On near approximations

S. Khodaii,[*] S.M. Anvariyeh,[†] B. Davvaz,[‡] S. Alikhani[§]
Department of Mathematical Sciences, Yazd University, 89195-741, Yazd, Iran.

S. Mirvakili[¶]
Department of Mathematics, Payame Noor University, Tehran, Iran.

This paper introduce the concept of a natural transformation on a nearness approximation space. We show that natural transformations and the nearness approximation spaces are categories. Also, we characterize an epimorphism and a monomorphisms in these categories. In addition, we prove that the category of a nearness approximation space does not necessarily have a product but for some families of an approximation spaces have a coproduct.

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1. INTRODUCTION

Pawlak [27, 28] introduced rough set theory as an extension of set theory, which is an important mathematical tool for dealing with uncertainty of data. A subset of a universe was defined by a pair of ordinary sets called the lower and upper approximations. Since the elements of a universe that have the same description are indiscernible with respect to the available information, an equivalence relation is defined such that two elements are equivalent if and only if they are indiscernible from each other. As a well-known result, every equivalence relation on a universe constructs a partition for it such that each equivalence class represents a piece of information about the elements, that is a form of classification.

The standard rough set theory was generalized by Davvaz and Mahdavipour [12]. They defined some new approximation operators based on the notion of covering induced by the notion of a property on the universe of discourse.

A spatial meaning of distance defined by Efremovi [14] is the starting point of the notion of nearness between sets (proximity relation between sets). Moreover, the notion of proximity was not limited to a spatial interpretation; see [19, 32]. For example, it was studied in [30] that near sets together with objects with affinities are considered perceptually near to each other. The main difference between a rough set and a near set is that every rough set has a nonempty boundary region, but the near set may be null. One of the important works in near sets is the approach introduced by Peters and Wasilewski [31] in the foundations of information science.

[*] khodaii.so@gmail.com
[†] Corresponding author, anvariyeh@yazd.ac.ir
[‡] davvaz@yazd.ac.ir
[§] alikhani@yazd.ac.ir
[¶] saeed_mirvakili@pnu.ac.ir

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For nearness relations, near sets, and a framework for classifying perceptual objects, perceptual information systems has a key role. After that, the notion of rough subgroups and a rough ideal in a semigroup was introduced by Biswas and Nanda [6] and Kuroki [24], respectively. Some properties of the lower and upper approximations with respect to the normal subgroups were studied in [26]. Also, Kuroki and Mordeson [25] studied the structure of rough sets and rough groups. Estaji et al. [15] investigated a relation between a rough set and lattice theory and introduced the concepts of upper and lower ideals (filters) in a lattice. Moreover, Estaji et al. [16] introduced the notion of θ-upper and θ-lower approximation of a fuzzy subset of the lattice; see also [17]. A connection between a rough set and ring theory was studied by Davvaz [8, 9] by considering a ring as a universal set. He [10] also introduced the notion of a rough ideal and a rough subring with respect to an ideal of a ring. The notions of a rough prime (primary) ideal and a rough fuzzy prime (primary) ideal in a ring and some properties of such ideals were studied by Kazanci et al. [23]. Davvaz et al [11] studied Rough modules.

The basic concepts of the algebraic structures of the near set theory and also the concept of near groups, weak cosets, near normal subgroups, and homomorphism of near groups on nearness approximation spaces were investigated by İnan and Öztürk [21, 22]. Moreover, the notion of near subsemigroups, nearideals, near bi-ideals, and homomorphisms of near semigroups on near approximation spaces was introduced by Bağirmaz [4]. Recently, an extended notion of a rough approximations in a group, a near approximations in a group, was introduced by Bagirmaz [3].

Some connections between the category theory and theoretical computer science were studied in [1, 2]. Indeed, some pure mathematical approaches such as categorical approach have been done. For example, the category ROUGH of Pawlak approximation spaces was introduced in [5], and it was proved that ROUGH is finitely complete but not a topos. The category **R-APR** is the power sets and pairs of rough set approximation operators. Also, for the category **cdrTex** whose objects are complemented textures and morphisms are complemented direlations, it was proved in [13] that **R-APR** is isomorphic to a full subcategory of **cdrTex**. Also, it was shown that **R-APR** and **cdrTex** are new examples of dagger symmetric monoidal categories.

Let (U, F, β_r) be a nearness approximation space. In this article, we assume that $r \neq |\beta|$. We recall from [7] that if θ is an equivalence relation on U and γ is an equivalence relation on V, then a function $\varphi : U \to V$ is called an *upper natural transformation* from (U, θ) into (V, γ), provided that the diagram

$$\begin{array}{ccc} \mathcal{P}(U) & \xrightarrow{\overline{apr}_\theta} & \mathcal{P}(U) \\ \overline{\varphi}\big\downarrow & & \big\downarrow\overline{\varphi} \\ \mathcal{P}(V) & \xrightarrow[\overline{apr}_\gamma]{} & \mathcal{P}(V) \end{array}$$

commutes, where $\overline{\varphi} : \mathcal{P}(U) \to \mathcal{P}(V)$ is the forward powerset operator induced by the mapping φ, that is, $\overline{\varphi}(A) := \varphi(A)$ for every $A \in \mathcal{P}(U)$. In the continuation, we show $\overline{\varphi}$ with φ. Also, approximation spaces and upper natural transformations form a category, which is denoted by $\overline{\textbf{AprS}}$. A lower natural transformation is defined similarly. Also, approximation spaces and lower natural transformations form a category, which is denoted by $\underline{\textbf{AprS}}$. The existence or non-existence of limits and colimits and injective objects of these two categories were studied in [7] and [18], respectively.

Many general theory about nearness of objects and applications of these to categories in the various sciences motivate us to study the category theory of nearness sets. Here we study some category theoretic properties of them.

This paper is organized as follows: After the introduction at Section 2, we present some examples and results about near sets, which will be used in other sections.

In Section 3, the concept of an upper natural transformation on an approximation space is introduced, and we show that approximation spaces and upper natural transformations form a category (see Proposition 3.2). In Theorem 3.7, upper natural transformations are characterized. Note that the proposed category has all coproducts and products. (see Proposition 3.12). At the end of this section, we introduce the concept equalizer (coequalizer) of a pair morphisms and then examine its features.

In Section 4, the concept of a lower natural transformation on a near approximation space is introduced. Near approximation spaces and lower natural transformations form a category.Lower natural transformations are characterized under stronger conditions in Theorems 4.6 and 4.7. Moreover, we show that this category does not necessarily have products and has coproducts under some conditions (see Proposition 4.9).

2. NEAR SETS AND ITS FEATURES

In this section, we present the main definitions and properties of near sets as in [31, 32].

Definition 2.1. [31] A perceptual information system $(\mathcal{O}, F)$, or, more concisely, perceptual system, is a real-valued total deterministic information system, where $\mathcal{O}$ is a non-empty set of perceptual objects, while $\mathcal{O}$ is a countable set of probe functions.

An object description is defined by means of a tuple of function values $\Phi(x)$ associated with an object $x \in A$. The important thing to note is the choice of functions $\varphi_i \in \beta$ used to describe an object of interest. Assume that $\beta \subseteq F$ (see Table I) is a given set of functions representing features of sample objects $A \subseteq \mathcal{O}$.

Definition 2.2. [31] Let $(\mathcal{O}, \mathcal{F})$ be a perceptual information system. For every $\beta \subseteq \mathcal{F}$, the indiscernibility relation $\sim_\beta$ is defined as follows:

$$\sim_\beta = \{(x, y) \in \mathcal{O} \times \mathcal{O} : \varphi_i(x) = \varphi_i(y), \text{ for all } \varphi_i \in \beta\}.$$

If $\beta = \{\varphi\}$, for some $\varphi \in \mathcal{F}$, then instead of $\sim_{\{\varphi\}}$, we write $\sim_\varphi$.

This relation defines a partition of $\mathcal{O}$ into non-empty, pairwise disjoint subsets, where they are equivalence classes and denoted by $[x]_\beta$, where $[x]_\beta = y \in \mathcal{O} : x \sim_\beta y$. These classes form the quotient set $O/\sim_\beta$, where $\mathcal{O}/\sim_\beta = \{[x]_\beta : x \in \mathcal{O}\}$.

Example 2.3. Let $(\mathcal{O}, \mathcal{F})$ be a perceptual system, where $\mathcal{O} = \{x_i : 1 \leq i \leq 5\}$ and $\mathcal{F} = \{f_1, f_2, f_3\}$ and the values of probe functions from $\mathcal{O}$ to percepts are defined in Table I.

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TABLE I. Values of probe functions

	f_1	f_2	f_3
x_1	1	0	3
x_2	4	2	6
x_3	1	5	3
x_4	1	5	6
x_5	4	0	3

Then, the partitions are as follows:

$$\mathcal{O}/\sim_{f_1}=\{\{x_1,x_3,x_4\},\{x_2,x_5\}\}=\{[x_1]_{f_1},[x_2]_{f_1}\},$$

$$\mathcal{O}/\sim_{f_2}=\{\{x_1,x_5\},\{x_2\}\{x_3,x_4\}\}=\{[x_1]_{f_2},[x_2]_{f_2},[x_3]_{f_2}\}\},$$
$$\mathcal{O}/\sim_{f_3}=\{\{x_1,x_3,x_5\},\{x_2,x_4\}\}=\{[x_1]_{f_3},[x_2]_{f_3}\}.$$

Then, for $r=1$, we conclude that $[x_1]_{\beta_1}=\{x_1,x_3,x_4,x_5\}$, $[x_2]_{\beta_1}=\{x_2,x_4,x_5\}$, $[x_3]_{\beta_1}=\{x_1,x_3,x_4,x_5\}$, $[x_4]_{\beta_1}=\{x_1,x_2,x_3,x_4\}$, and $[x_5]_{\beta_1}=\{x_1,x_2,x_3,x_5\}$.

Now let $r=2$. We have

$$\mathcal{O}/\sim_{f_1,f_2}=\{\{x_1\},\{x_2\},\{x_3,x_4\},\{x_5\}\}=\{[x_1]_{f_1,f_2},[x_2]_{f_1,f_2},[x_3]_{f_1,f_2},[x_5]_{f_1,f_2}\},$$

$$\mathcal{O}/\sim_{f_1,f_3}=\{\{x_1,x_3\},\{x_2\}\{x_4\}\}=\{[x_1]_{f_1,f_3},[x_2]_{f_{1},f_{3}},[x_4]_{f_2,f_3},[x_5]_{f_1,f_3}\},$$
$$\mathcal{O}/\sim_{f_2,f_3}=\{\{x_1,x_5\},\{x_3\},\{x_2\},\{x_4\}\}=\{[x_1]_{,f_2,f_3},[x_2]_{f_2,f_3}[x_3]_{f_2,f_3},[x_4]_{f_2,f_3}\}.$$

Then $[x_1]_{\beta_2}=\{x_1,x_3,x_5\}$, $[x_2]_{\beta_2}=\{x_2,x_4\}$, $[x_3]_{\beta_2}=\{x_1,x_3,x_4\}$, $[x_4]_{\beta_2}=\{x_2,x_3,x_4\}$, and $[x_5]_{\beta_2}=\{x_1,x_5\}$.

The basic idea in the near set approach to object recognition is to compare object descriptions. Sample perceptual objects $x,y\in\mathcal{O}$ $(x\neq y)$ are near each other if and only if x and y have similar descriptions. Let A and A' be two subsets of $\mathcal{O}$ and let $\beta\subseteq\mathcal{F}$. We say that A is near A' if there exist $x\in A$, $y\in A'$, and $f\in\beta$ such that $x\sim_f y$.

Although in rough sets, we have $[x]_r=\overline{Apr}_r([x]_r)=\underline{Apr}_r([x]_r)$, but the following example shows that, in the near sets, this feature does not exist in the general, that $[x]_{\beta_r}\subseteq\overline{N_r(\beta)}([x]_{\beta r})$, and that $\underline{N_r(\beta)}([x]_{\beta_r})\subseteq[x]_{\beta_r}$.

Example 2.4. In Example 2.3, let $x=x_1$ and $r=2$. We have $[x_1]_{\beta_2}=\{x_1,x_3,x_5\}$. Let $a=x_3\in[x_1]_{\beta_2}$, then $[a]_{\beta_2}=[x_3]_{\beta_2}=\{x_1,x_3,x_4\}\nsubseteq[x_1]_{\beta_2}$. Hence $x_3\in[x_1]_{\beta_2}$, but $x_3\notin\underline{N_2(\beta)}([x]_{\beta_2})$. This means that $[x_1]_{\beta_2}\nsubseteq\underline{N_2(\beta)}([x_1]_{\beta_2})$. Then $[x]_{\beta_r}\nsubseteq\underline{N_r(\beta)}([x]_{\beta_r})$ for every $r\neq|\beta|$ in general.

It is clear that $x_4\in\overline{N_2(\beta)}([x_1]_{\beta 2})$, but $x_4\notin[x_1]_{\beta 2}$. Hence $\overline{N_2(\beta)}([x_1]_{\beta 2})\nsubseteq[x_1]_{\beta 2}$ in general.

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Remark 2.5. Let (U, F, β_r) be a nearness approximation space. Then the following statements hold:

1. $\underline{N_r(\beta)}(A) \subseteq A \subseteq \overline{N_r(\beta)}(A)$ for every $A \subseteq U$.
2. $a \in [x]_{\beta_r}$ if and only if $x \in [a]_{\beta_r}$ and $[x]_{\beta_r} \neq [a]_{\beta_r}$ in general.
3. $[x]_{\beta_r} \subseteq \overline{N_r(\beta)}([x]_{\beta_r})$ and $\underline{N_r(\beta)}([x]_{\beta_r}) \subseteq [x]_{\beta_r}$.
4. $a \in [x]_{\beta_r}$ if and only if $x \in \overline{N_r(\beta)}(\{a\})$.

3. CATEGORY OF APPROXIMATION SPACES WITH UPPER NATURAL TRANSFORMATIONS

In this section we introduce the concept of a natural transformation on a nearness approximation space and show that natural transformations and the nearness approximation spaces both are categories. Also we characterize an epimorphism and a monomorphisms in these categories.

Remark 3.1. Let (U, F, β_r) and (V, G, α_r) be two nearness approximation spaces. Then the following statements hold:

(1) If $\varphi\colon U \to V$ is an upper natural transformation from (U, F, β_r) into (V, G, α_r), then $u \in \overline{N_r(\beta)}(\{x\})$ if and only if $x \in [u]_{\beta_r}$ for every $u, x \in U$.

(2) If $\varphi\colon U \to V$ is an upper natural transformation from (U, F, β_r) into (V, G, α_r), then $\{\, \varphi(x)\colon u \in U,\ x \in [u]_{\beta_r} \,\} = \{\, v \in V\colon \varphi(x) \in [v]_{\alpha_r} \,\}$ for every $x \in U$.

Let $\varphi\colon U \to V$ be an upper natural transformation from (U, F, β_r) into (V, G, α_r). Since

$$\varphi(\overline{N_r(\beta)}(\{x\})) = \{\varphi(u) : u \in U, x \in [u]_{\beta_r}\}$$

and

$$\overline{N_r(\alpha)}(\varphi(\{x\})) = \{v \in V : \varphi(x) \in [v]_{\alpha_r}\},$$

we conclude that

$$\{\varphi(u) : u \in U, x \in [u]_{\beta_r}\} = \{v \in V : \varphi(x) \in [v]_{\alpha_r}\}$$

for every $x \in U$.

Proposition 3.2. *The following properties hold:*

(1) *Nearness approximation spaces and right upper natural transformations form a category, which is denoted by* $\overline{\boldsymbol{RNEApr}}$.

(2) *Nearness approximation spaces and left upper natural transformations form a category, which is denoted by* $\overline{\boldsymbol{LNEApr}}$.

(3) *Nearness approximation spaces and upper natural transformations form a category, which is denoted by* $\overline{\boldsymbol{NEApr}}$.

Definition 3.3. Let (U, F, β) and (V, G, α) be two nearness approximation spaces, where $\overline{N_r(\beta)}\colon \mathcal{P}^*(U) \to \mathcal{P}^*(U)$ and $\overline{N_r(\alpha)}\colon \mathcal{P}^*(V) \to \mathcal{P}^*(V)$ are functions. Consider the function $\varphi\colon U \to V$.

(1) The function φ is an upper natural transformation from (U, F, β) into (V, G, α) if the following diagram commutes:

$$\begin{array}{ccc} \mathcal{P}(U) & \xrightarrow{\overline{N_r(\beta)}} & \mathcal{P}(U) \\ {\scriptstyle \overline{\varphi}}\downarrow & & \downarrow{\scriptstyle \overline{\varphi}} \\ \mathcal{P}(V) & \xrightarrow[\overline{N_r(\alpha)}]{} & \mathcal{P}(V). \end{array}$$

(2) The function φ is a right upper natural transformation from (U, F, β) into (V, G, α) if

$$\varphi\big(\overline{N_r(\beta)}(A)\big) \subseteq \overline{N_r(\alpha)}\big(\varphi(A)\big)$$

for every $A \in \mathcal{P}(U)$.

(3) The function φ is a left upper natural transformation from (U, F, β) into (V, G, α) if

$$\overline{N_r(\alpha)}\big(\varphi(A)\big) \subseteq \varphi\big(\overline{N_r(\beta)}(A)\big)$$

for every $A \in \mathcal{P}(U)$.

Proposition 3.4. *Let (U, F, β) and (V, G, α) be two near approximation spaces. Then $\varphi\colon U \to V$ is an upper natural transformation from (U, F, β) into (V, G, α) if and only if*

$$\varphi\big(\overline{N_r(\beta)}(\{x\})\big) = \overline{N_r(\alpha)}\big(\varphi(\{x\})\big)$$

for every $x \in U$.

Proof. Necessity. It is clear.

Sufficiency. Let $X \subseteq U$ be given. Since

$$\begin{aligned} y \in \varphi\big(\overline{N_r(\beta)}(X)\big) &\Rightarrow y = \varphi(u)\,\&\,[u]_{\beta_r} \cap X \neq \emptyset \text{ for some } u \in U \\ &\Rightarrow y = \varphi(u)\,\&\,x \in [u]_{\beta_r} \text{ for some } (u, x) \in U \times X \\ &\Rightarrow y = \varphi(u)\,\&\,u \in \overline{N_r(\beta)}(\{x\}) \text{ for some } (u, x) \in U \times X \\ &\Rightarrow y = \varphi(u) \in \varphi\big(\overline{N_r(\beta)}(\{x\})\big) \text{ for some } (u, x) \in U \times X \\ &\Rightarrow y \in \overline{N_r(\alpha)}\big(\varphi(\{x\})\big) \subseteq \overline{N_r(\alpha)}\big(\varphi(X)\big) \text{ for some } x \in X, \end{aligned}$$

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and

$$\begin{aligned} y \in \overline{N_r(\alpha)}\big(\varphi(X)\big) &\Rightarrow \varphi(x) \in [y]_{\alpha_r} \text{ for some } x \in X \\ &\Rightarrow y \in \overline{N_r(\alpha)}\big(\{\varphi(x)\}\big) \text{ for some } x \in X \\ &\Rightarrow y \in \varphi\big(\overline{N_r(\beta)}(\{x\})\big) \subseteq \varphi\big(\overline{N_r(\beta)}(X)\big) \text{ for some } x \in X, \end{aligned}$$

we conclude that $\varphi\big(\overline{apr}_t(X)\big) = \overline{apr}_s\big(\varphi(X)\big)$. □

Proposition 3.5. *Let (U, F, β_r) and (V, G, α_r) be two nearness approximation spaces. Then $\varphi : U \to V$ is a right upper natural transformation if and only if for every $x \in U$,*

$$\varphi([x]_{\beta_r}) \subseteq [\varphi(x)]_{\alpha_r}.$$

Proof. For every $x \in U$,

$$\begin{aligned} a \in \varphi([x]_{\beta_r}) &\Rightarrow a = \varphi(b) \text{ for some } b \in [x]_{\beta_r} \\ &\Rightarrow a = \varphi(b) \text{ for some } b \in U \text{ such that } x \in \overline{N_r(\beta)}(\{b\}) \\ &\Rightarrow a = \varphi(b) \,\&\, \varphi(x) \in \varphi\big(\overline{N_r(\beta)}(\{b\})\big) \subseteq \overline{N_r(\alpha)}\big(\varphi(\{b\})\big) \text{ for some } b \in U \\ &\Rightarrow a \in [\varphi(x)]_{\alpha_r}. \end{aligned}$$

Therefore, $\varphi([x]_{\beta_r}) \subseteq [\varphi(x)]_{\alpha_r}$ for every $x \in U$.

Conversely, let $\varphi([x]_{\beta_r}) \subseteq [\varphi(x)]_{\alpha_r}$ for every $x \in U$, let $A \subseteq U$, and let $a \in \varphi(\overline{N_r(\beta)}(A))$. Then there is $x \in \overline{N_r(\beta)}(A)$ such that $a = \varphi(x)$ and $[x]_{\beta_r} \cap A \neq \emptyset$. Let $y \in [x]_{\beta_r} \cap A$. Then

$$\varphi(y) \in \varphi([x]_{\beta_r}) \cap \varphi(A) \subseteq [\varphi(x)]_{\alpha_r} \cap \varphi(A) = [a]_{\alpha_r} \cap \varphi(A).$$

Hence $a \in \overline{N_r(\alpha)}\varphi(A)$. It follows that

$$\varphi(\overline{N_r(\beta)}(A)) \subseteq \overline{N_r(\alpha)}(\varphi(A)).$$

It implies that $\varphi(\overline{N_r(\beta)}(A)) \subseteq \overline{N_r(\alpha)}(\varphi(A))$ and means that φ is a right upper natural transformation. □

Proposition 3.6. *Let (U, F, β_r) and (V, G, α_r) be two nearness approximation spaces. Then $\varphi : U \to V$ is a left upper natural transformation if and only if for every $x \in U$,*

$$[\varphi(x)]_{\alpha_r} \subseteq \varphi([x]_{\beta_r}).$$

Proof. Let φ be a left upper natural transformation and let $y \in [\varphi(x)]_{\alpha_r}$. Then $\varphi(x) \in [y]_{\alpha_r}$ and $y \in \overline{N_r(\alpha)}(\{\varphi(x)\}) \subseteq \varphi(\overline{N_r(\beta)}(\{x\})$. On the other hand, there is $b \in \overline{N_r(\beta)}(\{x\})$ such that $y = \varphi(b)$, which implies that $x \in [b]_{\beta_r}$. Thus, $b \in [x]_{\beta_r}$. This means $y \in \varphi([x]_{\beta_r})$ and $[\varphi(x)]_{\alpha_r} \subseteq \varphi([x]_{\beta_r})$.

Conversely, let $a \in \overline{N_r(\alpha)}(\varphi(A))$. Then $[a]_{\alpha_r} \cap \varphi(A) \neq \emptyset$. Suppose that $b \in [a]_{\alpha_r} \cap \varphi(A)$. Then there exists $y \in A$ such that $b = \varphi(y)$ and $\varphi(y) \in [a]_{\alpha_r}$. It gives us $a \in [\varphi(y)]_{\alpha_r} \subseteq \varphi([y]_{\beta_r})$.

Hence $a = \varphi(z)$ for some $z \in [y]_{\beta_r}$. One can easily see that $[z]_{\beta_r} \cap A \neq \emptyset$ and so $z \in \overline{N_r(\beta)}(A)$ and $a = \varphi(z) \in \varphi(\overline{N_r(\beta)}(A))$. Thus

$$\overline{N_r(\alpha)}(\varphi(A)) \subseteq \varphi(\overline{N_r(\beta)}(A)).$$

□

Corollary 3.7. *Let (U, F, β_r) and (V, G, α_r) be two nearness approximation spaces. Then, $\varphi : U \to V$ is an upper natural transformation if and only if for every $x \in U$,*

$$[\varphi(x)]_{\alpha_r} = \varphi([x]_{\beta_r}).$$

Example 3.8. Let $U = \{x_1, x_2, x_3\}$, let $V = \{y_1, y_2, y_3, y_4\}$, let $\beta = \{f_1, f_2, f_3, \}$, let $\alpha = \{g_1, g_2, g_3, g_4\}$, and let the values of probe functions be defined by

	f_1	f_2	f_3
x_1	1	0	1
x_2	1	0	2
x_3	4	0	2

and

	g_1	g_2	g_3	g_4
y_1	1	0	2	1
y_2	1	0	2	1
y_3	7	5	0	2
y_4	0	6	3	5

If $\varphi \colon U \to V$ is given by $\varphi := \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_1 \end{pmatrix}$, then we can easily show that φ is an upper natural transformation and that $[\varphi(x)]_{\alpha_r} = \varphi([x]_{\beta_r})$ for every $x \in U$.

Corollary 3.9. *Let (U, F, β_r) and (V, G, α_r) be two nearness approximation spaces and let $\varphi : U \to V$ be an upper natural transformation. Then for every $a, b, x \in U$ and $y \in V$, the following statements hold:*

1. *If $\varphi(a) = \varphi(b)$, then $\varphi([a]_{\beta_r}) = \varphi([b]_{\beta_r})$.*

2. *$v \in \varphi(U)$ if and only if $N_r(\alpha)(\{v\}) \subseteq \varphi(U)$.*

Proof. (1) It is clear.

(2) Let $v \in \varphi(U)$ and let $a \in N_r(\alpha)(\{v\})$. Then there exists $u \in U$ such that $v = \varphi(u)$ and $v \in [a]_{\alpha_r}$. Thus $a \in [v]_{\alpha_r} = [\varphi(u)]_{\alpha_r} = \varphi([u]_{\beta_r}$. This means $N_r(\alpha)(\{v\}) \subseteq \varphi(U)$. □

Proposition 3.10. *Let (U, F, β_r) and (V, G, α_r) be nearness approximation spaces and let $\varphi : U \to V$ be a function. Then the following statements hold:*

(1) *If* φ *is an upper natural transformation from* (U, F, β_r) *into* (V, G, α_r)*, then* φ *is an epimorphism in* $\overline{\boldsymbol{RNAprS}}$ *if and only if* $\varphi : U \to V$ *is a surjective function.*

(2) *If* φ *is a right upper natural transformation from* (U, F, β_r) *into* (V, G, α_r)*, then* φ *is an epimorphism in* $\overline{\boldsymbol{RNAprS}}$ *if and only if* $\varphi : U \to V$ *is a surjective function.*

(3) *If* φ *is a left upper natural transformation from* (U, F, β_r) *into* (V, G, α_r)*, then* φ *is an epimorphism in* $\overline{\boldsymbol{LNAprS}}$ *if and only if* $\varphi : U \to V$ *is a surjective function.*

Proof. (1). *Necessity.* We proceed by contradiction. Assume that $\varphi(U) \neq V$. Given $W = \{w_1, w_2\}$ and $\Delta = \{f_1, f_2, f_3\}$, the values of probe functions are defined by

	f_1	f_2	f_3
w_1	1	2	3
w_2	0	2	1

Define $\phi, \psi : V \to W$ by $\phi(v) = w_1$ and

$$\psi = \begin{cases} w_1 & \text{if } v \in \varphi(U), \\ w_2 & \text{if } v \notin \varphi(U). \end{cases}$$

It is clear that for every $v \in V$, $\phi\big(\overline{N_r(\alpha)}(\{v\})\big) = \{w_1\} = \overline{N_r(\Delta)}\big(\phi(\{v\})\big)$, and we have

$$\psi\big(\overline{N_r(\alpha)}(\{v\})\big) = \begin{cases} \{w_1\} & \overline{N_r(\alpha)}(\{v\}) \subseteq \varphi(U), \\ \{w_2\} & \overline{N_r(\alpha)}(\{v\}) \subseteq V \setminus \varphi(U), \\ \{w_1, w_2\} & \text{otherwise,} \end{cases}$$

and

$$\overline{N_r(\Delta)}\big(\psi(\{v\})\big) = \begin{cases} \{w_1\} & v \in \varphi(U), \\ \{w_2\} & \text{otherwise.} \end{cases}$$

Hence, by the hypothesis, ϕ and ψ are upper natural transformations and $\phi\varphi = \psi\varphi$. Since the upper natural transformation φ is an epimorphism, then $\phi = \psi$, which is a contradiction.

Sufficiency. The proof is clear.

The proof of the other statements is similar to the proof of the first statement. □

Proposition 3.11. *Let* (U, F, β_r) *and* (V, G, α_r) *be nearness approximation spaces and let* $\varphi : U \to V$ *be a function. Then the following statements hold:*

(1) *Let* φ *be a right upper natural transformation from* (U, F, β_r) *into* (V, G, α_r)*. Then* φ *is a monomorphism in* $\overline{\boldsymbol{RNEApr}}$ *if and only if* $\varphi : U \to V$ *is an injection function.*

(2) *If φ is an upper natural transformation from (U, F, β_r) into (V, G, α_r) such that $[U]_{\beta_r}$ form a partition of $\bigcup[U]_{\beta_r}$, then φ is a monomorphism in $\overline{\boldsymbol{NEApr}}$ if and only if $\varphi : U \to V$ is an injection function.*

Proof. (1). *Necessity.* By the way of contradiction, assume that there exist $a, b \in U$ such that $\varphi(a) = \varphi(b)$ with $a \neq b$. Let $W := \{a, b\}$ and let $H = \{h_1, h_2, h_3\}$, and assume that the values of probe functions from W to percepts are defined as

	h_1	h_2	h_3
a	1	0	3
b	4	2	6

Then $[a]_{\Delta_r} = \{a\}$ and $[b]_{\Delta_r} = \{b\}$ for $\Delta \subseteq H$ with $r \neq |\Delta|$. Define $\alpha, \beta : W \to U$ by $\alpha := \begin{pmatrix} a & b \\ a & b \end{pmatrix}$, and $\beta := \begin{pmatrix} a & b \\ b & a \end{pmatrix}$. One can immediately see that α and β are right upper natural transformations and that $\varphi\alpha = \varphi\beta$. By the hypothesis, $\alpha = \beta$, which is a contradiction.

Sufficiency. The proof is clear.

(2). *Necessity.* Let $a, b \in U$ and let $\varphi(a) = \varphi(b)$. We assume that $|[a]_{\beta_r}| \leq |[b]_{\beta_r}|$. Let $W = [a]_{\beta_r}$ and let $[w]_{\alpha_r} = W$ for every $w \in W$. Let $h \in \prod_{x \in \varphi([a]_{\beta_r})} \varphi^{-1}(x)$ such that $h(x) \in [b]_{\beta_r}$ and $h(\varphi(a)) = b$. Define $\psi, \phi : W \to U$ by $\psi(x) = x$ and $\phi(x) = h(\varphi(x))$ for every $x \in W$. By Corollary 3.7, ψ and ϕ are upper natural transformations. Also we have $\varphi\psi(x) = \varphi\phi(x)$. By the hypothesis, $\psi = \phi$, which is a contradiction.
Sufficiency. The proof is clear. □

Proposition 3.12. *The following statements hold:*

(1) $\overline{\boldsymbol{NEApr}}$ *has all coproducts.*

(2) $\overline{\boldsymbol{LNEApr}}$ *has all coproducts.*

(3) $\overline{\boldsymbol{RNEApr}}$ *has all coproducts.*

Proof. (1). Let $\{(U_j, F_j, \beta_{j_r})\}_{j \in J}$ be a family of nearness approximation spaces and let $U = \bigcup_{j \in J}(Uj \times \{j\})$. Consider the inclusion map $\iota_j : U_j \to U$ with $\iota_j(a) = (a, j)$ for every $a \in U_j$. Define the relation $\beta_r \subseteq U \times U$ by $(x, y) \in \beta_r$ $(y \in [x]_{\beta_r})$ if and only if there exist $j \in J$ and $(a, b) \in U_j \times U_j$ such that $x = (a, j)$, $y = (b, j)$, and $(a, b) \in \beta_{j_r}$. It gives us

$$(b, j) \in [(a, j)]_{\beta_r} \Longleftrightarrow b \in [a]_{\beta_{j_r}}. \tag{3.1}$$

We claim that (U, F, β_r) together with $\{\iota_j\}_{j \in J}$, is a coproduct of the family $\{(U_j, F_j, \beta_{j_r})\}_{j \in J}$. First we show that for every $j \in J$, ι_j is an upper natural transformation from (U_j, F_j, β_{j_r}) into (U, F, β_r). By using (3.1), for every $x \in U_j$, we have $\iota_j([x]_{\beta_{j_r}}) = [(x, j)]_{\beta_r} = [\iota_j(x)]_{\beta_r}$. Then Propositions 3.7 implies that for $j \in J$, ι_j is an upper natural transformation.

We claim that (U, F, β_r) together with $\{\iota_j\}_{j\in J}$, is a coproduct of the family $\{(U_j, F_j, \beta_{j_r})\}_{j\in J}$. Let φ_j be an upper natural transformation from (U_j, F_j, β_{j_r}) into (W, G, α_r) for any $j \in J$.

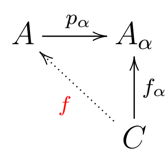
Then, $[\varphi_j(x)]_{\alpha_r} = \varphi j([x]_{\beta_{j_r}})$ for every $x \in U_j$, and also by the universal property of coproduct in the sets category, there exists a unique map $\varphi : U \to W$ such that $\varphi\iota_j = \varphi_j$. It is sufficient to show that φ is a unique upper natural transformation from (U, F, β_r) into (W, G, α_r). Let $a \in U$ be given, then there exist $j \in J$ and $a_j \in U_j$ such that $a = \iota_j(a_j)$, which implies that

$$\varphi([a]_{\beta_r}) = \varphi([\iota_j(a_j)]_{\beta_r} = \varphi\iota_j([a_j]_{\beta_{j_r}}) = \varphi_j([a_j]_{\beta_{j_r}}) = [\varphi_j(a_j)]_{\alpha_r} = [\varphi\iota_j(a_j)]_{\alpha_r} = [\varphi(a)]_{\alpha_r}.$$

It implies that φ is an upper natural transformation. It is clear that φ is unique.

The proof of the rest of statements is similar to the proof of the first statement. □

Let $\mathbb{C}$ be a category and let $(A_\alpha)_{\alpha\in I}$ be a family of objects in $\mathbb{C}$. Then a ***product*** of this family is an object A, denoted by $\prod_{\alpha\in I} A_\alpha$, together with a family of morphisms $(p_\alpha : A \to A_\alpha)_{\alpha\in I}$, called **projections**, such that for each object C and family of morphisms $(f_\alpha : C \to A_\alpha)_{\alpha\in I}$, there exists a unique morphism $f : C \to A$ such that $p_\alpha f = f_\alpha$ for each $\alpha \in I$. Hence, for every $\alpha \in I$, the following diagram is commutative:



Proposition 3.13. *Let $\{(U_j, F_j, \beta_{j_r})\}_{j\in J}$ be a family of nearness approximation spaces such that $|[u_j]_{\beta_{j_r}}| = |\{u_j\}| = 1$ for every $u_j \in U_j$. Then $\{(u_j, F_j, \beta_{j_r})$ has a product in $\overline{\textbf{\textit{RNEAprS}}}$.*

Proof. Consider the set theoretic Cartesian product $U = \prod_{j\in J} U_j$ and the projection map $\pi_j : U \to U_j$, that is, $\pi_j((a_j)_{j\in J}) = a_j$ for every $(a_j)_{j\in J} \in U$. Define the equivalence relation β_r on U by

$$((a_j)_{j\in J}, (b_j)_{j\in J}) \in \beta_r \Leftrightarrow (a_j, b_j) \in \beta_{j_r} \quad \text{for all } j \in J.$$

It is clear that $[(aj)_{j\in J}]_{\beta_r} = |\{(aj)_{j\in J}\}| = 1$. We claim that $\{(U, \beta_r), \{\pi_j\}_{j\in J}\}$ is a product of $\{(U_j, \beta_{j_r})\}_{j\in J}$. First we show that π_j, $j \in J$, is an upper natural transformation from (U, β_r) into (U_j, β_{j_r}). Suppose that $(a_j)_{j\in J} \in U$. It is easy to see that

$$\pi_j([(a_j)_{j\in J}]_{\beta_r}) = \pi_j(\{(a_j)_{j\in J}\}) = \{a_j\} = [a_j]_{\beta_{j_r}} = [\pi_j((a_j)_{j\in J})]_{\beta_{j_r}}.$$

Hence, Corollary 3.7 implies that, for $j \in J$, π_j is an upper natural transformation. Now, for any $j \in J$, let $\varphi_j : V \to U_j$, be an upper natural transformation from (V, G, α_r) into (U_j, F_j, β_j). Define

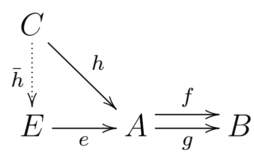
$$\begin{aligned} \varphi : V &\longrightarrow U \\ x &\longmapsto (\varphi_j(x))_{j\in J}. \end{aligned}$$

We show that φ is an upper natural transformation from (V, G, α_r) into (U, F, β_r). To see this, let $x \in V$. Then

$$\begin{aligned} \varphi([x]_{\alpha_r}) &= \{\varphi(y) | y \in [x]_{\alpha_r}\} = \{(\varphi_j(y))_{j\in J} | y \in [x]_{\alpha_r}\} = \{(\varphi_j(x))_{j\in J}\} = \{\varphi(x)\} \\ &= [\varphi(x)]_{\beta_r}. \end{aligned}$$

Thus, by Theorem 3.7, φ is an upper natural transformation. It is clear that $\pi_j\varphi = \varphi_j$ for every $j \in J$. Now, we prove that φ with this property is unique. Let ψ be an upper natural transformation from (V, G, α_r) into (U, F, β_r) such that $\pi_j\psi = \varphi_j$. Then It follows from the universal property of product in the sets category that $\psi = \varphi$. □

Let $\mathbb{C}$ be a category and let $f, g : A \to B$ be a pair of morphisms in $\mathbb{C}$. We recall from [1] that an object E, also denoted by $eq(f, g)$, together with a morphism $e : E \to A$ is called an ***equalizer*** of f and g if $f \circ e = g \circ e$ and for every morphism $h : C \to A$ with $f \circ h = g \circ h$, there exists a unique morphism $\bar{h} : C \to E$ such that $e \circ \bar{h} = h$. That is,



Proposition 3.14. *Let (U, F, β_r) and (V, G, α_r) be nearness approximation spaces and let $\varphi, \phi \colon U \to V$ be two functions. We set*

$$E := \{\, x \in U \colon \varphi(x) = \phi(x) \,\},$$

define $[x]_{\Delta_r} = [x]_{\beta_r} \cap E$, and assume that $E \neq \emptyset$. Then the following statements hold:

(1) *If $\varphi, \phi \colon U \to V$ are two right upper natural transformations from (U, F, β_r) to (V, G, α_r) such that $[x]_{\beta_r} \cap E \neq \emptyset$ if and only if $[x]_{\beta_r} \subseteq E$ for every $x \in U$, then $\psi \colon E \to U$ given by $\psi(x) = x$ is a right upper natural transformation from (E, H, Δ_r) to (U, F, β_r), and (E, H, Δ_r) together with ψ is an equalizer of φ and ϕ in $\overline{\textbf{RNEAprS}}$.*

(2) *If $\varphi, \phi \colon U \to V$ are two left upper natural transformations from (U, F, β_r) to (V, G, α_r), then $\psi \colon E \to U$ given by $\psi(x) = x$ is a left upper natural transformation from (E, H, Δ_r) to (U, F, β_r), and (E, H, Δ_r) together with ψ is an equalizer of φ and ϕ in $\overline{\textbf{LNEAprS}}$.*

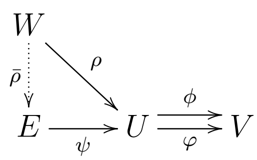
(3) *If $\varphi, \phi \colon U \to V$ are two upper natural transformations from (U, F, β_r) to (V, G, α_r) such that $[x]_{\beta_r} \cap E \neq \emptyset$ if and only if $[x]_{\beta_r} \subseteq E$ for every $x \in U$, then $\psi \colon E \to U$ given by $\psi(x) = x$ is an upper natural transformation from (E, H, Δ_r) to (U, F, β_r), and (E, H, Δ_r) together with ψ is an equalizer of φ and ϕ in $\overline{\textbf{NEAprS}}$.*

Proof. (1). First we shall prove that ψ is a right upper natural transformation. In order to approach this goal, let us assume that $x \in E$. In view of Proposition 3.5 and

$$\psi\big([x]_{\Delta_r}\big) = [x]_{\Delta_r} \subseteq [x]_{\beta_r} = [\psi(x)]_{\beta_r},$$

we infer that ψ is a right upper natural transformation from (E, H, Δ_r) to (U, F, β_r). Let $\rho \colon W \to U$ be a right upper natural transformation from (W, D, θ_r) to (U, F, β_r) such that $\varphi\rho = \phi\rho$, which implies that $\rho(W) \subseteq E$. We define $\bar{\rho} \colon W \to E$ by $\bar{\rho}(x) = \rho(x)$. Since $\rho(w) \in E$

for every $w \in W$, hence it is clear that $\bar{\rho}$ is a unique right upper natural transformation from (W, D, θ_r) to (E, H, Δ_r) such that the following diagram is commutative:

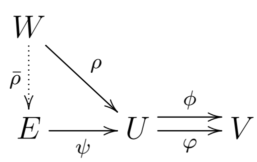


(2). Similar to the proof of expression (1), ψ is a left upper natural transformation from (E, H, Δ_r) to (U, F, β_r).

Let $\rho\colon W \to U$ be a left upper natural transformation from (W, D, θ_r) to (U, F, β_r) such that $\varphi\rho = \phi\rho$, which implies that $\rho(W) \subset E$. We define $\bar{\rho}\colon W \to E$ by $\bar{\rho}(x) = \rho(x)$. We have

$$[\bar{\rho}(w)]_{\Delta_r} = [\rho(w)]_{\Delta_r} \subseteq [\rho(w)]_{\beta_r} \subseteq \rho([w]_{\theta_r}) = \bar{\rho}([w]_{\theta_r}).$$

Then $\bar{\rho}$ is a unique left upper natural transformation from (W, D, θ_r) to (E, H, Δ_r) such that the following diagram is commutative:



(3). The proof is clear.

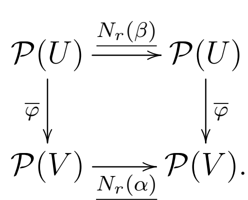
□

4. ON NEAR APPROXIMATION SPACES WITH THE LOWER NATURAL TRANSFORMATIONS

In this section, we prove that the category of a nearness approximation space does not necessarily have a product but that some families of an approximation spaces have a coproduct.

Definition 4.1. Let (U, F, β) and (V, G, α) be two nearness approximation spaces and let $\underline{N_r(\beta)}\colon \mathcal{P}^*(U) \to \mathcal{P}^*(U)$ and $\underline{N_r(\alpha)}\colon \mathcal{P}^*(V) \to \mathcal{P}^*(V)$ be functions. Consider a function $\varphi\colon U \to V$.

(1) The function φ is a lower natural transformation from (U, F, β) into (V, G, α) if the following diagram commutes:



(2) The function φ is a right lower natural transformation from (U,F,β) into (V,G,α) if

$$\varphi\big(\underline{N_r(\beta)}(A)\big) \subseteq \underline{N_r(\alpha)}\big(\varphi(A)\big)$$

for every $A \in \mathcal{P}(U)$.

(3) The function φ is a left lower natural transformation from (U,F,β) into (V,G,α) if

$$\underline{N_r(\alpha)}\big(\varphi(A)\big) \subseteq \varphi\big(\underline{N_r(\beta)}(A)\big)$$

for every $A \in \mathcal{P}(U)$.

Proposition 4.2. *The following statements hold:*

(1) *Nearness approximation spaces and right lower natural transformations form a category, which is denoted by* $\boldsymbol{\underline{RNE}Apr}$.

(2) *Nearness approximation spaces and left lower natural transformations form a category, which is denoted by* $\boldsymbol{\underline{LNE}Apr}$.

(3) *Nearness approximation spaces and lower natural transformations form a category, which is denoted by* $\boldsymbol{\underline{NE}Apr}$.

Proposition 4.3. *The function φ is a right lower natural transformation from (U,F,β) into (U,G,α) if and only if $[\varphi(x)]_{\alpha_r} \subseteq \varphi([x]_{\beta_r})$ for every $x \in U$.*

Proof. Necessity. For every $x \in U$, since $x \in \underline{N_r(\beta)}([x]_{\beta_r}$, we conclude that

$$\varphi(x) \in \varphi(\underline{N_r(\beta)}([x]_{\beta_r})) \subseteq \underline{N_r(\alpha)}(\varphi([x]_{\beta_r})),$$

which implies that $[\varphi(x)]_{\alpha_r} \subseteq \varphi([x]_{\beta_r})$.

Sufficiency. Let $A \subseteq U$ and $y \in \varphi(\underline{N_r(\beta)}(A))$ be given. Then there exists an element $x \in \underline{N_r(\beta)}(A)$ such that $y = \varphi(x)$, which implies that $[x]_{\beta_r} \subseteq A$. From the hypothesis, we have $[\varphi(x)]_{\alpha_r} \subseteq \varphi([x]_{\beta_r}) \subseteq \varphi(A)$. Thus $y = \varphi(x) \in \underline{N_r(\alpha)}(\varphi(A))$. Hence φ is a right lower natural transformation from (U,F,β) into (U,G,α).

□

Proposition 4.4. *Let (U,F,β) and (U,G,α) be two nearness approximation spaces. For every injection function $\varphi : U : U \to V$ and for every $x \in U$, if $[\varphi(x)]_{\alpha_r} = \varphi([x]_{\beta_r})$, then φ is a lower natural transformation from (U,F,β) into (U,G,α).*

Proof. By Proposition 4.3, it is sufficient to prove that $\underline{N_r(\alpha)}(\varphi(A)) \subseteq \varphi(\underline{N_r(\beta)}(A))$.

Let $a \in \underline{N_r(\alpha)}(\varphi(A))$. Then $[a]_{\alpha_r} \subseteq \varphi(A)$. Therefore there exists $b \in A$ such that $a = \varphi(b)$ and $[\varphi(b)]_{\alpha_r} \subseteq \varphi(A)$.

Then $\varphi([b]_{\beta_r}) = [\varphi(b)]_{\alpha_r} \subseteq \varphi(A)$. Since φ is an injection function, then $[b]_{\beta_r} \subseteq A$ and $b \in \underline{N_r(\beta)}(A)$. This means $a = \varphi(b) \in \varphi(\underline{N_r(\beta)}(A))$.

□

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The following example shows that the injection hypothesis in Proposition 4.4 cannot be removed.

Example 4.5. Let (U, F, β) and (V, G, α) be perceptual systems, where $U = \{x_1, x_2, x_3, x_4\}$, $F = \{f_1, f_2, f_3\}$, $V = \{y_1, y_2, y_3\}$, $G = \{g_1, g_2, g_3\}$, and the values of probe functions are defined, respectively, as follows:

	f_1	f_2	f_3
x_1	1	2	1
x_2	1	2	0
x_3	4	2	5
x_4	0	2	5

and

	g_1	g_2	g_3
y_1	1	0	1
y_2	2	0	3
y_3	3	0	3

If $\varphi : U \colon U \to V$ is given by $\varphi := \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_1 & y_2 & y_3, \end{pmatrix}$, then it is clear that $[\varphi(x)]_{\alpha_2} = \varphi([x]_{\beta_2})$ for every $x \in U$, but φ is not a lower natural transformation from (U, F, β) into (U, G, α).

Proposition 4.6. *Let φ is a lower natural transformation from (U, F, β) into (V, G, α). If $[U]_{\beta_r}$ form a partition of $\bigcup [U]_{\beta_r}$, then for every $a, b, x \in U$ and $y \in V$, the following statements hold:*

1. $[\varphi(x)]_{\alpha_r} = \varphi([x]_{\beta_r})$ *for every* $x \in U$.
2. $\varphi(x) \in \underline{N_r(\alpha)}(\varphi([x]_{\beta_r}))$ *for every* $x \in U$.
3. *If* $a, b \in [x]_{\beta_r}$ *and* $a \neq b$, *then* $\varphi(a) \neq \varphi(b)$ *for every* $x \in U$.
4. *The function* $\varphi|_{[x]_{\beta_r}} : [x]_{\beta_r} \to [\varphi(x)]_{\alpha_r}$ *is a one-to-one correspondence for every* $x \in U$.
5. *If* $\varphi(a) = \varphi(b)$, *then* $|[a]_{\beta_r}| = |[b]_{\alpha_r}|$.
6. *If* $|[a]_{\beta_r}| \geq 2$, *then* $\varphi^{-1}\varphi([a]_{\beta_r}) = [a]_{\beta_r}$.
7. *For every* $x \in U$ *and every* $A \subseteq U$, *if* $|[x]_{\beta_r}| \geq 2$ *and* $\varphi([x]_{\beta_r}) \subseteq \varphi(A)$, *then* $[x]_{\beta_r} \subseteq A$
8. *If* $|[a]_{\beta_r}| \geq 2$, *then* $\varphi([a]_{\beta_r}) = \varphi([b]_{\beta_r})$ *if and only if* $[a]_{\beta_r} = [b]_{\beta_r}$.
9. $[y]_{\alpha_r} \cap \varphi(U) \neq \emptyset$ *if and only if* $[y]_{\alpha_r} \subseteq \varphi(U)$.

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Proof. (1). Since $x \in \underline{N_r(\beta)}([x]_{\beta_r})$, we conclude that $\varphi(x) \in \varphi(\underline{N_r(\beta)}([x]_{\beta_r})) = \underline{N_r(\alpha)}(\varphi([x]_{\beta_r})$, which implies that$[\varphi(x)]_{\alpha_r} \subseteq \varphi([x]_{\beta_r})$. By the way of contradiction, assume that there exists an element $x \in U$ such that $\varphi([x]_{\beta_r}) \nsubseteq [\varphi(x)]_{\alpha_r}$. Then there exists an element $a \in \varphi([x]_{\beta_r}) \setminus [\varphi(x)]_{\alpha_r}$, which implies that there exists an element $b \in [x]_{\beta_r}$ such that $\varphi(b) = a$. Since $[\varphi(x)]_{\alpha_r} \subseteq \varphi([x]_{\beta_r}) \setminus \{a\} \subseteq \varphi\big([x]_{\beta_r} \setminus \{b\}\big)$, we conclude that

$$\varphi(x) \in \underline{N_r(\alpha)}\Big(\varphi\big([x]_{\beta_r} \setminus \{b\}\big)\Big) = \varphi\Big(\underline{N_r(\beta)}\big([x]_{\beta_r} \setminus \{b\}\big)\Big) = \varphi(\emptyset) = \emptyset,$$

which is a contradiction. Therefore, $\varphi([x]_{\beta_r}) \subseteq [\varphi(x)]_{\alpha_r}$, and the proof is now complete.

(2). By the first statement, it is clear.

(3). We argue by contradiction. Let us assume that $a, b \in [x]_{\beta_r}$ such that $\varphi(a) = \varphi(b)$ and $a \neq b$. Then, the second statement implies that

$$\varphi(x) \in \underline{N_r(\alpha)}(\varphi([x]_{\beta_r})) = \underline{N_r(\alpha)}(\varphi([x]_{\beta_r} \setminus \{b\})) = \varphi\Big(\underline{N_r(\beta)}\big([x]_{\beta_r} \setminus \{b\}\big)\Big) = \varphi(\emptyset) = \emptyset,$$

which is a contradiction.

Statements (1) and (3) implies that (4) and (5) hold.

(6). Let $|[a]_{\beta_r}| \geq 2$ and $x \in \varphi^{-1}\varphi([a]_{\beta_r})$, then $\varphi(x) \in \varphi([a]_{\beta_r})$, and statement (1) implies that

$$\varphi([x]_{\beta_r}) = [\varphi(x)]_{\alpha_r} = [\varphi(a)]_{\alpha_r} = \varphi([a]_{\beta_r}).$$

By statement (4), we have $[x]_{\beta_r} = [a]_{\beta_r}$. Therefore $\varphi^{-1}\varphi([a]_{\beta_r}) = [a]_{\beta_r}$.

(7). We proceed by contradiction. Assume that there exists an element $y \in [x]_{\beta_r} \setminus A$. Then there exists an element $z \in A$ such that $\varphi(z) = \varphi(y)$ and $z \neq y$. We put $B := \{z\} \cup \big([x]_{\beta_r} \setminus \{y\}\big)$. Since, by the second statement,

$$\varphi(x) \in \underline{N_r(\beta)}\Big(\varphi\big([x]_{\beta_r}\big)\Big) = \underline{N_r(\alpha)}(\varphi(B)) = \varphi(\underline{N_r(\beta)}(B)),$$

we conclude that there exists an element $u \in \underline{N_r(\beta)}(B)$ such that $\varphi(x) = \varphi(u)$. Statement (5) implies that $2 \leq |[x]_{\beta_r}| = |[u]_{\beta_r}|$. On the other hand, $[u]_{\beta_r} = \{z\}$, which is a contradiction.

(8). By statements (1), (3), and (5) it is clear.

(9). Let $[y]_{\alpha_r} \cap \varphi(U) \neq \emptyset$. Then there exists $x \in U$ such that $\varphi(x) \in [y]_{\alpha_r}$. We proceed by contradiction. Assume that there exists an element $a \in [y]_{\alpha_r} \setminus \varphi(U)$. Then there are no element $u \in U$ such that $a = \varphi(u)$. Thus $a \notin \varphi([u]_{\beta_r}) = [\varphi(u)]_{\alpha_r}$ for every $u \in U$. Since $a \in [y]_{\alpha_r}$, then $y \notin \varphi(U)$, but $y \in [\varphi(x)]_{\alpha_r} = \varphi([x]_{\beta_r})$, which is a contradiction. □

Theorem 4.7. *Let (U, F, β) and (V, G, α) be two nearness approximation spaces. A function $\varphi : U \to V$ is a lower natural transformation from $(U, F, \beta))$ into (V, G, α), if the following statements hold:*

1. *For every $x \in U$, $\varphi([x]_{\beta_r}) = [\varphi(x)]_{\alpha_r}$.*

2. *The function $\varphi|_{[x]_{\beta_r}} : [x]_{\beta_r} \to [\varphi(x)]_{\alpha_r}$ is a one-to-one correspondence for every $x \in U$.*

3. For every $a \in U$, if $|[a]_t| \geq 2$, then $\varphi^{-1}\varphi([a]_{\beta)r}) = [a]_{\beta_r}$.

Proof. Let us first show that for $A \subseteq U$,

$$\bigcup_{[x]_{\beta_r} \subseteq A} [\varphi(x)]_{\alpha_r} = \bigcup_{[\varphi(x)]_{\alpha_r} \subseteq \varphi(A)} [\varphi(x)]_{\alpha_r}.$$

Let $A \subseteq U$, then by statement (1),

$$\bigcup_{[x]_{\beta_r} \subseteq A} [\varphi(x)]_{\alpha_r} \subseteq \bigcup_{[\varphi(x)]_{\alpha_r} \subseteq \varphi(A)} [\varphi(x)]_{\alpha_r}.$$

Let $v \in \bigcup_{[\varphi(x)]_{\alpha_r} \subseteq \varphi(A)} [\varphi(x)]_{\alpha_r}$. Then there exists $a \in A$ such that

$$v \in \varphi([a]_{\beta_r}) = [\varphi(a)]_{\alpha_r} \subseteq \varphi(A)$$

and $v = \varphi(a)$. If $|[a]_{\beta_r}| = 1$, then $[a]_{\beta_r} \subseteq A$. It follows that $v \in \bigcup_{[x]_{\beta_r} \subseteq A} [\varphi(x)]_{\alpha_r}$. Now we assume that $|[a]_{\beta_r}| \geq 2$. Let $x \in [a]_{\beta_r}$. Since $\varphi(x) \in \varphi([a]_{\beta_r}) \subseteq \varphi(A)$, we conclude that there exists $y \in A$ such that $\varphi(x) = \varphi(y)$, it follows that

$$\varphi([x]_{\beta_r}) = [\varphi(x)]_{\alpha_r} = [\varphi(y)]_{\alpha_r} = \varphi([y]_{\beta_r}).$$

By statement (3), $[a]_{\beta_r} = [x]_{\beta_r} = [y]_{\beta_r}$, we infer from this and statement (2) that $x = y \in A$. Therefore $v \in \bigcup_{[x]_{\beta_r} \subseteq A} [\varphi(x)]_{\alpha_r}$. Now, we have

$$\begin{aligned}
\varphi(\underline{N_r(\beta)}(A)) &= \varphi(\textstyle\bigcup_{[x]_{\beta_r} \subseteq A} [x]_{\beta_r}) \\
&= \textstyle\bigcup_{[x]_{\beta_r} \subseteq A} \varphi([x]_{\beta_r}) \\
&= \textstyle\bigcup_{[x]_{\beta_r} \subseteq A} [\varphi(x)]_{\alpha_r} \\
&= \textstyle\bigcup_{[\varphi(x)]_{\alpha_r} \subseteq \varphi(A)} [\varphi(x)]_{\alpha_r} \\
&= \underline{N_r(\alpha)}(\varphi(A)).
\end{aligned}$$

Hence, the function $\varphi : U \to V$ is a lower natural transformation from $(U, F, \beta))$ into (V, G, α). □

Proposition 4.8. *Let φ be a lower natural transformation from (U, F, β) into (V, G, α). If $[U]_{\beta_r}$ form a partition of $\bigcup [U]_{\beta_r}$, then $\varphi : U \to V$ is a monomorphism if and only if is an injection.*

Proof. Suppose that φ is a lower natural transformation, while φ is not an injective map. Let $a, b \in U$ with $a \neq b$ and let $\varphi(a) = \varphi(b) = z$. Consider $W = \varphi(U)$ and

$$\Delta_r = \{(x, y) \in \alpha_r | x, y \in \varphi(U)\}.$$

It is clear from Corollary 4.6, that for every $x \in W$, $[x]_{\Delta_r} = [x]_{\alpha_r}$ and that

$$\frac{W}{\Delta}_r = \{[y]_{\alpha_r} | y \in \varphi(U)\}$$

is a partition of W. Let $\frac{W}{\Delta}_r = \{[y_\lambda]_{\alpha_r}\}_{\lambda\in\Lambda}$ and let $z = y_{\lambda_0}$ for some $\lambda_0 \in \Lambda$. Since for every $y \in W$, $\varphi^{-1}(y) \neq \emptyset$, we conclude that $\prod_{y\in W} \varphi^{-1}(y) \neq \emptyset$. Let us assume that $h \in \prod_{y\in W} \varphi^{-1}(y)$. Then $h(y) \in \varphi^{-1}(y)$ and $\varphi(h(y)) = y$ for every $y \in W$. We define $h_1, h_2 : W \to U$ by

$$h_1(v) = \begin{cases} h(v) & \text{if } v \neq z, \\ a & \text{if } v = z \end{cases}$$

and

$$h_2(v) = \begin{cases} h(v) & \text{if } v \neq z, \\ b & \text{if } v = z. \end{cases}$$

Now if $v \in W$, then there exists a unique $\lambda \in \Lambda$ such that $v \in [y_\lambda]_{\alpha_r}$, and we define $\psi(v) = \varphi|^{-1}_{[h_1(y_\lambda)]_{\beta_r}}(v)$ and $\phi(v) = \varphi|^{-1}_{[h_2(y_\lambda)]_{\beta_r}}(v)$. Hence $\psi, \phi : W \to U$ are two maps. Let $v \in [y_\lambda]_{\alpha_r}$. Then

$$[v]_{\Delta_r} = [v]_{\alpha_r} = [y_\lambda]_{\alpha_r} \subseteq \varphi(U)$$

and using Corollary 4.6, we have

$$\psi([v]_{\Delta_r}) = \psi([y_\lambda]_{\alpha_r}) = \varphi|^{-1}_{[h_1(y_\lambda)]_{\beta_r}}([y_\lambda]_{\alpha_r}) = [h_1(y_\lambda)]_{\beta_r}.$$

Now, since

$$\psi(v) = \varphi|^{-1}_{[h_1(y_\lambda)]_{\beta_r}}(v) \in [h_1(y_\lambda)]_{\beta_r},$$

we conclude that

$$[\psi(v)]_{\beta_r} = [h_1(y_\lambda)]_{\beta_r} = \psi([v]_{\Delta_r}).$$

Similarly, $[\phi(v)]_{\beta_r} = \phi([v]_{\Delta_r})$ for every $v \in W$. Let $v_1, v_2 \in W$ and let $\psi([v_1]_{\alpha_r}) = \psi([v_2]_{\alpha_r})$. Then there exist $\lambda_1, \lambda_2 \in \Lambda$ such that $v_1 \in [y_{\lambda_1}]_{\alpha_r}$ and $v_2 \in [y_{\lambda_2}]_{\alpha_r}$. Now, we have

$$\begin{aligned} [v_1]_{\alpha_r} &= [y_{\lambda_1}]_{\alpha_r} = \varphi([h(y_{\lambda_1})]_{\beta_r}) = \varphi\psi([v_1]_{\alpha_r}) = \varphi\psi([v_2]_{\alpha_r}) = \varphi([h(y_{\lambda_2})]_{\beta_r}) = [y_{\lambda_2}]_{\alpha_r} \\ &= [v_2]_{\alpha_r}. \end{aligned}$$

Similarly, for every $v_1, v_2 \in W$, if $\phi([v_1]_{\alpha_r}) = \phi([v_2]_{\alpha_r})$, then $[v_1]_{\alpha_r} = [v_2]_{\alpha_r}$. Let $v_1, v_2 \in [x]_{\Delta_r}$ and let $\psi(v_1) = \psi(v_2)$. Then there exists $\lambda_1, \lambda_2 \in \Lambda$ such that $v_1 \in [y_{\lambda_1}]$ and $v_2 \in [y_{\lambda_2}]$. Hence $\varphi|^{-1}_{[h(y_{\lambda_1})]_t}(v_1) = \varphi|^{-1}_{[h(y_{\lambda_2})]_t}(v_2)$, and it follows that $[h(y_{\lambda_1})]_{\beta_r} = [h(y_{\lambda_2})]_{\beta_r}$. So that

$$[y_{\lambda_1}]_{\alpha_r} = [\varphi h(y_{\lambda_1})]_{\alpha_r} = \varphi([h(y_{\lambda_1})]_{\beta_r}) = \varphi([h(y_{\lambda_2})]_{\beta_r}) = [\varphi h(y_{\lambda_2})]_{\alpha_r} = [y_{\lambda_2}]_{\alpha_r}.$$

Since $\frac{W}{\alpha}_r = \frac{W}{\Delta}_r = \{[y_\lambda]_{\alpha_r}\}_{\lambda\in\Lambda}$, we conclude that $y_{\lambda_1} = y_{\lambda_2}$ and that

$$\varphi|^{-1}_{[h(y_{\lambda_1})]_{\beta_r}}(v_1) = \psi(v_1) = \psi(v_2) = \varphi|^{-1}_{[h(y_{\lambda_2})]_{\beta_r}}(v_2) = \varphi|^{-1}_{[h(y_{\lambda_1})]_{\beta_r}}(v_2).$$

Hence, Corollary 4.6 implies that $v_1 = v_2$. Therefore, the function

$$\psi|_{[x]_{\Delta_r}} : [x]_r \longrightarrow [\psi(x)]_{\beta_r}$$

is a one-to-one correspondence. Similarly, for every $x \in W$, the function

$$\phi|_{[x]_{\Delta_r}} : [x]_r \longrightarrow [\psi(x)]_{\beta_r}$$

is a one-to-one correspondence. Therefore, by Theorem 4.7, $\phi, \psi : W \to U$ are lower natural transformations from (W, r) into (U, t). If $v \in [y_\lambda]_{\alpha_r}$, then

$$\varphi\psi(v) = \varphi(\varphi|^{-1}_{[h_1(y_\lambda)]_t}(v)) = v = \varphi(\varphi|^{-1}_{[h_2(y_\lambda)]_t}(v)) = \varphi\phi(v).$$

Hence $\varphi\phi = \varphi\psi$. Since

$$\psi(z) = \psi(y_{\lambda_0}) = \varphi|^{-1}_{[h_1(y_{\lambda_0})]_t}(y_{\lambda_0}) = \varphi|^{-1}_{[a]_{\beta_r}}(y_{\lambda_0}) = a$$

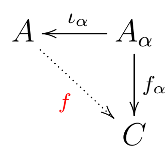
and

$$\phi(z) = \phi(y_{\lambda_0}) = \varphi|^{-1}_{[h_2(y_{\lambda_0})]_t}(y_{\lambda_0}) = \varphi|^{-1}_{[b]_{\beta_r}}(y_{\lambda_0}) = b,$$

the natural transformation φ is not a monomorphism, which is a contradiction.

The proof of converse is clear. □

Let $\mathbb{C}$ be a category and let $(A_\alpha)_{\alpha \in I}$ be a family of objects in $\mathbb{C}$. Then a ***coproduct*** of this family is an object A, denoted by $\coprod_{\alpha \in I} A_\alpha$, together with a family of morphisms $(\iota_\alpha : A_\alpha \to A)_{\alpha \in I}$, called **injections**, such that for each object C and family of morphisms $(f_\alpha : A_\alpha \to C)_{\alpha \in I}$, there exists a unique morphism $f : A \to C$ such that $f\iota_\alpha = f_\alpha$ for each $\alpha \in I$. Hence, for every $\alpha \in I$, the following diagram is commutative:



Proposition 4.9. *Let $\{(U_j, F_j, \beta_{j_r})\}_{j \in J}$ be a family of nearness approximation spaces and let $[U_j]_{\beta_{j_r}}$ form a partition of $\bigcup [U_j]_{\beta_{j_r}}$. Then $\{(U_j, F_j, \beta_{j_r})\}_{j \in J}$ has coproduct in* ***NE Apr****.*

Proof. Let $\{(U_j, F_j, \beta_{j_r})\}_{j \in J}$ be a family of nearness approximation spaces and let $U = \bigcup_{j \in J}(Uj \times \{j\})$. Consider the inclusion map $\iota_j : U_j \to U$ with $\iota_j(a) = (a, j)$ for every $a \in U_j$. Define the relation $\beta_r \subseteq U \times U$ by $(x, y) \in \beta_r$ ($y \in [x]_{\beta_r}$) if and only if there exist $j \in J$ and $(a, b) \in U_j \times U_j$ such that $x = (a, j)$, $y = (b, j)$, and $(a, b) \in \beta_{j_r}$. It gives us

$$(b, j) \in [(a, j)]_{\beta_r} \Longleftrightarrow b \in [a]_{\beta_{j_r}}. \tag{4.1}$$

We claim that (U, F, β_r) together with $\{\iota_j\}_{j \in J}$, is a coproduct of the family $\{(U_j, \beta_{j_r})\}_{j \in J}$. First, we show that for every $j \in J$, ι_j is a lower natural transformation from (U_j, F_j, β_{j_r}) into (U, F, β_r). By (4.1), for every $x \in U_j$, we have $\iota_j([x]_{\beta_{j_r}}) = [(x, j)]_{\beta_r} = [\iota_j(x)]_{\beta_r}$.

Also since ι_j is injective, Proposition 4.4 implies that ι_j, for $j \in J$, is a lower natural transformation.

Now let φ_j be a lower natural transformation from (U_j, F_j, β_{j_r}) into (W, G, α_r) for $j \in J$. Then on the one hand, $\underline{N_r(\alpha)}(\varphi_j) = \varphi_j(\underline{N_r(\beta_j)})$, and on the other hand, by the universal property of coproduct in the set category, there exists an unique map $\varphi : U \to W$ such that $\varphi\iota_j = \varphi_j$. It is sufficient to show that φ is a unique lower natural transformation from (U, F, β_r) into (W, G, α_r). To see that, we show that $\underline{N_r(\alpha)}(\varphi(A)) = \varphi(\underline{N_r(\beta)}(A))$ for $A \subseteq U$. We consider

$$A = \bigcup_{j\in J}(B_j \times \{j\}) = \bigcup_{j\in J} \iota_j(B_j),$$

where $B_j \subseteq U_j$. Then by the hypothesis, we have

$$\begin{aligned}
\underline{N_r(\alpha)}(\varphi(A)) &= \underline{N_r(\alpha)}(\varphi(\textstyle\bigcup_{j\in J}\iota_j(B_j))) = \underline{N_r(\alpha)}(\textstyle\bigcup_{j\in J}(\varphi(\iota_j(B_j))) = \textstyle\bigcup_{j\in J}\underline{N_r(\alpha)}(\varphi\iota_j(B_j))\\
&= \textstyle\bigcup_{j\in J}\underline{N_r(\alpha)}(\varphi_j(B_j)) = \textstyle\bigcup_{j\in J}\varphi_j(\underline{N_r(\beta_j)}(B_j)) = \textstyle\bigcup_{j\in J}\varphi(\iota_j\underline{N_r(\beta_j)}(B_j)\\
&= \textstyle\bigcup_{j\in J}\varphi\underline{N_r(\beta)}(\iota_j(B_j)) = \varphi(\textstyle\bigcup_{j\in J}\underline{N_r(\beta)}(\iota_j(B_j))) = \varphi(\underline{N_r(\beta)}((\textstyle\bigcup_{j\in J}\iota_j(B_j)))\\
&= \varphi(\underline{N_r(\beta)}(A))).
\end{aligned}$$

Thus the following diagram is commutative:

$$\begin{array}{ccc}
\mathcal{P}(U) & \xrightarrow{\underline{N_r(\beta)}} & \mathcal{P}(U)\\
\varphi\big\downarrow & & \big\downarrow\varphi\\
\mathcal{P}(W) & \xrightarrow[\underline{N_r(\alpha)}]{} & \mathcal{P}(W).
\end{array}$$

Now let τ be a lower natural transformation from (U, F, β_r) into (W, G, α_r) such that $\tau\iota_j = \varphi_j$. Then the universality φ implies that $\varphi = \tau$. □

Proposition 4.10. *It follows that* ***NE_Apr*** *does not necessarily have products.*

Proof. Consider $U = \{x_1, x_2\}, V = \{a, b\}$, $\beta_1 = \{f_1, f_2, f_3\}$, $\beta_2 = \{g_1, g_2, g_3\}$, and the values of probe functions from U and V to percepts are defined as

	f_1	f_2	f_3
x_1	1	0	3
x_2	4	2	1

and

	g_1	g_2	g_3
a	0	2	5
b	2	1	3

Then $[u]_{\beta_{1_r}} = \{u\}$ and $[v]_{\beta_{2_r}} = \{v\}$, for every $u \in U$ and $v \in V$. Let $\{(W, \beta_r), \varphi, \psi\}$ be a product of (U, β_{1_r}) and (V, β_{2_r}). Then, by Proposition 4.6, $|W| = 2$ and $\alpha_r = W \times W$. Let us assume that $W = \{w_1, w_2\}$, that $\varphi(w_1) = x_1$, that $\varphi(w_2) = x_2$, that $\psi(w_1) = a$, and that $\psi(w_2) = b$. Now suppose that $Z = \{z_1, z_2\}$ and that $\alpha_r = Z \times Z$. Consider $\tau : Z \to U$ by $\tau(z_1) = x_1$and $\tau(z_2) = x_2$, and $\sigma : Z \to V$ by $\sigma(z_1) = b$, $\sigma(z_2) = a$. One can easily see that τ

and σ are two lower natural transformation from (Z,α_r) into (U,β_{1_r}) and (V,β_{2_r}), respectively. Then there is a unique lower natural transformation ϕ from (Z,α_r) into (W,β_r) such that $\varphi\phi=\tau$ and $\psi\phi=\sigma$. Let $\phi(z_1)=w_1$; then $\psi\phi(z_1)=a$ and $\sigma(z_1)=b$, which is a contradiction. Hence $\phi(z_1)=w_2$. Then we have $\varphi\phi(z_1)=x_2$ and $\tau(z_1)=x_1$. It is a contradiction, too. □

Proposition 4.11. *Let $\{(U,\beta_r),\{\varphi_j\}_{j\in J}\}$ be a product of $\{(U_j,\beta_{j_r})\}_{j\in J}$ in* ***NE Apr*** *and let $[U_j]_{\beta_{j_r}}$ form a partition of $\bigcup[U_j]_{\beta_{j_r}}$ for every $j\in J$. If $|J|\geq 2$, then the following conditions hold:*

1. *For every $x\in\varphi_j(U)$, $|[x]_{\beta_{j_r}}|=1$.*
2. *$\beta_r=\Delta_U$, where Δ_U denotes the equality relation on U.*
3. *There is no $(a_j)_{j\in J}\in\prod_{j\in J}(U_j\setminus\varphi_j(U))$ such that $[a_j]_{\beta_{j_r}}$, for every $j\in J$, have the same cardinal.*

Proof. (1)Suppose that there exist $k\in J$ and $a\in\varphi_k(U)$ such that $|[a]_{\beta_{k_r}}|\geq 2$. Then there is $x\in U$ such that $a=\varphi_k(x)$. It follows that

$$[a]_{\beta_{k_r}}=[\varphi_k(x)]_{\beta_{k_r}}=\varphi_k([x]_{\beta_r})$$

and $[x]_t\geq 2$. Suppose that $y\in[x]_{\beta_r}\setminus\{x\}$. Since $|J|\geq 2$, we conclude that there exists $k'\in J$ such that $k\neq k'$. Let $W=[x]_{\beta_r}$, let $\alpha_r=W\times W$, and let $\psi_{k'}:W\to U_{k'}$ be the function given by

$$\psi_{k'}(w)=\begin{cases}\varphi_{k'}(w), & w\in W\setminus\{x,y\},\\ \varphi_{k'}(y), & w=x,\\ \varphi_{k'}(x), & w=y.\end{cases}$$

Also, for $j\in J\setminus\{k'\}$, we define $\psi_j:W\to U_j$ given by $\psi_j(w)=\varphi_j(w)$ for all $w\in W$. It is clear that ψ_j is a lower natural transformation from (W,r) into (U_j,t_j) for every $j\in J$. Then there exists a unique lower natural transformation ψ from (W,α_r) into (U,β_r) such that $\varphi_j\psi=\psi_j$. Since

$$\varphi_{k'}\psi(x)=\psi_{k'}(x)=\varphi_{k'}(y),$$

Proposition 4.6 implies that $\psi(x)=y$. Since

$$\varphi_k(y)=\varphi_k\psi(x)=\psi_k(x)=\varphi_k(x),$$

gain Proposition 4.6 implies that $x=y$, which is a contradiction.

(2). It is an immediate consequence of (1) and Proposition 4.6.

(3). Let $(a_j)_{j\in J}\in\prod_{j\in J}(U_j\setminus\varphi_j(U))$ such that $|[a_j]_{\beta_{j_r}}|=|[a_i]_{\beta_{j_r}}|$ for every $i,j\in J$. Let $W=U\cup X$, and let

$$r=t\cup\{(a,b)|a,b\in X\},$$

where X is a set such that $|X|=|[a_j]_{t_j}|$ and $U\cap X=\emptyset$. We assume that $f_j:X\to[a_j]_{\beta_{j_r}}$ is a bijection for every $j\in J$. Define $\psi_j:W\to U_j$, for $j\in J$, by

$$\psi_j(x)=\begin{cases}\varphi_j(x), & x\in U,\\ f_j(x), & x\in X.\end{cases}$$

where ψ_j, for every $j \in J$, is a lower natural transformation. Then there exists a unique lower natural transformation ψ from (W, r) into (U, t) such that $\varphi_j \psi = \psi_j$. Since

$$\varphi_j(\psi(X)) = \psi_j(X) = f_j(X) = [a_j]_{t_j},$$

Proposition 4.6 implies that $[a_j]_{\beta_{jr}} \subseteq \varphi_j(U)$, which is a contradiction. $\square$

5. DATA AVAILABILITY STATEMENT

NA

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