A SEMILINEAR PROBLEM GOVERNED BY THE LOGARITHMIC OPERATOR FROM STRONGLY DAMPED WAVE EQUATION

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Abstract. In this paper, we study a semilinear Cauchy problem governed by the logarithmic operator defined by a strongly damped wave operator in bounded Lipschitz domain in $\mathbb{R}^N$, in terms of properties of the logarithmic Dirichlet Laplacian operator.

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1. Introduction

In this paper, we consider a semilinear problem governed by the logarithmic operator associated with a strongly damped wave equation. It is well known the notion of logarithmic operators under different spectral conditions, see e.g. [1], [9], [10], [11], [17], [18], [20], [22] and [23]. In particular, we already have literature on the logarithm of sectorial operators (see e.g. [1], [12], [17] and [19]), but when it comes to matrix operators of elliptic operators with a certain spectral behavior (associated with the sectorial operator theory), we do not have much information available in the specialized literatures on how the notion of logarithmic operator behaves in this situation, to the best of our knowledge. Because of this, we present an explicit characterization of the matrix representation of logarithmic operator for strongly damped wave operators on bounded smooth domain in n-dimensional Euclidian spaces, in terms of properties of the logarithmic Dirichlet Laplacian operator, sometimes called 'spectral-theoretic logarithm of the Dirichlet Laplacian operator' (see e.g. [19]).

Inspired by [2], [3] and [17] we consider the strongly damped wave equation

$$\partial_t^2 u - \Delta_D u + 2(-\Delta_D)^{\frac{1}{2}}\partial_t u = f(u),\ t > 0,\ x \in \Omega, \tag{1.1}$$

with boundary and initial conditions given by

$$\begin{cases} u(0,x) = u_0(x),\ \partial_t u(0,x) = v_0(x), & x \in \Omega, \\ u(t,x) = 0, & t \geqslant 0,\ x \in \partial\Omega, \end{cases}$$

where Ω is a bounded Lipschitz domain in $\mathbb{R}^N$, $N \geqslant 3$, and $f \in C^1(\mathbb{R})$ satisfies

$$|f'(s)| \leqslant C(1 + |s|^{\rho-1}), \quad s \in \mathbb{R}, \tag{1.2}$$

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for some

$$1 < \rho < \frac{N}{N-2}.$$

Thanks to the theory of fractional powers of sectorial operators, in the sense of [16, Chapter I], the notion of logarithmic operators, in the sense of [1, Chapter III], and Balakrishnan's formula, we study the logarithmic operator defined by the strongly damped wave equation associated with (1.1) and the semilinear problem governed by it.

To better present our results, we introduce some notations. Let $X = L^2(\Omega)$ and let $A : D(A) \subset X \to X$ be the unbounded linear operator defined by

$$Au = -\Delta_D u \;\text{ for }\; u \in D(A) = H^2(\Omega) \cap H_0^1(\Omega), \tag{1.3}$$

then A is a positive self-adjoint sectorial operator and $-A$ generates a compact analytic C^0-semigroup in X.

Denote by X^α the fractional power spaces associated with operator A; that is, $X^\alpha = D(A^\alpha)$ with the norm $\|A^\alpha \cdot\|_X : X^\alpha \to \mathbb{R}^+$. For $\alpha > 0$ define also $X^{-\alpha}$ as the completion of X with the norm $\|A^{-\alpha} \cdot\|_X$. Observe that with this notation $X^{\frac{1}{2}} = H_0^1(\Omega)$ and $X^1 = H^2(\Omega) \cap H_0^1(\Omega)$.

It is well known that the problem (1.1) can be rewritten in $Y = X^{\frac{1}{2}} \times X$ as the abstract parabolic Cauchy problem

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} + \Lambda \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ f^e(u) \end{bmatrix}, \; t > 0, \qquad \begin{bmatrix} u \\ v \end{bmatrix}_{t=0} = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}, \tag{1.4}$$

where f^e is given by

$$f^e(\varphi)(\cdot) := f(\varphi(\cdot))$$

and the unbounded linear operator Λ is defined by

$$\begin{gathered} \Lambda : D(\Lambda) \subset X^{\frac{1}{2}} \times X \to X^{\frac{1}{2}} \times X, \\ \Lambda \begin{bmatrix} \varphi \\ \psi \end{bmatrix} := \begin{bmatrix} 0 & -I \\ A & 2A^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \;\text{ for }\; \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in D(\Lambda) := X^1 \times X^{\frac{1}{2}}. \end{gathered} \tag{1.5}$$

which is a sectorial operator in the sense of [16, Chapter I], see [3].

We can consider well-posed logarithmic counterpart, in some sense, of the form

$$\frac{d}{dt}\begin{bmatrix} u_\ell \\ v_\ell \end{bmatrix} + (\log \Lambda) \begin{bmatrix} u_\ell \\ v_\ell \end{bmatrix} = \begin{bmatrix} 0 \\ f^e(u) \end{bmatrix}, \; t > 0, \qquad \begin{bmatrix} u_\ell \\ v_\ell \end{bmatrix}_{t=0} = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}, \tag{1.6}$$

where $\log \Lambda = -(-\log \Lambda)$ denotes the logarithmic operator of Λ in the sense of [1, Chapter III, Section 4]. It is well known that we can consider $\Lambda^\alpha : D(\Lambda^\alpha) \subset Y \to Y$ be the fractional power associated with operator Λ with $\alpha \geqslant 0$ and the resolvent operators $\Lambda^{-\alpha} = (\Lambda^\alpha)^{-1}$ for any $\alpha \geqslant 0$. Moreover, the operator Λ has bounded imaginary powers. Namely, $-\log \Lambda$ denotes the infinitesimal generator of the strongly continuous semigroup $\{\Lambda^{-t} \in \mathcal{L}(Y) : \; t \geqslant 0\}$ which is analytic of angle $\pi/2$ defined on

$$D(-\log \Lambda) = \Big\{ u \in Y; \exists \lim_{t \searrow 0} \frac{1}{t}(\Lambda^{-t} - I)u \Big\}$$

with

$$-\log \varLambda u = \lim_{t\searrow 0} \frac{1}{t}(\varLambda^{-t} - I)u$$

for any $u \in D(-\log \varLambda)$.

Denote by Y^α the fractional power spaces associated to operator A; that is, $Y^\alpha = D(\varLambda^\alpha)$ with the norm $\|\varLambda^\alpha \cdot\|_Y : Y^\alpha \to \mathbb{R}^+$. We assume that $f : Y^{\frac{1}{2}} \to Y$ is a locally Lipschitz continuous function; that is, for every u in $Y^{\frac{1}{2}}$ there exists a neighborhood U of u in $Y^{\frac{1}{2}}$ and a constant $L_U > 0$ such that for $v, v' \in U$,

$$\|f(v) - f(v')\|_Y \leqslant L_U \|v - v'\|_{Y^{\frac{1}{2}}}.$$

This condition is natural in the applications to semilinear partial differential equations.

Here, we consider the following notion of solution for (1.6). Given $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in Y$ we say that $\begin{bmatrix} u \\ v \end{bmatrix}$ is a local mild solution of (1.6) provided that $\begin{bmatrix} u \\ v \end{bmatrix} \in C([0, \tau_{u_0;v_0}), Y)$ and $\begin{bmatrix} u \\ v \end{bmatrix}$ satisfies for $t \in (0, \tau_{u_0;v_0})$ the integral equation

$$\begin{bmatrix} u \\ v \end{bmatrix}(t) = \varLambda^{-t} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} + \int_0^t \varLambda^{t-s} \begin{bmatrix} 0 \\ f^e(u) \end{bmatrix}(s) ds$$

for some $\tau_{u_0;v_0} > 0$.

In a previous article [2], we considered the following unbounded linear operator defined by logarithmic stationary wave operator and an evolution equation governed by it

$$\log \begin{bmatrix} 0 & -I \\ A & 0 \end{bmatrix}$$

defined usually on

$$D\Big(-\log \begin{bmatrix} 0 & -I \\ A & 0 \end{bmatrix}\Big) = \Big\{ u \in X^{\frac{1}{2}} \times X; \exists \lim_{t\searrow 0} \frac{1}{t}\Big(\begin{bmatrix} 0 & -I \\ A & 0 \end{bmatrix}^{-t} - I \Big) u \Big\}$$

in terms of properties of the logarithmic Dirichlet Laplacian operator. Now, in the context of sectoriality of operators, we intend to go further.

The relationship between logarithmic operators and well-posed evolutionary equations also can be found in [9], [10], [11], [13], [14] and [15] where the authors consider results on existence, regularity and asymptotic behavior of solution for well-posed logarithmic approximations of evolution equations.

In the next section, we introduce the logarithmic operator of $\varLambda$ and we study the semilinear parabolic problem (1.6).

2. A second-order evolution equation

This section contains the main results of this article.

2.1. **Sectorial operators.** In order to better explain the results in this paper, we will introduce some concepts and terminology. Let Z be a Banach space with norm $\|\cdot\|$, and let $B : D(B) \subset Z \to Z$ be a positive sectorial operator with bounded imaginary powers, in the sense of [1, Chapter III, Section 4] and [16, Definition 1.3.1]. This allows us to define the fractional power $B^{-\alpha}$, with inverse B^{α}, which is given by

$$B^{-\alpha} = \frac{\sin(\alpha\pi)}{\pi} \int_0^\infty \lambda^{-\alpha}(\lambda I + B)^{-1} d\lambda, \quad \alpha \in (0,1),$$

see [16].

Denote by $Z^\alpha = D(B^\alpha)$ where $\alpha \in [0,1]$. Recall that Z^α is dense in Z for all $\alpha \in (0,1]$ (see [1]). The fractional power space Z^α endowed with the graphic norm $\|\cdot\|_\alpha := \|B^\alpha \cdot\|_Z$ is a Banach space.

Under our considerations on B, we can consider the logarithmic operator of B defined by $\log B := -(-\log B)$, where $-\log B$ denote the infinitesimal generator of the strongly continuous semigroup $\{B^{-t}; t \geqslant 0\}$ on Z which is analytic of angle $\pi/2$, see [1, Chapter III, p. 152], defined on

$$D(-\log B) = \Big\{ u \in Z; \exists \lim_{t \searrow 0} \frac{1}{t}(B^{-t} - I)u \Big\}$$

with

$$-\log Bu = \lim_{t \searrow 0} \frac{1}{t}(B^{-t} - I)u$$

for any $u \in D(-\log B)$.

From sectorial calculus, we also have

$$(\log B)\phi := \int_0^\infty \frac{1}{1+\lambda}(\lambda I + B)^{-1}(B\phi - \phi) d\lambda = \lim_{t \searrow 0} \frac{1}{t}(B^t - I)\phi$$

for any $\phi \in D(B) \cap R(B)$. Moreover, $D(B) \cap R(B)$ is a core for the $\log B$, see [20], [22, p. 317] and [23].

Thanks to [16] we have the next result.

Theorem 2.1. *Let $B : D(B) \subset Z \to Z$ be a positive sectorial operator and, for some $\alpha \in [0,1)$, $f : Z^\alpha \to Z$ be Lipschitz continuous on bounded subsets of Z^α. Then, for each $w_0 \in Z^\alpha$ there exists a unique $Z^\alpha-$solution $w = w(t, w_0)$ of*

$$\begin{cases} \dfrac{dw}{dt} + Bw = f(w), \ t > 0, \\ w(0) = w_0 \end{cases}$$

defined on its maximal interval of existence $[0, \tau_{w_0})$ and such that

$$w \in C([0,\tau_{w_0}), Z^\alpha) \cap C^1((0,\tau_{w_0}), Z^\beta) \cap C((0,\tau_{w_0}), Z^1), \quad \beta \in [0,1).$$

Thanks to [17, Theorem 6.1] we have the next result.

Theorem 2.2. *Let $B : D(B) \subset Z \to Z$ be a positive sectorial operator. Then the infinitesimal generator of the strongly continuous semigroup $\{B^{-t}; t \geqslant 0\}$ in Z denoted by $-\log B$*

is such that its additive inverse $-(-\log B) = \log B$ *is a sectorial operator, not necessarily positive, on* Z.

By Theorem 2.2, we have the following result.

Corollary 2.3. *Let* $B : D(B) \subset Z \to Z$ *be a positive sectorial operator such that* $\log B : D(\log B) \subset Z \to Z$ *is a positive sectorial operator and,* $f : Z \to Z$ *be Lipschitz continuous on bounded subsets of* Z*. Then, for each* $w_0 \in Z$ *there exists a unique* $Z-$*solution* $w_\ell = w_\ell(t, w_0)$ *of*

$$\begin{cases} \dfrac{dw_\ell}{dt} + (\log B)w_\ell = f(w_\ell), \ t > 0, \\ w_\ell(0) = u_0 \end{cases} \tag{2.1}$$

defined on its maximal interval of existence $[0, \tau_{w_0})$ *and such that*

$$w_\ell(t) = B^{-t}w_0 + \int_0^t B^{-(t-s)}f(w_\ell(s))ds.$$

and

$$w_\ell \in C([0, \tau_{w_0}), Z) \cap C^1((0, \tau_{w_0}), Z) \cap C((0, \tau_{w_0}), D(\log B)).$$

Proof: Since $B : D(B) \subset Z \to Z$ is a positive sectorial operator such that $\log B : D(\log B) \subset Z \to Z$ is a positive sectorial operator, the result follows from Theorem 2.1.

2.2. **Damped wave operator.** In this subsection, we begin to study the parabolic problem (1.6). Initially, we recall that the eigenvalues of $\log A$ are given merely as $\log \mu_n$, where $\{\mu_n; n \in \mathbb{N}\}$ denotes the ordered sequence of eigenvalues of A including their multiplicity. We also stress the fact that the eigenfunctions of $\log A$ are the same as those of the operator A, see e.g. [12], [13] and [19].

From [3, Lemma 8.1] we have the following lemma.

Lemma 2.4. *If* A *and* Λ *are as in* (1.3) *and in* (1.5) *respectively then we have all the following.*
i) $0 \in \rho(\Lambda)$ *and*

$$\Lambda^{-1} = \begin{bmatrix} 2A^{-\frac{1}{2}} & A^{-1} \\ -I & 0 \end{bmatrix}.$$

ii) The adjoint Λ^* *of* Λ *is given by*

$$\Lambda^* = \begin{bmatrix} 0 & I \\ -A & 2A^{\frac{1}{2}} \end{bmatrix}.$$

iii) Λ *is sectorial operator in* $X^{\frac{1}{2}} \times X$ *with* $Re\sigma(\Lambda) > 0$*. The semigroup* $\{e^{-\Lambda t} : t \geqslant 0\}$ *is analytic and compact in* $X^{\frac{1}{2}} \times X$.
iv) Fractional powers Λ^α *can be defined for* $\alpha \in (0,1)$ *through*

$$\Lambda^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{-\alpha}(\lambda I + \Lambda)^{-1} d\lambda.$$

v) For each $\alpha \in (0,1)$ the operator Λ^α is a negative generator of an analytic C^0-semigroup $\{e^{-\Lambda^\alpha t} : t \geqslant 0\}$.
vi) Given any $0 < \alpha < 1$ we have that

$$\Lambda^{-\alpha} = \begin{bmatrix} (1+\alpha)A^{-\frac{\alpha}{2}} & \alpha A^{\frac{-1-\alpha}{2}} \\ -\alpha A^{\frac{1-\alpha}{2}} & (1-\alpha)A^{-\frac{\alpha}{2}} \end{bmatrix},$$

and

$$\Lambda^{\alpha} = \begin{bmatrix} (1-\alpha)A^{\frac{\alpha}{2}} & -\alpha A^{\frac{-1+\alpha}{2}} \\ \alpha A^{\frac{1+\alpha}{2}} & (1+\alpha)A^{\frac{\alpha}{2}} \end{bmatrix}. \tag{2.2}$$

vii) For each $\alpha \in (0,1]$ the spectrum of $-\Lambda^\alpha$ is the point spectrum consisting of eigenvalues

$$\lambda_{\alpha,n} = -\mu_n^{\frac{\alpha}{2}}, \quad n \in \mathbb{N},$$

where $\{\mu_n\}_{n\in\mathbb{N}}$ denotes the ordered sequence of eigenvalues of A including their multiplicity.

It follows from [1, Theorem 2.1.3, p. 289] that the semigroup generated by $-\Lambda$, that we denote by $\{e^{-\Lambda t} : t \geqslant 0\}$ is analytic and compact in $X^{\frac{1}{2}} \times X$.

Remark 2.5. *Consider the fractional partial differential equation*

$$\begin{cases} \partial_t^2 u_\alpha + (-\Delta_D)^\alpha u_\alpha + 2(-\Delta_D)^{\frac{\alpha}{2}} \partial_t u_\alpha = \alpha(-\Delta_D)^{\frac{-1+\alpha}{2}} f(u_\alpha), \\ u_\alpha(0,x) = u_0(x) \in X^{\frac{1}{2}},\ u_{\alpha t}(0,x) = u_1(x) \in X^{\frac{1-\alpha}{2}}, \end{cases}$$

for positive time and $x \in \Omega$. This problem for variable u_α comes from fractional formulation of the initial value problem in (1.4) for variable $\begin{bmatrix} u_\alpha \\ v_\alpha \end{bmatrix}$ with

$$v_\alpha = \alpha^{-1}(1-\alpha)(-\Delta_D)^{\frac{1}{2}} u_\alpha + \alpha^{-1}(-\Delta_D)^{\frac{1-\alpha}{2}} \partial_t u_\alpha,$$

that is, it can be rewritten in $X^{\frac{1}{2}} \times X$ as the parabolic Cauchy problem

$$\frac{d}{dt}\begin{bmatrix} u_\alpha \\ v_\alpha \end{bmatrix} + \Lambda^\alpha \begin{bmatrix} u_\alpha \\ v_\alpha \end{bmatrix} = \begin{bmatrix} 0 \\ f^e(u_\alpha) \end{bmatrix}, \ t > 0, \quad \begin{bmatrix} u_\alpha \\ v_\alpha \end{bmatrix}_{t=0} = \begin{bmatrix} u_0 \\ \alpha^{-1}(1-\alpha)(-\Delta_D)^{\frac{1}{2}} u_0 + \alpha^{-1}(-\Delta_D)^{\frac{1-\alpha}{2}} u_1 \end{bmatrix} \tag{2.3}$$

where Λ^α is the operator given by (2.2), see e.g. [4].

Combining the growing conditions and sign on the f (see (1.2)); a dissipative condition of the type $\limsup_{|s|\to\infty} \frac{f(s)}{|s|} < \mu_1$, with μ_1 being the first eigenvalue of the negative Dirichlet Laplacian $-\Delta_D$ in $L^2(\Omega)$; spectral properties of the operator Λ and its fractional powers (2.2) ($-\Lambda$ is the infinitesimal generator of compact analytic semigroup with exponential decay), and the properties of embedding of the fractional powers spaces $X^\alpha = D(A^\alpha)$ we can conclude that the initial value problems (1.4) and (2.3) are globally well-posed and they possess a global attractor in some space in the scale of fractional powers spaces of the operator Λ, for more details see [3, 5, 6, 7, 8].

From now on we consider the logarithmic counterpart of the problem (1.4) continuing the analysis in [3].

2.3. **Logarithmic operators.** In this section, we study the spectral properties of the unbounded linear operator that we will understand as being the logarithmic operator of Λ. Using properties of the logarithmic Dirichlet Laplacian operator, we have the following result.

Theorem 2.6. *Let* A *and* Λ *be operators as in* (1.3)-(1.5), *respectively. For each* $\begin{bmatrix} u \\ v \end{bmatrix} \in D(\log \Lambda)$ *we have*

$$
\begin{aligned}
(\log \Lambda)\begin{bmatrix} u \\ v \end{bmatrix} &= \begin{bmatrix} \frac{1}{2}\log A - I & -A^{-\frac{1}{2}} \\ A^{\frac{1}{2}} & \frac{1}{2}\log A + I \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \\
&:= \begin{bmatrix} (\log A^{\frac{1}{2}} - I)u - A^{-\frac{1}{2}}v \\ A^{\frac{1}{2}}u + (\log A^{\frac{1}{2}} + I)v \end{bmatrix}.
\end{aligned} \tag{2.4}
$$

Proof: By definition

$$
D(-\log \Lambda) = \left\{ \begin{bmatrix} u \\ v \end{bmatrix} \in Y; \exists \lim_{t \searrow 0} \frac{1}{t}(\Lambda^{-t} - I)\begin{bmatrix} u \\ v \end{bmatrix} \right\}
$$

with

$$
-\log \Lambda \begin{bmatrix} u \\ v \end{bmatrix} = \lim_{t \searrow 0} \frac{1}{t}(\Lambda^{-t} - I)\begin{bmatrix} u \\ v \end{bmatrix}
$$

for any $\begin{bmatrix} u \\ v \end{bmatrix} \in D(-\log \Lambda)$. Using the explicit formula for Λ^{-t} given in Lemma 2.4 (vi), we obtain

$$
\begin{aligned}
-\log \Lambda \begin{bmatrix} u \\ v \end{bmatrix} &= \lim_{t \searrow 0} \begin{bmatrix} \frac{1}{t}[(1+t)A^{-t/2} - I] & A^{-(1+t)/2} \\ -A^{(1-t)/2} & \frac{1}{t}[(1+t)A^{-t/2} - I] \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \\
&= \begin{bmatrix} (-\log A^{\frac{1}{2}} + I)u + A^{-\frac{1}{2}}v \\ -A^{\frac{1}{2}}u + (-\log A^{\frac{1}{2}} + I)v \end{bmatrix}.
\end{aligned}
$$

Clearly, $D(\log \Lambda) = D(-\log \Lambda)$ and

$$
\log \Lambda \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\log A - I & -A^{-\frac{1}{2}} \\ A^{\frac{1}{2}} & \frac{1}{2}\log A + I \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}.
$$

□

Using properties of the logarithmic Dirichlet Laplacian operator and the characterizations we can prove the following result.

Theorem 2.7. *Let* A *and* Λ *be operators as in* (1.3)-(1.5), *respectively. The point spectrum of* $\log \Lambda$ *consists of eigenvalues*

$$
\lambda = \log \mu_n^{\frac{1}{2}}, \ n \in \mathbb{N},
$$

where $\{\mu_n; n \in \mathbb{N}\}$ *denotes the ordered sequence of eigenvalues of* A *including their multiplicity.*

Proof: Let

$$(\log \Lambda) \begin{bmatrix} u \\ v \end{bmatrix} = \lambda \begin{bmatrix} u \\ v \end{bmatrix}$$

be the eingevalue problem for the logarithmic operator $\log \Lambda$. Thanks to Theorem 2.6 we obtain

$$\begin{cases} (\log A^{\frac{1}{2}} - I)u - A^{-\frac{1}{2}}v = \lambda u, \\ A^{\frac{1}{2}}u + (\log A^{\frac{1}{2}} + I)v = \lambda v. \end{cases} \tag{2.5}$$

Using the first equation in (2.5) we have

$$v = A^{\frac{1}{2}}(\log A^{\frac{1}{2}} - I)u - \lambda A^{\frac{1}{2}}u,$$

and using the second equation in (2.5) we obtain

$$u + (\log^2 A^{\frac{1}{2}} - I)u - \lambda(\log A^{\frac{1}{2}} + I)u = \lambda(\log A^{\frac{1}{2}} - I)u - \lambda^2 u.$$

Then

$$\lambda^2 - 2\log \mu_n^{\frac{1}{2}}\lambda + \log^2 \mu_n^{\frac{1}{2}} = 0,$$

for $n \in \mathbb{N}$.

Therefore

$$\lambda = \log \mu_n^{\frac{1}{2}},\ n \in \mathbb{N},$$

where $\{\mu_n; n \in \mathbb{N}\}$ denotes the ordered sequence of eigenvalues of A including their multiplicity. □

2.4. **Logarithmic equations.** We now can write an initial-boundary value problem for u_ℓ derived from abstract Cauchy problem (1.6) using the explicit formula for $\log \Lambda$ given by (2.4) with

$$v_\ell = (-\Delta_D)^{\frac{1}{2}}(\log(-\Delta_D)^{\frac{1}{2}} - I)u_\ell + (-\Delta_D)^{\frac{1}{2}}\partial_t u_\ell$$

as follows

$$\partial_t^2 u_\ell + \frac{1}{4}\log^2(-\Delta_D)u_\ell + \log(-\Delta_D)\partial_t u_\ell = (-\Delta_D)^{-\frac{1}{2}}f(u_\ell),\ t > 0,\ x \in \Omega, \tag{2.6}$$

since $det(-\log \Lambda) = \frac{1}{4}\log^2 A$, $tr(-\log \Lambda) = -\log A$ and the term first row and second column of $-\log \Lambda$ is equal to $A^{-\frac{1}{2}}$, see e.g. [4, Corollary 2.5]. The boundary and initial conditions are given by

$$\begin{cases} u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x), & x \in \Omega, \\ u(t,x) = 0, & t \geqslant 0,\ x \in \partial\Omega. \end{cases}$$

The following results deal with the well-posedness of the problem (1.6) in $X^{\frac{1}{2}} \times X$, in other words, we will prove that (2.6) is locally well-posed in $H_0^1(\Omega)$.

Theorem 2.8. *The linear Cauchy problem associated with* (1.6) *(with f identically zero)*
(2.7)
$$\frac{d}{dt}\begin{bmatrix}u_\ell\\ v_\ell\end{bmatrix}+(\log \Lambda)\begin{bmatrix}u_\ell\\ v_\ell\end{bmatrix}=0,\ t>0,\qquad \begin{bmatrix}u_\ell\\ v_\ell\end{bmatrix}_{t=0}=\begin{bmatrix}u_0\\ (-\Delta_D)^{\frac12}(\log(-\Delta_D)^{\frac12}-I)u_0+(-\Delta_D)^{\frac12}u_1\end{bmatrix},$$
is well-posed in $X^{\frac12}\times X$; in other words, the problem associated with the logarithmic linear wave equation
$$\begin{cases}\partial_t^2 u_\ell+\dfrac14\log^2(-\Delta_D)u_\ell+\log(-\Delta_D)\partial_t u_\ell=0, & t>0, x\in\Omega,\\ u_\ell(0,x)=u_0(x)\in X^{\frac12},\ \partial_t u_\ell(0,x)=u_1(x)\in X^{\frac12}, & t>0, x\in\Omega,\end{cases}$$
is well-posed in $H_0^1(\Omega)$ and the explicit solution is given by
$$u_\ell(t)=(-\Delta_D)^{-\frac t2}(I+t\log(-\Delta_D)^{\frac12})u_0+t(-\Delta_D)^{-\frac t2}u_1 \tag{2.8}$$

Proof: The well-posedness follows from the theory of fractional powers of densely defined closed operators and geometric theory of (semi)linear equations, see e.g. [1, Chapter III, p. 152 and 153] and [16, Chapter 1]. For the explicit solution u_ℓ, we consider the first component of the equation
$$\begin{bmatrix}u_\ell\\ v_\ell\end{bmatrix}(t)=\Lambda^{-t}\begin{bmatrix}u_0\\ v_0\end{bmatrix},\quad t\geqslant 0,$$
since $-\log\Lambda$ is the infinitesimal generator of the strongly continuous semigroup $\{\Lambda^{-t}:\ t\geqslant 0\}$. Note that Λ^{-t} was explicitly obtained in Lemma 2.4(vi). □

The next two propositions were proved in [2]; namely, see [2, Proposition 2.9] and [2, Proposition 2.10], respectively.

Proposition 2.9. *Suppose that* (1.2) *holds. Then, for all $u_1,u_2\in X^{\frac12}$,*
$$\|f(u_1)-f(u_2)\|_X\leqslant c\|u_1-u_2\|_{X^{\frac12}}(1+\|u_1\|^{\rho-1}_{X^{\frac12}}+\|u_2\|^{\rho-1}_{X^{\frac12}}).$$
Consequently, given $\begin{bmatrix}u_1\\ v_1\end{bmatrix},\begin{bmatrix}u_2\\ v_2\end{bmatrix}\in X^{\frac12}\times X$,
$$\Big\|F\Big(\begin{bmatrix}u_1\\ v_1\end{bmatrix}\Big)-F\Big(\begin{bmatrix}u_2\\ v_2\end{bmatrix}\Big)\Big\|_{X^{\frac12}\times X}\leqslant c\Big\|\begin{bmatrix}u_1\\ v_1\end{bmatrix}-\begin{bmatrix}u_2\\ v_2\end{bmatrix}\Big\|_{X^{\frac12}\times X}\Big(1+\Big\|\begin{bmatrix}u_1\\ v_1\end{bmatrix}\Big\|^{\rho-1}_{X^{\frac12}\times X}+\Big\|\begin{bmatrix}u_2\\ v_2\end{bmatrix}\Big\|^{\rho-1}_{X^{\frac12}\times X}\Big).$$

Proposition 2.10. *Suppose that* (1.2) *holds. Then, for all $u\in X^{\frac12}$, we get*
$$\|f(u)\|_X\leqslant c(1+\|u\|^{\rho}_{X^{\frac12}}).$$
Consequently, given $\begin{bmatrix}u\\ v\end{bmatrix}\in X^{\frac12}\times X$, we have
$$\Big\|F\Big(\begin{bmatrix}u\\ v\end{bmatrix}\Big)\Big\|_{X^{\frac12}\times X}\leqslant c\Big(1+\Big\|\begin{bmatrix}u\\ v\end{bmatrix}\Big\|^{\rho}_{X^{\frac12}\times X}+\Big\|\begin{bmatrix}u\\ v\end{bmatrix}\Big\|_{X^{\frac12}\times X}\Big).$$

Finally, the following result guarantees the local well-posedness of the problem (1.6) in Y.

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Theorem 2.11. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$, $N \geqslant 3$, and let μ_1 denote the first eigenvalue of A. Assume* (1.2) *and $\mu_1 > 1$. Then we have*

i) For any $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in Y$ *there exists a unique mild solution* $\begin{bmatrix} u_\ell \\ v_\ell \end{bmatrix} \in C([0, \tau_{u_0,v_0}), Y)$ *of* (1.6) *defined on a maximal interval of existence* $[0, \tau_{u_0,v_0})$*; namely*

$$\begin{bmatrix} u_\ell \\ v_\ell \end{bmatrix}(t) = \Lambda^{-t}\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} + \int_0^t \Lambda^{-(t-s)}\begin{bmatrix} 0 \\ f^e(u_\ell) \end{bmatrix}(s)ds, \quad t \geqslant 0,$$

This solution depends continuously on the initial data and satisfies a blow up alternative in Y. In particular, if $\left\| \begin{bmatrix} u_\ell \\ v_\ell \end{bmatrix} \right\|_Y$*−norm remains bounded as long as the solution exists then* $\tau_{u_0,v_0} = \infty$.

ii) For any $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in D(\log \Lambda)$*, the solution in part i) above is a regular solution. Namely,*

$$\begin{bmatrix} u_\ell \\ v_\ell \end{bmatrix} \in C([0, \tau_{u_0,v_0}), D(\log \Lambda)) \cap C^1((0, \tau_{u_0,v_0}), Y) \tag{2.9}$$

and $\begin{bmatrix} u_\ell \\ v_\ell \end{bmatrix}$ *satisfies* (1.6).

Proof: The part *i*) follows from Proposition 2.9 and the classical [21, Theorem 1.4, p. 185]. The part *ii*) follows from Proposition 2.9, Proposition 2.10 and the classical Theorem 2.1 and [21, Theorem 1.5, p. 185]. □

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References

[1] H. Amann, Linear and quasilinear parabolic problems. Volume I: Abstract Linear Theory, Birkhäuser Verlag, Basel, 1995.

[2] F. D. M. Bezerra, A Second-Order Evolution Equation and Logarithmic Operators, Bull. Brazilian Math. Soc., New Series, **53** (2022) 571–593.

[3] F. D. M. Bezerra, A. N. Carvalho and M. J. D. Nascimento, Fractional approximations of abstract semilinear parabolic problems, Discrete Contin. Dyn. Syst., **450** (2017), 377-405.

[4] F. D. M. Bezerra and L. A. Santos, An extended version of the Cayley-Hamilton-Ziebur Theorem, arXiv:2210.01976, 2023. https://doi.org/10.48550/arXiv.2210.01976

[5] A. N. Carvalho and J. W. Cholewa, Local well posedness for strongly damped wave equations with critical nonlinearities, Bull. Austral. Math. Soc. **66** (2002), 443-463.

[6] A. N. Carvalho and J. W. Cholewa, Strongly damped wave equations in $W_0^{1,p}(\Omega) \times L^p(\Omega)$, Discrete and Continuous Dynamical Systems, Supplement 2007, 230-239.

[7] A. N. Carvalho, J. W. Cholewa and T. Dlotko, Damped wave equations with fast growing dissipative nonlinearities, Discrete Contin. Dyn. Syst. **24** (4) (2009), 1147-1165.

[8] A. N. Carvalho and S. Piskarev, A general approximation scheme for attractors of abstract parabolic problems, Numer. Funct. Anal. Optim. **27** (7-8) (2006), 785-829.

[9] R. C. Charão, M. D'Abbicco and R. Ikehata, Asymptotic profiles for a wave equation with parameter dependent logarithmic damping, Math. Methods Appl. Sci., bf 44 18 (2021) 14003–14024.

[10] R. C. Charão and R. Ikehata, Asymptotic profile and optimal decay of solutions of some wave equations with logarithmic damping, Z. Angew. Math. Phys. **71** (2020) 148.

[11] R. C. Charão, R. Ikehata and A. Piske, A dissipative logarithmic type evolution equation: asymptotic profile and optimal estimates, J. Math. Anal. Appl. **506** (2022) 125587.

[12] H. Chen and L. Veron, Bounds for eigenvalues of the Dirichlet problem for the logarithmic Laplacian, Adv. Calc. Var., 2022, https://doi.org/10.1515/acv-2021-0025.

[13] H. Chen and T. Weth, The Dirichlet problem for the logarithmic Laplacian, Commun. Partial. Differ. Equ., **44**, 11 (2019), 1100–1139.

[14] M.A. Fury, Logarithmic regularization of non-autonomous non-linear ill-posed problems in Hilbert spaces, Electron. J. Differ. Equ. 2018 (28) (2018) 1–11.

[15] M.A. Fury, Logarithmic well-posed approximation of the backward heat equation in Banach space, J. Math. Anal. Appl. 475 (2019) 1367–1384.

[16] D. Henry, Geometric theory of semilinear parabolic equations, Lecture Notes in Mathematics 840, Springer-Verlag, Berlin, 1981.

[17] M. Haase, Spectral properties of operator logarithms, Math. Z. 245 (2003) 761–779.

[18] G. L. Krabbe, On the logarithm of a uniform bounded opeartor, Trans. Am. Math. Soc. **81**, No. 1 (1956) 155–166.

[19] A. Laptev and T. Weth, Spectral properties of the logarithmic Laplacian, Anal. Math. Phys. (2021) 11:133.

[20] V. Nollau, Über den Logarithmus abgeschlossener Operatoren in Banaschen Raümen, Acta Sci. Math. (Szeged) 30 (1969), 161–174.

[21] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York, 1983.

[22] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. I: Functional Analysis*, 1972 hardcover published by Academic Press.

[23] J. Weilenmann, Continuity properties of fractional powers, of the logarithm, and of holomorphic semigroups, J. Funct. Anal., **27**, (1978) 1–20.

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