CONSTRUCTION OF JACOBI CUSP FORMS USING ADJOINT OPERATOR OF CERTAIN DIFFERENTIAL OPERATOR

SHIVANSH PANDEY

ABSTRACT. In this paper we construct Jacobi cusp forms by computing the adjoint of certain differential operator with respect to the Petersson scalar product in the case of Jacobi forms and Hermitian Jacobi forms. Jacobi cusp forms constructed by this method involve special values of certain Dirichlet series attached to the considered Jacobi cusp forms.

1. INTRODUCTION

Modular forms are important objects in number theory and it has wide range of applications in all other branches of Mathematics as well as in Physics. Construction of modular forms is one of the most important problem in the theory of modular forms. It is well-known that the derivative of a modular form is not necessarily a modular form. There are several ways to construct modular forms using derivatives, we mention two of them here (see [26] pp. 48 for more details). The first one is to modify the differential operator called Serre derivative. The second one is to take an appropriate linear combination of higher order derivatives called Rankin-Cohen brackets. The Rankin-Cohen bracket is generalization of the product. Kohnen [15] constructed cusp forms by computing the adjoint of the product map w.r.t. the Petersson scalar product. The Fourier coefficients of constructed cusp form involve special values of certain Dirichlet series. Herrero [9] extended the work of Kohnen [15] and computed the adjoint of certain linear maps constructed using Rankin-Cohen brackets. The work of Herrero has been generalized by several authors for various automorphic forms ([10, 11, 12, 14, 24]). Recently, Kumar [13] constructed cusp forms by computing the adjoint of Serre derivative.

Jacobi forms are natural generalization of modular forms to several variable case. Jacobi forms were first studied systematically by Eichler and Zagier in [7] and they played a key role in the proof of Saito-Kurokawa conjecture. In the case of Jacobi forms the classical heat operator plays an important role. One of the important uses of the heat operator is to construct certain bilinear holomorphic differential operators on the space of Jacobi cusp forms. For Jacobi forms, we have modified heat operator (analogue of Serre derivative)[2, 21] and the Rankin-Cohen brackets (defined in [2, 3, 5, 17]). These operators give rise to certain linear operators between spaces of Jacobi forms. Hermitian Jacobi forms are generalization of Jacobi forms. These were first introduced by Haverkamp [8] and studied by Richter and Senadheera [22].

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The main aim of this paper is to extend the work of Kumar[13] in the case of Jacobi forms, Jacobi forms of several variables and Hermitian Jacobi forms by computing the adjoint of the modified heat operator w.r. t. the Petersson scalar product.

2. Preliminaries

In this section we recall the basic definition and some properties of Jacobi forms and Hermitian-Jacobi forms which are required to state and prove our results.

2.1. **Jacobi forms.** Let $\mathbb{C}$ and $\mathcal{H}$ denote the complex plane and the complex upper half-plane, respectively. For a complex number a, we use the following notation: $e(a) := exp(2\pi i a)$. An $n \times n$ matrix $M = (m_{ij})_{1\leq i,j\leq n}$ is said to be half-integral matrix if $m_{ii}, 2m_{ij} \in \mathbb{Z},\ 1 \leq i, j \leq n$.

Let g be a fixed positive integer. The Jacobi group $\Gamma_g^J := SL_2(\mathbb{Z}) \ltimes (\mathbb{Z}^g \times \mathbb{Z}^g)$ acts on $\mathcal{H} \times \mathbb{C}^g$ as follows;

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu)\right) \cdot (\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d}\right).$$

Let $k \in \mathbb{Z}$ and M be a positive definite, symmetric, half-integral $g\times g$ matrix. For a complex-valued holomorphic function ϕ defined on $\mathcal{H} \times \mathbb{C}^g$ and $\gamma = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu)\right) \in \Gamma_g^J$, we define the slash operator as follows:

$$(\phi|_{k,M}\ \gamma)(\tau, z) := (c\tau + d)^{-k} e\left(\frac{-c}{c\tau + d} M[z + \lambda\tau + \mu] + M[\lambda]\tau + 2\lambda^t M z\right) \phi(\gamma \cdot (\tau, z)),$$

where $A[B] := B^t AB$, with A and B are matrices of appropriate size and B^t denotes the transpose of the matrix B.

Definition 2.1. *A complex-valued holomorphic function $\phi : \mathcal{H} \times \mathbb{C}^g \to \mathbb{C}$ is said to be a Jacobi form of weight k and index M if it satisfies $\phi|_{k,M}\ \gamma = \phi$, $\forall \gamma \in \Gamma_g^J$ and ϕ has a Fourier expansion of the form*

$$\phi(\tau, z) = \sum_{\substack{n\in\mathbb{Z},\ r\in\mathbb{Z}^g,\\ 4n\geq M^{-1}[r^t]}} c_\phi(n, r)\ e(n\tau + rz).$$

Here, $r \in \mathbb{Z}^g$, denotes a row vector. Further, we say ϕ is a cusp form if $c_\phi(n, r) = 0$ whenever $4n = M^{-1}[r^t]$.

We denote the space of all Jacobi forms (resp. Jacobi cusp forms) of weight k and index M on Γ_g^J by $J_{k,M}$ (resp. $J_{k,M}^{cusp}$). Let $\phi, \psi \in J_{k,M}$ be such that at least one of them is cusp form. Then the Petersson scalar product of ϕ and ψ is defined by

$$\langle \phi, \psi \rangle := \int\limits_{\Gamma_g^J \backslash \mathcal{H}\times\mathbb{C}^g} \phi(\tau, z)\ \overline{\psi(\tau, z)}\ v^k\ e(-4\pi M[y]v^{-1}) dV_g^J,$$

where $\tau = u + iv \in \mathcal{H}, z = x + iy \in \mathbb{C}^g$ and $dV_g^J = \dfrac{dudvdxdy}{v^{g+2}}$ is an invariant measure under the action of Γ_g^J on $\mathcal{H} \times \mathbb{C}^g$. The space $J_{k,M}^{cusp}$ with the Petersson scalar product defined above forms a finite dimensional Hilbert space. For more details on the theory of Jacobi forms of several variables, we refer to [27].

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2.2. **Poincaré series.** Let M be a fixed positive definite, symmetric, half-integral $g \times g$ matrix, $N \in \mathbb{Z}$, and $R \in \mathbb{Z}^g$ be such that $4N > M^{-1}[R^t]$. The $(N,R)^{th}$ Poincaré series of weight k and index M is defined by

$$P_{k,M}^{N,R}(\tau, z) := \sum_{\gamma \in \Gamma_{1,g,\infty}^J \backslash \Gamma_{1,g}^J} e(N\tau + Rz)|_{k,M}\ \gamma, \tag{1}$$

where $\Gamma_{g,\infty}^J := \left\{ \left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, (0,\mu) \right) : t \in \mathbb{Z}, \mu \in \mathbb{Z}^g \right\}$ is the stabilizer of $e(N\tau + Rz)$ in Γ_g^J. It is well-known that $P_{k,M}^{N,R} \in J_{k,\ M}^{cusp}$ for $k > g+2$. The space of Jacobi cusp forms is generated by Poincaré series. The Poincaré series has the following property which will be crucial in the proof of our result. For more details on Poincaré series we refer to [1].

Lemma 2.2. [1] *Let $\phi \in J_{k,\ M}^{cusp}$ with Fourier expansion $\phi(\tau,z) = \sum_{\substack{n \in \mathbb{Z},\ r \in \mathbb{Z}^g \\ 4n > M^{-1}[r^t]}} c_\phi(n,r) e(n\tau + rz)$. Then*

$$\langle \phi, P_{k,M}^{N,R} \rangle = \lambda_{\mathcal{K},M,D}\ c_\phi(N,R), \tag{2}$$

where $\lambda_{\mathcal{K},M,D} = 2^{\mathcal{K}(g-1)-g}\Gamma(\mathcal{K})\pi^{-\mathcal{K}}(det\ M)^{\mathcal{K}-\frac{1}{2}}D^{-\mathcal{K}}$, with $\mathcal{K} = k - \frac{g}{2} - 1,\ D = det(T),\ T = \begin{pmatrix} 2n & r \\ r^t & 2M \end{pmatrix}$.

2.3. **Heat Operator for Jacobi forms.** We now define the heat operator which acts on the space of Jacobi forms. For a positive definite, symmetric, half-integral $g \times g$ matrix M, we define the heat operator by

$$L_M := \frac{1}{(2\pi i)^2}\left(8\pi i |M|\ \frac{\partial}{\partial \tau} - \sum_{1 \le i,j \le g} M_{ij} \frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j} \right),$$

where $\tau \in \mathcal{H}$ and $z^t = (z_1, z_2, \cdots, z_g) \in \mathbb{C}^g$ and M_{ij} is the (i,j)-th cofactor of the matrix M.

Remark 2.1. Note that for $g = 1$, the above heat operator is the classical heat operator [7, 20] defined by

$$L_m := \frac{1}{(2\pi i)^2}\left(8\pi i m \frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial z^2} \right).$$

Also note that the action of L_M on $e(n\tau + rz)$ is given by

$$L_M(e(n\tau + rz)) = (4n|M| - \widetilde{M}[r^t])e(n\tau + rz),$$

where $\widetilde{M}$ denotes the matrix of cofactors M_{ij} of the matrix M.

Lemma 2.3. [18] *Let $\phi \in J_{k,M}$. Then for $k \in \mathbb{Z}^+,\ \nu \ge 0$ and $A = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, we have*

$$(L_M\phi)|_{k+2,M}A = L_M(\phi|_{k,M}A) + \frac{2|M|(\mathcal{K}+1)}{\pi i}\left(\frac{c}{c\tau + d} \right)(\phi|_{k,M}A). \tag{3}$$

It is immediate from the above lemma that for a Jacobi form ϕ of weight k and index m, $L_M\phi$ is not a Jacobi form of weight $k+2$ and index M. We now define the modified heat operator which maps Jacobi forms to Jacobi forms as:

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$$L_{k,M} := L_M - \frac{(k-\frac{g}{2})|M|}{3} E_2, \tag{4}$$

where $E_2 = 1 - 24\sum_{n=1}^{\infty}\sigma(n)q^n$ is the Eisenstein series . A routine calculation shows that:

Lemma 2.4. *The operator $L_{k,M}$ maps a Jacobi form (resp. Jacobi cusp forms) of weight k and index m to a Jacobi form (resp. Jacobi cusp forms) of weight $k+2$ and index M.*

2.4. **Hermitian-Jacobi forms.** Hermitian Jacobi forms are well studied in [8, 22]. In this section we recall the definition and basic properties of Hermitian-Jacobi forms.

Let $\mathcal{O}$ be ring of integers of $\mathbb{Q}(i)$, i.e. $\mathcal{O} = \mathbb{Z}[i]$. $\mathcal{O}^\times = \{\pm 1, \pm i\}$ be the set of units in $\mathcal{O}$. Let $\mathcal{O}^\sharp = \frac{i}{2}\mathcal{O}$. We define $\Gamma(\mathcal{O}) := \{\epsilon M | \epsilon \in \mathcal{O}^\times, M \in SL_2(\mathbb{Z})\}$ and the Hermitian-Jacobi group by

$$\Gamma^J(\mathcal{O}) := \Gamma(\mathcal{O}) \ltimes \mathcal{O}^2 = \{\gamma = (\epsilon M, X) | M \in \Gamma(\mathcal{O}), X \in \mathcal{O}^2\}.$$

The Hermitian-Jacobi group acts on $\mathcal{H} \times \mathbb{C}^2$ by

$$\gamma \cdot (\tau, z, w) = \left(\frac{a\tau+b}{c\tau+d}, \epsilon\frac{z+\lambda\tau+\mu}{c\tau+d}, \bar{\epsilon}\frac{w+\overline{\lambda}\tau+\overline{\mu}}{c\tau+d}\right),$$

where $\gamma = \left(\epsilon\begin{pmatrix} a & b \\ c & d\end{pmatrix}, [\lambda,\mu]\right)$.

Let $k \in \mathbb{Z}$ and m be a positive integer. For a complex-valued holomorphic function ϕ, $\delta \in \{+,-\}$ and $\gamma = \left(\epsilon\begin{pmatrix} a & b \\ c & d\end{pmatrix}, [\lambda,\mu]\right)$, we define the slash operator as follows:

$$\phi|_{k,m,\delta}\gamma := \sigma(\epsilon)\epsilon^{-k}(c\tau+d)^{-k}e^m\left(-\frac{c(z+\lambda\tau+\mu)(w+\overline{\lambda}\tau+\overline{\mu})}{c\tau+d} + \lambda\overline{\lambda}\tau + \overline{\lambda}z + \lambda w\right)\phi(\gamma\cdot(\tau,z,w)),$$

where $\sigma(\epsilon) = \begin{cases} 1, & \text{if } \delta = + \\ \epsilon^2 & \text{if } \delta = -. \end{cases}$

Definition 2.5. *A holomorphic function $\phi : \mathcal{H}\times\mathbb{C}^2 \to \mathbb{C}$ is said to be a Hermitian Jacobi form of weight k, index m and parity δ on $\Gamma^J(\mathcal{O})$ if for each $\gamma \in \Gamma^J(\mathcal{O})$ we have $\phi\mid_{k,m,\delta}\gamma = \phi$ and ϕ has Fourier expansion of the form*

$$\phi(\tau,z,w) = \sum_{\substack{n\in\mathbb{Z}, r\in\mathcal{O}^\sharp \\ nm-|r|^2\geqslant 0}} c_\phi(n,r)e(n\tau + rz + \overline{r}w).$$

If $c_\phi(n,r) = 0$ whenever $nm - |r|^2 = 0$, we say ϕ is a Hermitian Jacobi cusp forms.

We denote the space of all Hermitian-Jacobi forms (resp. Hermitian-Jacobi cusp forms) of weight k, index m and parity δ on $\Gamma^J(\mathcal{O})$ by $J_{k,m,\delta}(\mathcal{O})$ (resp. $J^{cusp}_{k,m,\delta}(\mathcal{O})$). We define the Petersson scalar product on $J^{cusp}_{k,m,\delta}(\mathcal{O})$

$$\langle\phi,\psi\rangle := \int_{\Gamma^J(\mathcal{O})\backslash\mathcal{H}\times\mathbb{C}^2} \phi(\tau,z,w)\overline{\psi(\tau,z,w)}v^k e^{\frac{-\pi m}{v}|w-\overline{z}|^2} dV_J,$$

where $\tau = u+iv, z = x_1+iy_1, w = x_2+iy_2$ and $dV_J = \dfrac{dudvdx_1dy_1dx_2dy_2}{v^4}$ is an invariant measure under the action of $\Gamma^J(\mathcal{O})$ on $\mathcal{H}\times\mathbb{C}^2$.

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Definition 2.6. *For fixed $m \in \mathbb{N}$, $N \in \mathbb{N}$ and $R \in \mathcal{O}^\sharp$ with $|R|^2 < mN$, the $(N,R)^{th}$ Poincaré series of weight k, index m and parity δ is defined as*

$$P^{N,R}_{k,m,\delta}(\tau,z,w) := \sum_{\gamma \in \Gamma^J_\infty(\mathcal{O}) \backslash \Gamma^J(\mathcal{O})} e(N\tau + Rz + \overline{R}w)|_{k,m,\delta}\gamma(\tau,z,w), \tag{5}$$

where $\Gamma^J_\infty(\mathcal{O}) := \left\{ \left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, (0,\mu) \right) | t, \mu \in \mathcal{O} \right\}$ *is the stabilizer of $e(N\tau + RZ + \overline{R}w)$ in Γ^J. It is well-known that $P^{N,R}_{k,m,\delta} \in J^{cusp}_{k,m,\delta}$ for $k > 4$.*

The space of Hermitian-Jacobi cusp forms is generated by Poincaré series. The Poincaré series has the following property which will be crucial in the proof of our result.

Lemma 2.7. *Let $\phi \in J^{cusp}_{k,m,\delta}(\mathcal{O})$ with Fourier expansion $\phi(\tau,z) = \sum_{\substack{n\in\mathbb{Z}, r\in\mathcal{O}^\sharp,\\ nm-|r|^2>0}} c_\phi(n,r)e(n\tau + rz + \overline{r}w)$. Then*

$$\langle \phi, P^{N,R}_{k,m,\delta} \rangle = \frac{m^{k-3}\Gamma(k-2)}{(4\pi)^{k-2}(Nm - |R|^2)^{k-2}} c_\phi(N,R). \tag{6}$$

2.5. **Heat Operator for Hermitian-Jacobi forms.** For a natural number m we define the heat operator

$$\mathcal{L}_m := \frac{1}{(2\pi i)^2}\left(8\pi i m \frac{\partial}{\partial \tau} - 4\frac{\partial^2}{\partial w \partial z} \right).$$

The operator $\mathcal{L}_m$ does not map Hermitian-Jacobi forms to Hermitian-Jacobi forms. Therefore, we modify this heat operator to obtain an operator $\mathcal{L}_{k,m}$ which maps Hermitian-Jacobi forms to Hermitian-Jacobi forms. The modified heat operator $\mathcal{L}_{k,m}$ is defined as

$$\mathcal{L}_{k,m} := \mathcal{L}_m - \frac{k-1}{3} m E_2. \tag{7}$$

The operator $\mathcal{L}_{k,m}$ satisfies the following functional equation

$$\mathcal{L}_{k,m}\phi \mid_{k+2,m,\delta} M = \mathcal{L}_{k,m}(\phi \mid_{k,m,-\delta}), M \in SL_2(\mathbb{Z}).$$

The above equation implies that the operator maps a Hermitian Jacobi (resp. Hermitian Jacobi cusp) to a Hermitian Jacobi form with opposite parity (Hermitian Jacobi cusp form).

3. Main Results

The modified heat operator $L_{k,M}$ defined in (4) is a $\mathbb{C}$-linear map between finite dimensional Hilbert spaces $J^{cusp}_{k,M}$ and $J^{cusp}_{k+2,M}$. Therefore it has an adjoint map $L^*_{k,M} : J^{cusp}_{k+2,M} \to J^{cusp}_{k,M}$ such that

$$\langle L^*_{k,M}(\phi), \psi \rangle = \langle \phi, L_{k,M}(\psi) \rangle \ \forall \phi \in J^{cusp}_{k+2,M} \text{ and } \psi \in J^{cusp}_{k,M}.$$

The next theorem gives us the image of a Jacobi cusp form ϕ under the map $L^*_{k,M}$.

Theorem 3.1. *Let $k > 4$ and m be natural number. Let $\phi \in J^{cusp}_{k+2,M}$ with Fourier expansion $\phi(\tau,z) = \sum_{\substack{n,r\in\mathbb{Z}^g,\\ 4n>M^{-1}[rt]>0}} c_\phi(n,r)q^N\zeta^R$. Then the image of ϕ under $L^*_{k,M}$ is given by*

$$L^*_{k,M}(\phi)(\tau,z) = \sum_{\substack{N,R\in\mathbb{Z}^g,\\ 4N-M^{-1}[R^t]>0}} a(N,R)q^n\zeta^r,$$

where

$$a(N,R) = \frac{|M|^{\frac{5-g}{2}}(\mathcal{K}+1)(\mathcal{K})(4N|M|-\tilde{M}[R])^{\mathcal{K}}}{\pi^2 2^{(g-1)(k-\frac{g}{2}-1)}}\Bigg[\frac{\left(4N|M|-\tilde{M}[R]-\frac{\mathcal{K}|M|}{3}\right)}{(4N|M|-\tilde{M}[R])^{+2}}c_\phi(N,R) + 8(\mathcal{K}+1)|M|\sum_{n\geqslant 1}\frac{c_\phi(n+N,R)\sigma(n)}{(4(n+N)|M|-\tilde{M}[R])^{\mathcal{K}+2}}\Bigg],$$

where $\mathcal{K} = k - \frac{g}{2} - 1$.

We now state the above result in the case of $g = 1$.

Corollary 3.2. *Let $k > 4$ and m be natural number. Let $\phi \in J^{cusp}_{k+2,m}$ with Fourier expansion $\phi(\tau,z) = \sum_{\substack{n,r\in\mathbb{Z},\\ 4mn-r^2>0}} c_\phi(n,r)q^n\zeta^r$. Then the image of ϕ under $L^*_{k,m}$ is given by*

$$L^*_{k,m}(\phi)(\tau,z) = \sum_{\substack{N,R\in\mathbb{Z},\\ 4mN-R^2>0}} a(N,R)q^N\zeta^R,$$

where

$$a(N,R) = \frac{(4mN-R^2)^{k-\frac{3}{2}}m^2(k-\frac{1}{2})(k-\frac{3}{2})}{\pi^2}\Bigg[\frac{(4Nm-R^2)-\frac{2k-1}{6}m}{(4Nm-R^2)^{k+\frac{1}{2}}}c_\phi(N,R) + 4m(2k-1)\sum_{n\geqslant 1}\frac{c_\phi(n+N,R)\sigma(n)}{(4(n+N)m-R^2)^{k+\frac{1}{2}}}\Bigg]. \tag{8}$$

Next, we consider the modified heat operator $\mathcal{L}_{k,m}$ defined in (7). The operator $\mathcal{L}_{k,m}$ is a $\mathbb{C}$-linear map between finite dimensional Hilbert spaces $J^{cusp}_{k,m,-\delta}(\mathcal{O})$ and $J^{cusp}_{k+2,m,\delta}(\mathcal{O})$. Therefore it has an adjoint map $\mathcal{L}^*_{k,m} : J^{cusp}_{k+2,m,\delta}(\mathcal{O}) \to J^{cusp}_{k,m,-\delta}(\mathcal{O})$ such that

$$\langle \mathcal{L}^*_{k,m}(\phi), \psi\rangle = \langle \phi, \mathcal{L}_{k,m}(\psi)\rangle = \ \forall \phi \in J^{cusp}_{k+2,m,\delta}(\mathcal{O}) \text{ and } \psi \in J^{cusp}_{k,m,\delta}(\mathcal{O}).$$

The next theorem gives us the image of a Jacobi cusp form ϕ under the map $\mathcal{L}^*_{k,m}$.

Theorem 3.3. *Let $k > 4$ and m be natural number. Let $\phi \in J^{cusp}_{k+2,m,\delta}(\mathcal{O})$ with Fourier expansion $\phi(\tau,z) = \sum_{\substack{n\in\mathbb{Z},r\in\mathcal{O}^\sharp,\\ 4mn-|r|^2>0}} c_\phi(n,r)e(n\tau+rz+\overline{r}w)$. Then the image of ϕ under $\mathcal{L}^*_{k,m}$ is given by*

$$\mathcal{L}^*_{k,m}(\phi)(\tau,z) = \sum_{\substack{N\in\mathbb{Z},R\in\mathcal{O},\\ 4mN-|R|^2>0}} a(N,R)e(N\tau+Rz+\overline{R}w),$$

where

$$a(N,R) = \frac{(mN-|R|^2)^{k-2}m^2(k-1)(k-2)}{(4\pi)^2}\Bigg[\frac{(4Nm-4|R|^2)-\frac{k-1}{3}m}{(Nm-|R|^2)^k}c_\phi(N,R) + 8m(k-1)\sum_{n\geqslant 1}\frac{c_\phi(n+N,R)}{((n+N)m-|R|^2)^k}\Bigg]. \tag{9}$$

4. Proof of Theorems

First we state a lemma which we shall use to prove Theorem 3.1.

Lemma 4.1. *Let* $\phi \in J^{cusp}_{k+2,M}$. *Then the sum*

$$\sum_{\gamma \in \Gamma^J_{g,\infty}\backslash \Gamma^J_g} \int_{\Gamma^J_g \backslash \mathcal{H}\times\mathbb{C}^g} \mid \phi(\tau,z)\overline{L_{k,M}\left(e^{2\pi i(N\tau+Rz)} \mid_{k,M} \gamma\right)} v^{k+2} e^{\frac{-4\pi M[y]}{v}} \mid dV_J$$

converges.

Proof. For a proof we refer to [10]. ☐

4.1. **Proof of Theorem 3.1.** Let $L^*_{k,M}(\phi)(\tau,z) = \sum_{\substack{n,r\in\mathbb{Z}^g,\\ 4n>M^{-1}[r^t]>0}} a(n,r)e(n\tau+rz)$. By Lemma 2.2 and property of adjoint map we have

$$a(N,R) = \frac{1}{\lambda_{\mathcal{K},M,D}} \langle \phi, L_{k,M}(P^{N,R}_{k,M})\rangle. \tag{10}$$

We now compute $\langle \phi, L_{k,M}(P^{N,R}_{k,M})\rangle$. A simple calculation using Lemma 4.1 and usual Rankin's unfolding argument shows that

$$\langle \phi, L_{k,M}(P^{N,R}_{k,M})\rangle = (4N|M| - \tilde{M}[R^t] - \frac{(k-\frac{g}{2})}{3})I_1 + 8(k-\frac{g}{2})mI_2, \tag{11}$$

where $I_1 = \int_{\Gamma^J_{g,\infty}\backslash\mathcal{H}\times\mathbb{C}^g} \phi(\tau,z)\overline{e(N\tau+Rz)}\ v^{k+2}e^{\frac{-4\pi M[y]}{v}} dV_J$ and

$I_2 = \int_{\Gamma^J_{g,\infty}\backslash\mathcal{H}\times\mathbb{C}^g} \phi(\tau,z)\overline{\left(\sum_{j\geq 1}\sigma(j)e(j\tau)\right)\ e(N\tau+Rz)}\ v^{k+2}e^{\frac{-4\pi M[y]}{v}} dV_J.$

We put $\tau = u+iv$ and $z = x+iy$, where $x = (x_1,x_2,...,x_g)$ and $y = (y_1,y_2,...,y_g)$. A fundamental domain for action of $\Gamma^J_{g,\infty}$ on $\mathcal{H}\times\mathbb{C}^g$ is given by $\{(\tau,z)\in\mathcal{H}\times\mathbb{C}^g : 0 \leqslant u \leqslant 1, v>0, x_i\in[0,1], y\in\mathbb{R}^g\}$. Integrating over this region we get

$$I_1 = \frac{|M|^{k+1-g}\Gamma(k-\frac{g}{2}+1)}{2^g\pi^{k-\frac{g}{2}+1}} \frac{c_\phi(N,R)}{(4N|M|-\tilde{M}[R^t])^{k-\frac{g}{2}+1}},$$

and

$$I_2 = \frac{|M|^{k+1-g}\Gamma(k-\frac{g}{2}+1)}{2^g\pi^{k-\frac{g}{2}+1}} \sum_{n\geqslant 1} \frac{c_\phi(n+N,R)\sigma(n)}{(4(n+N)|M|-\tilde{M}[R^t])^{k-\frac{g}{2}+1}}.$$

Substituting values of I_1 and I_2 in (11), finally from (10) we have the Fourier coefficient of the adjoint map of the heat operator

$$\begin{aligned} a(N,R) &= \frac{|M|^{\frac{5-g}{2}}(\mathcal{K}+1)\mathcal{K}(4N|M|-\tilde{M}[R])^{\mathcal{K}}}{\pi^2 2^{(g-1)\mathcal{K}}}\Bigg[\frac{\left(4N|M|-\tilde{M}[R]-\frac{\mathcal{K}|M|}{3}\right)}{(4N|M|-\tilde{M}[R])^{\mathcal{K}+2}}c_\phi(N,R) \\ &\quad +8(\mathcal{K}+1)|M|\sum_{n\geqslant 1}\frac{c_\phi(n+N,R)\sigma(n)}{(4(n+N)|M|-\tilde{M}[R])^{\mathcal{K}+2}}\Bigg]. \end{aligned}$$

4.2. **Another proof of Theorem 3.1.** We give another proof of Theorem 3.1 using Zagier's technique by representing $L_{k,m}(P_{k,m}^{N,R})$ in terms of Poincaré series. For simplicity we prove the theorem for $g=1$. For $g>1$ proof is similar. First we state a lemma which we will use in the proof.

Lemma 4.2. *The series* $v^{k+2}e^{-2pim\frac{y^2}{v}}\sum_{n\in\mathbb{Z},\ \gamma\in\Gamma_\infty^J\backslash\Gamma^J}\sigma(n)e^{2\pi i((n+N)\tau+rz)}|_{k+2,m}\gamma$ *is absolutely uniformly convergent on subsets* $V_{\epsilon,C}=\{(\tau,z)\in\mathcal{H}\times\mathbb{C}|v\geqslant\epsilon,\ |yv^{-1}\leqslant C,\ |x|\leqslant\frac{1}{\epsilon},\ u\leqslant\frac{1}{\epsilon}\}$ *for given* $\epsilon>0$ *and* $C>0$.

For a proof we refer to [4]. Now we prove the theorem. We first prove the following identity:

$$L_{k,m}(P_{k,m}^{N,R})=(4Nm-R^2-\frac{2k-1}{6})P_{k+2,m}^{N,R}+4(2k-1)m\sum_{n\geqslant1}\sigma(n)P_{k+2,m}^{n+N,R}. \qquad (12)$$

By definition $L_{k,m}(P_{k,m}^{N,R})$ equals

$$L_{k,m}(\sum_{\gamma\in\Gamma_\infty^J\backslash\Gamma^J}e^{2\pi i(N\tau+Rz)}|_{k,m}\gamma)=\sum_{\gamma\in\Gamma_\infty^J\backslash\Gamma^J}L_{k,m}(e^{2\pi i(N\tau+Rz)}|_{k,m}\gamma)$$

$$=(4Nm-R^2-\frac{(2k-1)m}{6})P_{k+2,m}^{N,R}+4(2k-1)m\sum_{n\geqslant1}\sigma(n)\left(\sum_{\gamma\in\Gamma_\infty^J\backslash\Gamma^J}e^{2\pi i((N+n)\tau+Rz)}|_{k+2,m}\gamma\right)$$

$$=\ (4Nm-R^2-\frac{(2k-1)m}{6})P_{k+2,m}^{N,R}+4(2k-1)m\sum_{n\geqslant1}\sigma(n)P_{k+2,m}^{n+N,R},$$

where in rearranging the sum we have used Lemma 4.2. Now theorem follows by taking inner product of a Jacobi cusp form $\phi\in J_{k+2,m}^{cusp}$ with both sides of eq.(12).

4.3. **Proof of Theorem 3.3.** Detailed proof of Theorem 3.3 is omitted. Key points in the proof are Rankin unfolding arguments for Hermitian Jacobi forms setup and the fact that a fundamental domain for action of $\Gamma_\infty^J(\mathcal{O})$ on $\mathbb{H}\times\mathbb{C}^2$ is given by $([0,1]\times[0,\infty])\times([0,1]\times[0,1])(\mathbb{R}\times\mathbb{R})$. One can calculate analogous integrals I_1 and I_2 as in case of Jacobi forms of matrix index to obtain the proof of Theorem 3.3.

5. Applications

As a consequence of Theorem 3.2 we get the following corollary.

Corollary 5.1. *Let* $\phi_{10,1}=\sum_{\substack{n,r\in\mathbb{Z},\\4mn-r^2>0}}C_{\phi_{10,1}}(n,r)q^n\zeta^r\in J_{10,1}^{cusp}$ *and* $\phi_{12,1}=\sum_{\substack{n,r\in\mathbb{Z},\\4mn-r^2>0}}C_{\phi_{12,1}}(n,r)q^n\in J_{12,1}^{cusp}$. *Then we have the following identity:*

$$-\frac{1}{6}\frac{||\phi_{12,1}||^2}{||\phi_{10,1}||^2}C_{\phi_{10,1}}(N,R)\ =\ \frac{323(D_{N,R})^{\frac{17}{2}}}{4\pi}\left[\frac{(D_{N,R}-\frac{19}{6})}{(D_{N,R})^{\frac{21}{2}}}C_{\phi_{12,1}}(N,R)+76L_{\phi_{12,1}}(N,R;\frac{21}{2})\right],$$

where $D_{N,R}=4N-R^2$ *and* $L_\phi(N,R;s)=\sum_{n\geqslant1}\frac{C_\phi(n+N,R)}{(D_{n+N,R})^s}$.

Proof: We know that $J_{10,1}^{cusp}$ and $J_{12,1}^{cusp}$ are one dimensional and $\mathcal{L}_{10,1}(\phi_{10,1}) \in J_{12,1}^{cusp}$. Hence by comparing Fourier coefficients we get $\mathcal{L}_{10,1}(\phi_{10,1}) = -\frac{1}{6}\phi_{12,1}$. Now let $\mathcal{L}_{10,1}^{*}(\phi_{12,1}) = \alpha\phi_{10,1}$, we have

$$\alpha||\phi_{10,1}||^2 \quad = \quad \langle \alpha\phi_{10,1}, \phi_{10,1}\rangle = \langle \mathcal{L}_{10,1}^{*}(\phi_{12,1}), \phi_{10,1}\rangle = \langle \phi_{12,1}, \mathcal{L}_{10,1}(\phi_{10,1})\rangle = -\frac{1}{6}||\phi_{12,1}||^2.$$

Now from Theorem 3.2 we get the desired identity.

Remark 5.1. One can obtain similar type of identities by computing the adjoint map between certain spaces of Hermitian Jacobi cusp forms.

Remark 5.2. Observe that $J_{4,1}$ and $J_{6,1}$ are one dimensional spaces generated by $E_{4,1}$ and $E_{6,1}$ respectively. Comparing the constant term we get $L_{4,1}(E_{4,1}) = -\frac{7}{6}E_{6,1}$. Hence we get relation between generalized class numbers

$$\frac{1}{\zeta(-5)}[(4n-r^2)H(3,4n-r^2) - \frac{7}{6}H(3,4n-r^2) - 28\sum_{\substack{n_1+n_2=n,\\ 4n_2-r^2\geqslant 0}}\sigma(n_1)H(3,4n_2-r^2)] = \frac{1}{\zeta(-9)}H(5,4n-r^2).$$

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School of Mathematical Sciences, National Institute of Science Education and Research, An OCC of Homi Bhabha National Institute, Bhubaneswar, Via: Jatni, Khurda, Odisha- 752 050, India

Email address: shivansh.pandey@niser.ac.in