WHEN TWO COMPATIBLE METRICS GENERATE THE SAME TOPOLOGIES OF UNIFORM CONVERGENCES ON FUNCTIONAL SPACES

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ABSTRACT. Let X be a Tychonoff space, Y be a metrizable one and $C(X,Y)$ be the space of continuous functions from X to Y. It is a classical result that two compatible metrics on Y generate the same topologies of uniform convergence on compacta on $C(X,Y)$. We extend the result for the space $U(X,Y)$ of upper semicontinuous nonempty compact-valued maps from X to Y. We also present characterizations of uniform equivalence of metrics via uniform convergence on functional spaces, as well as functional characterizations of locally compact and k-spaces. We give a partial answer to a question posed in [23] after Example 1.2.7, when two compatible metrics on Y generate the same topologies of uniform convergence on $C(X,Y)$.

1. Introduction

Let X be a Tychonoff space, Y be a metrizable one and $C(X,Y)$ be the space of continuous functions from X to Y. It is known that two compatible metrics on Y generate the same topologies of uniform convergence on compacta on $C(X,Y)$, in fact they generate compact-open topology on $C(X,Y)$. We prove that two compatible metrics on Y generate the same topologies of uniform convergence on compacta on the space $U(X,Y)$ of upper semicontinuous nonempty compact-valued maps from X to Y. Following Christensen [4] such maps are called usco.

Usco maps have applications in topology, approximation theory, optimization, differentiability theory of convex functions, variational analysis and the differentiation theory of Lipschitz functions [13, 26, 28, 30]. There are two important subclasses of usco maps: minimal usco and minimal cusco maps. Both these subclasses have many applications too [2, 3, 13, 26, 27]. There are many papers which study topologies of uniform convergence on compacta and uniform convergence on usco, minimal usco and minimal cusco maps [9, 10, 11, 14, 17, 18, 19, 20, 21]. Interesting results concerning selections of usco and minimal usco/cusco maps can be found in [13, 15, 16, 29].

2. Preliminaries

In what follows let X and Y be Hausdorff topological spaces, $\mathbb{N}$ be the set of positive integers, $\mathbb{R}$ be the space of real numbers with the usual Euclidean metric.

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The symbol $\overline{A}$ and Int$A$ will stand for the closure and interior of the set A in a topological space. Throughout the paper all spaces are assumed nontrivial; i.e. all spaces contain at least 2 different points.

A set-valued map, or a multifunction, from X to Y is a function that assigns to each element of X a subset of Y. Following [6] the term map is reserved for a set-valued map. If F is a map from X to Y, then its graph is the set $\{(x,y) \in X \times Y : y \in F(x)\}$. In our paper, we will identify maps with their graphs.

A map $F : X \to Y$ is upper semicontinuous at a point $x \in X$ if for every open set V containing $F(x)$, there exists an open set U such that $x \in U$ and

$$F(U) = \bigcup\{F(u) : u \in U\} \subset V.$$

F is upper semicontinuous if it is upper semicontinuous at each point of X. Following Christensen [4] we say, that a map F is usco if it is upper semicontinuous and takes nonempty compact values. Finally, a map F from a topological space X to a topological space Y is said to be minimal usco if it is a minimal element in the family of all usco maps (with the domain X and the range Y); that is, if it is usco and does not contain properly any other usco map from X into Y.

Let (Y,d) be a metric space. The open d-ball with center $z_0 \in Y$ and radius $\varepsilon > 0$ will be denoted by $S_\varepsilon(z_0)$ and the ε-parallel body $\bigcup_{a\in A} S_\varepsilon(a)$ for a subset A of Y will be denoted by $S_\varepsilon(A)$.
Denote by $CL(Y)$ the space of all nonempty closed subsets of Y and by $K(Y)$ the space of all nonempty compact subsets of Y.

If $A \in CL(Y)$, the *distance functional* $d(.,A) \ : Y \mapsto [0,\infty)$ is described by the familiar formula

$$d(z,A) = \inf\{d(z,a) : \ a \in A\}.$$

Let A and B be nonempty subsets of (Y,d). The *excess* of A over B with respect to d is defined by the formula

$$e_d(A,B) = \sup\{d(a,B) : \ a \in A\}.$$

The *Hausdorff metric* (extended-valued) H_d on $CL(Y)$ [Be] is defined by

$$H_d(A,B) = \max\{e_d(A,B), e_d(B,A)\}.$$

We will often use the following equality on $CL(Y)$:

$$H_d(A,B) = \inf\{\varepsilon > 0 : \ A \subset S_\varepsilon(B) \text{ and } B \subset S_\varepsilon(A)\}.$$

The topology generated by H_d is called the *Hausdorff metric topology.*

Let X be a topological space and (Y,d) be a metric space. Following [9] we will define the topology τ_p of pointwise convergence on $CL(Y)^X$. The *topology of pointwise convergence* τ_p on $CL(Y)^X$ is induced by the uniformity $\mathfrak{U}_p$ of pointwise convergence which has a base consisting of sets of the form

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$$W(A,\varepsilon) = \{(\Phi,\Psi) : \ \forall\, x \in A \ \ H_d(\Phi(x),\Psi(x)) < \varepsilon\},$$

where A is a finite set in X and $\varepsilon > 0$.

We will define the *topology of uniform convergence on compact sets* τ_{UC} on $CL(Y)^X$ [9]. This topology is induced by the uniformity $\mathfrak{U}_{UC}$ which has a base consisting of sets of the form

$$W(K,\varepsilon) = \{(\Phi,\Psi) : \ \forall\, x \in K \ \ H_d(\Phi(x),\Psi(x)) < \varepsilon\},$$

where $K \in K(X)$ and $\varepsilon > 0$. The general τ_{UC}-basic neighborhood of $\Phi \in CL(Y)^X$ will be denoted by $W(\Phi,K,\varepsilon)$, i.e. $W(\Phi,K,\varepsilon) = W(K,\varepsilon)[\Phi] = \{\Psi : H_d(\Phi(x),\Psi(x)) < \varepsilon$ for every $x \in K\}$.

Finally, we will define the *topology of uniform convergence* τ_U on $CL(Y)^X$ [9]. This topology is induced by the uniformity $\mathfrak{U}_U$ which has a base consisting of sets of the form

$$W(\varepsilon) = \{(\Phi,\Psi) : \ \forall\, x \in X \ \ H_d(\Phi(x),\Psi(x)) < \varepsilon\},$$

where $\varepsilon > 0$.

Some topological properties of the space $(U(X,Y),\tau_{UC})$ can be found in [17].

Let X be hemicompact and (Y,d) be a metric space. Let $\{K_n : \ n \in \mathbb{Z}^+\}$ be a countable cofinal subfamily in $K(X)$ with respect to the inclusion. It is easy to verify that the countable family $\{W(K_m,1/n) : m,n \in \mathbb{N}\}$ is a base of the uniformity $\mathfrak{U}_{UC}$ [17]. Thus the uniformity $\mathfrak{U}_{UC}$ is metrizable [22]. We will define a compatible metric ρ on $U(X,Y)$.

For every $K \in K(X)$ let p_K be the pseudometric on $U(X,Y)$ defined by

$$p_K(F,G) = \sup\{H_d(F(x),G(x)) : \ x \in K\}.$$

Notice that for every $F \in U(X,Y)$ and every $K \in K(X)$ the set $F(K)$ is compact [1].

Then for every $K \in K(X)$ we have the pseudometric h_K defined as

$$h_K(F,G) = \min\{1, p_k(F,G)\}.$$

We define a function $\rho : \ U(X,Y) \times U(X,Y) \to \mathbb{R}$ as follows

$$\rho(F,G) = \sum_{n=1}^{\infty} \frac{1}{2^n} h_{K_n}(F,G).$$

It is easy to see that ρ is a metric on $U(X,Y)$ and uniformity $\mathfrak{U}_{UC}$ is generated by ρ.

3. When two compatible metrics generate the same topologies of uniform convergences

If (Y,d) is a metric space, then the Vietoris topology and the Hausdorff metric topology generated by H_d coincide on $K(Y)$ [24]. Thus if d and e are two compatible metrics on Y, then the topology generated by H_d and the topology generated by H_e on $K(Y)$ are the same.

Let X be a topological space and Y be metrizable. If d is a compatible metric on Y denote by τ_p^d, τ_{UC}^d and τ_U^d the topology of pointwise convergence, the topology of uniform convergence on compacta and the topology of uniform convergence on $CL(Y)^X$ generated by the Hausdorff metric H_d.

We also denote the open d-ball with center $z_0 \in Y$ and radius $\varepsilon > 0$ by $S_\varepsilon^d(z_0)$, the ε-parallel body $\bigcup_{a\in A} S_\varepsilon^d(a)$ for a subset A of Y will be denoted by $S_\varepsilon^d(A)$ and $W^d(\Phi,K,\varepsilon) = \{\Psi : H_d(\Phi(x),\Psi(x)) < \varepsilon \text{ for every } x \in K\}$.

The following result is obvious.

Proposition 3.1. *Let X be a topological space and d and e be two compatible metrics on Y. Then the topologies τ_p^d and τ_p^e on the space $K(Y)^X$ are the same, so also on $U(X,Y)$.*

Theorem 3.2. *Let X be a Hausdorff topological space and d and e be two compatible metrics on Y. Then the topologies τ_{UC}^d and τ_{UC}^e on the space $U(X,Y)$ of usco maps from X to Y are the same.*

Proof. Let $\{F_\sigma : \sigma \in \Sigma\}$ converge to F in $(U(X,Y),\tau_{UC}^d)$ and suppose that it fails to converge to F in $(U(X,Y),\tau_{UC}^e)$. There is $K \in K(X)$ and $\epsilon > 0$ such that

$$\text{for every } \sigma \text{ there is } \eta > \sigma \text{ with } F_\eta \notin W^e(F,K,\epsilon).$$

Without loss of generality we can suppose that there is a net $\{x_a : a \in A\}$ in K and a subnet $\{F_a : a \in A\}$ of the net $\{F_\sigma : \sigma \in \Sigma\}$ such that $H_e(F_a(x_a),F(x_a)) \geq \epsilon$.

We have two possibilities: either i) $F(x_\lambda)$ is not contained in $S_\epsilon^e(F_\lambda(x_\lambda)$ for $\lambda \in \Lambda$, where Λ is a cofinal family in A or ii) $F_i(x_i)$ is not contained in $S_\epsilon^e(F(x_i)$ for $i \in I$, where I is a cofinal family in A.

Consider i). Without loss of generality we can suppose that $\Lambda = A$. For every $a \in A$ there is $y_a \in F(x_a) \setminus S_\epsilon^e(F_a(x_a))$. Let x be a cluster point of $\{x_a : a \in A\}$. There is a cluster point y of $\{y_a : a \in A\}$. There is $\eta > 0$ such that $S_\eta^d(y) \subset S_{\epsilon/2}^e(y)$. There is $a_0 \in A$ such that

$$F_a \in W^d(F,K,\eta/2) \text{ for every } a \geq a_0.$$

Let $a \geq a_0$ be such that $y_a \in S_{\eta/2}^d(y)$. Then $y_a \in F(x_a) \subset S_{\eta/2}^d(F_a(x_a))$. There is $z_a \in F_a(x_a)$ such that $d(y_a,z_a) < \eta/2$. Thus $d(y,z_a) < \eta$. Then $e(y,z_a) < \epsilon/2$. Thus $e(y_a,z_a) < \epsilon$, a contradiction.

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Consider ii). Without loss of generality we can suppose that $I = A$. For every $a \in A$ there is $y_a \in F_a(x_a) \setminus S^e_\epsilon(F(x_a))$. Let x be a cluster point of $\{x_a : a \in A\}$. Suppose first that there is a cluster point y of the net $\{y_a : a \in A\}$. There is $\eta > 0$ such that $S^d_\eta(y) \subset S^e_{\epsilon/2}(y)$. There is $a_0 \in A$ such that

$$F_a \in W^d(F, K, \eta/2) \text{ for every } a \geq a_0.$$

Let $a \geq a_0$ be such that $y_a \in S^d_{\eta/2}(y)$. Then $y_a \in F_a(x_a) \subset S^d_{\eta/2}(F(x_a))$. There is $z_a \in F(x_a)$ such that $d(y_a, z_a) < \eta/2$. Thus $d(y, z_a) < \eta$. Then $e(y, z_a) < \epsilon/2$. Thus $e(y_a, z_a) < \epsilon$, a contradiction.

Suppose now there is no cluster point of the net $\{y_a : a \in A\}$. Since $F(x)$ is compact there is $\eta > 0$ and $a_0 \in A$ such that $y_a \notin S^d_\eta(F(x))$ for every $a \geq a_0$. The upper semicontinuity of F at x implies that there is an open neighbourhood U of x such that $F(z) \subset S^d_{\eta/2}(F(x))$ for every $z \in U$. There is $a_1 \in A$ such that $a_1 \geq a_0$ and

$$F_a \in W^d(F, K, \eta/2) \text{ for every } a \geq a_1.$$

Let $a \geq a_1$ be such that $x_a \in U$. Then $y_a \in F_a(x_a) \subset S^d_{\eta/2}(F(x_a)) \subset S^d_\eta(F(x))$, a contradiction.

□

Concerning the topology of uniform convergence, Example 1.2.7 in [23] presents two compatible metrics ρ and σ on $\mathbb{R}$ such that $\tau^\rho_U \neq \tau^\sigma_U$ on $C(\mathbb{R}, \mathbb{R})$, the space of continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Of course the metrics ρ and σ in Example 1.2.7. in [23] are not uniformly equivalent.

Two metrics d and e on Y are called uniformly equivalent iff for every $\epsilon > 0$ there are positive numbers $\alpha_1 = \alpha_1(\epsilon)$ and $\alpha_2 = \alpha_2(\epsilon)$ such that

$$d(x, y) < \alpha_1 \Rightarrow e(x, y) < \epsilon \text{ and } e(x, y) < \alpha_2 \Rightarrow d(x, y) < \epsilon.$$

The following proposition is obvious.

Proposition 3.3. *Let X be a topological space and d and e be two compatible metrics on Y. If the identity mapping $id_Y : (Y, d) \to (Y, e)$ on Y is uniformly continuous, then the topology τ^e_U is weaker than τ^d_U on Y^X, thus also on $C(X, Y)$.*

Corollary 3.4. *Let X be a topological space d and e be two uniformly equivalent metrics on Y. Then $\tau^d_U = \tau^e_U$ on Y^X, thus also on $C(X, Y)$.*

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Remark 3.5. Let d be the Euclidean metric on $\mathbb{R}$. As was mentioned above, Example 1.2.7 in [23] presents two compatible metrics ρ and σ on $\mathbb{R}$ such that $\tau_U^\rho \neq \tau_U^\sigma$ on $C(\mathbb{R},\mathbb{R})$. In the Example 1.2.7 in [23] the metric ρ is uniformly equivalent to d and the identity mapping $id_{\mathbb{R}} : (\mathbb{R},\sigma) \to (\mathbb{R}, d)$ is not uniformly continuous. In the following Example we present a metric e on $\mathbb{R}$ compatible with d and such that the identity mapping $id_{\mathbb{R}} : (\mathbb{R}, d) \to (\mathbb{R}, e)$ is not uniformly continuous and $\tau_U^d \neq \tau_U^e$ on $C(\mathbb{R},\mathbb{R})$.

Example 3.6. Let d be the Euclidean metric on $\mathbb{R}$. Define a continuous function $f : \mathbb{R} \to \mathbb{R}$ as follows: $f(x) = 0$ for $x < 2$ and on every interval $[n, n+1]$, $n \geq 2$, f is the piecewise linear function whose graph connects the following points in the succession:

$$(n, 0), (n + 1/n), 1) \text{ and } (n + 1, 0).$$

Define the metric $e : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as follows: $e(x, y) = d(x, y) + |f(x) - f(y)|$. Of course, the metrics d and e are compatible and $id_{\mathbb{R}} : (\mathbb{R}, d) \to (\mathbb{R}, e)$ is not uniformly continuous, since $d(n, n + 1/n) = 1/n$ and $e(n, n + 1/n) > 1$ for every $n \geq 2$, $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ define a continuous function $g_n : \mathbb{R} \to \mathbb{R}$ as: $g_n(x) = x + 1/n$ for $x \in \mathbb{R}$. It is easy to verify that $\{g_n : n \in \mathbb{N}\}$ converges to $id_{\mathbb{R}}$ in $(C(\mathbb{R},\mathbb{R}), \tau_U^d)$, however it fails to converge to $id_{\mathbb{R}}$ in $(C(\mathbb{R},\mathbb{R}), \tau_U^e)$.

We have the following characterization.

Proposition 3.7. *Let X be a non compact metric space, Y be a locally convex space metrizable by a translation invariant metric d. Let e be a metric on Y compatible with d. The following are equivalent:*
(1) the identity mapping $id_Y : (Y, d) \to (Y, e)$ is uniformly continuous;
(2) τ_U^e is weaker than τ_U^d on $C(X, Y)$.

Proof. It is sufficient to prove that (2) ⇒ (1). Suppose that $id_Y : (Y, d) \to (Y, e)$ fails to be uniformly continuous, where id_Y is the identity function on Y. We can find $\epsilon > 0$ and sequences $\{y_n : n \in \mathbb{N}\}$ and $\{v_n : n \in \mathbb{N}\}$ such that for every $n \in \mathbb{N}$, $d(y_n, v_n) < 1/n$ and $e(y_n, v_n) \geq \epsilon$. Since X is a non compact metric space, X is not countably compact. Thus there is a countably infinite subset A of X which has no accumulation point. Enumerate A as $\{x_n : n \in \mathbb{N}\}$. Then A is a closed discrete set in X. We will define a continuous function $f : A \to Y$ as: $f(x_n) = y_n$, $n \in \mathbb{N}$. Let $n \in \mathbb{N}$. Define a function $f_n : A \to Y$ as follows: $f_n(x_n) = v_n$ and $f_n(x) = f(x)$ otherwise. Of course $\{f_n : n \in \mathbb{N}\}$ converges to f in $(C(A, Y), \tau_U^d)$. Notice that the uniformity generated by d on Y is the standard uniformity on Y (generated by a local base of absolutely convex neigbourhoods of the zero element in Y). By Theorem 7.1 in [25] there is a mapping

$$\Phi : (C(A, Y), \tau_U^d) \to (C(X, Y), \tau_U^d)$$

such that $\Phi(f)$ is an extension of f for every $f \in C(A, Y)$ and Φ is an isomorphism. Thus the sequence $\{\Phi(f_n) : n \in \mathbb{N}\}$ converges to $\Phi(f)$ in $(C(X, Y), \tau_U^d)$.

Since for every $n \in \mathbb{N}$ $e(\Phi(f_n)(x_n), \Phi(f)(x_n)) = e(v_n, y_n) \geq \epsilon$, the sequence $\{\Phi(f_n) : n \in \mathbb{N}\}$ fails to converge to $\Phi(f)$ in $(C(X,Y), \tau_U^e)$, a contradiction.

□

Michael wrote in his paper [25], that the mapping Φ in the proof of Proposition 3.7 was constructed by Dugundji in [7].

Corollary 3.8. *Let d be the usual Euclidean metric on $\mathbb{R}$ and e be a compatible metric on $\mathbb{R}$. The following are equivalent:*
(1) the identity mapping $id_{\mathbb{R}} : (\mathbb{R}, d) \to (\mathbb{R}, e)$ is uniformly continuous;
(2) τ_U^e is weaker than τ_U^d on $C(\mathbb{R}, \mathbb{R})$.

We have the following partial answer to a question posed in [23] after Example 1.2.7.

Theorem 3.9. *Let X be a zero-dimensional non compact metric space, Y be a metrizable one and d and e be two compatible metrics on Y. The following are equivalent:*
(1) d and e are uniformly equivalent;
(2) $\tau_U^d = \tau_U^e$ on $C(X,Y)$.

Proof. It is sufficient to prove that (2) $\Rightarrow$ (1). Suppose that $id_Y : (Y,d) \to (Y,e)$ fails to be uniformly continuous, where id_Y is the identity function on Y. We can find $\epsilon > 0$ and sequences $\{y_n : n \in \mathbb{N}\}$ and $\{v_n : n \in \mathbb{N}\}$ such that for every $n \in \mathbb{N}$, $d(y_n, v_n) < 1/n$ and $e(y_n, v_n) \geq \epsilon$. Since X is a non compact metric space, X is not countably compact. Thus there is a countably infinite subset A of X which has no accumulation point. Enumerate A as $\{x_n : n \in \mathbb{N}\}$. Then A is a closed discrete set in X. Let ρ be a compatible metric on X. There is a sequence $\{\epsilon_n : n \in \mathbb{N}\}$ such that $0 < \epsilon_n < 1/n$ for every $n \in \mathbb{N}$ and the family $\{S_{\epsilon_n}(x_n) : n \in \mathbb{N}\}$ is pairwise disjoint, where $S_{\epsilon_n}(x_n) = \{x \in X : \rho(x_n, x) < \epsilon_n\}$ for every $n \in \mathbb{N}$. Since X is zero-dimensional, it has a base consisting of clopen sets. Thus for every $n \in \mathbb{N}$ there is an open and closed set $O(x_n)$ such that $x_n \in O(x_n)$ and $O(x_n) \subset S_{\epsilon_n}(x_n)$. The set $L = \bigcup\{O(x_n) : n \in \mathbb{N}\}$ is also open and closed. Without loss of generality we can suppose that the set $X \setminus L$ is nonempty. We will define a continuous function $f : X \to Y$ as follows: $f(x) = y_n$, if $x \in O(x_n)$, $n \in \mathbb{N}$ and $f(x) = y_1$ if $x \in X \setminus L$. Let $n \in \mathbb{N}$. Define a continuous function $f_n : X \to Y$ as follows: $f_n(x) = v_n$ if $x \in O(x_n)$ and $f_n(x) = f(x)$ otherwise. Of course the sequence $\{f_n : n \in \mathbb{N}\}$ converges to f in $(C(X,Y), \tau_U^d)$ and fails to converge to f in $(C(X,Y), \tau_U^e)$.

□

If d and e are uniformly equivalent metrics on Y, then for every $\epsilon > 0$ there are positive numbers $\alpha_1 = \alpha_1(\epsilon)$ and $\alpha_2 = \alpha_2(\epsilon)$ such that for every nonempty set B in Y

$$\{y \in Y : d(y,B) < \alpha_1\} \subset \{y \in Y : e(y,B) < \epsilon\}$$

and

$$\{y \in Y : e(y, B) < \alpha_2\} \subset \{y \in Y : d(y, B) < \epsilon\}.$$

Thus uniform equivalence of d and e yields uniform equivalence of H_d and H_e on $CL(Y)$ (see the proof of Theorem 3.3.2 in [1]).

Thus we have the following proposition.

Proposition 3.10. *Let X be a topological space, d and e be two uniformly equivalent metrics on Y. Then $\tau_U^d = \tau_U^e$ on $CL(Y)^X$ and thus on $K(Y)^X$ and $U(X,Y)$.*

The following characterization holds.

Proposition 3.11. *Let X be an unbounded metric space, Y be a metrizable one and d and e be two compatible metrics on Y. The following are equivalent:*
(1) d and e are uniformly equivalent;
(2) $\tau_U^d = \tau_U^e$ on $U(X,Y)$.

Proof. It is sufficient to prove that (2) $\Rightarrow$ (1). Suppose that d and e are not uniformly equivalent. Let us say that $id_Y : (Y,d) \to (Y,e)$ fails to be uniformly continuous, where id_Y is the identity function on Y. We can find $\epsilon > 0$ and sequences $\{y_n : n \in \mathbb{N}\}$ and $\{v_n : n \in \mathbb{N}\}$ such that for every $n \in \mathbb{N}$, $d(y_n, v_n) < 1/n$ and $e(y_n, v_n) \geq \epsilon$. By the Efremovič lemma [1] and by passing to a subsequence we can suppose that

$$inf\{e(y_n, v_m) : n, m \in \mathbb{N}\} \geq \epsilon/4.$$

Let x_0 be any point in X. There is a sequence of positive integers $\{\alpha_n : n \in \mathbb{N}\}$ such that $\alpha_n < \alpha_{n+1}$, $n \in \mathbb{N}$, $\alpha_n \to \infty$ and

$$S_{\alpha_n}(x_0) \setminus S_{\alpha_{n-1}}(x_0) \neq \emptyset \text{ for every } n \in \mathbb{N}, n \geq 2.$$

We will define an usco map $F : X \to Y$ as follows: $F(x) = \{y_1\}$ if $x \in S_{\alpha_1}(x_0)$ and $F(x) = \{y_1, ... y_n\}$ for $x \in S_{\alpha_n}(x_0) \setminus S_{\alpha_{n-1}}(x_0)$, $n \geq 2$. Let $n \in \mathbb{N}$. We will define an usco map $F_n : X \to Y$. Put $F_1 = F$ and for $n \geq 2$ define F_n as follows: $F_n(x) = F(x)$ if $x \in S_{\alpha_n}(x_0)$,

$$F_n(x) = \{y_1, ..., y_n, v_{n+1}\} \text{ if } x \in S_{\alpha_{n+1}}(x_0) \setminus S_{\alpha_n}(x_0) \text{ and}$$

$$F_n(x) = \{y_1, ..., y_n, v_{n+1}, ... v_k\} \text{ if } x \in S_{\alpha_k}(x_0) \setminus S_{\alpha_{k-1}}(x_0),\ k > n+1.$$

It is easy to verify that the sequence $\{F_n : n \in \mathbb{N}\}$ converges to F in $(U(X,Y), \tau_U^d)$. For every $n \in \mathbb{N}, n \geq 2$ and $x \in S_{\alpha_{n+1}}(x_0) \setminus S_{\alpha_n}(x_0)$ we have

$$H_e(F_n(x), F(x)) = H_e(\{y_1, ..., y_n, v_{n+1}\}, \{y_1, ..., y_n, y_{n+1}\}) \geq \epsilon/4.$$

Thus the sequence $\{F_n : n \in \mathbb{N}\}$ fails to converge to F in $(U(X,Y), \tau_U^e)$, a contradiction.

□

Passing to set-valued maps with nonempty closed values we have the following result.

Proposition 3.12. *Let X be a topological space and Y be a metrizable space. Let d and e be compatible metrics. The following are equivalent:*
(1) d and e are uniformly equivalent;
(2) $\tau_U^d = \tau_U^e$ on $CL(Y)^X$;
(3) $\tau_{UC}^d = \tau_{UC}^e$ on $CL(Y)^X$;
(4) $\tau_p^d = \tau_p^e$ on $CL(Y)^X$;
(5) $\tau_{H_d} = \tau_{H_e}$ on $CL(Y)$, where τ_{H_d} (τ_{H_e}) is the topology generated by H_d (H_e).

Proof. (5) $\Rightarrow$ (1) is proved in Theorem 3.3.2 in [1].

□

4. Functional characterizations of k-spaces and locally compact spaces

A Hausdorff topological space X is a k-space if a set A is closed in X if and only if $A \cap K$ is closed in X for every compact K in X. It is known that if X is a k-space and (Y, d) is a metric space, then $C(X, Y)$ is a closed subset of (Y^X, τ_{UC}). It was mention in [17] (without a complete proof) that if X is a k-space and (Y, d) is a metric space, then $U(X, Y)$ is a closed subset of $(K(Y)^X, \tau_{UC})$. For a reader's convenience we present the proof of it.

Proposition 4.1. *Let X be a k-space and (Y, d) be a metric space. Then $U(X, Y)$ is a closed subset of $(K(Y)^X, \tau_{UC})$.*

Proof. Let G be in the closure of $U(X, Y)$ in $(K(Y)^X, \tau_{UC})$. We prove that $G \in U(X, Y)$. By [1] $G \in U(X, Y)$ if, and only if, for every closed set B in Y, the set $G^-(B) = \{x \in X : G(x) \cap B \neq \emptyset\}$ is closed in X. Since X is a k-space, it is sufficient to prove that $G^-(B) \cap C$ is a closed set in X for every compact set C in X. Suppose that there are a closed set B in Y and a compact set C in X such that $G^-(B) \cap C$ is not closed in X. Let

$$x \in \overline{G^-(B) \cap C} \setminus G^-(B); \text{ i.e. } G(x) \cap B = \emptyset.$$

Since $G(x)$ is compact, there is $\epsilon > 0$ such that $S_\epsilon(B) \cap G(x) = \emptyset$. Put $D = \overline{S_{\epsilon/4}(B)}$. Consider $W(G, C, \epsilon/4)$. Let $H \in U(X, Y) \cap W(G, C, \epsilon/4)$. Thus

$$H(x) \subset S_{\epsilon/4}(G(x)) \subset D^c \text{ and } x \in \overline{H^-(D) \cap C},$$

a contradiction, since $H \in U(X, Y)$.

□

We can characterize k-spaces as follows.

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Proposition 4.2. *Let X be a Hausdorff topological space and (Y,d) be a metric space. The following are equivalent:*
(1) X is a k-space;
(2) $U(X,Y)$ is a closed subset of $(K(Y)^X, \tau_{UC})$.

Proof. (1) $\Rightarrow$ (2) is proved in Proposition 4.1. To prove (2) $\Rightarrow$ (1) suppose that X is not a k-space. Let A be a non-closed set in X, such that $A \cap K$ is a closed set in X for every $K \in K(X)$. Let $x_0 \in \overline{A} \setminus A$ and let y_1, y_2 be two different points in Y. Put $\mathcal{K} = \{K \in K(X) : K \cap A \neq \emptyset\}$ and for every $K \in \mathcal{K}$ define a set-valued map $F_K : X \to Y$ as follows: $F_K(x) = \{y_1, y_2\}$ if $x \in K \cap A$ and $F_K(x) = \{y_1\}$ otherwise. It is easy to verify that F_K is usco for every $K \in \mathcal{K}$. Define a set-valued map $F : X \to Y$ as $F(x) = \{y_1, y_2\}$ if $x \in A$ and $F(x) = \{y_1\}$ otherwise. F is not upper semicontinuous at x_0. $\mathcal{K}$ is a directed set with respect to set inclusion; $C \leq K$ if and only if $C \subseteq K$. The net $\{F_K : K \in \mathcal{K})\}$ converges to F in $(K(Y)^X, \tau_{UC})$, a contradiction.

□

Denote by $MU(X,Y)$ the space of minimal usco maps from a topological space X into a metric space (Y,d). Of course, the following inclusions hold

$$C(X,Y) \subset MU(X,Y) \subset U(X,Y).$$

It is interesting to mention that if X is a k-space and (Y,d) a metric one, then $MU(X,Y)$ need not be a closed subset of $(K(Y)^X, \tau_{UC})$.

Proposition 4.3. *([11] Let X be a locally compact space and (Y,d) be a metric space. Then $MU(X,Y)$ is a closed subset of $(K(Y)^X, \tau_{UC})$.*

In the class of metric spaces we can even characterize local compactness via Proposition 4.3. We will need the following lemmas.

Lemma 4.4. *([13], Proposition 1.3.5) Let X and Y be topological spaces and $F : X \to Y$ be usco. Then F is minimal usco if, and only if, for each pair of open subsets U of X and W of Y with $F(U) \cap W \neq \emptyset$ there exists a nonempty open set $V \subset U$ such that $F(V) \subset W$.*

Lemma 4.5. *Let X be a topological space, Y be a compact topological space and D be a dense set in X. If $F : D \to Y$ is a minimal usco map, then $\overline{F} : X \to Y$ is a minimal usco map.*

Proof. By Corollary 1.1.15 from [13] $\overline{F}$ is usco. To prove that $\overline{F}$ is minimal usco we use Lemma 4.4. □

Proposition 4.6. *Let (X,ρ) and (Y,d) be metric spaces. The following are equivalent:*
(1) X is locally compact;
(2) $MU(X,Y)$ is a closed subset of $(K(Y)^X, \tau_{UC})$.

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Proof. It is sufficient to prove (2) $\Rightarrow$ (1). We will use an idea from [13]. Let $x_0 \in X$ fail to have a local base of compact sets. Let $\delta_1 = 1$. There is a sequence $\{x_i^1 : i \in \mathbb{N}\}$ of different points of $\{z \in X : 0 < \rho(x_0, z) < \delta_1\}$ with no cluster point in X. There exists $\varepsilon_1 > 0$ such that $\varepsilon_1 < \rho(x_0, x_i^1)$ for every $i \in \mathbb{N}$. Next, let $\delta_2 = \min\{\frac{1}{2}, \frac{\varepsilon_1}{2}\}$ and let $\{x_i^2 : i \in \mathbb{N}\}$ be a sequence of different points of $\{z \in X : 0 < \rho(x_0, z) < \delta_2\}$ with no cluster point in X. Chose $\varepsilon_2 > 0$ such that $\varepsilon_2 < \rho(x_0, x_i^2)$ for every $i \in \mathbb{N}$ and set $\delta_3 = \min\{\frac{1}{3}, \frac{\varepsilon_2}{2}\}$. Continuing we can produce for each $n \in \mathbb{N}$ a sequence $\{x_i^n : i \in \mathbb{N}\}$ of different points with no cluster point in X and sequences of positive real numbers $\{\delta_n : n \in \mathbb{N}\}$, $\{\epsilon_n : n \in \mathbb{N}\}$, such that $\delta_n = \min\{\frac{1}{n}, \frac{\varepsilon_{n-1}}{2}\}$, $0 < \varepsilon_n < \delta_n$ and $\{x_i^n : i \in \mathbb{N}\} \subset \{z \in X : \varepsilon_n < \rho(x_0, z) < \delta_n\}$.

Let $S_{\varepsilon_i^n}(x_i^n)$ be an open ball with the center x_i^n and radius $\varepsilon_i^n < 1/i$ for every $i \in \mathbb{N}$ and $n \in \mathbb{N}$ such that the family $\{\overline{S_{\varepsilon_i^n}(x_i^n)} : i \in \mathbb{N}\}$ is parwise disjoint and $\overline{S_{\varepsilon_i^n}(x_i^n)} \subset \{z \in X : \varepsilon_n < \rho(x_0, z) < \delta_n\}$. Let $\mathbb{N}^{\mathbb{N}}$ be partially ordered by the product order; i.e., $h \leq g$, if $h(n) \leq g(n)$ for every $n \in \mathbb{N}$.

Let a, b be two different points in Y. For every $g \in \mathbb{N}^{\mathbb{N}}$ define a dense set D_g in X and a minimal usco map $F_g : D_g \to \{a, b\}$ as follows. Put $D_g = \bigcup_{n \in N} S_{\varepsilon^n_{g(n)}}(x^n_{g(n)}) \cup (X \setminus \overline{\bigcup_{n \in N} S_{\varepsilon^n_{g(n)}}(x^n_{g(n)})})$ and

$$F_g(x) = \begin{cases} \{a\}, & x \in X \setminus \overline{\bigcup_{n \in N} S_{\varepsilon^n_{g(n)}}(x^n_{g(n)})}; \\ \{b\}, & x \in \bigcup_{n \in N} S_{\varepsilon^n_{g(n)}}(x^n_{g(n)}). \end{cases}$$

Evidently $F_g : D_g \to \{a, b\}$ is a minimal usco map. By Lemma 4.5 $\overline{F_g}$ is a minimal usco map from X to $\{a, b\}$. Thus $\overline{F_g}$ is also a minimal usco map from X to Y. Now define a map F as follows:

$$F(x) = \begin{cases} \{a, b\}, & x = x_0; \\ \{a\}, & \text{otherwise}. \end{cases}$$

Then $F \notin MU(X, Y)$ and the net $\{\overline{F_g} : g \in \mathbb{N}^{\mathbb{N}}\}$ converges in $(K(Y)^X, \tau_{UC})$ to F. Clearly, $\overline{F_g}(x_0) = \{a, b\}$ for every $g \in \mathbb{N}^{\mathbb{N}}$. For every $g \in \mathbb{N}^{\mathbb{N}}$ we have $\overline{\bigcup_{n \in N} S_{\varepsilon^n_{g(n)}}(x^n_{g(n)})} = (\bigcup_{n \in N} \overline{S_{\varepsilon^n_{g(n)}}(x^n_{g(n)})}) \cup \{x_0\}$.

Let K be a compact set in (X, ρ) and let $0 < \epsilon < 1$. Then for every $n \in \mathbb{N}$, there are only finitely many i such that $\overline{S_{\epsilon_i^n}(x_i^n)} \cap K \neq \emptyset$. For every $n \in \mathbb{N}$, let $k_n \in \mathbb{N}$ be such that $\overline{S_{\epsilon_k^n}(x_k^n)} \cap K = \emptyset$ for every $k \geq k_n$. Define $g : \mathbb{N} \to \mathbb{N}$ as follows: $g(n) = k_n$. Then for every $h \geq g$ we have $H_d(\overline{F_h}(x), F(x)) = 0$ for every $x \in K$. (Notice that if $x \in K$ and $x \neq x_0$, then $x \in X \setminus \overline{\bigcup_{n \in N} S_{\varepsilon^n_{g(n)}}(x^n_{g(n)})}$.) Thus $MU(X, Y)$ is not a closed subspace of $(K(Y)^X, \tau_{UC})$, a contradiction. $\square$

In the following propositions we will use the well-known fact that if (Y, d) is a complete metric space, then $(K(Y), H_d)$ is also a complete metric space [1].

Proposition 4.7. *Let X be a k-space and (Y, d) be a complete metric space. Then the uniform space $(U(X, Y), \mathfrak{U}_{UC})$ is complete.*

Proof. By Theorem 7.10 in [22] $(K(Y)^X, \mathfrak{U}_{UC})$ is complete. By Proposition 4.1 we are done.

□

The proof of the following Proposition is similar.

Proposition 4.8. *Let X be a locally compact space and (Y, d) be a complete metric space. Then the uniform space $(MU(X, Y), \mathfrak{U}_{UC})$ is complete.*

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