Convergence of solutions of nonautonomous perturbed singular systems via a refined integral inequality

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Abstract
The construction of a suitable Lyapunov function is still a difficult task. This paper mainly analyzes the practical uniform exponential stability of linear time–varying singular systems, which are transferable into standard canonical form. Our method used is based on the explicit solution form of the system via integral inequalities of the type of Gamidov under some restrictions on the perturbation term. An example is analyzed to verify the effectiveness of the proposed approach.

Keywords: Linear time–varying singular systems, Perturbed systems, Standard canonical form, Consistent initial conditions, Gamidov's inequality, Practical stability.

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1 Introduction

Singular systems are those dynamics of which are governed by a mixture of algebraic and differential equations. In that sense, singular systems represent the constraints to the solution of

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the differential part. These systems are referred to as degenerate systems, generalized systems, descriptor systems, semi–state systems, and differential–algebraic systems.

In [24], Rosenbrock proposed singular systems for the first time and handled the transformation of linear singular systems. Later on, singular systems representation has been used as a perfect tool to model a wide variety of problems, such as electrical engineering, aircraft dynamics, robotics, economics, optimization problems, chemical, biology, etc.

The complex nature of singular systems causes many difficulties in the analytical and numerical solutions. Because of the existence of the algebraic equations, the investigation of descriptor systems is more complicated than the study of systems in the classical form.

Because of the difficulty arising in analysis, few results are concerned with the stability of differential–algebraic equations (DAEs, in short).

The stability theory of linear differential–algebraic systems is an active research topic. Different authors tackled the question of stability of linear time–invariant singular systems. The interested reader refereed to see [8, 10, 11, 15, 23] and the references therein, for a complete overview on this subject. However, there have been few results on the stability of linear time–varying singular systems in the past few decades. Some developments toward this subject have been done in [4, 6, 9] and the references therein.

The method of Lyapunov is one of the most effective and efficient methods for the investigation of the stability of singular systems, without knowing the explicit solution form. Different authors developed the problem of stability of differential algebraic equations via Lyapunov techniques, see for instance [5, 16]. The construction of an appropriate Lyapunov function is not always possible, so that it is very important to look for other techniques to develop the problem of stability.

Few results are concerned with the stability based on the knowledge of the solution form via Gronwall type inequality, see [12, 19, 25].

The objective of our paper is to investigate the stability of linear time–varying singular systems under perturbation based on the explicit solution form via integral inequalities of the Gronwall type, in particular Gamidov's inequality.

Systems can naturally show a perturbed structure where the solutions of unperturbed equations are in general supposed to be stable and some restrictions are imposed on the uncertainties or disturbances like special growth conditions in order to derive conclusions about the behavior of solutions of the perturbed state equation, see [3].

Our results are related to the relation between a perturbed linear time–varying singular system and the associated unperturbed one. Given two solutions to the perturbed singular system and the associated unperturbed one with initial conditions that are close at the same value of time, these solutions will remain close over the entire time interval and not just at the initial time.

Indeed, the qualitative behavior of the solutions of linear time–varying perturbed singular systems is often analyzed by regarding the Lyapunov function candidate for the nominal system as an appropriate Lyapunov function candidate for the perturbed system.

Nevertheless, the construction of an appropriate Lyapunov function is still a difficult task. Hence, this encourages us to investigate the problem of stability of perturbed singular systems

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by utilizing integral inequalities of the Gronwall type under some constraints on the perturbation term. The usual property of the solutions concluded for such systems is ultimate boundedness. That means that the solutions stay in some neighborhoods of the origin after a sufficiently large time, called “Practical Stability, see [17, 22].

It is well known that asymptotic stability is more critical than stability. Even the selected system may oscillate near the origin. Therefore the concept of practical stability is more appropriate in different cases than asymptotic stability.

In this case, all state trajectories are bounded and approach a sufficiently small neighborhood of the origin. One also desires that the state approaches the origin (or some sufficiently small neighborhood of it) in a sufficiently fast manner especially in presence of perturbations.

The remainder of this paper is organized as follows. In Section 2, we introduce the linear time–varying singular systems transferable into standard canonical form, basic notations and some necessary properties. In Section 3, we present a class of linear time–varying perturbed singular system, where its associated unperturbed nominal system is transferable into standard canonical form. In Section 4, we investigate the practical stability of different classes of linear time–varying perturbed singular systems via non linear–integral inequalities of Gamidov's type. In Section 5, we present an example to show the effectiveness of the obtained results. Finally, some conclusions are included.

Notations

$\mathrm{ker A}$: Kernel of the matrix $\mathrm{A} \in \mathrm{R}^{d\times n}$.

$\mathrm{im A}$: Image of the matrix $\mathrm{A} \in \mathrm{R}^{d\times n}$.

$\mathrm{GL}_n(\mathrm{R})$: General linear group of degree n, i.e., the set of all invertible $n \times n$ matrices over R.

$\mathcal{C}(J, u)$: Set of continuous functions $f : J \to u$ from an open set $J \subseteq \mathrm{R}$ to a vector space u.

$\mathcal{C}^k(J, u)$: Set of k–times continuously differentiable functions $f : J \to u$ from an open set $J \subseteq \mathrm{R}$ to a vector space u.

dom f : Domain of the function f.

I_n : Identity matrix in $\mathrm{R}^{n\times n}$.

$||x|| := \sqrt{x^T x}$: Euclidean norm of $x \in \mathrm{R}^n$.

$||\mathrm{A}|| := \sup\{||\mathrm{A}x|| \mid ||x|| = 1\}$, induced matrix norm of $\mathrm{A} \in \mathrm{R}^{d\times n}$.

2 Preliminaries

In this section, we introduce all needed preliminaries about linear time–varying singular system transferable into a standard canonical form. Also, we remember some required properties for the sequel. For extra details, we refer the reader to see [4, 5].

Consider the following linear time–varying continuous singular system:

$$\mathrm{E}(t)\dot{x}(t) = \mathrm{A}(t)x(t), \quad x(t_0) = x_0, \tag{2.1}$$

where $x(t) \in \mathrm{R}^n$ is the system state vector, $x(t_0) = x_0 \in \mathrm{R}^n$ is the initial condition, with $\mathrm{E}, \mathrm{A} \in \mathcal{C}(\mathrm{R}_+, \mathrm{R}^{n\times n}) \times \mathcal{C}(\mathrm{R}_+, \mathrm{R}^{n\times n})$, and E is a singular matrix.

Definition 2.1. [4] A pair $(t_0, x_0) \in \mathrm{R}_+ \times \mathrm{R}^n$ is called a pair of consistent initial values of the linear time–varying singular system (2.1) if there exists a solution $x(\cdot)$ of (2.1), with $t_0 \in \operatorname{dom} x(\cdot)$ and $x(t_0) = x_0$.

In the sequel, we represent the set of all pairs of consistent initial values of the linear time–varying singular system (2.1) by $\mathcal{W}$. Furthermore, for $t_0 \in \mathrm{R}_+$,

$$\mathcal{W}(t_0) = \{x_0 \in \mathrm{R}^n : (t_0, x_0) \in \mathcal{W}\}.$$

$\mathcal{W}(t_0)$ is the linear subspace of initial values, which are consistent at time t_0.

Definition 2.2. [4] DAEs $(\mathrm{E}_1, \mathrm{A}_1)$ and $(\mathrm{E}_2, \mathrm{A}_2) \in \mathcal{C}(\mathrm{R}_+, \mathrm{R}^{n\times n})^2$ are said to be equivalent if there exist $\mathrm{S} \in \mathcal{C}(\mathrm{R}_+, \mathrm{GL}_n(\mathrm{R}))$ and $\mathrm{T} \in \mathcal{C}^1(\mathrm{R}_+, \mathrm{GL}_n(\mathrm{R}))$ such that

$$\mathrm{E}_2 = \mathrm{S}\mathrm{E}_1\mathrm{T}, \qquad \mathrm{A}_2 = \mathrm{S}\mathrm{A}_1\mathrm{T} - \mathrm{S}\mathrm{E}_1\dot{\mathrm{T}},$$

and we note $(\mathrm{E}_1, \mathrm{A}_1) \sim (\mathrm{E}_2, \mathrm{A}_2)$.

Definition 2.3. [4] The singular system (2.1) is said to be transferable into a standard canonical form (SCF) if there exist $\mathrm{S} \in \mathcal{C}(\mathrm{R}_+, \mathrm{GL}_n(\mathrm{R}))$, $\mathrm{T} \in \mathcal{C}^1(\mathrm{R}_+, \mathrm{GL}_n(\mathrm{R}))$, and $a > 0$ such that

$$(\mathrm{E}, \mathrm{A}) \sim \left(\begin{pmatrix} \mathrm{I}_a & 0 \\ 0 & \mathrm{N} \end{pmatrix}, \begin{pmatrix} \mathrm{J} & 0 \\ 0 & \mathrm{I}_{n-a} \end{pmatrix} \right),$$

where $\mathrm{J} : \mathrm{R}_+ \to \mathrm{R}^{a\times a}$ and $\mathrm{N} : \mathrm{R}_+ \to \mathrm{R}^{(n-a)\times(n-a)}$, which is a pointwise strictly lower triangular matrix.

Next, we characterize the set of consistent initial conditions of DAEs (E, A) transferable into a standard canonical form.

Proposition 2.1. *[4] Assume that a DAE* $(\mathrm{E}, \mathrm{A}) \in \mathcal{C}(\mathrm{R}_+, \mathrm{R}^{n\times n})^2$ *is transferable into a standard canonical form. Then we have*

$$(t_0, x_0) \in \mathcal{W} \Leftrightarrow x_0 \in \operatorname{im} \mathrm{T}(t_0) \begin{pmatrix} \mathrm{I}_a \\ 0 \end{pmatrix}.$$

We recall some needed properties of the generalized transition matrix.

Proposition 2.2. *[4] Let* $(\mathrm{E},\mathrm{A}) \in \mathcal{C}(\mathrm{R}_+, \mathrm{R}^{n\times n})^2$ *be transferable into SCF for* (S,T) *defined in (2.3). Then any solution to the initial value problem* $x(t_0) = x_0$ *where* $(t_0, x_0) \in \mathcal{W}$*, extends uniquely to a global solution* $x(\cdot)$*. This solution satisfies*

$$x(t) = \mathrm{U}(t, t_0)x_0,$$

where

$$\mathrm{U}(t, t_0) = \mathrm{T}(t)\begin{pmatrix} \phi_{\mathrm{J}}(t, t_0) & 0 \\ 0 & 0 \end{pmatrix}\mathrm{T}^{-1}(t_0),$$

and $\phi_{\mathrm{J}}(\cdot,\cdot)$ *denotes the transition matrix of* $\dot{y} = \mathrm{J}(t)y$*. If the linear time–varying singular system* (2.1) *is transferable into SCF, then for all* $t, \tau, s \in \mathrm{R}_+$*, the generalized transition matrix satisfies the following properties*

(i) $\mathrm{E}(t)\frac{d}{dt}\mathrm{U}(t, s) = \mathrm{A}(t)\mathrm{U}(t, s)$,

(ii) $\mathrm{im}\mathrm{U}(t, s) = \mathcal{W}(t)$,

(iii) $\mathrm{U}(t, s) = \mathrm{U}(t, \tau)\mathrm{U}(\tau, s)$,

(iv) $\mathrm{U}^2(t, t) = \mathrm{U}(t, t)$,

(v) $\forall x \in \mathcal{W}(t),\ \mathrm{U}(t, t)x = x$,

(vi) $\frac{d}{dt}\mathrm{U}(s, t) = -\mathrm{U}(s, t)\mathrm{T}(t)\mathrm{S}(t)\mathrm{A}(t)$.

Definition 2.4. [4] The linear time–varying singular system (2.1) is said to be uniformly exponentially stable if there exist two positive constants λ_1 and λ_2 such that for all $(t_0, x_0) \in \mathcal{W}$,

$$||x(t)|| \leq \lambda_1||x_0||e^{-\lambda_2(t-t_0)}, \quad \forall t \geq t_0.$$

Theorem 2.3. *[4] Let (2.1) be transferable into SCF, then the linear time–varying singular system (2.1) is uniformly exponentially stable, if and only if, there exist two positive constants* α *and* μ *such that for all* $(t_0, x_0) \in \mathcal{W}$*, we have*

$$\|\mathrm{U}(t, t_0)x_0\| \leq \alpha e^{-\mu(t-t_0)}||x_0||, \quad \forall t \geq t_0.$$

3 Linear time–varying perturbed singular systems

Assume that some parameters of the linear time–varying singular system (2.1) are excited or perturbed, and the perturbed singular system is defined as follows:

$$\mathrm{E}(t)\dot{x}(t) = \mathrm{A}(t)x(t) + \mathrm{E}(t)\Pi(t)f(t, x(t)), \quad x(0) = x_0, \tag{3.1}$$

where $\Pi \in \mathcal{C}(\mathrm{R}_+, \mathrm{R}^{n\times n})$ such that $\mathrm{im}\Pi(t) = \mathcal{W}(t)$ for all $t \in \mathrm{R}_+$, and $f : \mathrm{R}_+ \times \mathrm{R}^n \to \mathrm{R}^n$ is a continuous function in (t, x), Lipschitz in x, uniformly in t. Rather, we assume that (E, A) are transferable into a standard canonical form.

Remark 3.1. Various authors have attacked the technique of structured perturbation for linear differential–algebraic equations, see [6, 8, 9, 16]. Indeed, $\mathrm{E}\Pi f$ is a structured perturbation that guarantees "consistency" with the DAE. Thus,

$$\mathrm{E}(t)\Pi(t)f(t, x(t)) \in \mathcal{W}(t), \text{ for all } t \geq 0.$$

Assume that there exists t such that $f(t, 0) \neq 0$, i.e., the linear time–varying perturbed singular system (3.1) does not have the trivial solution $x \equiv 0$.

The notion of practical stability has been introduced and well studied by different researchers such as cited in [1, 18, 21]

Definition 3.1. The perturbed singular system (3.1) is said to be practically uniformly exponentially stable if there exist two positive constants λ_1, λ_2, and $r > 0$ such that for all $(0, x_0) \in \mathcal{W}$, the following inequality is

$$\| x(t, 0, x_0) \| \leq \lambda_1 \| x_0 \| \exp(-\lambda_2 t) + r, \quad \forall t \geq 0. \tag{3.2}$$

Remark 3.2. Eq. (3.2) implies that $x(t)$ will be bounded by a small bound $r > 0$, thus $||x(t)||$ will be small for sufficiently large t. That is to say, the solution provided in (3.2) will be uniformly ultimately bounded for sufficiently large t. The factor λ_2 in Eq. (3.2) is called the convergence speed, whereas the factor λ_1 is called the transient estimate.

It is even worth seeing that, in the earlier definition if we take $r = 0$, then we recover the standard concept of the uniform exponential stability of the origin viewed as an equilibrium point.

Remark 3.3. Several authors deal with the issue of the practical stability of singular systems within the method of Lyapunov, see [8, 9, 16]. The construction of an appropriate Lyapunov function is not always feasible, which encourages us to search for another method. Our process in this paper is to analyze the stability of the linear time–varying perturbed singular system based on the explicit solution form and Gamidov's inequality.

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Our principal purpose is to state sufficient conditions to provide the practical uniform exponential stability of the linear time–varying perturbed singular system (3.1). In fact if we suppose that the perturbation term is bounded, then the origin is not necessarily an equilibrium point. For that reason, we will analyze the convergence of the solutions toward a neighborhood of origin.

Theorem 3.4. *[16] A function $x : \mathrm{R}_+ \to \mathrm{R}^n$ is a solution of (3.1), if and only if, $x(\cdot)$ satisfies the following integral equation*

$$x(t) = \mathrm{U}(t,0)x_0 + \int_0^t \mathrm{U}(t,s)\Pi(s)f(s,x(s))ds, \quad t \geq 0.$$

We suppose the following assumptions which are required for the stability purposes.

$(\mathcal{H}_1)$ The unperturbed nominal singular system (2.1) is uniformly exponentially stable.

$(\mathcal{H}_2)$ The matrix Π is a continuous bounded matrix.

$(\mathcal{H}_3)$ There exists a continuous non–negative known function $\vartheta(t)$ such that

$$\|f(t,x)\| \leq \vartheta(t), \quad \forall (t,x) \in \mathcal{W}.$$

The bounds of the nonlinearities should be generally associated with the dynamic of the nominal system, and in our case, they should be small enough. We restrict the function $\vartheta(t)$ to study the stability of the linear time–varying perturbed singular system (3.1).

$(\mathcal{H}_4)$ The non–negative continuous function $\vartheta(t)$ is such that

$$\lim_{t\to\infty} \vartheta(t) = 0.$$

Theorem 3.5. *Under assumptions* $(\mathcal{H}_1) - (\mathcal{H}_4)$ *the linear time–varying perturbed singular system* (3.1) *is practically uniformly exponentially stable.*

Proof. By using Theorem 3.4, the solution with initial condition x_0 of the linear time–varying perturbed singular system (3.1) is expressed as follows:

$$x(t) = \mathrm{U}(t,0)x_0 + \int_0^t \mathrm{U}(t,s)\Pi(s)f(s,x(s))ds, \quad \forall t \geq 0.$$

Based on assumption $(\mathcal{H}_1)$ and Theorem 2.3, there exist two positive constants α and μ such that for all $(0,x_0) \in \mathcal{W}$, one obtains

$$\|\mathrm{U}(t,0)x_0\| \leq \alpha e^{-\mu t}||x_0||, \quad \forall t \geq 0.$$

Further, since $\Pi(s)f(s,x(s)) \in \mathcal{W}(s)$, it follows that for all $t \geq 0$,

$$\begin{aligned} ||x(t)|| &\leq \alpha e^{-\mu t}||x_0|| + \int_0^t \alpha e^{-\mu(t-s)}||\Pi(s)f(s,x(s))||ds \\ &\leq \alpha e^{-\mu t}||x_0|| + \int_0^t \alpha e^{-\mu(t-s)}||\Pi(s)||\ ||f(s,x(s))||ds. \end{aligned} \tag{3.3}$$

The assumption $(\mathcal{H}_2)$ yields that there exists $m > 0$ such that

$$||\Pi(t)|| \leq m, \quad \forall t \geq 0. \tag{3.4}$$

Substituting (3.4) and $(\mathcal{H}_3)$ into (3.3), yields

$$\begin{aligned} ||x(t)|| \leq& \alpha e^{-\mu t}||x_0|| + \int_0^t m\alpha e^{-\mu(t-s)}\vartheta(s)ds \\ &\leq \alpha e^{-\mu t}||x_0|| + m\alpha e^{-\mu t}\int_0^t e^{\mu s}\vartheta(s)ds. \end{aligned}$$

Assumption $(\mathcal{H}_4)$ yields that there exists $\widetilde{\vartheta} > 0$ such that

$$\vartheta(t) \leq \widetilde{\vartheta}, \quad \forall t \geq 0.$$

It follows that,

$$\begin{aligned} ||x(t)|| &\leq \alpha e^{-\mu t}||x_0|| + m\alpha\widetilde{\vartheta}e^{-\mu t}\int_0^t e^{\mu s}ds \\ &\leq \alpha e^{-\mu t}||x_0|| + m\alpha\widetilde{\vartheta}e^{-\mu t}\left(\frac{1}{\mu}e^{\mu t} - \frac{1}{\mu}\right) \\ &\leq \alpha e^{-\mu t}||x_0|| + \frac{m\alpha\widetilde{\vartheta}}{\mu}. \end{aligned}$$

That is, we arrive at

$$||x(t)|| \leq \alpha e^{-\mu t}||x_0|| + \frac{m\alpha\widetilde{\vartheta}}{\mu}, \quad \forall\ (0,x_0) \in \mathcal{W}, \quad \forall t \geq 0.$$

As a consequence, the linear time–varying perturbed singular system (3.1) is practically uniformly exponentially stable. □

A simple extension can be done if we replace the assumption $(\mathcal{H}_4)$ by the following:

$(\mathcal{H}_4')$ The continuous non–negative function $\vartheta(t)$ satisfies

$$\int_0^{\infty} \vartheta(t) < \infty,$$

or

$$\vartheta(t) \leq \bar{\vartheta} < \infty, \quad \forall t \geq 0.$$

Theorem 3.6. *Under assumptions* $(\mathcal{H}_1) - (\mathcal{H}_3)$ *and* $(\mathcal{H}_4')$ *the linear time–varying perturbed singular system* (3.1) *is practically uniformly exponentially stable.*

Proof. The solution with initial condition x_0 of the linear time–varying perturbed singular system (3.1) is the following

$$x(t) = \mathrm{U}(t,0)x_0 + \int_0^t \mathrm{U}(t,s)\Pi(s)f(s,x(s))ds, \quad \forall t \geq 0.$$

By using assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$, we see that

$$||x(t)|| \leq \alpha e^{-\mu t}||x_0|| + \int_0^t m\alpha e^{-\mu(t-s)}||f(s,x(s))||ds.$$

In light of assumption $(\mathcal{H}_3)$, we arrive at

$$\begin{aligned} ||x(t)|| &\leq \alpha e^{-\mu t}||x_0|| + \int_0^t m\alpha e^{-\mu(t-s)}\vartheta(s)ds \\ &\leq \alpha e^{-\mu t}||x_0|| + m\alpha e^{-\mu t}\int_0^t e^{\mu s}\vartheta(s)ds. \end{aligned}$$

Indeed, since the non–negative continuous function $\vartheta(t)$ satisfies assumption $(\mathcal{H}_3')$, there exists $\zeta > 0$ such that

$$e^{-\mu t}\int_0^t e^{\mu s}\vartheta(s)ds \leq \zeta, \quad \forall t \geq 0,$$

where $\zeta = \min\left(\dfrac{\bar{\vartheta}}{\mu}, ||\varphi||_1\right)$, which implies that

$$||x(t)|| \leq \alpha e^{-\mu t}||x_0|| + \zeta m\alpha, \quad \forall (0,x_0) \in \mathcal{W}, \quad \forall t \geq 0.$$

Hence, the linear time–varying perturbed singular system (3.1) is practically uniformly exponentially stable. $\square$

4 Gamido's type inequality and applications

In this section, we aim to prove the practical uniform exponential stability of the linear time–varying perturbed singular system (3.1), based on the generalized integral inequality of the Gamidov type.

The generalized integral inequality of the Gamidov type was proved by Gamidov [20].

Lemma 4.1. *[20] Assume that*

$$\mathcal{V}(t) \leq \tau(t) + \varsigma \int_0^t \chi(s)\mathcal{V}^q(s)ds,$$

where all functions are continuous and non–negative on $[0, h)$, $0 < q < 1$, *and* $h, \varsigma > 0$. *Then there exists a constant* $\varrho > 0$ *such that*

$$\mathcal{V}(t) \leq \tau(t) + \varsigma\varrho^q \left(\int_0^t \chi^{\frac{1}{1-q}}(s)ds \right)^{1-q}.$$

Later on Abdellatif et al. [2] improved the previous lemma for $h = \infty$.

Lemma 4.2. *[2] Assume that*

$$\mathcal{V}(t) \leq \tau(t) + \varsigma \int_0^t \chi(s)\mathcal{V}^q(s)ds,$$

where all functions are continuous and non–negative on $[0, \infty)$, $0 < q < 1$, *and* $\varsigma > 0$. *Then there exists a constant* $\varrho > 0$ *such that*

$$\mathcal{V}(t) \leq \tau(t) + \varsigma\varrho^q(t) \left(\int_0^t \chi^{\frac{1}{1-q}}(s)ds \right)^{1-q}.$$

In particular for $\chi(t) = e^{(1-q)\mu t}$ and $\tau(t) \leq \bar{m}$, they proved the following lemma.

Lemma 4.3. *[2] Assume that*

$$\mathcal{V}(t) \leq \bar{m} + \varsigma \int_0^t e^{(1-q)\mu s}\mathcal{V}^q(s)ds,$$

where $\mathcal{V}$ *is continuous and non–negative on* $[0, \infty)$, $0 < q < 1$, *and* $\bar{m}$, $\varsigma > 0$.
Then

$$\mathcal{V}(t) \leq 2^{\frac{q}{1-q}}\bar{m} + \left(\frac{2^q\varsigma}{\mu} \right)^{\frac{1}{1-q}} e^{\mu t}.$$

($\mathcal{A}_3$) There exists a continuous positive functions $\beta_1(t)$ and $\beta_2(t)$ such that

$$||f(t,x)|| \leq \beta_1(t)||x||^q + \beta_2(t), \quad 0 < q < 1, \ \forall (0,x) \in \mathcal{W},$$

where the function $\beta_1(\cdot)$ satisfies

$$\beta_1(t) \leq \widetilde{\beta}_1, \quad \forall t \geq 0,$$

and the function $\beta_2(\cdot)$ satisfies

$$\int_0^\infty e^{\mu s}\beta_2(s)ds = \bar{\beta} < \infty.$$

Theorem 4.4. *Suppose that assumptions* ($\mathcal{H}_1$) $-$ ($\mathcal{H}_2$) *and* ($\mathcal{A}_3$) *are satisfied, then the linear time–varying perturbed singular system* (3.1) *is practically uniformly exponentially stable.*

Proof By using Theorem 3.4, the solution with initial condition x_0 of the linear time–varying perturbed singular system (3.1) is expressed as follows:

$$x(t) = \mathrm{U}(t,0)x_0 + \int_0^t \mathrm{U}(t,s)\Pi(s)f(s,x(s))ds, \quad \forall t \geq 0.$$

From assumption ($\mathcal{H}_1$) and Theorem 2.3, we arrive at

$$\begin{aligned} ||x(t)|| &\leq \alpha e^{-\mu t}||x_0|| + \int_0^t \alpha e^{-\mu(t-s)}||\Pi(s)f(s,x(s))||ds \\ &\leq \alpha e^{-\mu t}||x_0|| + \int_0^t \alpha e^{-\mu(t-s)}||\Pi(s)|| \; ||f(s,x(s))||ds. \end{aligned}$$

It follows from assumption ($\mathcal{H}_2$) and ($\mathcal{A}_3$) that

$$\begin{aligned} ||x(t)|| &\leq \alpha e^{-\mu t}||x_0|| + \alpha m e^{-\mu t}\int_0^t e^{\mu s}\left(\beta_1(s)||x(s)||^q + \beta_2(s)\right)ds \\ &\leq \alpha e^{-\mu t}||x_0|| + \alpha m e^{-\mu t}\int_0^t \left(\widetilde{\beta}_1 e^{\mu s}||x(s)||^q + e^{\mu s}\beta_2(s)\right)ds. \end{aligned}$$

Multiplying both sides by $e^{\mu t}$, one obtains

$$e^{\mu t}||x(t)|| \leq \alpha||x_0|| + \alpha m\widetilde{\beta}_1 \int_0^t e^{\mu s}||x(s)||^q ds + \alpha m \int_0^t e^{\mu s}\beta_2(s)ds.$$

Setting $\mathcal{V}(t) = e^{\mu t}||x(t)||$, we obtain

$$\mathcal{V}(t) \leq \alpha||x_0|| + \alpha m\widetilde{\beta}_1 \int_0^t e^{(1-q)\mu s}\mathcal{V}^q(s)ds + \alpha m \int_0^t e^{\mu s}\beta_2(s)ds.$$

Since, $\int_0^{\infty} e^{\mu s}\beta_2(s)ds = \bar{\beta} < \infty$, we have

$$\mathcal{V}(t) \le \alpha||x_0|| + \alpha m\widetilde{\beta}_1 \int_0^t e^{(1-q)\mu s}\mathcal{V}^q(s)ds + \alpha m\bar{\beta}.$$

Thus,

$$\mathcal{V}(t) \le \bar{m} + \alpha\varsigma \int_0^t e^{(1-q)\mu s}\mathcal{V}^q(s)ds,$$

where $\bar{m} = \alpha||x_0||$ and $\varsigma = \alpha m\widetilde{\beta}_1$.

From the Gamidov inequality (Lemma 4.3), it follows that

$$\mathcal{V}(t) \le 2^{\frac{q}{1-q}}\bar{m} + \left(\frac{2^q\varsigma}{\mu}\right)^{\frac{1}{1-q}} e^{\mu t}.$$

Accordingly, we see

$$e^{\mu t}||x(t)|| \le 2^{\frac{q}{1-q}}\bar{m} + \left(\frac{2^q\varsigma}{\mu}\right)^{\frac{1}{1-q}} e^{\mu t}.$$

Then we arrive at

$$||x(t)|| \le 2^{\frac{q}{1-q}}\alpha||x_0||e^{-\mu t} + 2^{\frac{q}{1-q}}m\alpha\bar{\beta} + \left(\frac{2^q m\alpha\widetilde{\beta}_1}{\mu}\right)^{\frac{1}{1-q}}, \quad \forall\ (0,x_0) \in \mathcal{W}, \quad \forall t \ge 0,$$

which means that the linear time–varying perturbed singular system (3.1) is practically uniformly exponentially stable.
□

Finally, for the standard case, we suppose that the perturbation term $f(t,x)$ satisfies the following assumption.

($\mathcal{A}_3'$) There exists a continuous non–negative known function $\bar{\varphi}(t)$ such that

$$||f(t,x)|| \le \bar{\varphi}(t)||x||, \quad \forall (t,x) \in \mathcal{W},$$

where $\bar{\varphi}(t)$ is bounded and satisfies

$$\sup_{t\ge 0}\bar{\varphi}(t) \le \theta\mu, \quad 0 < \theta < \frac{1}{m\alpha}.$$

Theorem 4.5. *Suppose that assumptions* $(\mathcal{H}_1) - (\mathcal{H}_2)$ *and* $(\mathcal{A}_3')$ *are satisfied, then the linear time–varying perturbed singular system* (3.1) *is uniformly exponentially stable.*

Proof. The solution with initial condition x_0 of the linear time–varying perturbed singular system (3.1) is given by the following:

$$x(t) = \mathrm{U}(t,0)x_0 + \int_0^t \mathrm{U}(t,s)\Pi(s)f(s,x(s))ds, \quad \forall t \geq 0.$$

Based on the standing hypothesis, we see that

$$\begin{aligned}
||x(t)|| &\leq \alpha e^{-\mu t}||x_0|| + \int_0^t \alpha e^{-\mu(t-s)}||\Pi(s)f(s,x(s))||ds \\
&\leq \alpha e^{-\mu t}||x_0|| + \int_0^t \alpha e^{-\mu(t-s)}||\Pi(s)||\ ||f(s,x(s))||ds \\
&\leq \alpha e^{-\mu t}||x_0|| + \int_0^t \alpha m e^{-\mu(t-s)}||f(s,x(s))||ds \\
&\leq \alpha e^{-\mu t}||x_0|| + \int_0^t \alpha m e^{-\mu(t-s)}\bar{\varphi}(s)||x(s)||ds.
\end{aligned}$$

Multiplying both sides by $e^{\mu t}$, one obtains

$$||x(t)||e^{\mu t} \leq \alpha||x_0|| + \alpha m \int_0^t e^{\mu s}\bar{\varphi}(s)||x(s)||ds.$$

Suppose $u(t) = ||x(t)||e^{\mu t}$, it yields that

$$u(t) \leq \alpha||x_0|| + \alpha m \int_0^t \bar{\varphi}(s)u(s)ds.$$

By applying the Gronwall lemma [13], one gets

$$u(t) \leq \alpha||x_0||e^{\alpha m \int_0^t \bar{\varphi}(s)ds} \leq \alpha||x_0|| \exp(\alpha m \theta \mu t).$$

As a consequence, we derive that

$$||x(t)|| \leq \alpha||x_0|| \exp\left(-(1 - m\alpha\theta)\mu t\right), \quad \forall (0,x) \in \mathcal{W}(t).$$

That is, the linear time–varying perturbed singular system (3.1) is uniformly exponentially stable. $\square$

Next, we will consider the following linear time–varying perturbed singular system associated to the system (3.1):

$$\mathrm{E}(t)\dot{x}(t) = \mathrm{A}(t)x(t) + \mathrm{E}(t)\Pi(t)f(t,x(t)) + \mathrm{E}(t)\Pi(t)\mathrm{h}(t,x(t)), \quad x(0) = x_0, \tag{4.1}$$

where $\mathrm{h} : \mathrm{R}_+ \times \mathrm{R}^n \to \mathrm{R}^n$ is a continuous function in (t, x), Lipschitz in x, uniformly in t.

$(\mathcal{A}_4)$ There exists a continuous non–negative constant ρ such that

$$||\mathrm{h}(t,x)|| \leq \rho||x(t)||, \quad \forall (t,x) \in \mathcal{W}.$$

Theorem 4.6. *Suppose that assumptions $(\mathcal{H}_1) - (\mathcal{H}_2)$ and $(\mathcal{A}_3) - (\mathcal{A}_4)$ are satisfied, then the linear time–varying perturbed singular system (3.1) is practically uniformly exponentially stable.*

To prove the previous theorem, we need to recall a generalized integral inequality of the Gamidov type, proved in [2].

Lemma 4.7. *[2] Assume that*

$$\mathcal{V}(t) \leq \bar{m} + \int_0^t \left(\varsigma e^{(1-q)\mu s} \mathcal{V}^q(s) + \sigma \mathcal{V}(s) \right) ds,$$

where $\mathcal{V}$ is continuous and non–negative on $[0, \infty)$, $0 < q < 1$, and $\bar{m}$, $\varsigma > 0$, and μ, σ such that $0 \leq \sigma < \mu$.

Then

$$\mathcal{V}(t) \leq 2^{\frac{q}{1-q}} \bar{m} e^{\sigma t} + \left(\frac{2^q \varsigma}{\mu - \sigma} \right)^{\frac{1}{1-q}} e^{\mu t}.$$

Proof of Theorem 4.6. The solution of the perturbed singular system (4.1) is expressed as follows:

$$x(t) = \mathrm{U}(t,0)x_0 + \int_0^t \mathrm{U}(t,s)\Pi(s) \left(f(s,x(s)) + \mathrm{h}(s,x(s)) \right) ds, \quad \forall t \geq t_0.$$

Since the unperturbed nominal system (2.1) is uniformly exponentially stable, we have

$$\|\mathrm{U}(t,0)x_0\| \leq \alpha e^{-\mu t} ||x_0||, \quad \forall t \geq 0.$$

Further, we have $\Pi(s) \left(f(s,x(s)) + \mathrm{h}(s,x(s)) \right) \in \mathcal{W}(s)$, then for all $t \geq 0$, it yields that,

$$\begin{aligned} ||x(t)|| &\leq \alpha e^{-\mu t} ||x_0|| + \int_0^t \alpha e^{-\mu(t-s)} ||\Pi(s) \left(f(s,x(s)) + \mathrm{h}(s,x(s)) \right)|| ds \\ &\leq \alpha e^{-\mu t} ||x_0|| + \int_0^t \alpha e^{-\mu(t-s)} ||\Pi(s)|| \; ||f(s,x(s)) + \mathrm{h}(s,x(s))|| ds. \end{aligned}$$

From assumption $(\mathcal{H}_2)$, one obtains

$$||x(t)|| \leq \alpha e^{-\mu t} ||x_0|| + \int_0^t \alpha m e^{-\mu(t-s)} ||f(s,x(s)) + \mathrm{h}(s,x(s))|| ds.$$

Based on assumption $(\mathcal{A}_4)$, for $\rho = \dfrac{\Theta\mu}{\alpha}$, $0 < \Theta < \frac{1}{m}$, it yields that

$$||\mathrm{h}(t,x)|| \leq \frac{\Theta\mu}{\alpha}||x(t)||, \quad \forall (t,x) \in \mathcal{W}. \tag{4.2}$$

By assumption $(\mathcal{A}_3)$ and Eq. (4.2), we obtain

$$||x(t)|| \leq \alpha e^{-\mu t}||x_0|| + \alpha m e^{-\mu t}\int_0^t e^{\mu s}\left(\beta_1(s)||x(s)||^q + \frac{\Theta\mu}{\alpha}||x(s)|| + \beta_2(s)\right) ds.$$

Multiplying both sides by $e^{\mu t}$, yields

$$||x(t)||e^{\mu t} \leq \alpha||x_0|| + \alpha m \int_0^t e^{\mu s}\left(\beta_1(s)||x(s)||^q + \frac{\Theta\mu}{\alpha}||x(s)||\right) ds + \alpha m \int_0^t e^{\mu s}\beta_2(s) ds.$$

Let $\mathcal{V}(t) = e^{\mu t}||x(t)||$, and by using assumption $(\mathcal{A}_3)$, it comes that

$$\begin{aligned}
\mathcal{V}(t) &\leq \alpha||x_0|| + \int_0^t \left(e^{(1-q)\mu s}\alpha m\widetilde{\beta}_1\mathcal{V}^q(s) + \Theta\mu m\mathcal{V}(s)\right) ds + \alpha m \int_0^\infty e^{\mu s}\beta_2(s) ds \\
&\leq \alpha||x_0|| + \int_0^t \left(e^{(1-q)\mu s}\alpha m\widetilde{\beta}_1\mathcal{V}^q(s) + \Theta\mu m\mathcal{V}(s)\right) ds + \alpha m\bar{\beta}.
\end{aligned}$$

That is,

$$\mathcal{V}(t) \leq \bar{m} + \int_0^t \left(\varsigma e^{(1-q)\mu s}\mathcal{V}^q(s) + \sigma\mathcal{V}(s)\right) ds,$$

where $\bar{m} = \alpha||x_0|| + \alpha m\bar{\beta}$, $\varsigma = \alpha m\widetilde{\beta}_1$, and $\sigma = \Theta\mu m$.

By application of the Gamidov lemma (Lemma 4.7), one obtains

$$\mathcal{V}(t) \leq 2^{\frac{q}{1-q}}\bar{m}e^{\sigma t} + \left(\frac{2^q\varsigma}{\mu - \sigma}\right)^{\frac{1}{1-q}} e^{\mu t}.$$

That is,

$$e^{\mu t}||x(t)|| \leq 2^{\frac{q}{1-q}}\bar{m}e^{\sigma t} + \left(\frac{2^q\varsigma}{\mu - \sigma}\right)^{\frac{1}{1-q}} e^{\mu t}.$$

Thus, we obtain

$$||x(t)|| \leq 2^{\frac{q}{1-q}}\alpha||x_0||e^{-(1-\Theta m)\mu t} + \left(\frac{2^q\alpha m\widetilde{\beta}_1}{(1-\Theta m)\mu}\right)^{\frac{1}{1-q}} + 2^{\frac{q}{1-q}}\alpha m\bar{\beta}, \quad \forall (0,x) \in \mathcal{W}.$$

So, the system (4.1) is practically uniformly exponentially stable. $\square$

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5 Example

In this section, we present an example to illustrate the validity of the results.

Let's consider the following linear time–varying perturbed singular system:

$$\mathrm{E}(t)\dot{x}(t) = \mathrm{A}(t)x(t) + \mathrm{E}(t)\Pi(t)f(t, x(t)), \tag{5.1}$$

where $x = (x_1, x_2) \in \mathrm{R}^2$,

$$\mathrm{E}(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ \mathrm{A}(t) = \begin{pmatrix} -t & 0 \\ 0 & 1 \end{pmatrix}, \ t \in \mathrm{R}_+, \ f(t, x) = \begin{pmatrix} f_1(t, x) \\ f_2(t, x) \end{pmatrix},$$

with

$$\begin{cases} f_1(t, x) = \zeta\sqrt{|x_1|} + e^{-\iota t} \\ \\ f_2(t, x) = e^{-\nu t}, \quad \zeta, \ \nu > 0, \ \iota \geq 1. \end{cases}$$

The system (5.1) might be regarded as a perturbed singular system of the following linear time–varying system:

$$\mathrm{E}(t)\dot{x}(t) = \mathrm{A}(t)x(t). \tag{5.2}$$

Note that, with this term of perturbation f, the fact that the function $\sqrt{|x_1|}$ is not Lipschitzian around zero does not pose a problem for the uniqueness of the solutions because the concept of practical stability is based on the convergence of solutions to a neighborhood of the origin.

The nominal system (5.2) is transferable into a standard canonical form, where $\mathrm{S}(\cdot) = \mathrm{T}(\cdot) = \mathrm{I}_2$. As a consequence, by applying Proposition 2.1, one obtains

$$\mathcal{W} = \mathrm{R}_+ \times \mathrm{im} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

As we know that $\mathrm{im}\Pi(t) = \mathcal{W}(t)$, so we may select the matrix Π as follows:

$$\Pi(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

That is, we arrive at $\mathrm{E}\Pi f = \begin{pmatrix} f_1 \\ 0 \end{pmatrix}$.

Based on Theorem 2.3 the nominal system (5.2) is uniformly exponentially stable, as we can see in Figure 1. Actually, the transition matrix $\mathrm{U}(t, 0)$ related to the unperturbed system (5.2) fulfills the following expression:

$$\mathrm{U}(t, 0) = e^{-\frac{1}{2}t^2}.$$

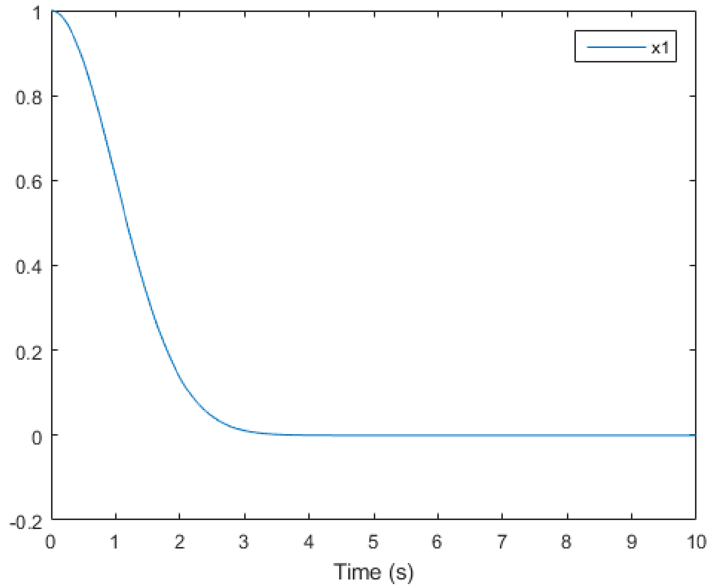

Figure 1: The initial response of the unperturbed nominal system (5.2), with the initial condition $x_0 = [1, 0]^T$

For their part, we have

$$||f(t,x)||^2 = f_1^2(t,x) + f_2^2(t,x) \leq \zeta |x_1| + e^{-2\iota t} + e^{-2\nu t}.$$

Accordingly, we achieve that for all $(t, x) \in \mathcal{W}$,

$$||f(t,x)|| \leq \zeta \sqrt{|x_1|} + e^{-\iota t}.$$

It is clear that assumption $(\mathcal{A}_3)$ is satisfied with $\beta_1(t) = \zeta$ and $\beta_2(t) = e^{-\iota t}$. Furthermore, the matrix Π is bounded. Hence, all the assumptions of Theorem 4.4 are satisfied, and then the linear time–varying perturbed singular system (5.1) is practically uniformly exponentially stable, as shown in Figure 2, for $\zeta = 2$, $\iota = \nu = 1$.

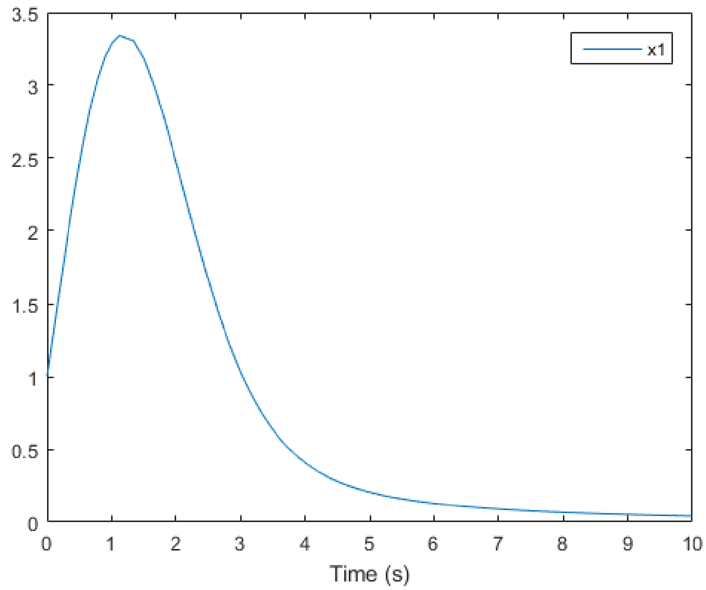

Figure 2: The initial response of the perturbed singular system (5.1), with the initial condition $x_0 = [1, 0]^T$

6 Conclusion

The construction of a suitable Lyapunov function is not always possible. In this paper, we investigate the problem of stability of linear time–varying perturbed singular systems, which are transferable into standard canonical form based on the explicit solution form through integral inequalities of Gamidov type under some restrictions on the perturbation term. An example is offered to demonstrate the utility of the conclusions reached.

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