Pseudo asymptotically Bloch periodic solutions with measures for some differential Equations.

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Abstract: In this work we focus on upon the (μ_1, μ_2)-Pseudo-asymptotically Bloch τ-periodicity and its applications. Firstly, we define a new notion of (μ_1, μ_2)-Pseudo-asymptotically Bloch τ-periodic functions and some fundamental properties. Then, the obtained results are applied to investigate the existence and uniqueness of (μ_1, μ_2)-Pseudo-asymptotically Bloch τ-periodic mild solutions to Semi-linear Evolution equation in Banach spaces. Finally, we gave an application that facilitates the work.

Keywords: Bloch periodic functions, (μ_1, μ_2)-Pseudo-asymptotically Bloch τ-periodic functions, Semi-linear Evolution Equation.

MSC CLASSIFICATION: 34K14; 35B15.

1 Introduction

This work is based on the principle of Bloch τ-periodicity, which introduces for the first time by Hasler and N'Guérékata in [14], where they treat τ-periodicity, τ-anti-periodicity cases, and some fundamental results of Bloch type τ-periodic functions, such as the completeness of space and the composition and convolution product theorems.
In [18], Oueama-Guengai and N'Guérékata introduced the existence and uniqueness of Bloch τ-periodic mild solutions to semi-linear fractional differential equation in Banach spaces.
On the other hand, we can speak of asymptotic S-asymptotic τ-periodicity which is an extension of classical τ-periodicity, for more results on S-asymptotic τ-periodicity and some applications , we can refer to [11, 13, 15, 18].
Recently, Yong-Kui Chang and Yangean Wei [9] introduced a new concept, the pseudo S-asymptotically Bloch τ-periodicity and give some applications to evolution equation.
In this paper, we generalize pseudo S-asymptotic Bloch τ-periodicity into (μ_1, μ_2)-Pseudo-asymptotically Bloch τ-periodic by measure theory.
A function $f \in \mathcal{C}_b(\mathbb{R}, X)$ is said to be (μ_1, μ_2)-Pseudo-asymptotically Bloch τ-periodic if for given $\tau, \rho \in \mathbb{R}$,

$$\lim_{r \to +\infty} \frac{1}{\mu_2(\Omega_r)} \int_{\Omega_r} \|f(s+\tau) - e^{i\rho\tau} f(s)\| d\mu_1(s) = 0,$$

where μ_1 and μ_2 are two positives measures that we will define then later (See [1, 4, 4]).
we say that $f \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2)$.

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The main purpose of this paper is to consider the following semi-linear integro-differential equation

$$u'(s) = Tu(s) + a\int_{-\infty}^{s} \frac{(s-t)^{m-1}}{\Gamma(m)} e^{-b(s-t)} Tu(t)dt + g(s, u(s)),\ s \in \mathbb{R}, \tag{1.1}$$

where $T : D(T) \subseteq X \to X$ is a closed linear operator on a Banach space X, $a \neq 0$, $b > 0$, $m \geq 1$, $g \in \mathcal{C}_b(\mathbb{R} \times X, X)$, and $\Gamma(.)$ is the Gamma function.
The equation (1.1) arises from the thermodynamics of materials with memory (as in [2]).

The next part of this article is outlined as follows: Section 2 is Preliminaries composing some basic definitions, remarks and notations. Section 3 we treat some results on (μ_1, μ_2)-Pseudo-asymptotically Bloch τ-periodic functions. Section 4 is concerned with the existence and uniqueness of (μ_1, μ_2)-Pseudo-asymptotically Bloch τ-periodic solutions to (1.1). Finally, in section 5, we give an application which explains the work.

2 Preliminaries

we consider the following notations:

- $(X, \|.\|)$: Banach space.

- $\|f\|_\infty = \sup_{t\in\mathbb{R}} \|f\|$.

- $\mathcal{C}_b(\mathbb{R}, X)$ (resp $\mathcal{C}_b(\mathbb{R} \times X, X)$) : Banach space of bounded continuous functions from $\mathbb{R}$ (resp $\mathbb{R} \times X$) to X with super-norm $\|.\|_\infty$.

- $\mathcal{B}(X)$: Space of all bounded linear operators from X into itself.

- $f_a(.) := f(. + a)$, with $a \in \mathbb{R}$ and $f \in \mathcal{C}_b(\mathbb{R}, X)$

- $\Omega_r := [-r, r]$, $r > 0$.

- $\Re(z)$: real part of z, with $z \in \mathbb{C}$.

Definition 2.1. [14] For given $\tau, \rho \in \mathbb{R}$, a function $f \in \mathcal{C}_b(\mathbb{R}, X)$ is said to be Bloch τ-periodic if for all $s \in \mathbb{R}$, $f(s + \tau) = e^{i\rho\tau} f(s)$.
We denote by $BP_{\tau,\rho}(\mathbb{R}, X)$, the space of all Bloch τ-periodic functions from $\mathbb{R}$ to X.

Remark 2.1. From definition 2.1, we can see that f is τ-periodic if $\rho\tau = 0$, and f is τ-anti-periodic if $\rho\tau = \pi$.

Definition 2.2. [13] A function $f \in \mathcal{C}_b(\mathbb{R}, X)$ is said to be S-asymptotically τ-periodic if for a given $\tau \in \mathbb{R}$,

$$\lim_{|s|\to\infty} \|f(s + \tau) - f(s)\| = 0, s \in \mathbb{R}.$$

we denote the set of its functions by $SAP_\tau(\mathbb{R}, X)$.

Definition 2.3. [20] A function $f \in \mathcal{C}_b(\mathbb{R}, X)$ is said to be pseudo-S-asymptotically τ-periodic if for a given $\tau \in \mathbb{R}$,

$$\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|f(s + \tau) - f(s)\|ds = 0.$$

The set of such functions will be denoted by $PSAP_\tau(\mathbb{R}, X)$.

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3 (μ_1, μ_2)-Pseudo-asymptotically Bloch τ-periodic functions

We denote by $\mathcal{B}$ the Lebesgue σ-field of $\mathbb{R}$ and by $\mathcal{M}$ the set of all positive measures μ on $\mathcal{B}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a_1, a_2]) < +\infty$, for all $a_1, a_2 \in \mathbb{R}$, $(a_1 \leq a_2)$.

Definition 3.1. Let $\mu_1, \mu_2 \in \mathcal{M}$. The measures μ_1 and μ_2 are said to be equivalent ($\mu_1 \sim \mu_2$) if there exist a constants $c_1, c_2 > 0$ and a bounded interval $\mathrm{J} \subset \mathbb{R}$ (eventually $\mathrm{J} = \emptyset$) such that

$$c_1\mu_2(A) \leq \mu_1(A) \leq c_2\mu_2(A)$$

for all $A \in \mathcal{B}$ satisfying $A \bigcap J = \emptyset$.

Let $\mu_1, \mu_2 \in \mathcal{M}$ and $r > 0$, suppose that

$$\Theta_r = \frac{\mu_1(\Omega_r)}{\mu_2(\Omega_r)}.$$

In this paper, we need the following hypotheses:

(**M1**) Let $\mu_1, \mu_2 \in \mathcal{M}$ such that

$$\limsup_{r\to+\infty} \Theta_r < +\infty.$$

(**M2**) For $\mu \in \mathcal{M}$, $\omega \in \mathbb{R}$, there exist $\gamma > 0$ and a bounded interval J such that

$\mu(a + \omega; a \in \mathcal{A}) \leq \gamma\mu(\mathcal{A})$, when $\mathcal{A} \in \mathcal{B}$ satisfy $\mathcal{A} \bigcap \mathrm{J} = \emptyset$.

Definition 3.2. Let $\mu_1, \mu_2 \in \mathcal{M}$. A function $f \in \mathcal{C}_b(\mathbb{R}, X)$ is said to be (μ_1, μ_2)-Pseudo-asymptotically Bloch τ-periodic if for given $\tau, \rho \in \mathbb{R}$,

$$\lim_{r\to+\infty} \frac{1}{\mu_2(\Omega_r)} \int_{\Omega_r} \|f(s+\tau) - e^{i\rho\tau} f(s)\| d\mu_1(s) = 0.$$

We denote the set of all such functions by $PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2)$.

- If $\rho\tau = 0$, we have

$$\lim_{r\to+\infty} \frac{1}{\mu_2(\Omega_r)} \int_{\Omega_r} \|f(s+\tau) - f(s)\| d\mu_1(s) = 0,$$

then f is called (μ_1, μ_2)-Pseudo-asymptotically τ-periodic denoted by $PSAP_\tau(\mathbb{R}, X, \mu_1, \mu_2)$.

- If $\rho\tau = \pi$, we have

$$\lim_{r\to+\infty} \frac{1}{\mu_2(\Omega_r)} \int_{\Omega_r} \|f(s+\tau) + f(s)\| d\mu_1(s) = 0,$$

then f is called (μ_1, μ_2)-Pseudo-asymptotically τ-anti-periodic denoted by $PSAAP_\tau(\mathbb{R}, X, \mu_1, \mu_2)$.

- If $\mu_1 = \mu_2 = \mu$, we have

$$\lim_{r\to+\infty} \frac{1}{\mu(\Omega_r)} \int_{\Omega_r} \|f(s+\tau) - e^{i\rho\tau} f(s)\| d\mu(s) = 0,$$

then f is called μ-Pseudo-asymptotically Bloch τ-periodic denoted by $PSABP_{\tau,\rho}(\mathbb{R}, X, \mu)$.

Lemma 3.1. *Let $\mu_1, \mu_2 \in \mathcal{M}$ satisfy* (**M1**) *and* (**M2**), $f, g, h \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2)$. *Then we have the following results:*

(i) $g + h \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2)$, $cf \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2)$ *for each* $c \in \mathbb{R}$.

(ii) The function $f_a \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2)$ *for any* $a \in \mathbb{R}$.

(iii)The space $PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2)$ *is a Banach space with the super-norm.*

Proof. (i) Hence,

$$\begin{aligned} & \frac{1}{\mu_2(\Omega_r)} \int_{\Omega_r} \|(g+h)(s+\tau) - e^{i\rho\tau}(g+h)(s)\| d\mu_1(s) \\ \leq & \frac{1}{\mu_2(\Omega_r)} \int_{\Omega_r} \|g(s+\tau) - e^{i\rho\tau} g(s)\| d\mu_1(s) \\ + & \frac{1}{\mu_2(\Omega_r)} \int_{\Omega_r} \|h(s+\tau) - e^{i\rho\tau} h(s)\| d\mu_1(s), \end{aligned}$$

and

$$\frac{1}{\mu_2(\Omega_r)} \int_{\Omega_r} \|(cf)(s+\tau) - e^{i\rho\tau}(cf)(s)\| d\mu_1(s) \leq |c| \frac{1}{\mu_2(\Omega_r)} \int_{\Omega_r} \|f(s+\tau) - e^{i\rho\tau} f(s)\| d\mu_1(s).$$

Then

$$g + h, cf \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2).$$

(ii) For each $a \in \mathbb{R}$, hence,

$$\begin{aligned} & \frac{1}{\mu_2(\Omega_r)} \int_{\Omega_r} \|f(s+a+\tau) - e^{i\rho\tau} f(s+a)\| d\mu_1(s) \\ & \leq \frac{\gamma}{\mu_2(\Omega_r)} \int_{-r+a}^{r+a} \|f(s+\tau) - e^{i\rho\tau} f(s)\| d\mu_1(s) \\ & \leq \frac{\gamma}{\mu_2(\Omega_r)} \int_{-r-|a|}^{r+|a|} \|f(s+\tau) - e^{i\rho\tau} f(s)\| d\mu_1(s) \\ & = \gamma \frac{\mu_2([-r-|a|, r+|a|])}{\mu_2([-r,r])} \Big(\frac{1}{\mu_2([-r-|a|, r+|a|])} \int_{-r-|a|}^{r+|a|} \|f(s+\tau) - e^{i\rho\tau} f(s)\| d\mu_1(s) \Big). \end{aligned}$$

For r sufficiently large we obtain $f_a \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2)$.

(iii) Let $\{f_n\}_n \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2)$ converge to f as $r \to \infty$.

Then for any $\epsilon > 0$, we can choose suitable constants $N > 0$ and r_ϵ such that

$\frac{1}{\mu_2(\Omega_r)} \int_{\Omega_r} \|f_n(s+\tau) - e^{i\rho\tau} f_n(s)\| d\mu_1(s) \leq \frac{\epsilon}{3}$; $\|f_n - f\|_\infty \leq \frac{\epsilon}{3\Theta_r}$, for $n > N$ and $r > r_\epsilon$.

$\frac{1}{\mu_2(\Omega_r)} \int_{\Omega_r} \|f(s+\tau) - e^{i\rho\tau} f(s)\| d\mu_1(s)$

$= \frac{1}{\mu_2(\Omega_r)} (\int_{\Omega_r} \|f(s+\tau) - f_n(s+\tau) + f_n(s+\tau) - e^{ip\tau} f_n(s) + e^{i\rho\tau} f_n(s) - e^{i\rho\tau} f(s)\| d\mu_1(s))$

$\leq \frac{1}{\mu_2(\Omega_r)} \int_{\Omega_r} \|f(s+\tau) - f_n(s+\tau)\| d\mu_1(s)$

$+ \frac{1}{\mu_2(\Omega_r)} \int_{\Omega_r} \|e^{i\rho\tau} f_n(s) - e^{i\rho\tau} f(s)\| d\mu_1(s)$

$+ \frac{1}{\mu_2(\Omega_r)} \int_{\Omega_r} \|f_n(s+\tau) - e^{i\rho\tau} f_n(s)\| d\mu_1(s)$

$\leq 2\Theta_r \|f_n - f\|_\infty + (\frac{1}{\mu_2(\Omega_r)} \int_{\Omega_r} \|f_n(s+\tau) - e^{i\rho\tau} f_n(s)\| d\mu_1(s))$

$\leq 2\frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$

Which gives that the space $PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2)$ is a closed sub-space of $\mathcal{C}_b(\mathbb{R}, X)$, it is therefore a Banach space equipped with super-norm. □

Theorem 3.2. *Let $\mu_1, \mu_2 \in \mathcal{M}$ satisfies* (**M1**) *and* J *be a bounded interval (eventually* $\mathrm{J} = \emptyset$*) and* $f \in \mathcal{C}_b(\mathbb{R}, X)$, *then the following assertions are equivalent:*

(i) $f \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2)$.

(ii)

$$\lim_{r\to\infty} \frac{1}{\mu_2(\Omega_r \backslash \mathrm{J})} \int_{\Omega_r \backslash \mathrm{J}} \|f(s+\tau) - e^{i\rho\tau} f(s)\| d\mu_1(s) = 0$$

(iii) For any $\epsilon > 0$,

$$\lim_{r\to\infty} \frac{\mu_1(s \in \Omega_r \backslash \mathrm{J}, \|f(s+\tau) - e^{i\rho\tau} f(s)\| > \epsilon)}{\mu_2(\Omega_r \backslash \mathrm{J})} = 0.$$

Proof. $(i) \Leftrightarrow (ii)$ Denote by $A_1 = \mu_2(\mathrm{J})$, $A_2 = \int_{\mathrm{J}} \|f(s+\tau) - e^{i\rho\tau} f(s)\| d\mu_1(s)$, $A_3 = \mu_1(\mathrm{J})$.

Since the interval J is bounded and $f \in \mathcal{C}_b(\mathbb{R}, X)$, then A_1, A_2 and A_3 are finite.

for $r > 0$ such tat $\mathrm{J} \subset \Omega_r$ and $\mu_2(\Omega_r \backslash \mathrm{J}) > 0$, we have

$\frac{1}{\mu_2(\Omega_r \backslash \mathrm{J})} \int_{\Omega_r \backslash \mathrm{J}} \|f(s+\tau) - e^{i\rho\tau} f(s)\| d\mu_1(s) = \frac{1}{\mu_2(\Omega_r) - A_1} (\int_{\Omega_r} \|f(s+\tau) - e^{i\rho\tau} f(s)\| d\mu_1(s) - A_2)$

$= \frac{\mu_2(\Omega_r)}{\mu_2(\Omega_r) - A_1} (\frac{1}{\mu_2(\Omega_r)} \int_{\Omega_r} \|f(s+\tau) - e^{i\rho\tau} f(s)\| d\mu_1(s) - \frac{A_2}{\mu_2(\Omega_r)}).$

Since $\mu_2(\mathbb{R}) = \infty$, we deduce that $(ii) \Leftrightarrow (i)$.

$(iii) \Rightarrow (ii)$ Denote by Φ_r^ϵ and Ψ_r^ϵ the following sets

$$\Phi_r^\epsilon = \{s \in \Omega_r \backslash \mathrm{J}, \|f(s+\tau) - e^{i\rho\tau} f(s)\| > \epsilon\},$$

and

$$\Psi_r^\epsilon = \{s \in \Omega_r \backslash \mathrm{J}, \|f(s+\tau) - e^{i\rho\tau} f(s)\| \leq \epsilon\}.$$

Assume that (iii) holds, that is

$$\lim_{r\to\infty} \frac{\mu_1(\Phi_r^\epsilon)}{\mu_2(\Omega_r \backslash \mathrm{J})} = 0.$$

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We have $\int_{\Omega_r\setminus \mathrm{J}} \|f(s+\tau)-e^{i\rho\tau}f(s)\|d\mu_1(s) = \int_{\Phi_r^\epsilon} \|f(s+\tau)-e^{i\rho\tau}f(s)\|d\mu_1(s)+\int_{\Psi_r^\epsilon} \|f(s+\tau)-e^{i\rho\tau}f(s)\|d\mu_1(s)$

$\frac{1}{\mu_2(\Omega_r\setminus \mathrm{J})}\int_{\Omega_r\setminus \mathrm{J}} \|f(s+\tau)-e^{i\rho\tau}f(s)\|d\mu_1(s)$

$\leq 2\|f\|_\infty \frac{\mu_1(\Phi_r^\epsilon)}{\mu_2(\Omega_r\setminus \mathrm{J})} + \frac{\mu_1(\Psi_r^\epsilon)}{\mu_2(\Omega_r\setminus \mathrm{J})}\epsilon$

$\leq 2\|f\|_\infty \frac{\mu_1(\Phi_r^\epsilon)}{\mu_2(\Omega_r\setminus \mathrm{J})} + \frac{\mu_1(\Omega_r\setminus \mathrm{J})}{\mu_2(\Omega_r\setminus \mathrm{J})}\epsilon$

$= 2\|f\|_\infty \frac{\mu_1(\Phi_r^\epsilon)}{\mu_2(\Omega_r\setminus \mathrm{J})} + \frac{\mu_1(\Omega_r)-A_3}{\mu_2(\Omega_r)-A_1}\epsilon$

$= 2\|f\|_\infty \frac{\mu_1(\Phi_r^\epsilon)}{\mu_2(\Omega_r\setminus \mathrm{J})} + \frac{\mu_1(\Omega_r)(1-\frac{A_3}{\mu_1(\Omega_r)})}{\mu_2(\Omega_r)(1-\frac{A_1}{\mu_2(\Omega_r)})}\epsilon$

$\leq 2\|f\|_\infty \frac{\mu_1(\Phi_r^\epsilon)}{\mu_2(\Omega_r\setminus \mathrm{J})} + \Theta_r\frac{1-\frac{A_3}{\mu_1(\Omega_r)}}{1-\frac{A_1}{\mu_2(\Omega_r)}}\epsilon$

For r sufficiently large, since $\mu_1(\mathbb{R}) = \mu_2(\mathbb{R}) = \infty$, then for all $\epsilon > 0$ we have

$\frac{1}{\mu_2(\Omega_r\setminus \mathrm{J})}\int_{\Omega_r\setminus \mathrm{J}} \|f(s+\tau)-e^{i\rho\tau}f(s)\|d\mu_1(s) \leq \text{cst } \epsilon$

Then

$$\lim_{r\to\infty}\frac{1}{\mu_2(\Omega_r\setminus \mathrm{J})}\int_{\Omega_r\setminus \mathrm{J}} \|f(s+\tau)-e^{i\rho\tau}f(s)\|d\mu_1(s) = 0$$

Consequently (ii) holds.

(ii) $\Rightarrow$ (iii) Assume that (ii) holds

$\frac{1}{\mu_2(\Omega_r\setminus \mathrm{J})}\int_{\Omega_r\setminus \mathrm{J}} \|f(s+\tau)-e^{i\rho\tau}f(s)\|d\mu_1(s)$
$\geq \frac{1}{\mu_2(\Omega_r\setminus \mathrm{J})}\int_{\Phi_r^\epsilon} \|f(s+\tau)-e^{i\rho\tau}f(s)\|d\mu_1(s)$
$\geq \epsilon\frac{\mu_1(\Phi_r^\epsilon)}{\mu_2(\Omega_r\setminus \mathrm{J})}$

For r sufficiently large, we obtain (iii). □

Corollary 3.3. *A continuous function $f : \mathbb{R} \to X$ satisfying*

$$\lim_{|s|\to\infty} f(s) = 0$$

Then $f \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2)$ for all $\mu_1, \mu_2 \in \mathcal{M}$ satisfies (**M1**).

Proof. By hypothesis, we have

$$\lim_{|s|\to\infty}(f(s+\tau)-e^{i\rho\tau}f(s)) = 0$$

for all $\epsilon > 0$, there exists $\upsilon > 0$ such that

$$|s| \geq \upsilon \Rightarrow \|f(s+\tau)-e^{i\rho\tau}f(s)\| \leq \epsilon$$

Then for all $r > \upsilon$

$$\{s \in \Omega_r\setminus(-\upsilon, \upsilon); \|f(s+\tau)-e^{i\rho\tau}f(s)\| > \epsilon\} = \emptyset.$$

We conclude by using theorem 3.2. □

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Proposition 3.4. *Assume that* (**M1**) *hold. If* $\mu_1, \mu_2, \nu_1, \nu_2 \in \mathcal{M}$ *and* $\mu_1 \sim \nu_1$*,* $\mu_2 \sim \nu_2$*, then* $PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2) = PSABP_{\tau,\rho}(\mathbb{R}, X, \nu_1, \nu_2)$.

Proof. Since $\mu_1 \sim \nu_1$, $\mu_2 \sim \nu_2$, then for all $\mathcal{A} \in \mathcal{B}$ satisfying $\mathcal{A} \bigcap \Omega_r = \emptyset$, by definition 3.1, there exists $\alpha_1, \alpha_2 > 0$, $\beta_1, \beta_2 > 0$ such that
$\alpha_1\nu_1(\mathcal{A}) \leq \mu_1(\mathcal{A}) \leq \beta_1\nu_1(\mathcal{A})$, and $\alpha_2\nu_2(\mathcal{A}) \leq \mu_2(\mathcal{A}) \leq \beta_2\nu_2(\mathcal{A})$.

For r sufficiently, we have

$$\frac{\alpha_1}{\beta_2} \times \frac{\nu_1(\{s \in \Omega_r \backslash \mathrm{J}; \|f(s+\tau) - e^{i\rho\tau}f(s)\| > \epsilon\})}{\nu_2(\Omega_r \backslash \mathrm{J})}$$

$$\leq \frac{\mu_1(\{s \in \Omega_r \backslash \mathrm{J}; \|f(s+\tau) - e^{i\rho\tau}f(s)\| > \epsilon\})}{\mu_2(\Omega_r \backslash \mathrm{J})}$$

$$\leq \frac{\beta_1}{\alpha_2} \times \frac{\nu_1(\{s \in \Omega_r \backslash \mathrm{J}; \|f(s+\tau) - e^{i\rho\tau}f(s)\| > \epsilon\})}{\nu_2(\Omega_r \backslash \mathrm{J})}.$$

Hence $PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2) = PSABP_{\tau,\rho}(\mathbb{R}, X, \nu_1, \nu_2)$, by theorem 3.2. □

Theorem 3.5. *Let* $\mu_1, \mu_2 \in \mathcal{M}$ *and* $\{S(s)\}_{s\geq 0} \subseteq \mathcal{B}(X)$ *be a uniformly integrable and strongly continuous family.*

If $f \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2)$*, then*

$$U(s) := \int_{-\infty}^{s} S(s-\xi)f(\xi)d\xi \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2).$$

Proof. Since $\{S(s)\}_{s\geq 0} \subseteq \mathcal{B}(X)$ is uniformly integrable, we have

$$\int_0^\infty \|S(s)\|ds < \infty.$$

It follows from $f \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2)$ that

$$\lim_{r\to\infty} \frac{1}{\mu_2(\Omega_r)} \int_{\Omega_r} \|f(s+\tau) - e^{i\rho\tau}f(s)\|d\mu_1(s) = 0.$$

On the other hand, by the Fubini theorem, we have

$\frac{1}{\mu_2(\Omega_r)} \int_{\Omega_r} \|U(s+\tau) - e^{i\rho\tau}U(s)\|d\mu_1(s)$
$= \frac{1}{\mu_2(\Omega_r)} \int_{\Omega_r} \| \int_{-\infty}^{s+\tau} S(s+\tau-\xi)f(\xi)d\xi - e^{i\rho\tau} \int_{-\infty}^{s} S(s-\xi)f(\xi)d\xi\|d\mu_1(s)$
$= \frac{1}{\mu_2(\Omega_r)} \int_{\Omega_r} \| \int_{-\infty}^{s} S(s-\xi)(f(\xi+\tau) - e^{i\rho\tau}f(\xi))d\xi\|d\mu_1(s)$

$\leq \frac{1}{\mu_2(\Omega_r)} \int_{\Omega_r} [\int_{-\infty}^{s} \|S(s-\xi)(f(\xi+\tau) - e^{i\rho\tau} f(\xi))\| d\xi] d\mu_1(s)$

$\leq \frac{1}{\mu_2(\Omega_r)} \int_{\Omega_r} \int_0^{\infty} \|S(\xi)(f(s-\xi+\tau) - e^{i\rho\tau} f(s-\xi))\| d\xi d\mu_1(s)$

$\leq \int_0^{\infty} \|S(\xi)\| (\frac{1}{\mu_2(\Omega_r)} \int_{\Omega_r} \|f(s-\xi+\tau) - e^{i\rho\tau} f(s-\xi)\| d\mu_1(s)) d\xi.$

Since $f \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2)$, we use Lemma 3.1 (ii) and the Lebesgue dominated convergence theorem, we obtain

$$\lim_{r\to\infty} \frac{1}{\mu_2(\Omega_r)} \int_{\Omega_r} \|U(s+\tau) - U(s)\| d\mu_1(s) = 0$$

ie: $U \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2)$.

□

Definition 3.3. Let $\mu_1, \mu_2 \in \mathcal{M}$.
A continuous function $f : \mathbb{R} \times Y \to X$ is said to be (μ_1, μ_2)-Pseudo-asymptotically Bloch τ-periodic in s uniformly with respect to $y \in Y$ if the following conditions are true:
(i) For all $y \in Y$, $f(., y) \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2)$.
(ii) f is uniformly continuous on each compact K in Y with respect to the second variable y.
ie: for each compact K in X, for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $y_1, y_2 \in K$, we have

$$\|y_1 - y_2\| \leq \delta \Rightarrow \sup_{s\in\mathbb{R}} \|f(s, y_1) - f(s, y_2)\| \leq \epsilon$$

The collection of such function is denoted by $PSABP_{\tau,\rho}U(\mathbb{R} \times Y, X, \mu_1, \mu_2)$.

We use the following assumptions which will be applied in the rest of this work.

(A1) Let $g \in \mathcal{C}_b(\mathbb{R} \times X, X)$, for all $(s, x) \in \mathbb{R} \times X$, $g(s+\tau, x) = e^{i\rho\tau} g(s, e^{-i\rho\tau} x)$.

(A2) Let $L_g > 0$ and $g \in \mathcal{C}_b(\mathbb{R} \times X, X)$, such that, for all $x_1, x_2 \in X$, $s \in \mathbb{R}$, we have:

$$\|g(s, x_1) - g(s, x_2)\| \leq L_g \|x_1 - x_2\|.$$

(A3) Let $\mu \in \mathcal{M}$ and $g \in \mathcal{C}_b(\mathbb{R} \times X, X)$, such that for all $x_1, x_2 \in X$, $s \in \mathbb{R}$, we have:

$$\|g(s, x_1) - g(s, x_2)\| \leq L_g(s) \|x_1 - x_2\|,$$

where $p \geq 1$, $L_g : \mathbb{R} \to \mathbb{R} \in \mathcal{L}^p(\mathbb{R}, d\mu) \bigcap \mathcal{L}^p(\mathbb{R}, dx)$.

Lemma 3.6. *Let $\mu_1, \mu_2 \in \mathcal{M}$ verifying hypothesis* **(M1)**, *then for all $p \geq 1$ we have*

$$\mathcal{L}^p(\mathbb{R}, d\mu_1) \cap \mathcal{C}_b(\mathbb{R}, X) \subset PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2).$$

Proof. • If $p > 1$, posing $q = \frac{p}{p-1}$, then $\frac{1}{p} + \frac{1}{q} = 1$, and if $f \in \mathcal{L}^p(\mathbb{R}, d\mu_1) \cap \mathcal{C}_b(\mathbb{R}, X)$, we have :

$\frac{1}{\mu_2(\Omega_r)} \int_{\Omega_r} \|f(s+\tau) - e^{i\rho\tau} f(s)\| d\mu_1(s)$

$\leq \frac{1}{\mu_2(\Omega_r)} (\int_{\Omega_r} \|f(s+\tau) - e^{i\rho\tau} f(s)\|^p d\mu_1(s))^{\frac{1}{p}} \mu_1(\Omega_r)^{\frac{1}{q}}$

$\leq 2 \frac{\mu_1(\Omega_r)^{\frac{1}{q}}}{\mu_2(\Omega_r)} \|f\|_{\mathcal{L}^p(\mathbb{R}, d\mu_1)}$

$\leq 2\Theta_r \frac{\|f\|_{\mathcal{L}^p(\mathbb{R}, d\mu_1)}}{\mu_1(\Omega_r)^{\frac{1}{p}}}.$

Since

$$\lim_{r \to +\infty} (\frac{\|f\|_{\mathcal{L}^p(\mathbb{R}, d\mu_1)}}{\mu_1(\Omega_r)^{\frac{1}{p}}}) = 0,$$

there $f \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2)$.

• If $p = 1$, taking $f \in \mathcal{L}^1(\mathbb{R}, d\mu_1) \cap \mathcal{C}_b(\mathbb{R}, X)$, then

$$\frac{1}{\mu_2(\Omega_r)} \int_{\Omega_r} \|f(s+\tau) - e^{i\rho\tau} f(s)\| d\mu_1(s) \leq \frac{2\|f\|_{\mathcal{L}^1(\mathbb{R}, d\mu_1)}}{\mu_2(\Omega_r)}$$

Since

$$\lim_{r \to +\infty} (\frac{\|f\|_{\mathcal{L}^1(\mathbb{R}, d\mu_1)}}{\mu_2(\Omega_r)}) = 0,$$

then $f \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2)$.

□

In the rest of this article, we will distinguish two cases: L_g constant and for this we will consider hypothesis (A2), or L_g variable and we will consider hypothesis (A3).

$*$**Case1** : $\mathbf{L_g}$**constant**

Theorem 3.7. *Let $\mu_1, \mu_2 \in \mathcal{M}$ and $g \in \mathcal{C}_b(\mathbb{R} \times X, X)$ satisfy (A1) and (A2). Then for each $\varphi \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2)$, $g(., \varphi(.)) \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2)$.*

Proof. Let $\varphi \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2)$, $s \in \mathbb{R}$, we have

$$\lim_{r \to +\infty} \frac{1}{\mu_2(\Omega_r)} \int_{\Omega_r} \|\varphi(s+\tau) - e^{i\rho\tau} \varphi(s)\| d\mu_1(s) = 0$$

On the other hand,

$\frac{1}{\mu_2(\Omega_r)} \int_{\Omega_r} \|g(s+\tau, \varphi(s+\tau)) - e^{i\rho\tau} g(s, \varphi(s))\| d\mu_1(s)$

$= \frac{1}{\mu_2(\Omega_r)} \int_{\Omega_r} \|e^{i\rho\tau} g(s, e^{-i\rho\tau} \varphi(s+\tau)) - e^{i\rho\tau} g(s, \varphi(s))\| d\mu_1(s)$

$\leq \frac{L_g}{\mu_2(\Omega_r)} \int_{\Omega_r} \|e^{-i\rho\tau} \varphi(s+\tau) - \varphi(s)\| d\mu_1(s)$

$= L_g(\frac{1}{\mu_2(\Omega_r)} \int_{\Omega_r} \|\varphi(s+\tau) - e^{i\rho\tau} \varphi(s)\| d\mu_1(s)).$

Thus

$$\lim_{r\to+\infty}\frac{1}{\mu_2(\Omega_r)}\int_{\Omega_r}\|g(s+\tau,\varphi(s+\tau))-e^{i\rho\tau}g(s,\varphi(s))\|d\mu_1(s)=0$$

ie: $g(.,\varphi(.))\in PSABP_{\tau,\rho}(\mathbb{R},X,\mu_1,\mu_2)$.

□

$*$**Case2** : $\mathbf{L_g}$ **Variable**

Theorem 3.8. *Let $\mu_1,\mu_2\in\mathcal{M}$ satisfies* (**M1**) *and $g\in\mathcal{C}_b(\mathbb{R}\times X,X)$ satisfy (A1) and (A3). For each $\varphi\in PSABP_{\tau,\rho}(\mathbb{R},X,\mu_1,\mu_2)$, then $g(.,\varphi(.))\in PSABP_{\tau,\rho}(\mathbb{R},X,\mu_1,\mu_2)$.*

Proof. Let $\varphi\in PSABP_{\tau,\rho}(\mathbb{R},X,\mu_1,\mu_2)$, $s\in\mathbb{R}$, we have

$$\frac{1}{\mu_2(\Omega_r)}\int_{\Omega_r}\|\varphi(s+\tau)-e^{i\rho\tau}\varphi(s)\|d\mu_1(s)=0$$

On the other hand,

$\frac{1}{\mu_2(\Omega_r)}\int_{\Omega_r}\|g(s+\tau,\varphi(s+\tau))-e^{i\rho\tau}g(s,\varphi(s))\|d\mu_1(s)$

$=\frac{1}{\mu_2(\Omega_r)}\int_{\Omega_r}\|e^{i\rho\tau}g(s,e^{-i\rho\tau}\varphi(s+\tau))-e^{i\rho\tau}g(s,\varphi(s))\|d\mu_1(s)$

$\leq\frac{1}{\mu_2(\Omega_r)}\int_{\Omega_r}L_g(s)\|e^{-i\rho\tau}\varphi(s+\tau)-\varphi(s)\|d\mu_1(s)$

$\leq\frac{1}{\mu_2(\Omega_r)}(\int_{\Omega_r}\|L_g\|^pd\mu_1(s))^{\frac{1}{p}}(\int_{\Omega_r}\|\varphi(s+\tau)-e^{i\rho\tau}\varphi(s)\|^qd\mu_1(s))^{\frac{1}{q}}$

$\leq\frac{\|L_g\|_{\mathcal{L}^p(\mathbb{R},d\mu_1)}}{\mu_2(\Omega_r)}2\|\varphi\|_\infty(\mu_1(\Omega_r))^{\frac{1}{q}})$.

$=2\Theta_r\|\varphi\|_\infty(\frac{\|L_g\|_{\mathcal{L}^p(\mathbb{R},d\mu_1)}}{\mu_1(\Omega_r))^{\frac{1}{p}}})$.

Since

$$\lim_{r\to+\infty}\frac{\|L_g\|_{\mathcal{L}^p(\mathbb{R},d\mu_1)}}{\mu_1(\Omega_r)^{\frac{1}{p}}}=0.$$

Thus

$$\lim_{r\to+\infty}\frac{1}{\mu_2(\Omega_r)}\int_{\Omega_r}\|g(s+\tau,\varphi(s+\tau))-e^{i\rho\tau}g(s,\varphi(s))\|d\mu_1(s)=0$$

ie: $g(.,\varphi(.))\in PSABP_{\tau,\rho}(\mathbb{R},X,\mu_1,\mu_2)$. □

4 Semi-Linear Evolution Equation

In this section we consider the following semi-linear integro-differential equation

$u^{'}(s)=Tu(s)+a\int_{-\infty}^{s}\frac{(s-t)^{m-1}}{\Gamma(m)}e^{-b(s-t)}Tu(t)dt+g(s,u(s)),\ s\in\mathbb{R}.$

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Where $T : D(T) \subseteq X \to X$ is a closed linear operator on a Banach space X, $a \neq 0$, $b > 0$, $m \geq 1$, $g \in \mathcal{C}_b(\mathbb{R} \times X, X)$, and $\Gamma(.)$ is the Gamma function.

In the remainder of this work we assume that the following hypotheses are satisfied:

(H1) The operator T generates an immediately norm continuous C_0-semigroup on a Banach space X.

(H2) $\Re((-a)^{\frac{1}{m}} - b) < 0$ and

$$\sup\{\Re(\lambda_T), \lambda_T \in \mathbb{C} : \lambda_T(\lambda_T + b)^m((\lambda_T + b)^m + a)^{-1} \in \sigma(T)\} < 0.$$

Using [[8], Proposition 3.1], we obtain a uniformly exponentially stable and strongly continuous family of operators $\{S(s)\}_{s\geq 0} \subseteq \mathcal{B}(X)$ verifying conditions (H1) and (H2),
i.e : there exist constants $\delta > 0$, $M > 0$ such that for all $s \geq 0$,

$$\|S(s)\| \leq Me^{-\delta s}. \tag{4.1}$$

For [[8], Theorem 3.2], the following linear equation,

$$u^{'}(s) = Tu(s) + a\int_{-\infty}^{s} \frac{(s-t)^{m-1}}{\Gamma(m)} e^{-b(s-t)} Tu(t)dt + g(s), s \in \mathbb{R}, \tag{4.2}$$

admits the mild solutions given by

$$u(s) = \int_{-\infty}^{s} S(s-\xi)g(\xi)d\xi, s \in \mathbb{R}. \tag{4.3}$$

Lemma 4.1. *If $g \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2)$, then the mild solution u(s) given by (4.3) belongs to $PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2)$.*

Proof. We use Theorem 3.5. □

Definition 4.1. We say that the function $u : \mathbb{R} \to X$ is a mild solution to the equation (4.2) if the function $\xi \mapsto S(s-\xi)g(\xi, u(\xi))$ is integrable on $(-\infty, s]$ for all $s \in \mathbb{R}$ and

$$u(s) = \int_{-\infty}^{s} S(s-\xi)g(\xi, u(\xi))d\xi, s \in \mathbb{R}.$$

Theorem 4.2. *Let $\mu_1, \mu_2 \in \mathcal{M}$ satisfies* (**M1**) *and* (**M1**), *assume that (H1) and (H2) hold, we have:*

(i) If $g \in \mathcal{C}_b(\mathbb{R} \times X, X)$ satisfies (A1).

Then equation (1.1) *admits a unique mild solution $u \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2)$ whenever*

$$\frac{ML_g}{\delta} < 1.$$

(ii) If $g \in \mathcal{C}_b(\mathbb{R} \times X, X)$ satisfies (A2).

Then system (1.1) *admits a unique mild solution* $u \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2)$ *whenever*

$$\|L_g\|_{\mathcal{L}^p(\mathbb{R},dx)} < \frac{(\delta q)^{\frac{1}{q}}}{M}.$$

Proof. We define the operator $\mathcal{F} : PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2) \to PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2)$ by

$(\mathcal{F}u)(s) := \int_{-\infty}^{s} S(s-\xi)g(\xi, u(\xi))d\xi,\ s \in \mathbb{R},$

where $\{S(s)\}_{s\geq 0}$ verifies the relation (4.1).

For each $u \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2)$, using Theorem 3.7 and Theorem 3.8, the function $\xi \mapsto g(\xi, u(\xi))$ belongs to $PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2)$. From Lemma 4.1 we have $\mathcal{F}u \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2)$, which give $\mathcal{F}$ is well defined.

(i) For $u_1, u_2 \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2)$ and $s \in \mathbb{R}$, we have

$\|(\mathcal{F}u_1)(s) - (\mathcal{F}u_2)(s)\| \leq \int_{-\infty}^{s} \|S(s-\xi)[g(\xi, u_1(\xi)) - g(\xi, u_2(\xi))]\|d\xi$

$\leq \int_{-\infty}^{s} L_g\|S(s-\xi)\|\|u_1(\xi) - u_2(\xi)\|d\xi$

$\leq L_g\|u_1 - u_2\|_\infty \int_0^\infty \|S(\xi)\|d\xi$

$\leq \frac{L_g M}{\delta}\|u_1 - u_2\|_\infty.$

Therefore,

$$\|\mathcal{F}u_1 - \mathcal{F}u_2\|_\infty \leq \frac{ML_g}{\delta}\|u_1 - u_2\|_\infty,$$

which gives that $\mathcal{F}$ is contractive for the assumption $\frac{ML_g}{\delta} < 1$.

So there is a unique $u \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2)$, such that $\mathcal{F}(u) = u$ via the Banach fixed point theorem.

(ii) For $u_1, u_2 \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2)$ and $s \in \mathbb{R}$, we have

$\|(\mathcal{F}u_1)(s) - (\mathcal{F}u_2)(s)\| \leq \int_{-\infty}^{s} \|S(s-\xi)[g(\xi, u_1(\xi)) - g(\xi, u_2(\xi))]\|d\xi$

$\leq M\int_{-\infty}^{s} e^{-\delta(s-\xi)}L_g(\xi)\|u_1(\xi) - u_2(\xi)\|d\xi$

$\leq M\|u_1 - u_2\|_\infty(\int_{-\infty}^{s} \|L_g(\xi)\|^p d\xi)^{\frac{1}{p}}(\int_{-\infty}^{s}(e^{-\delta(s-\xi)})^q d\xi)^{\frac{1}{q}}$

$\leq \frac{M\|L_g\|_{\mathcal{L}^p(\mathbb{R},dx)}}{(\delta q)^{\frac{1}{q}}}\|u_1 - u_2\|_\infty.$

Therefore,

$$\|\mathcal{F}u_1 - \mathcal{F}u_2\|_\infty \leq \frac{M\|L_g\|_{\mathcal{L}^p(\mathbb{R},dx)}}{(\delta q)^{\frac{1}{q}}}\|u_1 - u_2\|_\infty,$$

As well as the first result (i), the system (1.1) admits a unique mild solution $u \in PSABP_{\tau,\rho}(\mathbb{R}, X, \mu_1, \mu_2)$ for $\|L_g\|_{\mathcal{L}^p(\mathbb{R},d\mu_1)} < \frac{(\delta q)^{\frac{1}{q}}}{M}$.

□

5 Application:

We consider the problem

$$\begin{cases} \frac{\partial u}{\partial s}(s,x) = \frac{\partial^2 u}{\partial x^2}(s,x) - \int_{-\infty}^{s} \frac{(s-t)}{2\Gamma(2)} e^{-(s-t)} \frac{\partial^2 u}{\partial x^2}(t,x)dt + g(s,u(s)) \\ u(0,s) = u(\pi,s) = 0, \end{cases} \tag{5.1}$$

with $x \in [0,\pi]$, $t \in \mathbb{R}$.
Let $X := L^2([0,\pi])$, $T := \frac{d^2}{dx^2}$, with domain $D(T) = \{h \in H^2([0,\pi]), h(0) = h(\pi) = 0\}$.

Then we can write the problem (5.1) in the abstract form (1.1), with $a = -\frac{1}{2}$; $b = 1$ and $m = 2$. Since T generate an immediately norm continuous, with $\sigma(T) = \{-n^2, n \in \mathbb{N}\}$, then the solutions to equation

$$\frac{\lambda_T(\lambda_T+1)^2}{(\lambda_T+1)^2 - \frac{1}{2}} = -n^2,$$

are given by

$$\lambda_T^{n,1} = \frac{-(n^2+2)}{3} - \frac{6a_n}{c_n} + \frac{1}{6}c_n,$$
$$\lambda_T^{n,2} = \frac{-(n^2+2)}{3} + 3\frac{a_n}{c_n} - \frac{c_n}{12} + \frac{\sqrt{3}}{2}(\frac{1}{6}c_n + 6\frac{a_n}{c_n})i$$
$$\lambda_T^{n,3} = \frac{-(n^2+2)}{3} + 3\frac{a_n}{c_n} - \frac{c_n}{12} - \frac{\sqrt{3}}{2}(\frac{1}{6}c_n + 6\frac{a_n}{c_n})i$$

for all $n \geq 1$, where

$a_n := -\frac{1}{9} + \frac{2}{9}n^2 - \frac{1}{9}n^4$, $c_n := (8 + 30n^2 + 24n^4 - 8n^6 + 6n\sqrt{24 + 9n^2 + 72n^4 - 24n^6})^{\frac{1}{3}}$.

An easy computation shows that

$$\sup\{\Re(\lambda_T); \lambda_T(\lambda_T+1)^2((\lambda_T+1)^2 - \frac{1}{2})^{-1} \in \sigma(T)\} < 0.$$

Therefore, from $([2], proposition 3.1)$ we conclude that there exists a strongly continuous family of operators $\{S(s)\}_{s\geq 0} \subset \mathcal{B}(X)$ such that $\|S(s)\| \leq Me^{-\delta s}$ for some $M, \delta > 0$.

We take $g(s+\tau, \varphi)(t) := \eta(s)\varphi(t)$.
Assume that $\eta(s)$ is a bounded continuous τ-periodic function, i.e. $\eta(s+\tau) = \eta(s)$, then we have

$$g(s+\tau,\varphi)(t) = \eta(s+\tau)e^{i\rho\tau}e^{-i\rho\tau}\varphi(t) = e^{i\rho\tau}\eta(s)e^{-i\rho\tau}\varphi(t) = e^{i\rho\tau}g(s, e^{-i\rho\tau}\varphi)(t).$$

We also have

$$\|g(s,\varphi_1) - g(s,\varphi_2)\|^2_{L^2([0,\pi])} \leq |\eta(s)|^2 \|\varphi_1(t) - \varphi_2(t)\|^2_{L^2([0,\pi])}.$$

Then the equation (5.1) has a unique (μ_1, μ_2)-Pseudo-asymptotically Bloch τ-periodic mild solution on $\mathbb{R}$ provided that $\delta > \|\eta\|_\infty M$ by Theorem 4.2 (i).

6 Conflict of interest statement

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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