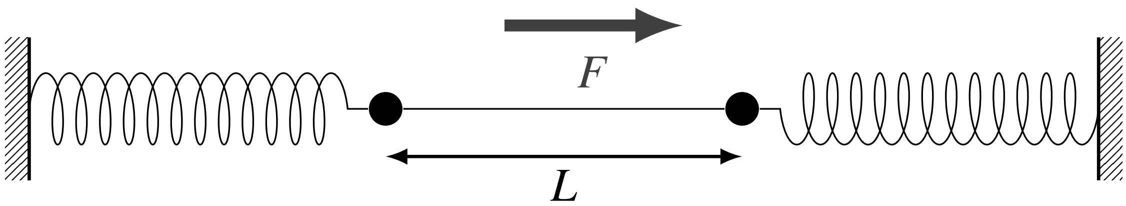


FIGURE 1. A particle along a straight line restrained by two linear springs with equal spring constant.

Herein, we give some examples to show the validity of pur main results for the problem (1.1) and (1.2). In this way, we give a computational technique for checking the problems.

Example 4.1. *Consider the $\mathbb{FNF}q\mathbb{DE}$ of the form:*

$$ {}^{c}\mathscr{D}_q^{4/5}\left(y(t) - \frac{2t^2 \sin y(t)}{5\left(t+\sqrt{10}\right)^2}\right) = \frac{\cos y(t)}{\sqrt[3]{10t^2+1}}, \qquad t \in \overline{J}_0 = [0, 2\pi], \tag{4.1}$$

for $q \in \left\{\frac{1}{5}, \frac{1}{2}, \frac{7}{8}\right\}$, under bounded delay $y_0 = \dot{\theta} \in \mathscr{C}_0 = C([-\pi, 0], \mathbb{R}^n)$. Clearly $\sigma = \frac{4}{5} \in (0,1)$, $\delta = \pi$, $\vartheta_0 = 0$, $\tau = 2\pi$ and

$$\Upsilon_{2\pi}(\varepsilon) = \left\{ y \in C([-\pi, 2\pi], \mathbb{R}^n) \,:\, y_0 = \dot{\theta}_0, \sup_{t \in \overline{J}_0} \left| y(t) - \dot{\theta}(0) \right| \leq \varepsilon \right\}.$$

Also, the functions $f, w : [0, \infty) \times \mathscr{C}_0 \to \mathbb{R}^n$ define by

$$f(t, y_t) = \frac{2t^2 \sin y(t)}{5(t+\sqrt{10})^2}, \qquad w(t, \rho) = \frac{\rho(t)}{\sqrt[3]{10t^2+1}},$$

and $y_t(u) = y(t+u)$ for $u \in [-\pi, 0]$. One can see that $w(t, \rho(t))$ is measurable with respect to $t \in \overline{J}_0$ and is continuous with respect to $\rho(t)$ on $\mathscr{C}_0$. If put $\sigma_1 = \frac{2}{5} < \frac{4}{5}$ then $\beta = (\frac{4}{5} - 1)/(1 - \frac{2}{5}) = -\frac{1}{3} \in (-1, 0)$. By taking $m(t) = \frac{2}{\sqrt{10}} t \in L^{5/2}\left(\overline{J}_0\right)$ and $f(t, y_t) = \frac{2}{5(t+\sqrt{10})^2} t^2 \sin(y_t)$. We have

$$w(t, y_t) = \frac{1}{\sqrt{10}} t \cos(y_t) \leq \frac{2}{\sqrt{10}} t = m(t),$$

for any $k \in \Upsilon_{2\pi}(\varepsilon)$ and for almost all $t \in \overline{J}_0$. Also, we have

$$\begin{aligned} |f(t, y_t) - f(t, z_t)| &= \left| \frac{2t^2 \sin y(t)}{5(t+\sqrt{10})^2} - \frac{2t^2 \sin z(t)}{5(t+\sqrt{10})^2} \right| \\ &\leq \left| \frac{2t^2}{5(t+\sqrt{10})^2} \right| |\sin y(t) - \sin z(t)| \leq \frac{2}{5} |y(t) - z(t)| \leq \ell \|y - z\|, \end{aligned}$$

here, $\ell = \frac{2}{5}$. *On the other hand, function* f *is completely continuous and for any bounded set* $\mathfrak{B} \in \Upsilon_{2\pi}(\varepsilon)$, *the set* $\{t \to f(t, y_t) : y \in \mathfrak{B}\}$, *is equicontinuous in* $C(\overline{J}_0, \mathbb{R}^n)$. *Therefore all hypothesis* (HE1), (HE2) *and* (HE3) *hold. Now, if* $r = \frac{12}{7}\pi \in (0, 2\pi)$ *then ,*

$$M = \|m(t)\|_{L^{1/\sigma_1}(\overline{J}_0)} = \|m(t)\|_{L^{5/2}(\overline{J}_0)} = \left[\int_0^{2\pi} \left|\frac{2}{\sqrt{10}}\xi\right|^{5/2} \mathrm{d}\xi\right]^{2/5} = 12.4841.$$

TABLE 1. Numerical results of η for three different values of $q = \frac{1}{5}, \frac{1}{2}, \frac{7}{8}$ and $\varepsilon = 0.95$ in Example 4.1.

	$q=\frac{1}{7}$			$q=\frac{1}{2}$			$q=\frac{7}{8}$		
n	$\Gamma_q(\sigma)$	Λ	η	$\Gamma_q(\sigma)$	Λ	η	$\Gamma_q(\sigma)$	Λ	η
1	1.0737	0.0080	0.0080	1.0759	0.0081	0.0081	0.8929	0.0051	0.0051
2	1.0770	0.0081	0.0081	1.0993	0.0085	0.0085	0.9448	0.0058	0.0058
3	1.0776	0.0081	0.0081	1.1103	0.0087	0.0087	0.9825	0.0064	0.0064
4	1.0777	0.0081	0.0081	1.1157	0.0088	0.0088	1.0113	0.0069	0.0069
5	1.0778	0.0081	0.0081	1.1183	0.0089	0.0089	1.0341	0.0073	0.0073
6	1.0778	0.0081	0.0081	1.1196	0.0089	0.0089	1.0525	0.0076	0.0076
7	1.0778	0.0081	0.0081	1.1203	0.0089	0.0089	1.0677	0.0079	0.0079
8	1.0778	0.0081	0.0081	1.1206	0.0089	0.0089	1.0802	0.0081	0.0081
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
29	1.0778	0.0081	0.0081	1.1209	0.0089	0.0089	1.1511	0.0095	0.0095
30	1.0778	0.0081	0.0081	1.1209	0.0089	0.0089	1.1516	0.0095	0.0095
31	1.0778	0.0081	0.0081	1.1209	0.0089	0.0089	1.1520	0.0095	0.0095
32	1.0778	0.0081	0.0081	1.1209	0.0089	0.0089	1.1524	0.0096	0.0096
33	1.0778	0.0081	0.0081	1.1209	0.0089	0.0089	1.1528	0.0096	0.0096

Now, we choose $\varepsilon = 6.45$ *then the exists* $r' = \frac{6\pi}{7}$, *such that from Eq.* (3.6), *we have*

$$\Lambda = \left[\frac{\varepsilon\Gamma_q(\sigma)(1+\beta)^{1-\sigma_1}}{3M}\right]^{1/(1+\beta)(1-\sigma_1)} \simeq \begin{cases} 0.0081, & q = \frac{1}{5}, \\ 0.0089, & q = \frac{1}{2}, \\ 0.0096, & q = \frac{7}{8}, \end{cases}$$

and so,

$$\begin{aligned} \eta &= \min\left\{r, r', \left[\frac{\varepsilon\Gamma_q(\sigma)(1+\beta)^{1-\sigma_1}}{3M}\right]^{1/(1+\beta)(1-\sigma_1)}\right\} \\ &= \min\left\{\frac{12\pi}{7}, \frac{6\pi}{7}, \left[\frac{6.45\Gamma_q\left(\frac{4}{5}\right)\left(\frac{2}{3}\right)^{3/5}}{3M}\right]^{5/2}\right\} \simeq \begin{cases} 0.0081, & q = \frac{1}{5}, \\ 0.0089, & q = \frac{1}{2}, \\ 0.0096, & q = \frac{7}{8}, \end{cases} \end{aligned}$$

Table 1 shows the results of η *for* $q = \frac{1}{5}, \frac{1}{2}, \frac{7}{8}$. *It nicely shows that the value of* η *when* q *is close to zero, in less repetitions, is obtained with four decimal places. As seen in Figure 2, with the data of the problem, Theorem 3.2 is valid for different values of* q. *Further, from the above facts, in view of Theorem 3.2, we conclude that Problem* (4.1) *has at least one solution on*

$$[0,\ 0.0081], \quad [0,\ 0.0089], \quad [0,\ 0.0096],$$

for $q = \frac{1}{5}, \frac{1}{2}, \frac{7}{8}$, *respectively, provided that* $y(t) = \dot{\theta}(0) + l(t)$.

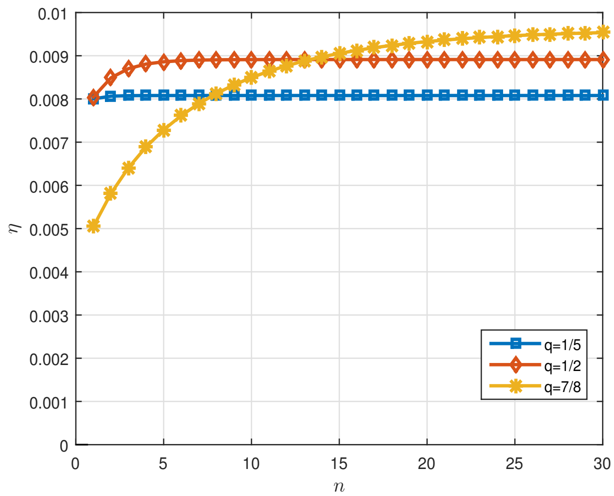


FIGURE 2. Graphical representation of η in Example 4.1 with different values of q.

In the next example, we discuss different values of order σ.

Example 4.2. *Consider the* $\mathbb{FNF}q\mathbb{DE}$ (4.1) *with different values of order* $\sigma \in \left\{\frac{8}{11}, \frac{9}{11}, \frac{10}{11}\right\}$ *and constant* $q = \frac{5}{9} \in (0,1)$ *as form:*

$$ {}^{c}\mathscr{D}^{\sigma}_{5/9}\left(y(t) - \frac{2t^2 \sin y(t)}{5\left(t+\sqrt{10}\right)^2}\right) = \frac{\cos y(t)}{\sqrt[3]{10t^2+1}}, \qquad t \in \overline{J}_0 = [0, 2\pi], \tag{4.2}$$

under bounded delay $y_0 = \dot{\theta} \in \mathscr{C}_0 = C([-\pi, 0], \mathbb{R}^n)$*. If we put*

$$\sigma_1 = \frac{7}{11} < \frac{8}{11} = \sigma, \quad \sigma_1 = \frac{8}{11} < \frac{9}{11} = \sigma, \quad \sigma_1 = \frac{9}{11} < \frac{10}{11} = \sigma,$$

then we have

$$\beta = \frac{\sigma - 1}{1 - \sigma_1} = \begin{cases} \frac{\frac{8}{11}-1}{1-\frac{7}{11}} = -\frac{3}{4}, & \sigma = \frac{1}{8}, \\ \frac{\frac{9}{11}-1}{1-\frac{8}{11}} = -\frac{2}{3}, & \sigma = \frac{1}{2}, \\ \frac{\frac{10}{11}-1}{1-\frac{9}{11}} = -\frac{1}{2}, & \sigma = \frac{10}{11}. \end{cases}$$

Clearly $m(t) = \frac{2}{\sqrt{10}}t$ *belongs to* $L^{1/\sigma_{i1}}(\overline{J}_0)$, $f(t, y_t) = \frac{2}{5(t+\sqrt{10})^2}t^2 \sin(y_t)$*, and according to Example 4.1, we have*

$$|f(t,y_t) - f(t,z_t)| = \left|\frac{t^2 \sin y(t)}{5(t+\sqrt{10})^2} - \frac{t^2 \sin z(t)}{5(t+\sqrt{10})^2}\right| \le \ell \|y - z\|,$$

where $\ell = \frac{2}{5}$*. On the other hand, function* f *is completely continuous and for any bounded set* $\mathfrak{B} \in \Upsilon_{2\pi}(\varepsilon)$*, the set* $\{t \to f(t, y_t) : y \in \mathfrak{B}\}$*, is equicontinuous in* $C(\overline{J}_0, \mathbb{R}^n)$*. Therefore all hypothesis* (HE1), (HE2) *and* (HE3) *hold. Now, if* $r = \frac{12}{7}\pi \in (0, 2\pi)$ *then by Eq.* (3.3)*, we obtain*

$$M = \|m(t)\|_{L^{1/\sigma_{i1}}(\overline{J}_0)} = \left[\int_0^{2\pi} \left|\frac{2}{\sqrt{10}}\xi\right|^{\sigma_{i1}} \mathrm{d}\xi\right]^{1/\sigma_{i1}} \simeq \begin{cases} 12.4841, & \sigma = \frac{8}{11}, \\ 12.4841, & \sigma = \frac{9}{11}, \\ 12.4841, & \sigma = \frac{10}{11}. \end{cases}$$

TABLE 2. Numerical results of η for three different values of $\sigma \in \{\frac{8}{11}, \frac{9}{11}, \frac{10}{11}\}$ when $q = \frac{4}{9}$ and $\varepsilon = 43$ in Example **??**.

	$\sigma = \frac{8}{11}$			$\sigma = \frac{9}{11}$			$\sigma = \frac{10}{11}$		
n	$\Gamma_q(\sigma)$	Λ	η	$\Gamma_q(\sigma)$	Λ	η	$\Gamma_q(\sigma)$	Λ	η
1	1.1104	0.0565	0.0565	1.0578	0.3140	0.3140	1.0224	1.4578	1.4578
2	1.1518	0.0845	0.0845	1.0831	0.4072	0.4072	1.0342	1.6530	1.6530
3	1.1733	0.1035	0.1035	1.0961	0.4644	0.4644	1.0402	1.7618	1.7618
4	1.1848	0.1153	0.1153	1.1031	0.4978	0.4978	1.0434	1.8224	1.8224
5	1.1911	0.1222	0.1222	1.1068	0.5169	0.5169	1.0451	1.8560	1.8560
6	1.1945	0.1261	0.1261	1.1089	0.5277	0.5277	1.0461	1.8747	1.8747
7	1.1964	0.1283	0.1283	1.1101	0.5337	0.5337	1.0466	1.8851	1.8851
8	1.1975	0.1296	0.1296	1.1107	0.5371	0.5371	1.0469	1.8909	1.8909
9	1.1981	0.1303	0.1303	1.1111	0.5390	0.5390	1.0470	1.8941	1.8941
10	1.1984	0.1307	0.1307	1.1112	0.5400	0.5400	1.0471	1.8959	1.8959
11	1.1986	0.1309	0.1309	1.1114	0.5406	0.5406	1.0472	1.8969	1.8969
12	1.1987	0.1310	0.1310	1.1114	0.5409	0.5409	1.0472	1.8975	1.8975
13	1.1987	0.1311	0.1311	1.1114	0.5411	0.5411	1.0472	1.8978	1.8978
14	1.1988	0.1311	0.1311	1.1115	0.5412	0.5412	1.0472	1.8979	1.8979
15	1.1988	0.1311	0.1311	1.1115	0.5412	0.5412	1.0472	1.8980	1.8980
16	1.1988	0.1312	$\underline{0.1312}$	1.1115	0.5413	$\underline{0.5413}$	1.0472	1.8981	$\underline{1.8981}$
17	1.1988	0.1312	0.1312	1.1115	0.5413	0.5413	1.0472	1.8981	1.8981
18	1.1988	0.1312	0.1312	1.1115	0.5413	0.5413	1.0472	1.8981	1.8981
19	1.1988	0.1312	0.1312	1.1115	0.5413	0.5413	1.0472	1.8981	1.8981

Now, we choose $\varepsilon = 43$ then the exists $r' = \frac{6\pi}{7}$, such that from Eq. (3.6), *we have*

$$\Lambda = \left[\frac{\varepsilon \Gamma_q(\sigma)(1+\beta)^{1-\sigma_1}}{3M}\right]^{1/(1+\beta)(1-\sigma_1)} \simeq \begin{cases} 0.1312, & \sigma = \frac{8}{11}, \\ 0.5413, & \sigma = \frac{9}{11}, \\ 1.8981, & \sigma = \frac{10}{11}. \end{cases}$$

and so,

$$\eta = \min\left\{r, r', \left[\frac{\varepsilon \Gamma_q(\sigma)(1+\beta)^{1-\sigma_1}}{3M}\right]^{1/(1+\beta)(1-\sigma_1)}\right\} \simeq \begin{cases} 0.1312, & \sigma = \frac{8}{11}, \\ 0.5413, & \sigma = \frac{9}{11}, \\ 1.8981, & \sigma = \frac{10}{11}. \end{cases}$$

Table 2 shows the results of η for fifferent values of order $\sigma = \frac{8}{11}, \frac{9}{11}, \frac{10}{11}$ when $q = \frac{4}{9}$. In fact, if the values of the order σ tend to zero, the values of η also decrease. As seen in Figure 3, with the data of the problem (4.2)*, Theorem 3.2 is valid for different values of σ. Thus, Theorem 3.2 implies that Problem* (4.2) *has at least one solution on*

$$[0,\ 0.1312], \quad [0,\ 0.5413], \quad [0,\ 1.8981],$$

for $\sigma = \frac{8}{11}, \frac{9}{11}, \frac{10}{11}$, respectively, provided that $y(t) = \dot{\theta}(0) + l(t)$.

In the next interesting example, we present the obtained results for a system of $\mathbb{FNF}q\mathbb{DE}$.

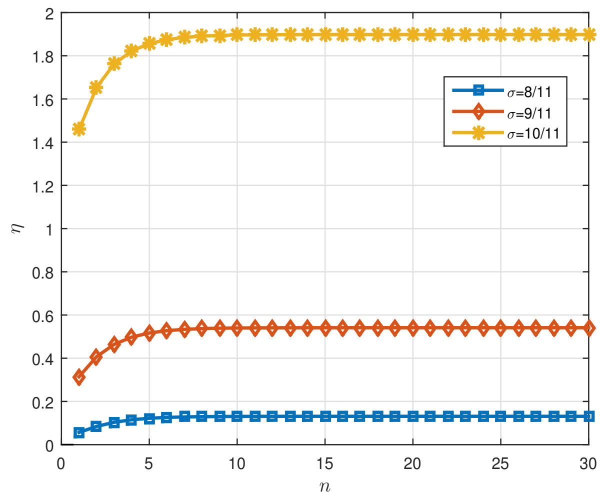


FIGURE 3. Graphical representation of η in Example 4.1 with different values of order σ.

Example 4.3. *Consider the follow 3-dimensional system of $\mathbb{FNF}q\mathbb{DE}$:*

$$(4.3)\qquad \begin{cases} {}^{c}\mathscr{D}_q^{5/6}\left(y_1(t)-\frac{5|t||\mathbf{y}_t|}{8(|t|+5)}\right)=\frac{t\sin^2(\mathbf{y}_t)}{\sqrt[3]{3t^2}+\sqrt[3]{5}},\\[2mm] {}^{c}\mathscr{D}_q^{3/5}\left(y_2(t)-\frac{2\exp(-t^2)|\mathbf{y}_t|}{9(|\mathbf{y}_t|+2)}\right)=\frac{|t|\cos(\mathbf{y}_t)}{10|t|+4.25},\\[2mm] {}^{c}\mathscr{D}_q^{8/9}\left(y_3(t)-\frac{\sqrt[3]{t}\sin^2(\mathbf{y}_t)}{10\left(\sin^2(\mathbf{y}_t)+1\right)}\right)=\frac{t\,|\mathbf{y}_t|}{\exp(t^2)|\mathbf{y}_t|+2}, \end{cases}$$

for $t\in\left[0,\frac{\pi}{2}\right]$ *under infinite delay* $y_{1_0}=\dot{\theta}_1$, $y_{2_0}=\dot{\theta}_2$, $y_{3_0}=\dot{\theta}_3\in\mathscr{C}_0=C\left(\left[-\frac{\pi}{2},0\right],\mathbb{R}^n\right)$. *Clearly* $\sigma_1=\frac{5}{6}$, $\sigma_2=\frac{3}{5}$, $\sigma_3=\frac{8}{9}$ *belong to* $(0,1)$, $\delta=\frac{\pi}{2}$, $\vartheta_0=0$, $\tau=\pi$, $\overline{J}_0=[0,\pi]$ *and*

$$\Upsilon_r(\varepsilon)=\left\{(y_1,y_2,y_3)\,\middle|\,y_{i_0}=\dot{\theta}_i,\ \sup_{t\in\overline{J}_0}\left|y_i(t)-\dot{\theta}_i(0)\right|\leq\varepsilon\ \text{for } i=1,2,3\right\}.$$

Also, the functions $f_i, w_i:[0,\infty)\times\mathscr{C}_0\to\mathbb{R}^n$ *define by*

$$f_1(t,\mathbf{y}_t)=\frac{5|t||\mathbf{y}_t|}{3t^2+1},\quad f_2(t,\mathbf{y}_t)=\frac{2\exp(-t^2)|\mathbf{y}_t|}{9(|\mathbf{y}_t|+2)},\quad f_3(t,\mathbf{y}_t)=\frac{\sqrt[3]{t}\sin^2(\mathbf{y}_t)}{10(\sin^2(\mathbf{y}_t)+1)},$$

$$w_1(t,\rho)=\frac{t\sin^2(\rho(t))}{\sqrt[3]{3t^2}+\sqrt[3]{5}},\quad w_2(t,\rho)=\frac{|t|\cos(\rho(t))}{10|t|+4.25},\quad w_3(t,\rho)=\frac{t\,|\rho(t)|}{\exp(t^2)|\mathbf{y}_t|+2},$$

and

$$y_{i_t}(u)=y_i(t+u),\qquad \forall u\in\left[-\frac{\pi}{2},\,0\right],\ i=1,2,3.$$

One can see that $w_i(t,\rho(t))$ *is measurable with respect to* $t\in\overline{J}_0$ *and is continuous with respect to* $\rho(t)$ *in* $\mathscr{C}_0$. *If put* $\sigma_{11}=\frac{2}{3}<\frac{5}{6}=\sigma_1$, $\sigma_{21}=\frac{2}{5}<\frac{3}{5}=\sigma_2$, $\sigma_{31}=\frac{7}{9}<\frac{8}{9}=\sigma_3$ *then*

$$\beta_1=\frac{\frac{5}{6}-1}{1-\frac{2}{3}}=-\frac{1}{2},\quad \beta_2=\frac{\frac{3}{5}-1}{1-\frac{2}{5}}=-\frac{2}{3},\quad \beta_3=\frac{\frac{8}{9}-1}{1-\frac{7}{9}}=-\frac{1}{2},$$

respectively. By taking

$$m_1(t) = \frac{t}{\sqrt[3]{5}} \in L^{2/3}(\bar{J}_0), \quad m_2(t) = \frac{|t|}{4.25} \in L^{5/2}(\bar{J}_0), \quad m_3(t) = \frac{t}{2} \in L^{9/7}(\bar{J}_0),$$

and

$$f_{11}(t,\mathbf{y}_t) = f_{12}(t,\mathbf{y}_t) = \frac{5|t||\mathbf{y}_t|}{16(|t|+5)},$$
$$f_{21}(t,\mathbf{y}_t) = f_{22}(t,\mathbf{y}_t) = \frac{\exp(-t^2)|\mathbf{y}_t|}{9(|\mathbf{y}_t|+2)},$$
$$f_{31}(t,\mathbf{y}_t) = f_{32}(t,\mathbf{y}_t) = \frac{\sqrt[3]{t}\,\sin^2(\mathbf{y}_t)}{20\left(\sin^2(\mathbf{y}_t)+1\right)},$$

we have

$$w_1(t,\mathbf{y}_t) = \frac{t\,\sin^2(\mathbf{y}_t)}{\sqrt[3]{3t^2+1}} \le \frac{t}{\sqrt[3]{5}} = m_1(t),$$
$$w_2(t,\mathbf{y}_t) = \frac{|t|\cos(\mathbf{y}_t)}{10|t|+4.25} \le \frac{|t|}{4.25} = m_2(t),$$
$$w_3(t,\mathbf{y}_t) = \frac{|\mathbf{y}_t|}{\exp(t^2)t\,|\mathbf{y}_t|+2} \le \frac{t}{2} = m_3(t),$$

for $\mathbf{y} \in \Upsilon_\pi(\varepsilon)$ *and for almost all* $t \in [0,\pi]$. *Let* $y,z \in \Upsilon_\pi(\varepsilon)$, *then we have*

$$\begin{aligned}|f_{11}(t,y_t) - f_{11}(t,z_t)| &= \left|\frac{5|t||y(t)|}{16(|t|+5)} - \frac{5|t||z(t)|}{16(|t|+5)}\right| \\ &\le \frac{5|t|}{16(|t|+5)}\,|y(t)-z(t)| \le \frac{5}{16}\,|y(t)-z(t)| \le \ell_1\|y-z\|,\end{aligned}$$

$$\begin{aligned}|f_{21}(t,y_t) - f_{21}(t,z_t)| &= \left|\frac{\exp(-t^2)|y(t)|}{9(|y(t)|+2)} - \frac{\exp(-t^2)|z(t)|}{9(|z(t)|+2)}\right| \\ &\le \frac{2\exp(-t^2)}{9}\,|y(t)-z(t)| \le \frac{2}{9}\,|y(t)-z(t)| \le \ell_2\|y-z\|,\end{aligned}$$

$$\begin{aligned}|f_{31}(t,y_t) - f_{31}(t,z_t)| &= \left|\frac{\sqrt[3]{t}\,\sin^2 y(t)}{20(\sin^2 y(t)+1)} - \frac{\sqrt[3]{t}\,\sin^2 z(t)}{20(\sin^2 z(t)+1)}\right| \\ &= \left|\frac{\sqrt[3]{t}\,\sin^2 y(t) - \sqrt[3]{t}\,\sin^2 z(t)}{20(\sin^2 y(t)+1)(\sin^2 z(t)+1)}\right| \\ &\le \left|\frac{\sqrt[3]{t}}{20}\right| |\sin y(t) - \sin z(t)|\,|\sin y(t) + \sin z(t)| \\ &\le \frac{2\sqrt[3]{t}}{20}\,|y(t)-z(t)| \le \frac{2\sqrt[3]{\pi}}{20}\,|y(t)-z(t)| \le \ell_3\|y-z\|.\end{aligned}$$

So, $\ell_1 = \frac{5}{16}$, $\ell_2 = \frac{2}{9}$, $\ell_3 = \frac{2\sqrt[3]{\pi}}{20}$. *On the other hand, function* f_2 *is completely continuous and for any bounded set* $\mathfrak{B} \in \Upsilon_\pi(\varepsilon)$, *the set* $\{t \to f_{i2}(t,\mathbf{y}_t) : y \in \mathfrak{B}\}$ *for* $i = 1,2,3$, *are equicontinuous in* $C(\bar{J}_0,\mathbb{R}^n)$. *Therefore* *all hypothesis* (HS1), (HS2) *and* (HS3) *hold. Now, if* $r = \frac{5\pi}{9} \in (0,\pi)$ *then by using Eq.* (3.3)*, we obtain*

$$M_1 = \left(\int_0^{\pi} \left| \frac{\xi}{\sqrt[3]{5}} \right|^{3/2} \mathrm{d}\xi \right)^{2/3} = 2.8858,$$

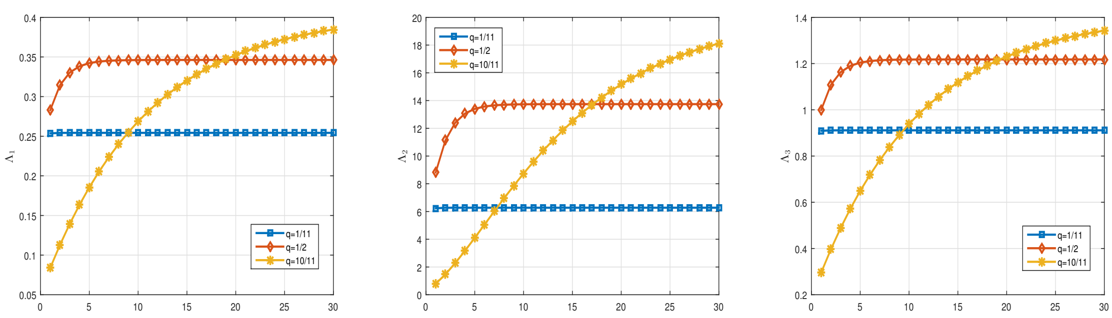
$$M_2 = \left(\int_0^{\pi} \left| \frac{\xi}{4.25} \right|^{2/5} \mathrm{d}\xi \right)^{5/2} = 1.1611,$$

$$M_3 = \left(\int_0^{\pi} \left| \frac{\xi}{2} \right|^{9/7} \mathrm{d}\xi \right)^{7/9} = 2.4674.$$

Now, we can choose $\varepsilon = 16.5 \leq \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$*. Then there exists* $r' = \frac{\pi}{5}$ *such that from Eq.* (3.9)*, we have*

$$\eta = \min \left\{ r, r', \left[\frac{\varepsilon \Gamma_{q_i}(\sigma_1)(1+\beta_1)^{1-\sigma_{11}}}{3M_1} \right]^{1/(1+\beta_1)(1-\sigma_{11})}, \left[\frac{\varepsilon \Gamma_{q_i}(\sigma_2)(1+\beta_2)^{1-\sigma_{21}}}{3M_2} \right]^{1/(1+\beta_2)(1-\sigma_{21})}, \left[\frac{\varepsilon \Gamma_{q_i}(\sigma_3)(1+\beta_3)^{1-\sigma_{31}}}{3M_3} \right]^{1/(1+\beta_3)(1-\sigma_{31})} \right\} \simeq \begin{cases} 0.2542, & q = \frac{1}{11}, \\ 0.3462, & q = \frac{1}{2}, \\ 0.4049, & q = \frac{10}{11}, \end{cases}$$

Figures 4a*,* 4b*,* 4c *and* 5 *show numerical results of* Λ_i*,* $i = 1,2,3$ *and* η *for* $q = \frac{1}{11}, \frac{1}{2}, \frac{10}{11}$ *respectively. Also,*



(A) $q = \frac{1}{11}$ (B) $q = \frac{1}{2}$ (C) $q = \frac{10}{11}$

FIGURE 4. Graphical representation of Λ_i, $i = 1,2,3$ for $q = \frac{1}{11}, \frac{1}{2}, \frac{10}{11}$ in Example 4.3.

Table 3 *shows these results (See MatLab lines* 1*). Further, from the above facts, in view of Theorem* 3.6*, we conclude that the system* (4.3) *has at least one solution on*

$$[0,\ 0.2542], \quad [0,\ 0.3462], \quad [0,\ 0.4049],$$

for different values of q*, respectively.*

5. Conclusion

In this paper, we first gave some properties of the fractional q–derivative and integral, and then using the proposed properties we established the existence of sultions for the single and multi-dimensional fractional neutral functional q–differential equations (1.1) and (1.2) on a time scale and show the perfect numerical

TABLE 3. Numerical results of Λ_i, $i = 1,2,3$ and η for three different values of $q \in \left\{\frac{1}{11}, \frac{1}{2}, \frac{10}{11}\right\}$ when $\varepsilon = 8.35$ in Example 4.3.

	$q=\frac{1}{11}$				$q=\frac{1}{2}$				$q=\frac{10}{11}$			
n	Λ_1	Λ_2	Λ_3	η	Λ_1	Λ_2	Λ_3	η	Λ_1	Λ_2	Λ_3	η
1	0.254	6.232	0.909	0.254	0.283	8.819	0.999	0.283	0.084	0.789	0.296	0.084
2	0.254	6.270	0.911	0.254	0.315	11.139	1.108	0.315	0.113	1.473	0.397	0.113
3	0.254	6.274	0.911	0.254	0.330	12.406	1.163	0.330	0.139	2.281	0.489	0.139
4	0.254	6.274	0.911	0.254	0.338	13.065	1.190	0.338	0.164	3.169	0.573	0.164
5	0.254	6.274	0.911	0.254	0.342	13.402	1.204	0.342	0.185	4.103	0.649	0.185
6	0.254	6.274	0.911	0.254	0.344	13.571	1.211	0.344	0.205	5.057	0.719	0.205
7	0.254	6.274	0.911	0.254	0.345	13.657	1.214	0.345	0.224	6.012	0.782	0.224
8	0.254	6.274	0.911	0.254	0.346	13.700	1.216	0.346	0.240	6.951	0.839	0.240
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
68	0.254	6.274	0.911	0.254	0.346	13.742	1.218	0.346	0.404	20.044	1.411	0.404
69	0.254	6.274	0.911	0.254	0.346	13.742	1.218	0.346	0.404	20.049	1.412	0.404
70	0.254	6.274	0.911	0.254	0.346	13.742	1.218	0.346	0.405	20.053	1.412	0.405
71	0.254	6.274	0.911	0.254	0.346	13.742	1.218	0.346	0.405	20.057	1.412	0.405
72	0.254	6.274	0.911	0.254	0.346	13.742	1.218	0.346	0.405	20.061	1.412	0.405
73	0.254	6.274	0.911	0.254	0.346	13.742	1.218	0.346	0.405	20.065	1.412	0.405

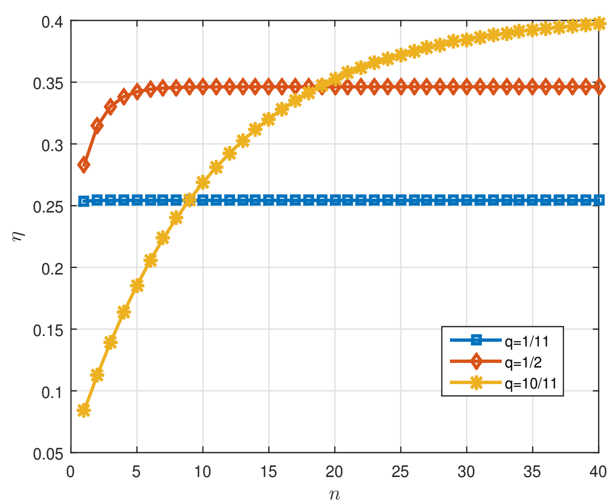


FIGURE 5. Graphical representation of η in Example 4.3.

effects for the problem which it confirmed our results. Compared to published articles in the literature, this article has new and numerical computational techniques.

Declarations

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Competing interests: The authors declare that they have no competing interests.

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Authors' contributions: **MES**: Actualization, methodology, formal analysis, validation, investigation, software, simulation, initial draft and was a major contributor in writing the manuscript. **AAI**: Actualization and initial draft. **MKAK**: Actualization, methodology, validation, investigation, initial draft, formal analysis, and supervision of the original draft and editing. **ZS**: Actualization, validation, methodology, formal analysis, investigation, and initial draft. **JA**: Actualization, validation, methodology, formal analysis, investigation, and initial draft. **AAT**: Actualization, validation, methodology, formal analysis, investigation, and initial draft. **MK**: Actualization, validation, methodology, formal analysis, investigation, and initial draft. All authors read and approved the final manuscript.

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Supporting Information

Algorithm 1: MATLAB lines for calculating the numerical results of variable in the multi-dimensional problem (1.2)

```
clear;
format long;
syms v;
q=[1/5 1/2 7/8];
epsilon=6.45;
tau=pi;
sigma=4/5;
sigma_1=2/5;
beta=(sigma-1)/(1-sigma_1);
M=eval(int( ((abs(2*v/sqrt(10)))^(1/sigma_1))^(sigma_1),v,0,2*tau));
r=12*pi/7;
rprime=8*pi/7;
column=1;
for i=1:3
    for n=1:120
        Parammatrix(n,column)=n;
        G1=qGamma(q(i), sigma, n);
        Parammatrix(n,column+1)=G1;
        Lambda=((epsilon*G1...
            *(1+beta)^(1-sigma_1))/(3*M))^(1/((1+beta)*(1-sigma_1)));
        Parammatrix(n,column+2)=Lambda;
        eta=min(min(r,rprime),Lambda);
```

```
            Parammatrix(n,column+3)=eta;
        end;
        column=column+4;
    end;
```

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