Existence of solutions for the fractional hybrid differential equation via measure of noncompactness

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Abstract. In this paper with the help of a newly defined contraction operator, a fixed point theorem is established and studied the solvability of fractional hybrid differential equation in a Banach space. Also, with the help of proper examples, we investigate our findings.

Key Words: Measure of noncompactness; fractional hybrid differential equation; Fixed point theorem.

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1. Introduction

Fractional differential equations play a significant role to study different types of real life phenomenon. Also, fractional integral equations are extremely useful in solving different real-world situations. Because of the relevance of integral equations of fractional order, it is necessary to understand such equations. The idea of a measure of noncompactness ($\mathcal{MNC}$) plays a relevant role in fixed point theory. Kuratowski [23] pioneered the concept of $\mathcal{MNC}$ in 1930. In the year 1955, G. Darbo [10] developed a result demonstrating the presence of a fixed point, that is, called condensing operators, utilizing the idea of the $\mathcal{MNC}$.

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Fixed point theory and the $\mathcal{MNC}$ have several applications in the analysis of various integral equations and differential equations that arise in many real-world scenarios, for example readers can see [7, 8, 12, 17, 18, 19, 20, 21, 29, 30] and references therein.

The hybrid differential equations is a quadratic perturbations of nonlinear differential equations. Many researchers have been works on the theory of hybrid differential equations. The theory of fractional hybrid differential equations ($FHDE$) was investigated by Zhao et al. [33]. Dhage and Lakshmikantham [16] discussed some basic results on hybrid differential equations of first order. Lu et al. [26] and Dhage and Jadhav [15] obtained basic results of hybrid differential equations with linear perturbations of second type. Recently, Li et al. [24] solved boundary value problems for Hadamard sequential fractional hybrid differential inclusions and Das et al. [9] investigated existence of solution of an infinite system of $FHDE$ in a tempered sequence space. Recently, Devi and Borah [14], discussed the existence of solution for a nonlinear hybrid functional fractional differential equation. For details on $FHDE$ and its applications, one can see [1, 3, 13, 25, 28, 31] and references therein.

In the literature an ample amount of work may be seen on the topic of fractional hybrid differential equations connecting $\mathcal{MNC}$, for convenient one can see [11, 27, 32]. So based on these articles, we are motivated to discuss the existence of solution of hybrid differential equations using $\mathcal{MNC}$.

The goal of this article is to obtain the generalizations of Darbo's fixed point theorem using alternating distance function and apply it to test the solvability of $FHDE$ in Banach space.

Let $\mathbb{D}$ be a real Banach space with the norm $\| \, . \, \|$. Assume $\mathrm{B}(\theta, r) = \{t \in \mathbb{D} : \| t - \theta \| \leq r\}$. If $\mathcal{W}(\neq \emptyset) \subseteq \mathbb{D}$. Therefore, $\bar{\mathcal{W}}$ and Conv$\mathcal{W}$ indicate the closure and convex closure of $\mathcal{W}$.

- $\mathbb{R}$ = Real numbers = $(-\infty, \infty)$,
- $\mathbb{R}_+ = [0, \infty)$,
- $\mathbb{N}$ = Natural numbers,
- $\mathfrak{M}_{\mathbb{D}}$ = Collection of all nonempty and bounded subsets of $\mathbb{D}$,
- $\mathfrak{N}_{\mathbb{D}}$ = Collection of all relatively compact sets.

The following definition of a $\mathcal{MNC}$ is as shown in [5].

Definition 1.1. *A mapping $\mathcal{H} : \mathfrak{M}_{\mathbb{D}} \to \mathbb{R}_+$ is called a $\mathcal{MNC}$ in $\mathbb{D}$, if it satisfies the following axioms:*

(i) *for all $\mathcal{W} \in \mathfrak{M}_{\mathbb{D}}$, we get $\mathcal{H}(\mathcal{W}) = 0$ implies $\mathcal{W}$ is relatively compact.*
(ii) *ker $\mathcal{H} = \{\mathcal{W} \in \mathfrak{M}_{\mathbb{D}} : \mathcal{H}(\mathcal{W}) = 0\} \neq \emptyset$ and ker $\mathcal{H} \subset \mathfrak{N}_{\mathbb{D}}$.*
(iii) $\mathcal{W} \subseteq \mathcal{W}_1 \implies \mathcal{H}(\mathcal{W}) \leq \mathcal{H}(\mathcal{W}_1)$.

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(iv) $\mathcal{H}\left(\bar{\mathcal{W}}\right)=\mathcal{H}\left(\mathcal{W}\right).$
(v) $\mathcal{H}\left(Conv\mathcal{W}\right)=\mathcal{H}\left(\mathcal{W}\right).$
(vi) $\mathcal{H}\left(\mathbb{A}\mathcal{W}+(1-\mathbb{A})\,\mathcal{W}_1\right)\leq \mathbb{A}\mathcal{H}\left(\mathcal{W}\right)+(1-\mathbb{A})\,\mathcal{H}\left(\mathcal{W}_1\right)$ *for* $\mathbb{A}\in[0,1].$
(vii) *if* $\mathcal{W}_l\in\mathfrak{M}_{\mathbb{D}}$, $\mathcal{W}_l=\bar{\mathcal{W}}_l$, $\mathcal{W}_{l+1}\subset\mathcal{W}_l$ *for* $l=1,2,3,4,\ldots$ *and* $\lim\limits_{l\to\infty}\mathcal{H}\left(\mathcal{W}_l\right)=0$, *so* $\bigcap_{l=1}^{\infty}\mathcal{W}_l\neq\emptyset$.

Now, the family ker$\mathcal{H}$ is called the *kernel of measure* $\mathcal{H}$. So, $\mathcal{W}_\infty=\bigcap_{l=1}^{\infty}\mathcal{W}_l\in\ker\mathcal{H}$. Since $\mathcal{H}(\mathcal{W}_\infty)\leq\mathcal{H}(\mathcal{W}_l)$ for any l, we conclude $\mathcal{H}(\mathcal{W}_\infty)=0$.

The following fundamental theorems are useful for our discussion.

Theorem 1.2. [2, Schauder] *Let* $\mathcal{W}$ *be a nonempty, bounded, closed and convex subset* ($N\mathbb{B}\mathbb{C}\mathbb{C}S$) *of a Banach space* $\mathbb{D}$. *Then* $\Im:\mathcal{W}\to\mathcal{W}$ *possesses at least one fixed point, provided that* $\Im$ *is a compact and continuous mapping.*

Theorem 1.3. [10, Darbo] *Let* $\mathfrak{O}$ *be a* $N\mathbb{B}\mathbb{C}\mathbb{C}S$ *of a Banach space* $\mathbb{D}$ *and let* $\Im:\mathfrak{O}\to\mathfrak{O}$. *Assume that a constant* $B\in[0,1)$ *such that*

$$\mathcal{H}(\Im\mathcal{C})\leq B\mathcal{H}(\mathcal{C}),\ \mathcal{C}\subseteq\mathfrak{O}.$$

Then, there is a fixed point in $\mathfrak{O}$ *for* $\Im$ *provided that* $\Im$ *is a continuous mapping.*

For an extension of Darbo's theorem, we consider the following functions.

Definition 1.4. [7] *Suppose* Υ *be a collection of functions* $\mathbb{S}:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ *satisfying:*

(1) $\max\{\mathbb{E},\mathbb{F}\}\leq\mathbb{S}(\mathbb{E},\mathbb{F})$ *for* $\mathbb{E},\mathbb{F}\geq 0$.
(2) $\mathbb{S}$ *is continuous and nondecreasing.*
(3) $\mathbb{S}(\mathbb{E}_1+\mathbb{E}_2,\mathbb{F}_1+\mathbb{F}_2)\leq\mathbb{S}(\mathbb{E}_1,\mathbb{F}_1)+\mathbb{S}(\mathbb{E}_2,\mathbb{F}_2)$.

For example, $\mathbb{S}(\mathbb{E},\mathbb{F})=\mathbb{E}+\mathbb{F}$.

Definition 1.5. [4] *Let* $f:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ *be the continuous mapping of the* $\mathcal{C}$*-class if the following axioms are holds:*

(1) $f(g,k)\leq g$,
(2) $f(g,k)=g$ *implies that either* $g=0$ *or* $k=0$.

Also, $f(0,0)=0$. *Note that the* $\mathcal{C}$*-class mapping is symbolized by* $\mathcal{C}$.

As an illustration:

(a) $f(g,k)=g-k$,
(b) $f(g,k)=ng,\ 0<n<1$.

Definition 1.6. [22] *A mapping* $\omega:\mathbb{R}\to\mathbb{R}$ *is an alternating distance mapping, if*

(1) $\omega(s) = 0 \Leftrightarrow s = 0$.

(2) ω *is continuous and increasing.*

We use Ψ *to denote this class of functions.*

For example, $\omega(s) = (1-a)s,\ 0 \leq a < 1$.

Definition 1.7. [4] *An continuous function* $\upsilon : \mathbb{R} \to \mathbb{R}$ *is an ultra altering distance mapping if* $\upsilon(0) \geq 0$ *and* $\upsilon(s) > 0$ *for* $s > 0$.

We use Φ to denote this class of functions.

Definition 1.8. *Let* $h : \mathbb{R} \to \mathbb{R}$ *be a continuous mapping of* $\mathcal{A}-$*class. If* $h(s) > s$ *for* $s \in (0, \infty)$. *Also* $h(0) = 0$.

For example, $h(s) = \bar{m}s,\ \bar{m} > 1$.

2. Main Theorems

Theorem 2.1. *Let* Ψ *be a* $N\mathbb{BCC}S$ *of a Banach space* $\mathbb{D}$. *If* $\Im : \Psi \to \Psi$ *is a continuous mapping such that*

$$h\left[\omega\left[S\left(\mathcal{H}\left(\Im\mathcal{C}\right), \gamma\left(\mathcal{H}\left(\Im\mathcal{C}\right)\right)\right)\right]\right] \leq f\left[\omega\left\{S\left(\mathcal{H}\left(\mathcal{C}\right), \gamma\left(\mathcal{H}\left(\mathcal{C}\right)\right)\right)\right\}, \upsilon\left\{S\left(\mathcal{L}\left(\mathcal{C}\right), \gamma\left(\mathcal{L}\left(\mathcal{C}\right)\right)\right)\right\}\right], \tag{2.1}$$

where $\mathcal{C} \subset \Psi$ *and* $\mathcal{H}$ *is an arbitrary* $\mathcal{MNC}$ *and* $S \in \Upsilon,\ \upsilon \in \Phi, \omega \in \Psi,\ f \in \mathcal{C},\ h \in \mathcal{A}$. *Also,* $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ *is a non-decreasing and continuous mapping. Then there exists at least one fixed point for* $\Im$ *in* Ψ.

Proof. Let us consider a sequence $\{\Psi_p\}_{p=1}^{\infty}$ with $\Psi_1 = \Psi$ and $\Psi_{p+1} = \mathrm{Conv}(\Im\Psi_p)$ for $p \in \mathbb{N}$. Also $\Im\Psi_1 = \Im\Psi \subseteq \Psi = \Psi_1,\ \Psi_2 = \mathrm{Conv}(\Im\Psi_1) \subseteq \Psi = \Psi_1$. By proceeding in the same manner gives $\Psi_1 \supseteq \Psi_2 \supseteq \Psi_3 \supseteq \ldots \supseteq \Psi_p \supseteq \Psi_{p+1} \supseteq \ldots$.

If $\mathcal{H}(\Psi_{p_0}) = 0$ for some $p_0 \in \mathbb{N}$. So Ψ_{p_0} is a compact set. In this instance, $\Im$ has a fixed point in Ψ, according to Schauder's Theorem.

Again, if $\mathcal{H}(\Psi_p) > 0$ for all $p \in \mathbb{N}$.

Now, for all $p \in \mathbb{N}$,

$$\begin{aligned}
&h\left[\omega\left[S\left(\mathcal{H}\left(\Psi_{p+1}\right), \gamma\left(\mathcal{H}\left(\Psi_{p+1}\right)\right)\right)\right]\right] \\
&= h\left[\omega\left[S\left(\mathcal{H}\left(\mathrm{Conv}\Im\Psi_p\right), \gamma\left(\mathcal{H}\left(\mathrm{Conv}\Im\Psi_p\right)\right)\right)\right]\right] \\
&= h\left[\omega\left[S\left(\mathcal{H}\left(\Im\Psi_p\right), \gamma\left(\mathcal{H}\left(\Im\Psi_p\right)\right)\right)\right]\right] \\
&\leq f\left[\omega\left\{S\left(\mathcal{H}\left(\Psi_p\right), \gamma\left(\mathcal{H}\left(\Psi_p\right)\right)\right)\right\}, \upsilon\left\{S\left(\mathcal{H}\left(\Psi_p\right), \gamma\left(\mathcal{H}\left(\Psi_p\right)\right)\right)\right\}\right] \\
&\leq \omega\left\{S\left(\mathcal{H}\left(\Psi_p\right), \gamma\left(\mathcal{H}\left(\Psi_p\right)\right)\right)\right\}.
\end{aligned}$$

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Also,

$$h\left\{\omega\left\{S\left(\mathcal{H}\left(\Psi_{p+1}\right),\gamma\left(\mathcal{H}\left(\Psi_{p+1}\right)\right)\right)\right\}\right\} \geq \omega\left\{S\left(\mathcal{H}\left(\Psi_{p+1}\right),\gamma\left(\mathcal{H}\left(\Psi_{p+1}\right)\right)\right)\right\}.$$

Hence

$$\omega\left\{S\left(\mathcal{H}\left(\Psi_{p+1}\right),\gamma\left(\mathcal{H}\left(\Psi_{p+1}\right)\right)\right)\right\} \leq \omega\left\{S\left(\mathcal{H}\left(\Psi_{p}\right),\gamma\left(\mathcal{H}\left(\Psi_{p}\right)\right)\right)\right\}.$$

Clearly $\{\omega\left\{S\left(\mathcal{H}\left(\Psi_{p}\right),\gamma\left(\mathcal{H}\left(\Psi_{p}\right)\right)\right)\right\}\}_{p=1}^{\infty}$ is a non-negative and non-increasing sequence, hence there exists $\sigma \geq 0$ such that

$$\lim_{p\to\infty} \omega\left\{S\left(\mathcal{H}\left(\Psi_{p}\right),\gamma\left(\mathcal{H}\left(\Psi_{p}\right)\right)\right)\right\} = \sigma.$$

If possible, let $\sigma > 0$. As $p \to \infty$, we get

$$h(\sigma) \leq \sigma$$

which is a contradiction.

Thus, $\sigma = 0$.

i.e.,

$$\omega\left\{\lim_{p\to\infty} S\left(\mathcal{H}\left(\Psi_{p}\right),\gamma\left(\mathcal{H}\left(\Psi_{p}\right)\right)\right)\right\} = 0,$$

i.e.,

$$\lim_{p\to\infty} S\left(\mathcal{H}\left(\Psi_{p}\right),\gamma\left(\mathcal{H}\left(\Psi_{p}\right)\right)\right) = 0$$

which gives

$$\lim_{p\to\infty} \mathcal{H}\left(\Psi_{p}\right) = 0.$$

Since $\Psi_p \supseteq \Psi_{p+1}$. By Definition 1.1, we obtain $\Psi_\infty = \bigcap_{p=1}^{\infty} \mathbb{U}_p$ is non-empty, closed and convex subset of $\mathbb{U}$ and $\mathbb{U}_\infty$ is $\Im$ invariant.

We conclude that $\Im$ has a fixed point in Ψ based on Theorem 1.2. This completes the proof of the theorem. ☐

Theorem 2.2. *Let Ψ be a $N\mathbb{BCCS}$ of a Banach space $\mathbb{D}$. If $\Im : \Psi \to \Psi$ is a continuous mapping such that*

$$h\left[\omega\left[\mathcal{H}\left(\Im\mathcal{C}\right)+\gamma\left(\mathcal{H}\left(\Im\mathcal{C}\right)\right)\right]\right] \leq f\left[\omega\left\{\mathcal{H}\left(\mathcal{C}\right)+\gamma\left(\mathcal{H}\left(\mathcal{C}\right)\right)\right\},\upsilon\left\{\mu\left(\mathcal{C}\right)+\gamma\left(\mu\left(\mathcal{C}\right)\right)\right\}\right], \tag{2.2}$$

where $\mathcal{C} \subset \Psi$ and $\mathcal{H}$ is an arbitrary $\mathcal{MNC}$, $\upsilon \in \Phi$, $\omega \in \Psi, f \in \mathcal{C}$, $h \in \mathcal{A}$. Also, $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing with continuous function. Then there exists at least one fixed point for $\Im$ in Ψ.

Proof. Taking the result leads to $S(\upsilon, w) = \upsilon + w$ in Theorem 2.1. ☐

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Theorem 2.3. *Let Ψ be a $N\mathbb{B}\mathbb{C}\mathbb{C}S$ of a Banach space $\mathbb{D}$. If $\Im : \Psi \to \Psi$ is a continuous mapping such that*

$$h\left[\omega\left[\mathcal{H}\left(\Im\mathcal{C}\right)\right]\right] \leq f\left[\omega\left\{\mathcal{H}\left(\mathcal{C}\right)\right\}, \upsilon\left\{\mathcal{H}\left(\mathcal{C}\right)\right\}\right], \tag{2.3}$$

where $\mathcal{C} \subset \Psi$ and $\mathcal{H}$ is an arbitrary $\mathcal{MNC}$, $\upsilon \in \Phi$, $\omega \in \Psi$, $f \in \mathcal{C}$, $h \in \mathcal{A}$. Then there exists at least one fixed point for $\Im$ in Ψ.

Proof. Taking the result leads to $\gamma(t) = 0$ in Theorem 2.2. □

Theorem 2.4. *Let Ψ be a $N\mathbb{B}\mathbb{C}\mathbb{C}S$ of a Banach space $\mathbb{D}$. If $\Im : \Psi \to \Psi$ is a continuous mapping such that*

$$h\left[\omega\left[S\left(\mathcal{H}\left(\Im\mathcal{C}\right), \gamma\left(\mathcal{H}\left(\Im\mathcal{C}\right)\right)\right)\right]\right] \leq \omega\left[S\left(\mathcal{H}\left(\Im\mathcal{C}\right), \gamma\left(\mathcal{H}\left(\Im\mathcal{C}\right)\right)\right)\right], \tag{2.4}$$

where $\mathcal{C} \subset \Psi$ and $\mathcal{H}$ is an arbitrary $\mathcal{MNC}$ and $S \in \Upsilon$, $\omega \in \Psi$, $h \in \mathcal{A}$. Also, $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing with continuous function. Then there exists at least one fixed point for $\Im$ in Ψ.

Proof. Using $f(g, h) \leq g$ in Theorem 2.1. □

Corollary 2.5. *Taking $\mathbb{S}(v, w) = v+w$, $\gamma(s) = 0$, $\omega(s) = s$, $f(g, k) = ng$ and $h(s) = \bar{m}s$, where $0 < m < 1$, $\bar{m} > 1$ in Theorem 2.1, one can obtain*

$$\mathcal{H}\left(\Im\mathcal{C}\right) \leq \lambda\mathcal{H}\left(\mathcal{C}\right), \ \lambda = \frac{m}{\bar{m}} \in (0, 1).$$

It can be observed that our fixed point theorem is a generalization of Darbo's fixed point theorem.

Remark 2.6. *We have extended Darbo's fixed point theorem using a new contraction operator that includes a $\mathcal{MNC}$ in order to investigate operators whose properties may be described as intermediate between those of contraction and compact mapping. The significant advantage of this generalization based on a $\mathcal{MNC}$ is that the compactness of the operator's domain, which is crucial to Schauder's theorem, has been extended.*

Definition 2.7. [6] *An element $(v, w) \in \mathcal{W} \times \mathcal{W}$ is called a coupled fixed point of the function $\mathfrak{J} : \mathcal{W} \times \mathcal{W} \to \mathcal{W}$ if $\mathfrak{J}(v, w) = v$ and $\mathfrak{J}(w, v) = w$.*

Theorem 2.8. [5] *Assume that $\rho_1, \rho_2, \ldots, \rho_n$ be the $\mathcal{MNC}$ in $\mathbb{D}_1, \mathbb{D}_2, \ldots, \mathbb{D}_n$, respectively. Moreover, let the mapping $\mathcal{W} : \mathbb{R}_+^n \to \mathbb{R}_+$ be convex with $\Upsilon\left(v_1, v_2, \ldots, v_n\right) = 0$ if and only if $v_k = 0$ for $k = 1, 2, 3, \ldots, n$, then $\rho(\mathcal{W}) = \Upsilon\left(\rho_1(\mathcal{W}_1), \rho_2(\mathcal{W}_2), \ldots, \rho_n(\mathcal{W}_n)\right)$ define a $\mathcal{MNC}$ in $\mathbb{D}_1 \times \mathbb{D}_2 \times \ldots \times \mathbb{D}_n$, where $\mathcal{W}_k$ denotes the natural projection of $\mathcal{W}$ into $\mathbb{D}_k$ for $k = 1, 2, 3, \ldots, n$.*

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Example 2.9. [5] *Let ρ be a $\mathcal{MNC}$ on $\mathbb{D}$. Define $\Upsilon(q,y) = q + y,\ q, y \in \mathbb{R}_+$. Then Υ has all the properties mentioned in the Theorem 2.8. Thus $\rho^{cf}(\mathcal{W}) = \rho(\mathcal{W}_1) + \rho(\mathcal{W}_2)$ is a $\mathcal{MNC}$ in the space $\mathbb{D} \times \mathbb{D}$, where $\mathcal{W}_k,\ k = 1, 2$ denote the natural projections of $\mathcal{W}$.*

Theorem 2.10. *Let Ψ be a $N\mathbb{BCCS}$ of a Banach space $\mathbb{D}$. Also $\mathcal{R} : \Psi \times \Psi \to \Psi$ is a continuous mapping such that*

$$
\begin{aligned}
& h\left[\omega\left[S\left(\mathcal{H}\left(\mathcal{R}\left(\Im_1 \times \Im_2\right)\right), \gamma\left(\mathcal{H}\left(\mathcal{R}\left(\Im_1 \times \Im_2\right)\right)\right)\right)\right]\right] \\
& \leq \frac{1}{2} \omega\left[S\left(\mathcal{H}\left(\Im_1\right) + \mathcal{H}\left(\Im_2\right), \gamma\left(\mathcal{H}\left(\Im_1\right) + \mathcal{H}\left(\Im_2\right)\right)\right)\right]
\end{aligned}
$$

for any nonempty $\Im_1, \Im_2 \subseteq \Psi$, where $\mathcal{H}$ is an arbitrary $\mathcal{MNC}$ and $S \in \Upsilon,\ h \in \mathcal{A},\ \omega \in \Psi$. Also, $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing with continuous mapping. Furthermore, $\gamma(q + y) \leq \gamma(q) + \gamma(y),\ \omega(q + y) \leq \omega(q) + \omega(y),\ h(q + y) \leq h(q) + h(y)$. Then $\mathcal{R}$ has at least one coupled fixed point in $\Psi \times \Psi$.

Proof. We observe that $\mathcal{H}^{cf}(\Im) = \mathcal{H}(\Im_1) + \mathcal{H}(\Im_2)$ is a $\mathcal{MNC}$ on $\mathbb{D} \times \mathbb{D}$ for any bounded subset $\Im \subseteq \mathbb{D} \times \mathbb{D}$, where $\Im_1, \Im_2$ denote the natural projection of $\Im$.

Consider a mapping $\mathcal{R}^{cf} : \Psi \times \Psi \to \Psi \times \Psi$ by $\mathcal{R}^{cf}(q, y) = (\mathcal{R}(q, y), \mathcal{R}(q, y))$.

It is trivial that $\mathcal{R}^{cf}$ is a continuous. Let $\Im \subseteq \Psi \times \Psi$ and we obtain

$$
\begin{aligned}
& h\left[\omega\left[S\left(\mathcal{H}^{cf}\left(\mathcal{R}^{cf}\left(\Im\right)\right), \gamma\left(\mathcal{H}^{cf}\left(\mathcal{R}^{cf}\left(\Im\right)\right)\right)\right)\right]\right] \\
& \leq h\left[\omega\left[S\left(\mathcal{H}^{cf}\left(\mathcal{R}\left(\Im_1 \times \Im_2\right) \times \mathcal{R}\left(\Im_2 \times \Im_1\right)\right), \gamma\left(\mathcal{H}^{cf}\left(\mathcal{R}\left(\Im_1 \times \Im_2\right) \times \mathcal{R}\left(\Im_1 \times \Im_2\right)\right)\right)\right)\right]\right] \\
& = h\left[\omega\left[S\left(\mathcal{H}\left(\mathcal{R}\left(\Im_1 \times \Im_2\right)\right) + \mathcal{H}\left(\mathcal{R}\left(\Im_2 \times \Im_1\right)\right), \gamma\left(\mathcal{H}\left(\mathcal{R}\left(\Im_1 \times \Im_2\right)\right) + \mathcal{H}\left(\mathcal{R}\left(\Im_2 \times \Im_1\right)\right)\right)\right)\right]\right] \\
& \leq h\left[\omega\left[S\left(\mathcal{H}\left(\mathcal{R}\left(\Im_1 \times \Im_2\right)\right) + \mathcal{H}\left(\mathcal{R}\left(\Im_2 \times \Im_1\right)\right), \gamma\left(\mathcal{H}\left(\mathcal{R}\left(\Im_1 \times \Im_2\right)\right)\right) + \gamma\left(\mathcal{H}\left(\mathcal{R}\left(\Im_2 \times \Im_1\right)\right)\right)\right)\right]\right] \\
& \leq h\left[\omega\left[S\left(\mathcal{H}\left(\mathcal{R}\left(\Im_1 \times \Im_2\right)\right), \gamma\left(\mathcal{H}\left(\mathcal{R}\left(\Im_1 \times \Im_2\right)\right)\right)\right)\right]\right] + h\left[\omega\left[S\left(\mathcal{H}\left(\mathcal{R}\left(\Im_2 \times \Im_1\right)\right), \gamma\left(\mathcal{H}\left(\mathcal{R}\left(\Im_2 \times \Im_1\right)\right)\right)\right)\right]\right] \\
& \leq \omega\left[S\left(\mathcal{H}\left(\Im_1\right) + \mathcal{H}\left(\Im_2\right), \gamma\left(\mathcal{H}\left(\Im_1\right) + \mathcal{H}\left(\Im_2\right)\right)\right)\right] \\
& = \omega\left[S\left(\mathcal{H}^{cf}\left(\Im\right), \gamma\left(\mathcal{H}^{cf}\left(\Im\right)\right)\right)\right].
\end{aligned}
$$

By Theorem 2.4, we conclude that $\mathcal{R}^{cf}$ has at least one fixed point in $\Psi \times \Psi$. That is, $\mathcal{R}$ has at least one coupled fixed point. □

3. Solvability fractional hybrid differential equation

Suppose $\mathbb{D} = C(I)$ represents the space of continuous real functions on $I = [0, T]$. Therefore, equipped with

$$
\| \mathcal{W} \| = \sup \left\{ |\mathcal{W}(\sigma)| : \sigma \in I \right\}, \ \mathcal{W} \in \mathbb{D}.
$$

Let $\mathcal{Z}(\neq \emptyset) \subseteq \mathbb{D}$ be bounded. For $\mathcal{W} \in \mathcal{Z}$ with $\delta > 0$, denote by $G(\mathcal{W}, \delta)$ the modulus of the continuity of $\mathcal{W}$, i.e.,

$$G(\mathcal{W}, \delta) = \sup \{|\mathcal{W}(\sigma_1) - \mathcal{W}(\sigma_2)| : \sigma_1, \sigma_2 \in I, |\sigma_2 - \sigma_1| \leq \delta\}.$$

In addition, we define

$$G(\mathcal{Z}, \delta) = \sup \{G(\mathcal{W}, \delta) : \mathcal{W} \in \mathcal{Z}\};\ G_0(\mathcal{Z}) = \lim_{\delta \to 0} G(\mathcal{Z}, \delta).$$

It is widely known that the mapping G_0 is a $\mathcal{MNC}$ in $\mathbb{D}$, with $\Theta(\mathcal{W}) = \frac{1}{2} G_0(\mathcal{W})$ (see [5]) functioning as the Hausdorff $\mathcal{MNC}$.

For any $\varpi \in \mathbb{R}$ with $0 < c < 1$, the mapping $\mu \in \mathbb{R}$ is a solution of the $FHDE$ [26]

$$D^c[\mu(v)e(v, \mu(v))] = \varpi(v),\ v \in [v_0, v_0 + a] = I \text{ and } \mu(v_0) = \mu_0 \tag{3.1}$$

iff μ satisfies the hybrid integral equation is

$$\mu(v) = \mu_0 - e(v_0, \mu_0) + e(v, \mu(v)) + \frac{1}{\Gamma(c)} \int_{v_0}^{v} (v - l)^{c-1} \varpi(l) dl, \tag{3.2}$$

where $0 < c < 1$.

Let

$$\mathbb{Q}_{r_0} = \{\mu \in \mathbb{D} : \parallel \mu \parallel \leq r_0\}.$$

Assume that

(A_1) $e : I \times \mathbb{R}_+ \to \mathbb{R}$ be a continuous and there exists a constant $D \geq 0$ satisfying

$$|e(v, \mu) - e(v, s)| \leq D |\mu - s|,$$

for $v \in I$ and $\mu, s \in \mathbb{R}$.

Also, for all $v \in I$

$$e(v, 0) = 0.$$

(A_2) $\varpi : I \to \mathbb{R}$ be continuous and $\parallel \varpi \parallel \leqslant H$, $H \geq 0$. Also

$$|\mu_0 - e(v_0, s_0)| \leq X_0 \text{ (say)}.$$

(A_3) There exists $r_0 > 0$ such that

$$X_0 + Dr_0 + \frac{Ha^c}{\Gamma(c+1)} \leq r_0$$

and

$$D < 1.$$

Theorem 3.1. *If the assupmtions (A_1)-(A_3) hold, then the equation (3.1) has a solution in $\mathbb{D} = C(I)$.*

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Proof. The operator$\Im : \mathbb{D} \to \mathbb{D}$ is defined as follows

$$(\Im\mu)(v) = \mu_0 - e(v_0, \mu_0) + e(v, \mu(v)) + \frac{1}{\Gamma(c)} \int_{v_0}^{v} (v-l)^{c-1} \varpi(l) dl.$$

Phase (1): We show that the operator $\Im$ maps $\mathbb{Q}_{r_0}$ into $\mathbb{Q}_{r_0}$. Let $\mu \in \mathbb{Q}_{r_0}$. We now have

$$\begin{aligned}
&|(\Im\mu)(v)| \\
&\leq |\mu_0 - e(v_0, \mu_0)| + |e(v, \mu(v))| + \frac{1}{\Gamma(c)} \int_{v_0}^{v} (v-l)^{c-1} |\varpi(l)| \, dl \\
&\leq X_0 + |e(v, \mu(v)) - e(v, 0)| + |e(v, 0)| + \frac{H}{\Gamma(c)} \int_{v_0}^{v} (v-l)^{c-1} \, dl \\
&\leq X_0 + D \parallel \mu \parallel + \frac{H}{\Gamma(c)} \left[\frac{-(v-l)^c}{c} \right]_{v_0}^{v} \\
&\leq X_0 + D \parallel \mu \parallel + \frac{H}{\Gamma(c+1)} (v - v_0)^c \\
&\leq X_0 + D \parallel \mu \parallel + \frac{H a^c}{\Gamma(c+1)}.
\end{aligned}$$

Hence $\parallel \mu \parallel \leq r_0$ gives

$$\parallel \Im \parallel \leq X_0 + D r_0 + \frac{H a^c}{\Gamma(c+1)} \leq r_0.$$

Thus $\Im$ maps $\mathbb{Q}_{r_0}$ to $\mathbb{Q}_{r_0}$.

Phase (2): We will show that $\Im$ is continuous on $\mathbb{Q}_{r_0}$. Let $\delta > 0$ and $\mu, s \in \mathbb{Q}_{r_0}$ such that $\parallel \mu - s \parallel < \delta$ and $s(v_0) = s_0$. We have

$$\begin{aligned}
&|(\Im\mu)(v) - (\Im s)(v)| \\
&= |\mu(v_0) - s(v_0)| + |e(v_0, \mu(v_0)) - e(v_0, s(v_0))| + |e(v, \mu(v)) - e(v, s(v))| \\
&\leq |\mu(v_0) - s(v_0)| + D \, |\mu(v_0) - s(v_0)| + D \, |\mu(v) - s(v)| \\
&< \delta + D\delta + D\delta \\
&= (1 + 2D)\delta,
\end{aligned}$$

i.e., as $\delta \to 0$, we obtain $|(\Im\mu)(v) - (\Im s)(v)| \to 0$.

Therefore, $\Im$ is continuous on $\mathbb{Q}_{r_0}$.

Phase (3): An estimate of $\Im$ with respect to G_0. Now, assuming $\Omega_\mu \subseteq \mathbb{Q}_{r_0}$. Let $\delta > 0$ be arbitrary and choosing $\mu \in \Omega_\mu$ and $v_1, v_2 \in I$ such as $|v_2 - v_1| \leq \delta$ with $v_2 \geq v_1$.

We have

$$
\begin{aligned}
&|(\Im\mu)(v_2) - (\Im\mu)(v_1)| \\
&\leq |e(v_2,\mu(v_2)) - e(v_1,\mu(v_1))| \\
&+ \frac{1}{\Gamma(c)} \mid \int_{v_0}^{v_2} (v_2 - l)^{c-1} \varpi(l)dl - \int_{v_0}^{v_1} (v_1 - l)^{c-1} \varpi(l)dl \mid \\
&\leq |e(v_2,\mu(v_2)) - e(v_1,\mu(v_2))| + |e(v_1,\mu(v_2)) - e(v_1,\mu(v_1))| \\
&+ \frac{1}{\Gamma(c)} \mid \int_{v_0}^{v_2} (v_2 - l)^{c-1} \varpi(l)dl - \int_{v_0}^{v_1} (v_1 - l)^{c-1} \varpi(l)dl \mid \\
&\leq \gamma_{r_0}(e,\delta) + DG(\mu,\delta) + \frac{1}{\Gamma(c)} \mid \int_{v_0}^{v_2} (v_2 - l)^{c-1} \varpi(l)dl - \int_{v_0}^{v_1} (v_1 - l)^{c-1} \varpi(l)dl \mid,
\end{aligned}
$$

where

$$\gamma_{r_0}(e,\delta) = \sup\left\{|e(v_2,\mu) - e(v_1,\mu)| : |v_2 - v_1| \leq \delta,\ v_1, v_2 \in I,\ \parallel \mu \parallel \leq r_0\right\}.$$

Now,

$$
\begin{aligned}
&\mid \int_{v_0}^{v_2} (v_2 - l)^{c-1} \varpi(l)dl - \int_{v_0}^{v_1} (v_1 - l)^{c-1} \varpi(l)dl \mid \\
&\leq \mid \int_{v_0}^{v_2} (v_2 - l)^{c-1} \varpi(l)dl - \int_{v_0}^{v_1} (v_2 - l)^{c-1} \varpi(l)dl \mid \\
&+ \mid \int_{v_0}^{v_1} (v_2 - l)^{c-1} \varpi(l)dl - \int_{v_0}^{v_1} (v_1 - l)^{c-1} \varpi(l)dl \mid \\
&\leq \int_{v_1}^{v_2} (v_2 - l)^{c-1} \mid \varpi(l) \mid dl + \int_{v_0}^{v_1} \left\{(v_1 - l)^{c-1} - (v_2 - l)^{c-1}\right\} \mid \varpi(l) \mid dl \\
&\leq H \int_{v_1}^{v_2} (v_2 - l)^{c-1} dl + H \int_{v_0}^{v_1} \left\{(v_1 - l)^{c-1} - (v_2 - l)^{c-1}\right\} dl \\
&= \frac{H}{c} (v_2 - v_1)^c + \frac{H}{c}[(v_2 - v_1)^c + (v_1 - v_0)^c - (v_2 - v_0)^c] \\
&\leq \frac{2H\delta^c}{c} + \frac{H}{c}[(v_1 - v_0)^c - (v_2 - v_0)^c].
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&|(\Im\mu)(v_2) - (\Im\mu)(v_1)| \\
&\leq \gamma_{r_0}(e,\delta) + DG(\mu,\delta) + \frac{2H\delta^c}{\Gamma(c+1)} + \frac{H}{\Gamma(c+1)}[(v_1 - v_0)^c - (v_2 - v_0)^c].
\end{aligned}
$$

As $\delta \to 0$, $v_2 \to v_1$ so

$$|(\Im\mu)(v_2) - (\Im\mu)(v_1)| \leq DG_0(\mu),$$

i.e.,

$$G(\Im\mathcal{C}_\mu, \delta) \le DG_0(\mathcal{C}_\mu).$$

As $\delta \to 0$,

$$G(\Im\mathcal{C}_\mu) \le DG_0(\mathcal{C}_\mu).$$

Thus by Corollary 2.5, $\Im$ has a fixed point in $\mathbb{Q}_{r_0}$. i.e., the equation (3.2) has a solution in $\mathbb{D}$. □

Example 3.2. Consider the fractional hybrid differential equation as follows

$$D^{\frac{1}{2}}\left[\mu(v) - \frac{\mu(v)}{6v}\right] = v^2,\ \mu(1) = 1 \tag{3.3}$$

for $v \in [1, 3] = I$.

Solution: Here $c = \frac{1}{2},\ a = 2,\ v_0 = 1,\ \mu(v) = 1 = \mu_0$. Also,

$$e(v, \mu(v)) = \frac{\mu(v)}{6v},\ \varpi(v) = v^2,\ e(v, 0) = 0$$

and

$$H = 9.$$

Therefore

$$|e(v, \mu(v)) - e(v, s(v))| \le \frac{|\mu(v) - s(v)|}{6},$$

and

$$D = \frac{1}{6} < 1.$$

Also,

$$\mid \mu_0 - e(v_0, \mu_0) \mid = \frac{5}{6} = X_0.$$

Substituting these values in the inequality of assumption (A_3), we get

$$\frac{5}{6} + \frac{r_0}{6} + \frac{9(2)^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} \le r_0$$

$$\implies \frac{5r_0}{6} \ge \frac{5}{6} + \frac{9(2)^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}$$

$$\implies r_0 \ge 1 + \frac{54(2)^{\frac{1}{2}}}{5\Gamma(\frac{3}{2})}.$$

However, assumption (A_3) is also fulfilled for $r_0 = 1 + \frac{54(2)^{\frac{1}{2}}}{5\Gamma(\frac{3}{2})}$.

We see that all of the assumptions from (A_1) to (A_3) in Theorem 3.1 are achieved. Therefore by Theorem 3.1, we conclude that equation (3.3) has a solution in $\mathbb{D} = C(I)$.

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Example 3.3. Consider another fractional hybrid differential equation as follows

$$D^{\frac{1}{3}}\left[\mu(v) - \frac{\mu(v)}{3v^2+1}\right] = v^3,\ \mu(1) = 1 \tag{3.4}$$

for $v \in [1,2] = I$.

Solution: Here $c = \frac{1}{2},\ a = 1,\ v_0 = 1,\ \mu(v) = 1 = \mu_0$. Also,

$$e(v,\mu(v)) = \frac{\mu(v)}{3v^2+1},\ \varpi(v) = v^3,\ e(v,0) = 0$$

and

$$H = 8.$$

Therefore

$$|e(v,\mu(v)) - e(v,s(v))| \le \frac{|\mu(v) - s(v)|}{4},$$

and

$$D = \frac{1}{4} < 1.$$

Also,

$$|\ \mu_0 - e(v_0,\mu_0)\ | = \frac{3}{4} = X_0.$$

Substituting these values in the inequality of assumption (A_3), we get

$$\frac{3}{4} + \frac{r_0}{4} + \frac{8}{\Gamma(\frac{4}{3})} \le r_0$$
$$\implies \frac{3r_0}{4} \ge \frac{3}{4} + \frac{8}{\Gamma(\frac{4}{3})}$$
$$\implies r_0 \ge 1 + \frac{32}{3\Gamma(\frac{4}{3})}.$$

However, assumption (A_3) is also fulfilled for $r_0 = 1 + \frac{32}{3\Gamma(\frac{4}{3})}$.

We see that all of the assumptions from (A_1) to (A_3) in Theorem 3.1 are achieved. By Theorem 3.1, we can conclude that equation (3.3) haa a solution in $\mathbb{D} = C(I)$.

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