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LOCALIZATION OF ZEROS OF POLAR POLYNOMIALS ON THE UNIT DISK

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ABSTRACT. In the paper we derive useful result about the zeros of the k-polar polynomials on the unit circle, in particular we obtain a circular region containing all the zeros of these polynomials. Some examples are presented.

1. Introduction

Finding the roots of polynomials is a problem of interest in both mathematics and in areas of application such as physical systems, which can be reduced to solving certain equation. There are very interesting geometric relationships between the roots of a polynomial $f_n(z)$ and those of $f_n'(z)$. The most important result is the following.

Theorem 1.1 (The Gauß-Lucas theorem [12]). *Let $f_n(z) \in \mathbb{C}[z]$ be a polynomial of degree at least one. All zeros of $f_n'(z)$ lie in the convex hull of the zeros of the zeros of $f_n(z)$.*

The location of zeros, or critical points, of polynomials has many physical and geometrical interpretations. For example, F. Gauß in 1816 showed that the roots of $f_n'(z)$ are the positions of equilibrium in the field of force due to equal particles situated at each root of $f_n(z)$, if each particle repels with a force equal to the inverse distance being the inverse distance law.

Many results exist concerning the location of the zeros of a polynomial of a complex variable as a function of the coefficients of the polynomial. One is the well-known Eneström-Kakeya theorem, [19]. Another one, useful to obtain more precise about the zeros of a polynomial, was obtained by J. H. Grace.

Theorem 1.2 (The Grace's theorem [9]). *Let $a(z)$ and $b(z)$ the following polynomials:*

$$a(z) = \sum_{\ell=0}^{n} a_\ell \binom{n}{\ell} z^\ell, \quad b(z) = \sum_{\ell=0}^{n} b_\ell \binom{n}{\ell} z^\ell.$$

If the zeros of both polynomials lie in the unit disk, then the zeros of the "composition" of the two

$$c(z) = \sum_{\ell=0}^{n} a_\ell b_\ell \binom{n}{\ell} z^\ell,$$

also lie in the unit disk.

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The zeros of orthogonal polynomials has been an intensively studied subject since the beginning of the 20th century, and several breakthroughs have been made in the recent years. In particular, Barry Simon proved in [19] that if is μ a finite positive measure defined on the Borelian σ-algebra of $\mathbb{C}$, μ is absolutely continuous with respect to the Lebesgue measure $d\theta/(2\pi)$ on $[-\pi,\pi]$; and $(L_n(z))$ is the system of monic orthogonal polynomials with respect to μ, i.e.

$$\int_{-\pi}^{\pi} L_n(z) z^{-j}\, d\mu(\theta) = 0, \quad j = 0,1,\ldots,n-1, \tag{1}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} L_n(z)\overline{L_n(z)}\, d\mu(\theta) = \|L_n\|^2 \neq 0, \quad n = 0,1,\ldots,$$

where $z = \exp(i\theta)$ and $d\mu(\theta) = \rho(\theta)d\theta$ with $\rho \in L^1([-\pi,\pi], d\theta)$, is a measure supported on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

Then, all the zeros of $L_n(z)$ are contained in the open unit disk $\mathbb{D} = \{z : |z| < 1\}$

In this contribution we are going to introduce monic k-polar polynomials.

Definition 1.3. *Let μ be a finite measure defined on the Borelian σ-algebra of $\mathbb{C}$ such that it contains an infinite number of points and let $(L_n(z))$ the system of monic orthogonal polynomials with respect to μ. Let ξ be a fixed complex number. Let k be a positive integer, the k-polar polynomial related to μ, which will be denoted by $Q_{n;k}(z;\xi)$, is the polynomial solution of degree n of the k-th order linear differential equation*

$$\frac{d^k}{dz^k}(z-\xi)^k P(z) = (n+1)\cdots(n+k)L_n(z).$$

Remark 1.4. *Notice that, by construction, $Q_{n;0}(z) = L_n(z)$ for all n.*

In the last years some attention has been paid to the so called polar orthogonal polynomials. Fundora, Pijeira, and Urbina [6] have studied 1-polar orthogonal polynomial sequences associated with a measure supported on the segment. A similar study has been done by Pijeira, Bello, and Urbina [18], in the case of 1-polar Legendre polynomials.

The main result in this paper is to study the zero location of k-polar orthogonal polynomials on the unit circle, in short OPUC, with respect to a generic measure μ.

The structure of this work is the following: in Section 2 we present some preliminaries and basic results we need to obtain the main result. In Section 3 we state the main result of this work about the location of the zeros of polar polynomials. In Section 4 we present three interesting examples of k-polar orthogonal polynomials on the unit circle. Since extensive calculations indicate that these polynomials often have complex zeros and there exists a ring-shaped region containing all the zeros of polar orthogonal we also present in this Section numerical calculations to see if Sendov's conjecture [14, p. 267] holds true or not for such examples.

2. Preliminaries

Given a complex number $z_0 \in \mathbb{C}$ and a radius $r > 0$, we define the open disk

$$D(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\},$$

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the closed disk
$$\overline{D(z_0,r)} := \{z \in \mathbb{C} : |z - z_0| \le r\},$$
and the circle
$$\partial D(z_0,r) := \{z \in \mathbb{C} : |z - z_0| = r\}.$$
Let μ be a finite positive Borel measure that is absolutely continuous with respect to the Lebesgue measure $d\theta/(2\pi)$ on $[-\pi,\pi]$. Let $(L_n(z))$ be the system of monic polynomials orthogonal with respect to μ. It is known that the following mutually equivalent recursions relations hold for these polynomials:
$$L_n(z) = zL_{n-1}(z) + L_n(0)L^*_{n-1}(z),$$
$$L^*_n(z) = L^*_{n-1}(z) + z\overline{L_n(0)}L_{n-1}(z),$$
where
$$L^*_n(z) = z^n\overline{L_n(1/\bar{z})} = 1 + z\sum_{\ell=0}^{n-1}\overline{L_{\ell+1}(0)}L_\ell(z), \quad z \neq 0. \tag{2}$$
Moreover, for $|z| = 1$, we have
$$\left|\frac{L^*_{n+1}(z)}{L^*_n(z)} - 1\right| = \left|\frac{L_{n+1}(z)}{L_n(z)} - z\right| = |L_{n+1}(0)|, \quad n = 0,1,\ldots \tag{3}$$
for more details about these former identities see [11, 17, 19, 20, 21].

Along this work we are going to deal with the zeros of polynomials with complex coefficients, therefore it is convenient to state the following results.

Theorem 2.1 (Szegő's theorem [20], [2])**.** *Let $a(z)$, $b(z)$ and $c(z)$ the polynomials defined in Theorem 1.2. If all the zeros of $a(z)$ lie in a closed disk $\overline{D}$ and λ_1, ..., λ_n are the zeros of $b(z)$, then every zero of $c(z)$ has the form $\lambda_\ell\gamma_\ell$, where $\gamma_\ell \in \overline{D}$.*

Lemma 2.2 (Cauchy's. Theorem (27,2) in [13])**.** *If $P(z) = a_nz^n + \cdots + a_1z + a_0$ is a complex polynomial of degree at least one, then all the zeros of P lie in a closed circle*
$$|z| \le 1 + A,$$
where $A = \max\{|a_0|,\ldots,|a_{n-1}|\}/|a_n|$.

Lemma 2.3 (Datt and Govil [3])**.** *If $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ is a complex polynomial of degree at least one, then all the zeros of P lie in a ring shaped region*
$$\frac{|a_0|}{2(1+B)^{n-1}(1+nB)} \le |z| \le 1 + \lambda_0 B,$$
where $B = \max\{|a_0|,\ldots,|a_{n-1}|\}$, and λ_0 is the unique root of the equation $(x-1)(1+Bx)^n + 1 = 0$ in the interval $[0,1]$.

For more information about inequalities that satisfy the zeros of complex polynomials, read the survey [16] and the references therein.

Recall that the convex hull of a set G is the smallest convex set containing G, that is, the intersection of all closed containing G.

28 Mar 2023 01:54:06 PDT
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Sendov's conjecture [15, p. 267] asserts that if a polynomial $f(z)$ of degree $n \geq 2$ has all of its zeros in $\mathbb{D}$, then for each such zero z_0 there is a zero of the derivative $f'(z)$ in the $\overline{D(z_0,1)}$.

We are going to handle with k-polar polynomials. In the $k=1$ case it is straightforward to obtain

$$(n+1)\int_{\xi}^{z} L_n(t)dt = (z-\xi)Q_{n;1}(z;\xi), \tag{4}$$

therefore it is logical to call $Q_{n;1}(z;\xi)$ the n-th first order polar polynomial of $L_n(z)$ (see [6, 18]). As a consequence of (4) we obtain

$$(n+1)L_n(z) = Q_{n;1}(z;\xi) + (z-\xi)Q'_{n;1}(z;\xi). \tag{5}$$

Remark 2.4. *Observe that, by construction, $Q_{n;1}(z;\xi)$ is a monic polynomial of degree n and the pole of this polynomial is not irregular. In fact,*

$$\lim_{z\to\xi} Q_{n;1}(z;\xi) = \lim_{z\to\xi} \frac{(n+1)\int_{\xi}^{z} L_n(t)dt}{z-\xi} = (n+1)L_n(\xi).$$

Analogous calculations can be done in order to see that the k-polar monic polynomial $Q_{n;k}(z;\xi)$ has degree n and the pole of this polynomial is not irregular.

3. Localization of zeros of polar polynomials

Our next propose is to prove that all the zeros of the k-polar monic polynomial $Q_{n;k}(z;\xi)$ are contained in a disk which radius is independent of n. First, let us express the polynomials $L_n(z)$ and $Q_{n;k}(z;\xi)$ in terms of power of $z-\xi$, that is

$$L_n(z) = \sum_{\ell=0}^{n} a_{n,\ell}(z-\xi)^{\ell}, \quad Q_{n;k}(z) = \sum_{\ell=0}^{n} b_{n,\ell;k}(z-\xi)^{\ell}, \quad k=1,2,..., \tag{6}$$

where $a_{n,n} = b_{n,n;k} = 1$ for all $k = 1,2,...$

Lemma 3.1. *Let k be a positive integer. Let $\xi \in \mathbb{C}$. The coefficients of $L_n(z)$ and $Q_{n;k}(z;\xi)$ are fulfill the following relation:*

$$b_{n,\ell;k} = \frac{(n+k)\cdots(n+1)}{(\ell+k)\cdots(\ell+1)} a_{n,\ell}, \quad \ell = 0,1,...,n-1. \tag{7}$$

Proof. By Definition 1.3 we have

$$\frac{d^k}{dz^k}(z-\xi)^k Q_{n;k}(z;\xi) = (n+k)\cdots(n+1)L_n(z). \tag{8}$$

Start with (8), use the power expansion (6) and taking into account the linearity of the derivative. If we compare the power coefficients in these expressions the result holds. □

By using this result we obtain the first main result.

Theorem 3.2. *Let μ be a finite measure defined on the Borelian σ-algebra of $\mathbb{C}$ such that it contains an infinite number of points and let $(L_n(z))$ the system of monic orthogonal polynomials with respect to μ. Let ξ be a fixed complex number. Let k be a positive integer.*

All the zeros of $Q_{n;k}(z;\xi)$ are contained in the closed disk $\overline{D(0,|\xi|+(k+1)(1+|\xi|)}$.

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Proof. Since L_n is OPUC, then by [19] we know the zeros of $L_n(z)$ lie in $\mathbb{D}$. let us define $\omega := z-\xi$ and, taking into account Lemma 3.1, let us consider the following polynomials:

$$f_n(\omega) = L_n(z) = \sum_{\ell=0}^{n} \frac{a_{n,\ell}}{\binom{n}{\ell}} \binom{n}{\ell} \omega^\ell,$$

and

$$g_{n;k}(\omega) = \sum_{\ell=0}^{n} \frac{(n+k)\cdots(n+1)}{(\ell+k)\cdots(\ell+1)} \binom{n}{\ell} \omega^\ell = \sum_{\ell=0}^{n} \binom{n+k}{\ell+k} \omega^\ell = \frac{(n+k)\cdots(k+1)}{n!} F(-n,1;k+1;-\omega),$$

where $F(a,b;c;z)$ is the Gauß function [4, 15.2.2].

The "composition" of f_n and g_n leads to

$$h_{n;k}(\omega) = \sum_{\ell=0}^{n} \frac{b_{n,\ell;k}}{\binom{n}{\ell}} \binom{n}{\ell} \omega^\ell = Q_{n;k}(z;\xi).$$

Due to Theorem 3.2 in [5] we know that $g_{n;k}(z)$ has non-real zeros for n even and $n-1$ non-real zeros for n odd. Moreover, it is known this function is defined for $|\omega| < 1$.

Remark 3.3. *Observe that by using* [1, Corollary 2] *we obtain*

$$(1-z)^{n+k} F(n+k+1,k;k+1;z) = F(-n,1;k+1;z) = \frac{n!}{(k+1)\cdots(k+n)} P_n^{(k,-k-n)}(1-2z), \tag{9}$$

where $P_n^{(\alpha,\beta)}(z)$ is the Jacobi polynomial of degree n and parameters α and β (see c.f. [4, 18.5.7] *we can express the polynomial $g_{n;k}(z)$ in terms of the Jacobi polynomials.*

Remark 3.4. *Notice that if we take the k-th derivative of $\omega^k g_{n;k}(\omega)$ we obtain*

$$\frac{d^k}{d\omega^k} \omega^k g_{n;k}(\omega) = \sum_{\ell=0}^{n} (n+k)\cdots(n+1) \binom{n}{\ell} \omega^\ell = (n+k)\cdots(n+1)(1+\omega)^n. \tag{10}$$

Therefore for one side we know that if $z_{n,0}$ is a zero of $L_n(z)$ then $|z_{n,0}| \le 1$, so $|w_{n,0}| \le 1+|\xi|$. On the other hand, since we know that if $w_{n,1}$ is a root of $g_{n;k}(z)$ then $|w_{n,1}| \le R_{n,k}$. With these two inequalities we can claim that, using Szegő's Theorem, for any root of $h_{n;k}(z)$, namely $|z_{n,3}|$, and since $|\omega_{n,3}| = |z_{n,3}+\xi|$, the following inequality holds:

$$|z_{n,3}| \le (1+|\xi|)R_{n,k} + |\xi|. \tag{11}$$

Since $P_1^{(k,-k-n)}(1+\omega) = \omega+k+1$ and the zeros of the Jacobi polynomials $P_n^{(k,-k-n)}(1+\omega)$ tend to the circle $\overline{D(-1,1)}$ when $n \to \infty$ (see [10]), we can assume $R_{n,k} \le k+1$. Hence the result follows. □

4. The examples

Along this section we will consider different examples which let us to explain why sometimes we can consider a ring shape region (strictly speaking) where the zeros are located in, and some other situations we can not. Of course the region depends on different parameters such as the value ξ, as well as the integers k and n, among others. The examples we are going to consider can be consider as canonical.

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4.1. ***First example. The Bernstein-Szegő Polynomials.*** Let $\beta \in \mathbb{D}$, and let us consider the measure

$$d\mu(z) = \frac{1}{|z+\beta|^2}\frac{d\theta}{2\pi}. \tag{12}$$

Notice that if $\beta = r\exp(-i\phi)$, then (12) becomes [19, (1.6.2)]

$$d\mu(z) = P_r(\theta,\phi)\frac{d\theta}{2\pi},$$

where P_r is the Poisson kernel of

$$\int \Re(g(z))z^{-n}\frac{d\theta}{2\pi}.$$

Let $(L_n(z))$ be the monic orthogonal polynomials with respect to μ. These polynomials can be expressed as follows [7, 8]:

$$L_n(z) = z^n + \beta z^{n-1}, \quad n = 1,2,\ldots. \tag{13}$$

From this expression their first order monic polar polynomials are defined as:

$$Q_{n;1}(z;\xi) = (n+1)\frac{\int_\xi^z L_n(t)dt}{z-\xi} = \frac{z^n(nz+(n+1)\beta) - \xi^n(n\xi+(n+1)\beta)}{n(z-\xi)}. \tag{14}$$

Moreover, the second order monic polar polynomial of degree n is

$$Q_{n;2}(z;\xi) = \frac{z^{n+1}(nz+\beta n+2\beta) + \xi^n\left(n(n+1)\xi^2 + n(n+2)\xi(\beta - z) - \beta(n+1)(n+2)z\right)}{n(n+1)(n+2)(z-\xi)^2}. \tag{15}$$

In the figures 1 and 2 we show the zeros of these polynomials under different settings (in the first case $|\xi_1| + 2(1+|\xi_1|) = 3$, and $|\xi_2| + 3(1+|\xi_2|) = 13/3$ in the second one).

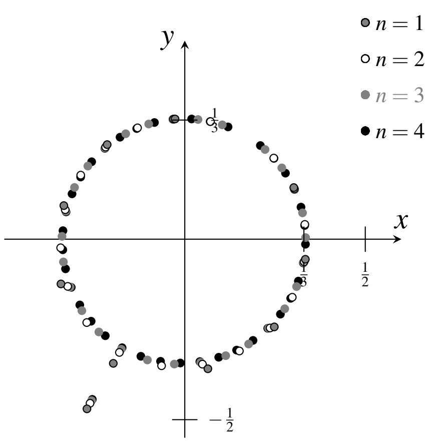

FIGURE 1. Zeros of $Q_{10n;1}(z;1/3\exp(i\pi/3))$ for $n = 1,2,3,4$ in the window $[-0.5,0.7]\times[-0.5,0.7]$, with $\beta = 1/2\exp(i\pi/3)$.

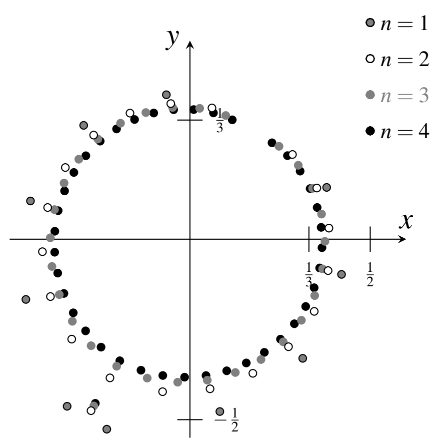

FIGURE 2. Zeros of $Q_{10n;2}(z;1/3\exp(i\pi/3))$ for $n = 1,2,3,4$ in the window $[-0.5,0.7]\times[-0.5,0.7]$, with $\beta = 1/2\exp(i\pi/3)$.

28 Mar 2023 01:54:06 PDT
230118-Costas Version 3 - Submitted to Rocky Mountain J. Math.

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In order to show the Sendov's conjecture remains valid in the table 1 we present the maximum distance between each zero of $Q_{n,1}(z,\xi)$ and the closest zero of $Q'_{n,1}(z,\xi)$ for different values of n.

n	zero	distance
2	$-0.04549-0.59920i$	0.2602
3	$0.25331-0.28868i$	0.41997
4	$0.32559-0.13863i$	0.65268
5	$0.34111-0.04380i$	0.68222
6	$0.33895+0.01995i$	0.74277
7	$0.33098+0.06499i$	0.75514
8	$0.32137+0.09813i$	0.77831
9	$-0.04905+0.33164i$	0.78565
10	$-0.02743+0.33361i$	0.79638
11	$-0.00958+0.33426i$	0.80117
12	$0.00536+0.33412i$	0.80687
13	$0.01804+0.33350i$	0.81013
14	$0.02892+0.33261i$	0.81350
15	$0.03835+0.33155i$	0.81577
16	$0.04660+0.33042i$	0.81795
17	$0.05388+0.32925i$	0.81957
18	$0.06033+0.32808i$	0.82106
19	$0.06610+0.32693i$	0.82225
20	$0.07129+0.32581i$	0.82332

n	zero	distance
2	$-0.89538-0.44530i$	0.5528
3	$0.40206-0.49827i$	0.85603
4	$0.47459-0.25270i$	0.89688
5	$-0.57262-0.15872i$	0.85890
6	$0.45849-0.01783i$	0.9069
7	$-0.35212-0.60990i$	0.88012
8	$0.41566+0.08718i$	0.90725
9	$-0.32375-0.56075i$	0.88470
10	$-0.06540+0.40007i$	0.90141
11	$-0.04147+0.39636i$	0.88964
12	$-0.02178+0.39227i$	0.89539
13	$-0.00532+0.38809i$	0.88893
14	$0.00861+0.38400i$	0.89015
15	$0.02054+0.38006i$	0.88620
16	$0.03086+0.37633i$	0.88568
17	$0.03986+0.37280i$	0.88297
18	$0.04779+0.36949i$	0.88182
19	$0.05481+0.36639i$	0.87977
20	$0.06106+0.36348i$	0.87845

TABLE 1. For every $2 \le n \le 20$, $\beta = 1/2\exp(i\pi/3)$, we obtain the zero of $Q_{n;1}(z;1/3\exp(i\pi/3))$ (left) and $Q_{n;2}(z;1/3\exp(i\pi/3))$ (right) which produces the maximum distance with respect to the zeros of their corresponding derivatives.

4.2. *Second example.* Fix $m \ge 0$. Let

$$d\mu_1(\theta) = \frac{d\theta}{2\pi} + m\delta(z-1),$$

where $z = \exp(i\theta)$ and

$$\int f(z)\,\delta(z-1)d\theta = f(1), \quad f \in \mathbb{P}[z].$$

The monic associated orthogonal polynomials with respect to μ_1 on $\mathbb{T}$ are [2, 7]:

$$L_n(z) = z^n - \frac{m}{1+nm}\sum_{k=0}^{n-1} z^k = z^n - \frac{m}{1+nm}\,\frac{z^n-1}{z-1}, \quad n = 1,2,\ldots \tag{16}$$

From this expression their first order monic polar polynomials are defined as:

$$Q_{n;1}(z;\xi) = \frac{z^{n+1}-\xi^{n+1}}{z-\xi} - \frac{m(n+1)}{1+nm}\sum_{k=0}^{n-1}\frac{z^{k+1}-\xi^{k+1}}{(k+1)(z-\xi)}. \tag{17}$$

28 Mar 2023 01:54:06 PDT
230118-Costas Version 3 - Submitted to Rocky Mountain J. Math.

Remark 4.1. *Notice that the zeros of these polynomials tend to accumulate around the unit circle $\mathbb{T}$ whenever $|\xi| \leq 1$, and around the closed circle $\overline{D(0,|\xi|)}$ whenever $|\xi| > 1$. Moreover, since we added a mass point at $z = 1$ then it is expected that some of zeros the polar polynomials close to $z = 1$ move outside of such boundary.*

Moreover, due to Theorem 3.2 we know all the zeros of these polynomials lie inside of $\overline{D(0, 2+3|\xi|)}$.

In the figures 3 and 4 we show the zeros of these polynomials under different settings (in the first case $|\xi_1| + 2(1+|\xi_1|) = 3$ and $|\xi_2| + 2(1+|\xi_2|) = 6$ in the second one).

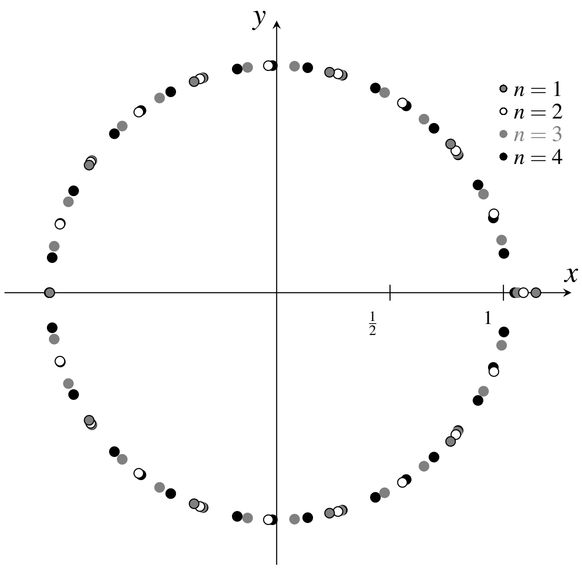

FIGURE 3. Zeros of $Q_{10n;1}(z;1/3)$ for $n = 1,2,3,4$ in the window $[-1,1.2] \times [-1,1]$, with $m = 2/3$.

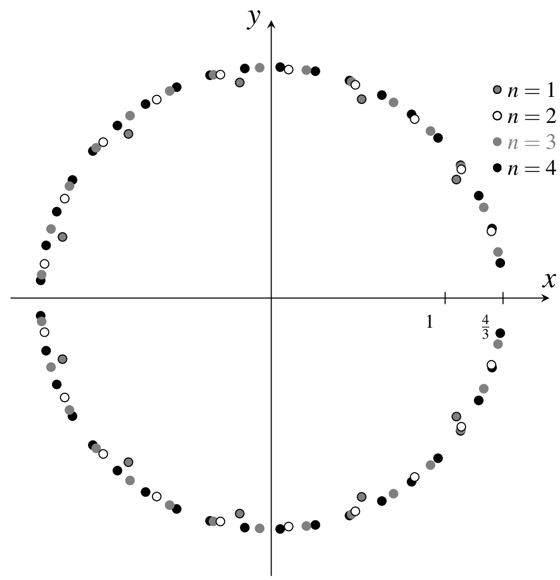

FIGURE 4. Zeros of $Q_{10n;1}(z;4/3)$ for $n = 1,2,3,4$ in the window $[-1.5,1.5] \times [-1.5,1.5]$, with $m = 2/3$.

In order to show the Sendov's conjecture remains valid in the table 2 we present the maximum distance between each zero of $Q_{n,1}(z,\xi)$ and the closet zero of $Q'_{n,1}(z,\xi)$ for different values of n.

4.3. ***Third Example.*** Let

$$d\mu_2(\theta) = |z-1|^2 \frac{d\theta}{2\pi}.$$

The monic associated orthogonal polynomials with respect to μ_2 on $\mathbb{T}$ are [2, 7]:

$$L_n(z) = \sum_{k=0}^{n} \frac{k+1}{n+1} z^k = \frac{(n+1)z^{n+2} - (n+2)z^{n+1} + 1}{(n+1)(z-1)^2}, \quad n = 1,2,\ldots \tag{18}$$

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n	zero	distance	n	zero	distance
2	0.99163	0.9440	2	$-0.45238+0.38021i$	0.38021
3	1.1388	1.5309	3	$-0.24822+0.77992i$	1.2234
4	-0.97769	1.5490	4	$0.13378+0.95290i$	1.6349
5	1.1848	1.7297	5	$0.44282+0.96439i$	1.7519
6	-0.99320	1.7641	6	$-0.28585-1.04862i$	1.9307
7	1.1711	1.7968	7	$-0.81931-0.79845i$	2.0409
8	-0.99834	1.8563	8	$-0.60642-1.03052i$	2.1514
9	$-0.94557-0.32389i$	1.8579	9	$-0.99493-0.71892i$	2.2245
10	-1.0005	1.9036	10	$-0.82147-0.94486i$	2.2841
11	$-0.96404-0.26952i$	1.9012	11	$-1.09907-0.63389i$	2.3315
12	-1.0015	1.9310	12	$-0.95942-0.84975i$	2.3671
13	$-0.97483-0.23056i$	1.9275	13	$-1.16310-0.56021i$	2.3993
14	-1.0020	1.9483	14	$-1.05017-0.76336i$	2.4222
15	$-0.98163-0.20134i$	1.9446	15	$-1.20452-0.49917i$	2.4452
16	-1.0022	1.9598	16	$-1.11214-0.68904i$	2.4609
17	$-0.98617-0.17864i$	1.9563	17	$-1.23260-0.44888i$	2.4779
18	-1.0023	1.9679	18	$-1.15604-0.62599i$	2.4892
19	$-0.98934-0.16051i$	1.9647	19	$-1.25243-0.40715i$	2.5022
20	-1.0023	1.9738	20	$-1.18814-0.57248i$	2.5107

TABLE 2. For every $2 \leq n \leq 20$, $m = 2/3$, we obtain the zero of $Q_{n;1}(z;1/3)$ (left) and $Q_{n;1}(z;4/3)$ (right) which produces the maximum distance with respect to the zeros of their corresponding derivatives.

From this expression their first order monic polar polynomials are defined as:

$$Q_{n;1}(z;\xi) = \frac{z\left(z^{n+1}-1\right)(\xi-1)-\xi\left(\xi^{n+1}-1\right)(z-1)}{(\xi-1)(z-\xi)(z-1)}, \quad z \neq 1, z \neq \xi. \tag{19}$$

In the figures 5 and 6 we show the zeros for the $k=1$ and $k=4$ polar polynomials. In the figures 5 and 6 we show the zeros of these polynomials under different settings (in the first case $|\xi_1|+2(1+|\xi|_1)=3$ and $|\xi_2|+5(1+|\xi_2|)=13$ in the second one).

In the $k=4$ case, and due of the length and difficulty of these expressions for the polynomials, are not presented in the manuscript.

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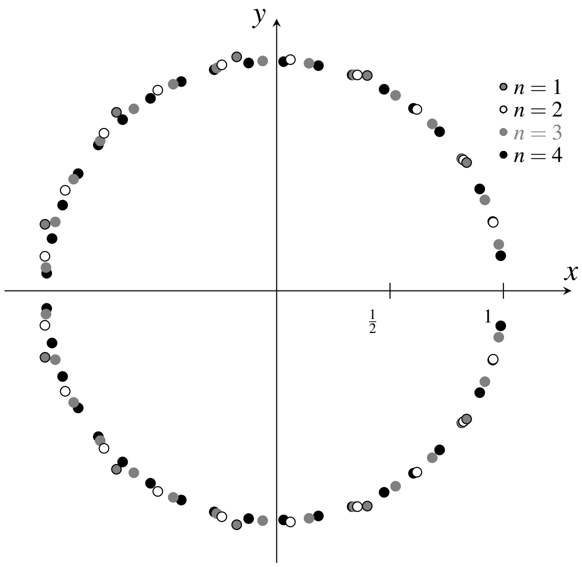

FIGURE 5. Zeros of $Q_{10n;1}(z;1/3)$ for $n=1,2,3,4$ in the window $[-1,1.2]\times[-1,1]$.

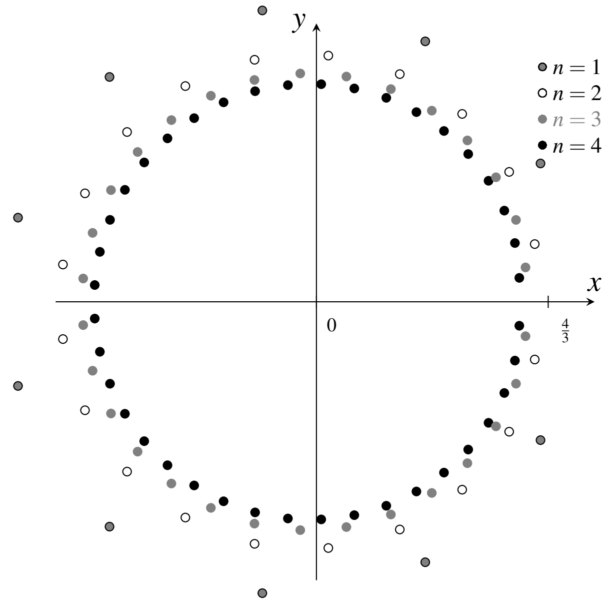

FIGURE 6. Zeros of $Q_{10n;4}(z;4/3)$ for $n=1,2,3,4$ in the window $[-1.5,1.5]\times[-1.5,1.5]$.

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