Weighted sum formula of multiple L-values and its applications

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Abstract

In this paper, we study the multiple L-values and the multiple zeta values of level N. We set up the algebraic framework for the double shuffle relations of the multiple zeta values of level N. Using the regularized double shuffle relations of multiple L-values, we give a sum formula and a weighted sum formula of multiple L-values. As applications, we give sum formulas and weighted sum formulas of double zeta values of level 2 and 3.

Keywords multiple L-values, multiple zeta values, weighted sum formulas.
2020 Mathematics Subject Classification 11M32, 11M06, 11M99.

1 Introduction

In this paper, we study both the multiple L-values and the multiple zeta values of level N. Let us recall the definition of multiple L-values from [1]. Let N be a fixed positive integer. Set $R = R_N = \mathbb{Z}/N\mathbb{Z}$. Fix a primitive Nth root of unity $\omega = \omega_N = \exp(2\pi i/N)$. For positive integers $n, k_1, k_2, \ldots, k_n$ and $a_1, a_2, \ldots, a_n \in R$, the multiple L-values are defined by

$$L_{N,*}(k_1, \ldots, k_n; a_1, \ldots, a_n) = \sum_{m_1 > \cdots > m_n > 0} \frac{\omega^{a_1 m_1 + \cdots + a_n m_n}}{m_1^{k_1} \cdots m_n^{k_n}},$$

$$L_{N,ш}(k_1, \ldots, k_n; a_1, \ldots, a_n) = \sum_{m_1 > \cdots > m_n > 0} \frac{\omega^{a_1(m_1 - m_2) + \cdots + a_{n-1}(m_{n-1} - m_n) + a_n m_n}}{m_1^{k_1} \cdots m_n^{k_n}}.$$

The above series are convergent when $k_1 \geq 2$ or $k_1 = 1$ and $a_1 \neq 0$. We have the following relations

$$L_{N,*}(k_1, \ldots, k_n; a_1, \ldots, a_n) = L_{N,ш}(k_1, \ldots, k_n; a_1, a_1 + a_2, \ldots, a_1 + \cdots + a_n),$$
$$L_{N,ш}(k_1, \ldots, k_n; a_1, \ldots, a_n) = L_{N,*}(k_1, \ldots, k_n; a_1, a_2 - a_1, \ldots, a_n - a_{n-1}).$$

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We also have the iterated integral representation

$$L_{N,ш}(k_1,\dots,k_n;a_1,\dots,a_n)=\int_0^1\left(\frac{dt}{t}\right)^{k_1-1}\frac{\omega^{a_1}dt}{1-\omega^{a_1}t}\cdots\left(\frac{dt}{t}\right)^{k_n-1}\frac{\omega^{a_n}dt}{1-\omega^{a_n}t},$$

where for one forms $\omega_i=f_i(t)dt$, we define

$$\int_0^1\omega_1\omega_2\cdots\omega_k=\int\limits_{1>t_1>t_2>\cdots>t_k>0}f_1(t_1)f_2(t_2)\cdots f_k(t_k)dt_1dt_2\cdots dt_k.$$

If $N=1$, the multiple L-values become multiple zeta values

$$\zeta(k_1,\dots,k_n)=\sum_{m_1>\cdots>m_n>0}\frac{1}{m_1^{k_1}\cdots m_n^{k_n}},$$

where $k_1,\dots,k_n\in\mathbb{N}$ with $k_1\geq 2$. If $N=2$, then $R=\{0,1\}$ and $\omega=-1$. Therefore the multiple L-values are just the alternating multiple zeta values:

$$L_{2,*}(k_1,\dots,k_n;a_1,\dots,a_n)=\sum_{m_1>\cdots>m_n>0}\frac{(-1)^{a_1m_1+\cdots+a_nm_n}}{m_1^{k_1}\cdots m_n^{k_n}}.$$

As usual, the above value is denoted by $\zeta^{(2)}\left(\boxed{k_1},\dots,\boxed{k_n}\right)$, where

$$\boxed{k_i}=\begin{cases}k_i & \text{if } a_i=0,\\ \overline{k_i} & \text{if } a_i=1.\end{cases}$$

Hence for example, we have $L_{2,*}(k_1,k_2;0,1)=\zeta^{(2)}(k_1,\overline{k_2})$. If $N=3$, then $R=\{0,1,2\}$, and we will denote $L_{3,*}(k_1,\dots,k_n;a_1,\dots,a_n)$ by $\zeta^{(3)}\left(\boxed{k_1},\dots,\boxed{k_n}\right)$ with

$$\boxed{k_i}=\begin{cases}k_i & \text{if } a_i=0,\\ \overline{k_i} & \text{if } a_i=1,\\ \widetilde{k_i} & \text{if } a_i=2.\end{cases}$$

For example, we have $L_{3,*}(k_1,k_2,k_3;0,1,2)=\zeta^{(3)}(k_1,\overline{k_2},\widetilde{k_3})$. Note that for positive integers $k_1,\dots,k_n$ with $k_1\geq 2$,

$$L_{N,*}(k_1,\dots,k_n;0,\dots,0)=\zeta(k_1,\dots,k_n).$$

In [4], Guo and Xie gave a weighted sum formula of multiple zeta values with the help of the regularized double shuffle relations. Using the regularized double shuffle relations of multiple L-values, we obtain a sum formula and a weighted sum formula of multiple L-values in this paper.

In [9], Xu and Zhao studied a variant of multiple zeta values of level 2 (which is called multiple mixed values therein), which forms a subspace of the space of alternating multiple zeta values. This variant includes both Hoffman's multiple t-values [5] and Kaneko-Tsumura's multiple T-values [6] as special cases. The multiple zeta values of

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any level are introduced by Yuan and Zhao in [10]. For positive integers $n, k_1, k_2, \ldots, k_n$ with $k_1 \geq 2$ and $a_1, a_2, \ldots, a_n \in R$, the multiple zeta values of level N are defined by

$$\zeta_N(k_1, \ldots, k_n; a_1, \ldots, a_n) = \sum_{\substack{m_1 > \cdots > m_n > 0 \\ m_i \equiv a_i \pmod N}} \frac{N^n}{m_1^{k_1} \cdots m_n^{k_n}}.$$

We will use the same tilde/bar notation for multiple zeta values of level 2 and 3 as well. For example,

$$\zeta_2(k_1, \overline{k_2}) = \zeta_2(k_1, k_2; 0, 1), \quad \zeta_3(k_1, \overline{k_2}, \widetilde{k_3}) = \zeta_3(k_1, k_2, k_3; 0, 1, 2).$$

In this paper, we set up the algebra framework for the double shuffle relations of multiple zeta values of level N.

In Section 2, we recall the double shuffle relations of multiple L-values. In Section 3, we study the double shuffle relations of multiple zeta values of level N. In Section 4, we give a sum formula and a weighted sum formula of multiple L-values. As applications, we provide some sum formulas and weighted sum formulas of double zeta values of level 2 and level 3. In particular, we reprove the weighted sum formula of double T-values appeared in [6].

2 Double shuffle relations of multiple L-values

We recall the double shuffle relations of multiple L-values from [1]. Let $\mathcal{A} = \mathcal{A}_N = \mathbb{Q}\langle x, y_a \mid a \in R_N \rangle$ be the non-commutative polynomial algebra generated by the alphabet $\{x, y_a \mid a \in R_N\}$. Define the subalgebras

$$\mathcal{A}^1 = \mathcal{A}_N^1 = \mathbb{Q} + \sum_{a \in R_N} \mathcal{A} y_a$$

and

$$\mathcal{A}^0 = \mathcal{A}_N^0 = \mathbb{Q} + \sum_{a \in R_N} x\mathcal{A} y_a + \sum_{a,b \in R_N, b \neq 0} y_b \mathcal{A} y_a.$$

For any $k \in \mathbb{N}$ and $a \in R$, set $z_{k,a} = x^{k-1} y_a$. Define the evaluation maps $\mathcal{L}_{N,*} : \mathcal{A}^0 \longrightarrow \mathbb{C}$ and $\mathcal{L}_{N,Ш} : \mathcal{A}^0 \longrightarrow \mathbb{C}$ by $\mathbb{Q}$-linearities and

$$\mathcal{L}_{N,*}(z_{k_1,a_1} \cdots z_{k_n,a_n}) = L_{N,*}(k_1, \ldots, k_n; a_1, \ldots, a_n),$$
$$\mathcal{L}_{N,Ш}(z_{k_1,a_1} \cdots z_{k_n,a_n}) = L_{N,Ш}(k_1, \ldots, k_n; a_1, \ldots, a_n).$$

Let $\mathcal{I}_N : \mathcal{A}^1 \longrightarrow \mathcal{A}^1$ be the $\mathbb{Q}$-linear endomorphism of $\mathcal{A}^1$ determined by

$$\mathcal{I}_N(z_{k_1,a_1} z_{k_2,a_2} \cdots z_{k_n,a_n}) = z_{k_1,a_1} z_{k_2,a_1+a_2} \cdots z_{k_n,a_1+\cdots+a_n}.$$

The linear map $\mathcal{I}_N$ is invertible, and the inverse $\mathcal{I}_N^{-1} : \mathcal{A}^1 \longrightarrow \mathcal{A}^1$ satisfies

$$\mathcal{I}_N^{-1}(z_{k_1,a_1} z_{k_2,a_2} \cdots z_{k_n,a_n}) = z_{k_1,a_1} z_{k_2,a_2-a_1} \cdots z_{k_n,a_n-a_{n-1}}.$$

Hence we have $\mathcal{L}_{N,*} = \mathcal{L}_{N,Ш} \circ \mathcal{I}_N$, or equivalently $\mathcal{L}_{N,Ш} = \mathcal{L}_{N,*} \circ \mathcal{I}_N^{-1}$. More precisely, for any $w \in \mathcal{A}^0$, we have

$$\mathcal{L}_{N,*}(w) = \mathcal{L}_{N,Ш}(\mathcal{I}_N(w)) \quad \text{and} \quad \mathcal{L}_{N,Ш}(w) = \mathcal{L}_{N,*}\left(\mathcal{I}_N^{-1}(w)\right).$$

The harmonic shuffle product $*$ on $\mathcal{A}^1$ is defined by $\mathbb{Q}$-bilinearity and the rules

$$
\begin{aligned}
&1 * w = w * 1 = w,\\
&z_{k,a}w_1 * z_{l,b}w_2 = z_{k,a}(w_1 * z_{l,b}w_2) + z_{l,b}(z_{k,a}w_1 * w_2) + z_{k+l,a+b}(w_1 * w_2),
\end{aligned}
$$

for all $k, l \geq 1$, $a, b \in R_N$, and any words $w, w_1, w_2 \in \mathcal{A}^1$. The harmonic shuffle product $*$ is associative and commutative. Hence we get the commutative algebra $(\mathcal{A}^1, *)$ and its subalgebra $(\mathcal{A}^0, *)$, which are denoted by $\mathcal{A}^1_*$ and $\mathcal{A}^0_*$, respectively. The shuffle product $ш$ on $\mathcal{A}$ is defined by $\mathbb{Q}$-bilinearity and the rules

$$
\begin{aligned}
&1 \,ш\, w = w \,ш\, 1 = w,\\
&uw_1 \,ш\, vw_2 = u(w_1 \,ш\, vw_2) + v(uw_1 \,ш\, w_2),
\end{aligned}
$$

for any words $w, w_1, w_2 \in \mathcal{A}$ and $u, v \in \{x, y_a \mid a \in R_N\}$. Then we have the commutative algebra $\mathcal{A}_ш$ and its subalgebras $\mathcal{A}^1_ш$ and $\mathcal{A}^0_ш$. For any $w_1, w_2 \in \mathcal{A}^0$, we have

$$\mathcal{L}_{N,*}(w_1 * w_2) = \mathcal{L}_{N,*}(w_1)\mathcal{L}_{N,*}(w_2) \quad \text{and} \quad \mathcal{L}_{N,ш}(w_1 * w_2) = \mathcal{L}_{N,ш}(w_1)\mathcal{L}_{N,ш}(w_2),$$

which induce the finite double shuffle relations

$$\mathcal{L}_{N,ш}\left(\mathcal{I}_N(w_1) \,ш\, \mathcal{I}_N(w_2) - \mathcal{I}_N(w_1 * w_2)\right) = 0, \quad (w_1, w_2 \in \mathcal{A}^0),$$

or equivalently

$$\mathcal{L}_{N,*}\left(\mathcal{I}_N^{-1}(w_1) * \mathcal{I}_N^{-1}(w_2) - \mathcal{I}_N^{-1}(w_1 \,ш\, w_2)\right) = 0, \quad (w_1, w_2 \in \mathcal{A}^0).$$

Notice that $\mathcal{A}^1_* = \mathcal{A}^0_*[y_0]$, then for any $w \in \mathcal{A}^1$, we can write w uniquely as

$$w = w_0 + w_1 * y_0 + \cdots + w_n * y_0^{*n},$$

where $w_0, w_1, \ldots, w_n \in \mathcal{A}^0$. Also, since $\mathcal{A}^1_ш = \mathcal{A}^0_ш[y_0]$, we can write w uniquely as

$$w = w'_0 + w'_1 \,ш\, y_0 + \cdots + w'_{n'} \,ш\, y_0^{ш n'},$$

where $w'_0, w'_1, \ldots, w'_{n'} \in \mathcal{A}^0$. Then one can define the regularization maps $\mathrm{reg}_* : \mathcal{A}^1_* \longrightarrow \mathcal{A}^0_*$ and $\mathrm{reg}_ш : \mathcal{A}^1_ш \longrightarrow \mathcal{A}^0_ш$ by $\mathrm{reg}_*(w) = w_0$ and $\mathrm{reg}_ш(w) = w'_0$ respectively. It is easy to see that the maps reg_* and $\mathrm{reg}_ш$ are algebraic morphisms. Hence we have the regularized double shuffle relations

$$\mathcal{L}_{N,ш}\left(\mathrm{reg}_ш\left(\mathcal{I}_N(w_0 * w_1) - \mathcal{I}_N(w_0) \,ш\, \mathcal{I}_N(w_1)\right)\right) = 0, \quad (w_0 \in \mathcal{A}^0, w_1 \in \mathcal{A}^1)$$

and

$$\mathcal{L}_{N,*}\left(\mathrm{reg}_*\left(\mathcal{I}_N^{-1}(w_0 \,ш\, w_1) - \mathcal{I}_N^{-1}(w_0) * \mathcal{I}_N^{-1}(w_1)\right)\right) = 0, \quad (w_0 \in \mathcal{A}^0, w_1 \in \mathcal{A}^1).$$

3 Double shuffle relations of multiple zeta values of level N

3.1 Multiple zeta values of level N

Recall that $\omega = \omega_N = \exp(2\pi i/N)$ is a fixed primitive Nth root of unity. The following lemma indicates that the multiple zeta values of level N can be expressed in terms of multiple L-values.

Lemma 3.1. *For positive integers $n, k_1, k_2, \ldots, k_n$ with $k_1 \geq 2$ and $a_1, a_2, \ldots, a_n \in R$, we have*

$$\begin{aligned}&\zeta_N(k_1, \ldots, k_n; a_1, \ldots, a_n)\\ &= \sum_{m_1 > m_2 > \cdots > m_n > 0} \frac{\prod_{i=1}^n (1 + \omega^{m_i - a_i} + \omega^{2(m_i - a_i)} + \cdots + \omega^{(N-1)(m_i - a_i)})}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}}.\end{aligned}$$

Proof. As

$$1 + \omega^r + \omega^{2r} + \cdots + \omega^{(N-1)r} = \begin{cases} N & \text{if } N \mid r, \\ 0 & \text{if } N \nmid r, \end{cases}$$

we get the result. □

Using the series representations, we get the harmonic shuffle structure of the multiple zeta values of level N as displayed in the following simple example:

$$\begin{aligned}\zeta_N(k;a)\zeta_N(l;b) &= \sum_{\substack{m=1\\ m\equiv a \pmod N}}^{\infty} \sum_{\substack{n=1\\ n\equiv b \pmod N}}^{\infty} \frac{N^2}{m^k n^l}\\ &= \sum_{\substack{m>n>0\\ m\equiv a, n\equiv b \pmod N}} \frac{N^2}{m^k n^l} + \sum_{\substack{n>m>0\\ m\equiv a, n\equiv b \pmod N}} \frac{N^2}{n^l m^k} + \delta_{a,b} \sum_{\substack{m=1\\ m\equiv a \pmod N}}^{\infty} \frac{N^2}{m^{k+l}}\\ &= \zeta_N(k,l;a,b) + \zeta_N(l,k;b,a) + \delta_{a,b} N \zeta_N(k+l;a),\end{aligned}$$

where $k, l \geq 2$, $a, b \in R$ and $\delta_{a,b}$ is the Kronecker symbol.

To study the shuffle structure among the multiple zeta values of level N, we introduce a map $r : \mathbb{Z} \longrightarrow \{1, 2, \ldots, N\}$, which is defined by

$$r(a) \equiv a \pmod N \quad \text{and} \quad r(a) \in \{1, 2, \ldots, N\}$$

for any $a \in \mathbb{Z}$. We also define one forms

$$\Omega_0 = \frac{dt}{t}, \quad \Omega_a = \frac{N t^{a-1} dt}{1 - t^N},$$

where $a \in \{1, 2, \ldots, N\}$. Then we have the iterated integral representation.

Lemma 3.2. *Let $k_1, k_2, \ldots, k_n$ be positive integers with $k_1 \geq 2$.*

(1) *For any $b_1, \ldots, b_n \in \{1, 2, \ldots, N\}$, we have*

$$\int_0^1 \Omega_0^{k_1 - 1} \Omega_{b_1} \cdots \Omega_0^{k_n - 1} \Omega_{b_n} = \zeta_N(k_1, \ldots, k_n; b_1 + \cdots + b_n, \ldots, b_{n-1} + b_n, b_n).$$

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(2) *For any* $a_1, a_2, \cdots, a_n \in R$, *we have*

$$\zeta_N(k_1,\ldots,k_n;a_1,\ldots,a_n)=\int_0^1 \Omega_0^{k_1-1}\Omega_{r(a_1-a_2)}\Omega_0^{k_2-1}\Omega_{r(a_2-a_3)}\cdots \\ \times \Omega_0^{k_{n-1}-1}\Omega_{r(a_{n-1}-a_n)}\Omega_0^{k_n-1}\Omega_{r(a_n)}.$$

Proof. It is sufficient to prove (1). Note that the depth one case was already given in [10]. As

$$\int_0^t \frac{t^{b_n-1}dt}{1-t^N}=\sum_{l=0}^\infty \int_0^t t^{lN+b_n-1}dt=\sum_{l=0}^\infty \frac{t^{lN+b_n}}{lN+b_n},$$

we get

$$\int_0^t \left(\frac{dt}{t}\right)^{k_n-1}\frac{t^{b_n-1}dt}{1-t^N}=\sum_{l=0}^\infty \frac{t^{lN+b_n}}{(lN+b_n)^{k_n}}. \tag{3.1}$$

Similarly, as

$$\begin{aligned}\int_0^t \frac{t^{b_{n-1}-1}dt}{1-t^N}\left(\frac{dt}{t}\right)^{k_n-1}\frac{t^{b_n-1}dt}{1-t^N}&=\sum_{l_2=0}^\infty \frac{1}{(l_2N+b_n)^{k_n}}\sum_{l_1=0}^\infty\int_0^t t^{(l_1+l_2)N+b_{n-1}+b_n-1}dt\\ &=\sum_{l_1,l_2\geq 0}\frac{t^{(l_1+l_2)N+b_{n-1}+b_n}}{(l_2N+b_n)^{k_n}((l_1+l_2)N+b_{n-1}+b_n)},\end{aligned}$$

we find

$$\int_0^t \left(\frac{dt}{t}\right)^{k_{n-1}-1}\frac{t^{b_{n-1}-1}dt}{1-t^N}\left(\frac{dt}{t}\right)^{k_n-1}\frac{t^{b_n-1}dt}{1-t^N} \\ =\sum_{l_1,l_2\geq 0}\frac{t^{(l_1+l_2)N+b_{n-1}+b_n}}{(l_2N+b_n)^{k_n}((l_1+l_2)N+b_{n-1}+b_n)^{k_{n-1}}}.$$

Then one can easily get the result by induction. □

3.2 Algebraic setup

Let $\mathcal{U}=\mathcal{U}_N=\mathbb{Q}\langle x_0,x_1,\ldots,x_N\rangle$ be the non-commutative polynomial algebra generated by the alphabet $\{x_a \mid a=0,1,\ldots,N\}$. Define the subalgebras

$$\mathcal{U}^1=\mathcal{U}_N^1=\mathbb{Q}+\sum_{a=1}^N \mathcal{U}x_a$$

spanned by words not ending in x_0 and

$$\mathcal{U}^0=\mathcal{U}_N^0=\mathbb{Q}+\sum_{a=1}^N x_0\mathcal{U}x_a$$

spanned by words beginning with x_0 and not ending in x_0. We set $y_{k,a}=x_0^{k-1}x_a$, where $k\in\mathbb{N}$ and $a\in\{1,2,\ldots,N\}$.

We define the $\mathbb{Q}$-linear map (called the evaluation map) $\zeta_N : \mathcal{U}^0 \longrightarrow \mathbb{R}$ by $\zeta_N(1_x) = 1$ and

$$\zeta_N(y_{k_1,a_1}\cdots y_{k_n,a_n}) = \zeta_N(k_1,\ldots,k_n;a_1,\ldots,a_n),$$

where 1_x is the empty word, $k_1,\ldots,k_n \in \mathbb{N}$, $k_1 \geq 2$ and $a_1,\cdots,a_n \in \{1,2,\cdots,N\}$.

We define the stuffle product $*$ on $\mathcal{U}^1$ by $\mathbb{Q}$-bilinearity and the rules:

$$\begin{aligned}
&1_x * w = w = w * 1_x,\\
&y_{k,a}w_1 * y_{l,b}w_2 = y_{k,a}(w_1 * y_{l,b}w_2) + y_{l,b}(y_{k,a}w_1 * w_2) + \delta_{a,b}Ny_{k+l,a}(w_1 * w_2),
\end{aligned}$$

where w, w_1, w_2 are words in $\mathcal{U}^1$, $k, l \in \mathbb{N}$, and $a, b \in \{1,2,\cdots,N\}$. The stuffle product $*$ is commutative and associative. Therefore $\mathcal{U}^1$ is a commutative $\mathbb{Q}$-algebra with respect to $*$. We denote it by $\mathcal{U}^1_*$. The subspace $\mathcal{U}^0$ is a subalgebra of $\mathcal{U}^1_*$ and we denote it by $\mathcal{U}^0_*$. Then from the infinite series representations of multiple zeta values of level N, we have the following result.

Proposition 3.3. *The map $\zeta_N : \mathcal{U}^0_* \longrightarrow \mathbb{R}$ is an algebra homomorphism. More precisely, for any $w_1, w_2 \in \mathcal{U}^0$, we have*

$$\zeta_N(w_1 * w_2) = \zeta_N(w_1)\zeta_N(w_2).$$

The shuffle product ш on $\mathcal{U}$ is defined by $\mathbb{Q}$-bilinearity and the rules

$$\begin{aligned}
&1 \,\text{ш}\, w = w \,\text{ш}\, 1 = w,\\
&uw_1 \,\text{ш}\, vw_2 = u(w_1 \,\text{ш}\, vw_2) + v(uw_1 \,\text{ш}\, w_2),
\end{aligned}$$

where w, w_1, w_2 are words in $\mathcal{U}$ and $u, v \in \{x_a \mid a = 0,1,\ldots,N\}$. Then we have the commutative algebra $\mathcal{U}_{\text{ш}}$ and its subalgebras $\mathcal{U}^1_{\text{ш}}$ and $\mathcal{U}^0_{\text{ш}}$.

Let $\mathcal{J}_N$ be the $\mathbb{Q}$-linear endomorphism of $\mathcal{U}^1$ determined by

$$\mathcal{J}_N(y_{k_1,a_1}\cdots y_{k_n,a_n}) = y_{k_1,r(a_1-a_2)}y_{k_2,r(a_2-a_3)}\cdots y_{k_{n-1},r(a_{n-1}-a_n)}y_{k_n,r(a_n)},$$

where $k_1,\ldots,k_n \in \mathbb{N}$ and $a_1,\ldots,a_n \in \{1,2,\cdots,N\}$. It is obvious that $\mathcal{J}_N$ is invertible, and its inverse $\mathcal{J}_N^{-1}$ satisfies

$$\mathcal{J}_N^{-1}(y_{k_1,a_1}\cdots y_{k_n,a_n}) = y_{k_1,r(a_1+\cdots+a_n)}y_{k_2,r(a_2+\cdots+a_n)}\cdots y_{k_{n-1},r(a_{n-1}+a_n)}y_{k_n,r(a_n)}.$$

Then from the iterated integral representations of multiple zeta values of level N, we have the following result.

Proposition 3.4. *For any $w_1, w_2 \in \mathcal{U}^0$, we have*

$$\zeta_N(\mathcal{J}_N^{-1}(w_1 \,\text{ш}\, w_2)) = \zeta_N(\mathcal{J}_N^{-1}(w_1))\zeta_N(\mathcal{J}_N^{-1}(w_2)).$$

Finally, we get the finite double shuffle relations of multiple zeta values of level N.

Theorem 3.5 (Finite double shuffle relation). *For any $w_1, w_2 \in \mathcal{U}^0$, we have*

$$\zeta_N\left(\mathcal{J}_N^{-1}(w_1) * \mathcal{J}_N^{-1}(w_2) - \mathcal{J}_N^{-1}(w_1 \,\text{ш}\, w_2)\right) = 0.$$

4 Sum formulas and weighted sum formulas

In this section, using the regularized double shuffle relations, we derive a sum formula and a weighted sum formula of multiple L-values. As applications, we (re)obtain some sum formulas and weighted sum formulas of double zeta values of level 2 and level 3.

4.1 Sum and weighted sum formulas of multiple L-values

Let N be a positive integer. We first compute the stuffle products.

Lemma 4.1. *For positive integers k, n with $k \geq n+1$, $n \geq 2$, and $a, a_1, \ldots, a_{n-1} \in R$, we have*

$$\begin{aligned}
&\sum_{\substack{k_1+\cdots+k_{n-1}=k-1\\ k_j\geq 1, k_1\geq 2}} \mathcal{I}_N^{-1}(z_{1,a}) * \mathcal{I}_N^{-1}\left(z_{k_1,a_1}\cdots z_{k_{n-1},a_{n-1}}\right)\\
={}&\sum_{\substack{k_1+\cdots+k_{n-1}=k-1\\ k_j\geq 1, k_1\geq 2}} z_{1,a} z_{k_1,a_1} z_{k_2,a_2-a_1}\cdots z_{k_{n-1},a_{n-1}-a_{n-2}}\\
&+\sum_{i=2}^{n}\sum_{\substack{k_1+\cdots+k_n=k\\ k_j\geq 1, k_1\geq 2, k_i=1}} z_{k_1,a_1}\cdots z_{k_{i-1},a_{i-1}-a_{i-2}} z_{k_i,a} z_{k_{i+1}.a_i-a_{i-1}}\cdots z_{k_n,a_{n-1}-a_{n-2}}\\
&+\sum_{\substack{k_1+\cdots+k_{n-1}=k\\ k_j\geq 1, k_1\geq 3}} z_{k_1,a+a_1} z_{k_2,a_2-a_1}\cdots z_{k_{n-1},a_{n-1}-a_{n-2}}\\
&+\sum_{i=2}^{n-1}\sum_{\substack{k_1+\cdots+k_{n-1}=k\\ k_j\geq 1, k_1,k_i\geq 2}} z_{k_1,a_1}\cdots z_{k_{i-1},a_{i-1}-a_{i-2}} z_{k_i,a+a_i-a_{i-1}} z_{k_{i+1},a_{i+1}-a_i}\cdots z_{k_{n-1},a_{n-1}-a_{n-2}}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{\substack{l+k_1+\cdots+k_{n-1}=k\\ l,k_j\geq 1, k_1\geq 2}} \mathcal{I}_N^{-1}(z_{l,a}) * \mathcal{I}_N^{-1}(z_{k_1,a_1}\cdots z_{k_{n-1},a_{n-1}})\\
={}&\sum_{\substack{k_1+\cdots+k_n=k\\ k_j\geq 1, k_2\geq 2}} z_{k_1,a} z_{k_2,a_1} z_{k_3,a_2-a_1}\cdots z_{k_n,a_{n-1}-a_{n-2}}\\
&+\sum_{i=1}^{n-1}\sum_{\substack{k_1+\cdots+k_n=k\\ k_j\geq 1, k_1\geq 2}} z_{k_1,a_1} z_{k_2,a_2-a_1}\cdots z_{k_i,a_i-a_{i-1}} z_{k_{i+1},a} z_{k_{i+2},a_{i+1}-a_i}\cdots z_{k_n,a_{n-1}-a_{n-2}}\\
&+\sum_{\substack{k_1+\cdots+k_{n-1}=k\\ k_j\geq 1, k_1\geq 2}} (k_1-2) z_{k_1,a+a_1} z_{k_2,a_2-a_1}\cdots z_{k_{n-1},a_{n-1}-a_{n-2}}\\
&+\sum_{i=2}^{n-1}\sum_{\substack{k_1+\cdots+k_{n-1}=k\\ k_j\geq 1, k_1\geq 2}} (k_i-1) z_{k_1,a_1} z_{k_2,a_2-a_1}\cdots z_{k_{i-1},a_{i-1}-a_{i-2}} z_{k_i,a+a_i-a_{i-1}}\\
&\qquad\times z_{k_{i+1},a_{i+1}-a_i}\cdots z_{k_{n-1},a_{n-1}-a_{n-2}}.
\end{aligned}$$

Proof. As

$$
\begin{aligned}
&\mathcal{I}_N^{-1}(z_{l,a}) * \mathcal{I}_N^{-1}(z_{k_1,a_1}\cdots z_{k_{n-1},a_{n-1}}) = z_{l,a} * z_{k_1,a_1} z_{k_2,a_2-a_1}\cdots z_{k_{n-1},a_{n-1}-a_{n-2}}\\
=&\sum_{i=0}^{n-1} z_{k_1,a_1}z_{k_2,a_2-a_1}\cdots z_{k_i,a_i-a_{i-1}}z_{l,a}z_{k_{i+1},a_{i+1}-a_i}\cdots z_{k_{n-1},a_{n-1}-a_{n-2}}\\
&+\sum_{i=1}^{n-1} z_{k_1,a_1}z_{k_2,a_2-a_1}\cdots z_{k_{i-1},a_{i-1}-a_{i-2}}z_{l+k_i,a+a_i-a_{i-1}}z_{k_{i+1},a_{i+1}-a_i}\cdots z_{k_{n-1},a_{n-1}-a_{n-2}},
\end{aligned}
$$

we get the result. $\square$

For shuffle products, we have

Lemma 4.2. *For positive integers k, n with $k \geq n+1$ and $n \geq 2$, $a, a_1, \ldots, a_{n-1} \in R$, we have*

$$
\begin{aligned}
&\sum_{\substack{k_1+\cdots+k_{n-1}=k-1\\ k_j\geq 1, k_1\geq 2}} z_{1,a} \,ш\, z_{k_1,a_1}\cdots z_{k_{n-1},a_{n-1}}\\
=&\sum_{\substack{k_1+\cdots+k_n=k\\ k_j\geq 1, k_1+k_2\geq 3}} z_{k_1,a}z_{k_2,a_1}\cdots z_{k_n,a_{n-1}} + \sum_{\substack{k_1+\cdots+k_{n-1}=k-1\\ k_j\geq 1, k_1\geq 2}} z_{k_1,a_1}\cdots z_{k_{n-1},a_{n-1}}z_{1,a}\\
&+\sum_{i=2}^{n-1}\sum_{\substack{k_1+\cdots+k_n=k\\ k_j\geq 1, k_1\geq 2}} z_{k_1,a_1}z_{k_2,a_2}\cdots z_{k_{i-1},a_{i-1}}z_{k_i,a}z_{k_{i+1},a_i}\cdots z_{k_n,a_{n-1}}
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{\substack{l+k_1+\cdots+k_{n-1}=k\\ l,k_j\geq 1, k_1\geq 2}} z_{l,a} \,ш\, z_{k_1,a_1}\cdots z_{k_{n-1},a_{n-1}}\\
=&\sum_{\substack{k_1+\cdots+k_n=k\\ k_j\geq 1, k_2\geq 2}} 2^{k_1-1}z_{k_1,a}z_{k_2,a_1}\cdots z_{k_n,a_{n-1}} + \sum_{\substack{k_1+\cdots+k_n=k\\ k_j\geq 1, k_2=1}} (2^{k_1-1}-1)z_{k_1,a}z_{k_2,a_1}\cdots z_{k_n,a_{n-1}}\\
&+\sum_{i=2}^{n-1}\sum_{\substack{k_1+\cdots+k_n=k\\ k_j\geq 1}} (2^{k_1+\cdots+k_i-i}-2^{k_2+\cdots+k_i-(i-1)})z_{k_1,a_1}\cdots z_{k_{i-1},a_{i-1}}z_{k_i,a}z_{k_{i+1},a_i}\cdots z_{k_n,a_{n-1}}\\
&+\sum_{\substack{k_1+\cdots+k_n=k\\ k_j\geq 1}} (2^{k_1+\cdots+k_{n-1}-(n-1)}-2^{k_2+\cdots+k_{n-1}-(n-2)})z_{k_1,a_1}\cdots z_{k_{n-1},a_{n-1}}z_{k_n,a}.
\end{aligned}
$$

Proof. As

$$
\begin{aligned}
&z_{1,a} \,ш\, z_{k_1,a_1}\cdots z_{k_{n-1},a_{n-1}}\\
=&\sum_{j=1}^{k_1} z_{j,a}z_{k_1+1-j,a_1}z_{k_2,a_2}\cdots z_{k_{n-1},a_{n-1}}\\
&+\sum_{i=2}^{n-1}\sum_{j=1}^{k_i} z_{k_1,a_1}z_{k_2,a_2}\cdots z_{k_{i-1},a_{i-1}}z_{j,a}z_{k_i+1-j,a_i}z_{k_{i+1},a_{i+1}}\cdots z_{k_{n-1},a_{n-1}}\\
&+z_{k_1,a_1}\cdots z_{k_{n-1},a_{n-1}}z_{1,a},
\end{aligned}
$$

we get the first equation.

In general, similarly as in [7], we have

$$\begin{aligned}
&z_{l,a} ш z_{k_1,a_1}\cdots z_{k_{n-1},a_{n-1}}\\
=&\sum_{i=1}^{n-1}\sum_{\substack{\alpha_1+\cdots+\alpha_{i+1}\\=l+k_1+\cdots+k_i,\alpha_j\geq 1}}\prod_{j=1}^{i-1}\binom{\alpha_j-1}{k_j-1}\binom{\alpha_i-1}{k_i-\alpha_{i+1}}z_{\alpha_1,a_1}\cdots z_{\alpha_{i-1},a_{i-1}}z_{\alpha_i,a}z_{\alpha_{i+1},a_i}\\
&\qquad\times z_{k_{i+1},a_{i+1}}\cdots z_{k_{n-1},a_{n-1}}\\
&+\sum_{\substack{\alpha_1+\cdots+\alpha_n\\=l+k_1+\cdots+k_{n-1},\alpha_j\geq 1}}\prod_{j=1}^{n-1}\binom{\alpha_j-1}{k_j-1}z_{\alpha_1,a_1}\cdots z_{\alpha_{n-1},a_{n-1}}z_{\alpha_n,a}.
\end{aligned}$$

Hence we get

$$\sum_{\substack{l+k_1+\cdots+k_{n-1}=k\\ l,k_j\geq 1,k_1\geq 2}} z_{l,a} ш z_{k_1,a_1}\cdots z_{k_{n-1},a_{n-1}}=S_1+S_2+S_3,$$

where

$$\begin{aligned}
S_1=&\sum_{\substack{l+k_1+\cdots+k_{n-1}=k\\ l,k_j\geq 1,k_1\geq 2}}\sum_{\substack{\alpha_1+\alpha_2=l+k_1\\ \alpha_j\geq 1}}\binom{\alpha_1-1}{k_1-\alpha_2}z_{\alpha_1,a}z_{\alpha_2,a_1}z_{k_2,a_2}\cdots z_{k_{n-1},a_{n-1}},\\
S_2=&\sum_{i=2}^{n-1}\sum_{\substack{l+k_1+\cdots+k_{n-1}=k\\ l,k_j\geq 1,k_1\geq 2}}\sum_{\substack{\alpha_1+\cdots+\alpha_{i+1}\\=l+k_1+\cdots+k_i,\alpha_j\geq 1}}\prod_{j=1}^{i-1}\binom{\alpha_j-1}{k_j-1}\binom{\alpha_i-1}{k_i-\alpha_{i+1}}z_{\alpha_1,a_1}\cdots z_{\alpha_{i-1},a_{i-1}}\\
&\qquad\times z_{\alpha_i,a}z_{\alpha_{i+1},a_i}z_{k_{i+1},a_{i+1}}\cdots z_{k_{n-1},a_{n-1}}\\
S_3=&\sum_{\substack{l+k_1+\cdots+k_{n-1}=k\\ l,k_j\geq 1,k_1\geq 2}}\sum_{\substack{\alpha_1+\cdots+\alpha_n\\=l+k_1+\cdots+k_{n-1},\alpha_j\geq 1}}\prod_{j=1}^{n-1}\binom{\alpha_j-1}{k_j-1}z_{\alpha_1,a_1}\cdots z_{\alpha_{n-1},a_{n-1}}z_{\alpha_n,a}.
\end{aligned}$$

For S_1, we have

$$S_1=\sum_{\substack{\alpha_1+\alpha_2+k_2+\cdots+k_{n-1}=k\\ \alpha_j,k_p\geq 1}}\sum_{\substack{k_1\geq 2\\ k_1\geq\alpha_2}}\binom{\alpha_1-1}{k_1-\alpha_2}z_{\alpha_1,a}z_{\alpha_2,a_1}z_{k_2,a_2}\cdots z_{k_{n-1},a_{n-1}}.$$

If $\alpha_2=1$, we get

$$\sum_{\substack{k_1\geq 2\\ k_1\geq\alpha_2}}\binom{\alpha_1-1}{k_1-\alpha_2}=\sum_{k_1\geq 2}\binom{\alpha_1-1}{k_1-1}=2^{\alpha_1-1}-1.$$

While if $\alpha_2\geq 2$, we have

$$\sum_{\substack{k_1\geq 2\\ k_1\geq\alpha_2}}\binom{\alpha_1-1}{k_1-\alpha_2}=\sum_{k_1\geq\alpha_2}\binom{\alpha_1-1}{k_1-\alpha_2}=2^{\alpha_1-1}.$$

Hence we find

$$S_1=\sum_{\substack{k_1+\cdots+k_n=k\\ k_j\geq 1,k_2\geq 2}}2^{k_1-1}z_{k_1,a}z_{k_2,a_1}\cdots z_{k_n,a_{n-1}}+\sum_{\substack{k_1+\cdots+k_n=k\\ k_j\geq 1,k_2=1}}(2^{k_1-1}-1)z_{k_1,a}z_{k_2,a_1}\cdots z_{k_n,a_{n-1}}.$$

For S_2, we have

$$S_2=\sum_{i=2}^{n-1}\sum_{\substack{\alpha_1+\cdots+\alpha_{i+1}+k_{i+1}\\ +\cdots+k_{n-1}=k,\alpha_j,k_p\geq 1}}\sum_{k_1=2}^{\alpha_1}\binom{\alpha_1-1}{k_1-1}\prod_{j=2}^{i-1}\sum_{k_j=1}^{\alpha_j}\binom{\alpha_j-1}{k_j-1}\sum_{k_i=\alpha_{i+1}}^{\alpha_i+\alpha_{i+1}-1}\binom{\alpha_i-1}{k_i-\alpha_{i+1}}$$
$$\times z_{\alpha_1,a_1}\cdots z_{\alpha_{i-1},a_{i-1}}z_{\alpha_i,a}z_{\alpha_{i+1},a_i}z_{k_{i+1},a_{i+1}}\cdots z_{k_{n-1},a_{n-1}}.$$

Since

$$\sum_{k_1=2}^{\alpha_1}\binom{\alpha_1-1}{k_1-1}=2^{\alpha_1-1}-1,\quad \sum_{k_j=1}^{\alpha_j}\binom{\alpha_j-1}{k_j-1}=2^{\alpha_j-1},\quad \sum_{k_i=\alpha_{i+1}}^{\alpha_i+\alpha_{i+1}-1}\binom{\alpha_i-1}{k_i-\alpha_{i+1}}=2^{\alpha_i-1},$$

we find

$$S_2=\sum_{i=2}^{n-1}\sum_{\substack{k_1+\cdots+k_n=k\\ k_j\geq 1}}(2^{k_1+\cdots+k_i-i}-2^{k_2+\cdots+k_i-(i-1)})z_{k_1,a_1}\cdots z_{k_{i-1},a_{i-1}}z_{k_i,a}z_{k_{i+1},a_i}\cdots z_{k_n,a_{n-1}}.$$

Similarly, for S_3, we have

$$\begin{aligned}S_3&=\sum_{\substack{\alpha_1+\cdots+\alpha_n=k\\ \alpha_j\geq 1}}\sum_{k_1=2}^{\alpha_1}\binom{\alpha_1-1}{k_1-1}\prod_{j=2}^{n-1}\sum_{k_j=1}^{\alpha_j}\binom{\alpha_j-1}{k_j-1}z_{\alpha_1,a_1}\cdots z_{\alpha_{n-1},a_{n-1}}z_{\alpha_n,a}\\ &=\sum_{\substack{k_1+\cdots+k_n=k\\ k_j\geq 1}}(2^{k_1+\cdots+k_{n-1}-(n-1)}-2^{k_2+\cdots+k_{n-1}-(n-2)})z_{k_1,a_1}\cdots z_{k_{n-1},a_{n-1}}z_{k_n,a}.\end{aligned}$$

Then we get the desired result. □

Lemma 4.1 and Lemma 4.2 induce the following sum formula and weighted sum formula.

Theorem 4.3. *For positive integers k,n with $k\geq n+1$, $n\geq 2$, and $a,a_1,\ldots,a_{n-1}\in R$, we have*

$$\sum_{\substack{k_1+\cdots+k_n=k\\ k_j\geq 1,k_1\geq 2}}L_{N,*}(k_1,\ldots,k_n;a,a_1-a,a_2-a_1,\ldots,a_{n-1}-a_{n-2})$$
$$+\sum_{i=2}^{n-1}\sum_{\substack{k_1+\cdots+k_n=k\\ k_j\geq 1,k_1\geq 2}}L_{N,*}(k_1,\ldots,k_n;a_1,a_2-a_1,\ldots,a_{i-1}-a_{i-2},$$
$$a-a_{i-1},a_i-a,a_{i+1}-a_i,\ldots,a_{n-1}-a_{n-2})$$
$$+\sum_{\substack{k_1+\cdots+k_n=k\\ k_1\geq 2,k_n=1}}L_{N,*}(k_1,\ldots,k_n;a_1,a_2-a_1,\ldots,a_{n-1}-a_{n-2},a-a_{n-1})$$

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$$= \sum_{\substack{k_1+\cdots+k_n=k\\ k_1=1,k_2\geq 2}} \mathcal{L}_{N,*}\left(z_{k_1,a}z_{k_2,a_1}z_{k_3,a_2-a_1}\cdots z_{k_n,a_{n-1}-a_{n-2}} - z_{k_1,a}z_{k_2,a_1-a}z_{k_3,a_2-a_1}\cdots z_{k_n,a_{n-1}-a_{n-2}}\right)$$
$$+\sum_{i=2}^{n}\sum_{\substack{k_1+\cdots+k_n=k\\ k_j\geq 1,k_1\geq 2,k_i=1}} L_{N,*}(k_1,\ldots,k_n;a_1,a_2-a_1,\ldots,a_{i-1}-a_{i-2},a,a_i-a_{i-1},\ldots,a_{n-1}-a_{n-2})$$
$$+\sum_{\substack{k_1+\cdots+k_{n-1}=k\\ k_j\geq 1,k_1\geq 3}} L_{N,*}(k_1,\ldots,k_{n-1};a+a_1,a_2-a_1,\ldots,a_{n-1}-a_{n-2})$$
$$+\sum_{i=2}^{n-1}\sum_{\substack{k_1+\cdots+k_{n-1}=k\\ k_j\geq 1,k_1,k_i\geq 2}} L_{N,*}(k_1,\ldots,k_{n-1};a_1,a_2-a_1,\ldots,a_{i-1}-a_{i-2},$$
$$a+a_i-a_{i-1},a_{i+1}-a_i,\ldots,a_{n-1}-a_{n-2}).$$

Theorem 4.4. *For positive integers k,n with $k\geq n+1$ and $n\geq 2$, $a,a_1,\ldots,a_{n-1}\in R$, we have*

$$\sum_{\substack{k_1+\cdots+k_n=k\\ k_j\geq 1,k_2\geq 2}} \mathcal{L}_{N,*}\left(2^{k_1-1}z_{k_1,a}z_{k_2,a_1-a}z_{k_3,a_2-a_1}\cdots z_{k_n,a_{n-1}-a_{n-2}}\right.$$
$$\left.-z_{k_1,a}z_{k_2,a_1}z_{k_3,a_2-a_1}\cdots z_{k_n,a_{n-1}-a_{n-2}}\right)$$
$$+\sum_{\substack{k_1+\cdots+k_n=k\\ k_j\geq 1,k_1\geq 2,k_2=1}} (2^{k_1-1}-1)L_{N,*}(k_1,\ldots,k_n;a,a_1-a,a_2-a_1,\ldots,a_{n-1}-a_{n-2})$$
$$+\sum_{i=2}^{n-1}\sum_{\substack{k_1+\cdots+k_n=k\\ k_j\geq 1,k_1\geq 2}} (2^{k_1+\cdots+k_i-i}-2^{k_2+\cdots+k_i-(i-1)})L_{N,*}(k_1,\ldots,k_n;a_1,a_2-a_1,\ldots,a_{i-1}-a_{i-2},$$
$$a-a_{i-1},a_i-a,a_{i+1}-a_i,\ldots,a_{n-1}-a_{n-2})$$
$$+\sum_{\substack{k_1+\cdots+k_n=k\\ k_j\geq 1,k_1\geq 2}} (2^{k_1+\cdots+k_{n-1}-(n-1)}-2^{k_2+\cdots+k_{n-1}-(n-2)})L_{N,*}(k_1,\ldots,k_n;a_1,a_2-a_1,$$
$$\ldots,a_{n-1}-a_{n-2},a-a_{n-1})$$
$$=\sum_{i=1}^{n-1}\sum_{\substack{k_1+\cdots+k_n=k\\ k_j\geq 1,k_1\geq 2}} L_{N,*}(k_1,\ldots,k_n;a_1,a_2-a_1,\ldots,a_i-a_{i-1},a,a_{i+1}-a_i,\ldots,a_{n-1}-a_{n-2})$$
$$+\sum_{\substack{k_1+\cdots+k_{n-1}=k\\ k_j\geq 1,k_1\geq 2}} (k_1-2)L_{N,*}(k_1,\ldots,k_{n-1};a+a_1,a_2-a_1,\ldots,a_{n-1}-a_{n-2})$$
$$+\sum_{i=2}^{n-1}\sum_{\substack{k_1+\cdots+k_{n-1}=k\\ k_j\geq 1,k_1\geq 2}} (k_i-1)L_{N,*}(k_1,\ldots,k_{n-1};a_1,a_2-a_1,\ldots,a_{i-1}-a_{i-2},a+a_i-a_{i-1},$$
$$a_{i+1}-a_i,\ldots,a_{n-1}-a_{n-2}).$$

Let $n=2$. From Theorem 4.3, we get the following sum formula of double L-values.

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Corollary 4.5. *For an integer k with $k \geq 3$, and $a_1, a_2 \in R$, we have*

$$\sum_{j=2}^{k-1} L_{N,*}(j, k-j; a_1, a_2) = L_{N,*}(k-1, 1; a_1+a_2, a_1) - L_{N,*}(k-1, 1; a_1+a_2, -a_2) + \mathcal{L}_{N,*}(z_{1,a_1} z_{k-1,a_1+a_2} - z_{1,a_1} z_{k-1,a_2}) + L_{N,*}(k, 2a_1+a_2).$$

From Theorem 4.4, we get the following weighted sum formula of double L-values.

Corollary 4.6. *For an integer k with $k \geq 3$, and any $a_1, a_2 \in R$, we have*

$$\begin{aligned}
&\sum_{j=2}^{k-1} \big(2^{j-1} L_{N,*}(j, k-j; a_1, a_2 - a_1) + (2^{j-1}-1) L_{N,*}(j, k-j; a_2, a_1 - a_2) \\
&\qquad - L_{N,*}(j, k-j; a_1, a_2) - L_{N,*}(j, k-j; a_2, a_1)\big) \\
=& L_{N,*}(k-1, 1; a_1, a_2 - a_1) - L_{N,*}(k-1, 1; a_1, a_2) \\
&+ \mathcal{L}_{N,*}(z_{1,a_1} z_{k-1,a_2} - z_{1,a_1} z_{k-1,a_2-a_1}) + (k-2) L_{N,*}(k; a_1+a_2).
\end{aligned}$$

4.2 Sum and weighted sum formulas of double zeta values of level 2

Setting $N = 2$ in Corollary 4.5, and taking all possible values of (a_1, a_2), we get the sum formulas of alternating double zeta values as displayed in the following corollary.

Corollary 4.7. *For an integer k with $k \geq 3$, we have*

$$\sum_{j=2}^{k-1} \zeta(j, k-j) = \zeta(k), \tag{4.1}$$

$$\sum_{j=2}^{k-1} \zeta^{(2)}(j, \overline{k-j}) = \zeta^{(2)}(\overline{k-1}, 1) - \zeta^{(2)}(\overline{k-1}, \overline{1}) + \zeta^{(2)}(\overline{k}), \tag{4.2}$$

$$\sum_{j=1}^{k-1} \zeta^{(2)}(\overline{j}, \overline{k-j}) = \zeta^{(2)}(\overline{1}, k-1) + \zeta^{(2)}(\overline{k}), \tag{4.3}$$

$$\sum_{j=1}^{k-1} \zeta^{(2)}(\overline{j}, k-j) = \zeta^{(2)}(\overline{k-1}, \overline{1}) - \zeta^{(2)}(\overline{k-1}, 1) + \zeta^{(2)}(\overline{1}, \overline{k-1}) + \zeta(k). \tag{4.4}$$

Note that formula (4.1) is attributed to Euler [3]. Sum relations (4.2)-(4.4) can also be found in [2, Theorem 3.2] with alternative proofs based on generating functions.

Now let $N = 2$ in Corollary 4.6. In the case of $(a_1, a_2) = (0, 0)$, we get

$$\sum_{j=2}^{k-1} (2^j - 3) \zeta(j, k-j) = (k-2) \zeta(k). \tag{4.5}$$

In the case of $(a_1, a_2) = (0, 1)$, we have

$$\sum_{j=2}^{k-1} (2^{j-1} - 1) \zeta^{(2)}(j, \overline{k-j}) + \sum_{j=2}^{k-1} (2^{j-1} - 1) \zeta^{(2)}(\overline{j}, \overline{k-j})$$

$$-\sum_{j=2}^{k-1}\zeta^{(2)}(\overline{j},k-j)=(k-2)\zeta^{(2)}(\overline{k}). \tag{4.6}$$

In the case of $(a_1,a_2)=(1,0)$, we get

$$\begin{aligned}&\sum_{j=1}^{k-1}2^{j-1}\zeta^{(2)}(\overline{j},\overline{k-j})+\sum_{j=2}^{k-1}(2^{j-1}-2)\zeta^{(2)}(j,\overline{k-j})-\sum_{j=1}^{k-2}\zeta^{(2)}(\overline{j},k-j)\\ =&\zeta^{(2)}(\overline{k-1},\overline{1})+(k-2)\zeta^{(2)}(\overline{k}). \end{aligned}\tag{4.7}$$

In the case of $(a_1,a_2)=(1,1)$, we get

$$\begin{aligned}&\sum_{j=2}^{k-1}(2^{j}-1)\zeta^{(2)}(\overline{j},k-j)-2\sum_{j=2}^{k-1}\zeta^{(2)}(\overline{j},\overline{k-j})\\ =&\zeta^{(2)}(\overline{k-1},1)-\zeta^{(2)}(\overline{k-1},\overline{1})+\zeta^{(2)}(\overline{1},\overline{k-1})-\zeta^{(2)}(\overline{1},k-1)+(k-2)\zeta(k). \end{aligned}\tag{4.8}$$

Then using the sum formulas of alternating double zeta values and (4.5)-(4.8), we get the following weighted sum formulas of alternating double zeta values.

Corollary 4.8. *For an integer k with $k\geq 3$, we have*

$$\sum_{j=2}^{k-1}2^{j}\zeta(j,k-j)=(k+1)\zeta(k), \tag{4.9}$$

$$\sum_{j=2}^{k-1}2^{j}\zeta^{(2)}(j,\overline{k-j})+\sum_{j=2}^{k-1}2^{j}\zeta^{(2)}(\overline{j},\overline{k-j})=2\zeta(k)+2k\zeta^{(2)}(\overline{k}), \tag{4.10}$$

$$\sum_{j=2}^{k-1}2^{j}\zeta^{(2)}(\overline{j},k-j)=(k-1)\zeta(k)+2\zeta^{(2)}(\overline{k}). \tag{4.11}$$

Note that (4.9) was first proved by Ohno and Zudilin [8, Theorem 3]. Relations (4.10) and (4.11) can also be found in [2, proof of Theorem 4.4].

By Lemma 3.1 with $N=2$ and $\omega=-1$, the double zeta values of level 2 can be represented by the alternating double zeta values in the following way:

$$\begin{pmatrix}\zeta_2(k,l)\\ \zeta_2(\overline{k},l)\\ \zeta_2(k,\overline{l})\\ \zeta_2(\overline{k},\overline{l})\end{pmatrix}=\begin{pmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{pmatrix}\begin{pmatrix}\zeta(k,l)\\ \zeta^{(2)}(\overline{k},l)\\ \zeta^{(2)}(k,\overline{l})\\ \zeta^{(2)}(\overline{k},\overline{l})\end{pmatrix}. \tag{4.12}$$

Then by Corollary 4.7, (4.12) and the fact $\zeta^{(2)}(\overline{k})=(2^{1-k}-1)\zeta(k)$, we get the sum formulas of double zeta values of level 2.

Corollary 4.9. *For an integer k with $k\geq 3$, we have*

$$\sum_{j=2}^{k-1}\zeta_2(j,k-j)=\frac{1}{2^{k-2}}\zeta(k),$$

$$\sum_{j=2}^{k-1}\zeta_2(\overline{j},k-j)=2\left(\zeta^{(2)}(\overline{k-1},1)-\zeta^{(2)}(\overline{k-1},\overline{1})\right)$$

$$\sum_{j=2}^{k-1}\zeta_2(j,\overline{k-j})=2\left(\zeta^{(2)}(\overline{k-1},\overline{1})+\zeta^{(2)}(\overline{1},\overline{k-1})-\zeta^{(2)}(\overline{k-1},1)-\zeta^{(2)}(\overline{1},k-1)\right)$$
$$+4\left(1-\frac{1}{2^k}\right)\zeta(k),$$

$$\sum_{j=2}^{k-1}\zeta_2(\overline{j},\overline{k-j})=2\left(\zeta^{(2)}(\overline{1},k-1)-\zeta^{(2)}(\overline{1},\overline{k-1})\right).$$

For an alternative proof of Corollary 4.9 based on generating functions, see [2, Corollary 3.3]. Similarly, using Corollary 4.7, (4.12) and (4.9)-(4.11), we get the weighted sum formulas of double zeta values of level 2.

Corollary 4.10. *For an integer k with $k \geq 3$, we have*

$$\sum_{j=2}^{k-1}2^j\zeta_2(j,k-j)=\frac{k+1}{2^{k-2}}\zeta(k), \tag{4.13}$$

$$\sum_{j=2}^{k-1}2^j\zeta_2(j,\overline{k-j})=4(k-1)\left(1-\frac{1}{2^k}\right)\zeta(k). \tag{4.14}$$

Proof. We get the first equation by (4.9)+(4.10)+(4.11), and the second equation by (4.9)+(4.11)−(4.10). □

Remark 4.11. *Since $\zeta_2(k_1,k_2)=2^{2-k_1-k_2}\zeta(k_1,k_2)$, one can easily find that (4.13) is a variant of (4.9). Also, since the Kaneko-Tsumura's double T-values are given by $T(k_1,k_2)=\zeta_2(k_1,\overline{k_2})$, (4.14) recovers [6, Theorem 3.2].*

4.3 Sum and weighted sum formulas of double zeta values of level 3

We consider the case of $N=3$. Taking all possible values of (a_1,a_2) in Corollary 4.5, we get the following sum formulas of multiple L-values of level 3.

Corollary 4.12. *For an integer k with $k \geq 3$, we have*

$$\sum_{j=2}^{k-1}\zeta(j,k-j)=\zeta(k),$$

$$\sum_{j=2}^{k-1}\zeta^{(3)}(\overline{j},k-j)=\zeta^{(3)}(\overline{1},\overline{k-1})-\zeta^{(3)}(\overline{1},k-1)+\zeta^{(3)}(\overline{k-1},\overline{1})-\zeta^{(3)}(\overline{k-1},1)+\zeta^{(3)}(\widetilde{k}),$$

$$\sum_{j=2}^{k-1}\zeta^{(3)}(\widetilde{j},k-j)=\zeta^{(3)}(\widetilde{1},\widetilde{k-1})-\zeta^{(3)}(\widetilde{1},k-1)+\zeta^{(3)}(\widetilde{k-1},\widetilde{1})-\zeta^{(3)}(\widetilde{k-1},1)+\zeta^{(3)}(\overline{k}),$$

$$\sum_{j=2}^{k-1}\zeta^{(3)}(j,\overline{k-j})=\zeta^{(3)}(\overline{k-1},1)-\zeta^{(3)}(\overline{k-1},\widetilde{1})+\zeta^{(3)}(\overline{k}),$$

$$\sum_{j=2}^{k-1}\zeta^{(3)}(\overline{j},\overline{k-j})=\zeta^{(3)}(\overline{1},\widetilde{k-1})-\zeta^{(3)}(\overline{1},\overline{k-1})+\zeta^{(3)}(\widetilde{k-1},\overline{1})-\zeta^{(3)}(\widetilde{k-1},\widetilde{1})+\zeta(k),$$

$$\sum_{j=2}^{k-1}\zeta^{(3)}(\widetilde{j},\overline{k-j})=\zeta^{(3)}(\widetilde{1},k-1)-\zeta^{(3)}(\widetilde{1},\overline{k-1})+\zeta^{(3)}(\widetilde{k}),$$

$$\sum_{j=2}^{k-1}\zeta^{(3)}(j,\widetilde{k-j})=\zeta^{(3)}(\widetilde{k-1},1)-\zeta^{(3)}(\widetilde{k-1},\overline{1})+\zeta^{(3)}(\widetilde{k}),$$

$$\sum_{j=2}^{k-1}\zeta^{(3)}(\overline{j},\widetilde{k-j})=\zeta^{(3)}(\overline{1},k-1)-\zeta^{(3)}(\overline{1},\widetilde{k-1})+\zeta^{(3)}(\overline{k}),$$

$$\sum_{j=2}^{k-1}\zeta^{(3)}(\widetilde{j},\widetilde{k-j})=\zeta^{(3)}(\widetilde{1},\overline{k-1})-\zeta^{(3)}(\widetilde{1},\widetilde{k-1})+\zeta^{(3)}(\overline{k-1},\widetilde{1})-\zeta^{(3)}(\overline{k-1},\overline{1})+\zeta(k).$$

Recall that ω is a primitive cubic root of unity. For positive integers k_1, k_2 with $k_1 \geq 2$ and $a_1, a_2 \in R_3$, from Lemma 3.1, we have

$$\begin{aligned}\zeta_3(k_1,k_2;a_1,a_2)=&\sum_{m_1>m_2>0}\frac{\left(1+\omega^{m_1-a_1}+\omega^{2(m_1-a_1)}\right)\left(1+\omega^{m_2-a_2}+\omega^{2(m_2-a_2)}\right)}{m_1^{k_1}m_2^{k_2}}\\ =&\zeta(k_1,k_2)+\omega^{-a_1}\zeta^{(3)}(\overline{k_1},k_2)+\omega^{-2a_1}\zeta^{(3)}(\widetilde{k_1},k_2)\\ &+\omega^{-a_2}\zeta^{(3)}(k_1,\overline{k_2})+\omega^{-a_1-a_2}\zeta^{(3)}(\overline{k_1},\overline{k_2})+\omega^{-2a_1-a_2}\zeta^{(3)}(\widetilde{k_1},\overline{k_2})\\ &+\omega^{-2a_2}\zeta^{(3)}(k_1,\widetilde{k_2})+\omega^{-a_1-2a_2}\zeta^{(3)}(\overline{k_1},\widetilde{k_2})+\omega^{-2a_1-2a_2}\zeta^{(3)}(\widetilde{k_1},\widetilde{k_2}).\end{aligned} \tag{4.15}$$

Hence we can get sum formulas of multiple zeta values of level 3 from Corollary 4.12. To state the results, we introduce some notations. For an integer k with $k \geq 3$, we set

$$\zeta_3^{0,1}(1,\overline{k-1})=\sum_{m_1>m_2>0}\frac{(1-\omega)\omega^{m_1+2}(\omega^{m_1-2}-1)(1+\omega^{m_2-1}+\omega^{2(m_2-1)})}{m_1m_2^{k-1}},$$

$$\zeta_3^{1,2}(1,\overline{k-1})=\sum_{m_1>m_2>0}\frac{(1-\omega)\omega^{m_1+1}(\omega^{m_1}-1)(1+\omega^{m_2-1}+\omega^{2(m_2-1)})}{m_1m_2^{k-1}},$$

$$\zeta_3^{2,0}(1,\overline{k-1})=\sum_{m_1>m_2>0}\frac{(1-\omega)\omega^{m_1}(\omega^{m_1+2}-1)(1+\omega^{m_2-1}+\omega^{2(m_2-1)})}{m_1m_2^{k-1}},$$

$$\zeta_3^{1,0}(1,\widetilde{k-1})=\sum_{m_1>m_2>0}\frac{(1-\omega)\omega^{m_1+2}(1-\omega^{m_1-2})(1+\omega^{m_2+1}+\omega^{2(m_2+1)})}{m_1m_2^{k-1}},$$

$$\zeta_3^{0,2}(1,\widetilde{k-1})=\sum_{m_1>m_2>0}\frac{(1-\omega)\omega^{m_1}(1-\omega^{m_1+2})(1+\omega^{m_2+1}+\omega^{2(m_2+1)})}{m_1m_2^{k-1}},$$

$$\zeta_3^{2,1}(1,\widetilde{k-1})=\sum_{m_1>m_2>0}\frac{(1-\omega)\omega^{m_1+1}(1-\omega^{m_1})(1+\omega^{m_2+1}+\omega^{2(m_2+1)})}{m_1m_2^{k-1}}.$$

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Using Corollary 4.12 and (4.15), we get the following sum formulas of double zeta values of level 3.

Corollary 4.13. *For an integer k with $k \geq 3$, we have*

$$\sum_{j=2}^{k-1} \zeta_3(j, k-j) = 3\zeta_3(k),$$

$$\sum_{j=2}^{k-1} \zeta_3(j, \overline{k-j}) = \zeta_3^{2,0}(1, \overline{k-1}) + \zeta_3(\overline{k-1}, \widetilde{1}) - \zeta_3(k-1, \widetilde{1}),$$

$$\sum_{j=2}^{k-1} \zeta_3(j, \widetilde{k-j}) = \zeta_3^{1,0}(1, \widetilde{k-1}) + \zeta_3(\widetilde{k-1}, \overline{1}) - \zeta_3(k-1, \overline{1}),$$

$$\sum_{j=2}^{k-1} \zeta_3(\overline{j}, k-j) = \zeta_3(k-1, \overline{1}) - \zeta_3(\overline{k-1}, \overline{1}),$$

$$\sum_{j=2}^{k-1} \zeta_3(\overline{j}, \overline{k-j}) = \zeta_3^{0,1}(1, \overline{k-1}),$$

$$\sum_{j=2}^{k-1} \zeta_3(\overline{j}, \widetilde{k-j}) = \zeta_3^{2,1}(1, \widetilde{k-1}) + \zeta_3(\widetilde{k-1}, \widetilde{1}) - \zeta_3(\overline{k-1}, \widetilde{1}) + 3\zeta_3(\widetilde{k}),$$

$$\sum_{j=2}^{k-1} \zeta_3(\widetilde{j}, k-j) = \zeta_3(k-1, \widetilde{1}) - \zeta_3(\widetilde{k-1}, \widetilde{1}),$$

$$\sum_{j=2}^{k-1} \zeta_3(\widetilde{j}, \overline{k-j}) = \zeta_3^{1,2}(1, \overline{k-1}) + \zeta_3(\overline{k-1}, \overline{1}) - \zeta_3(\widetilde{k-1}, \overline{1}) + 3\zeta_3(\overline{k}),$$

$$\sum_{j=2}^{k-1} \zeta_3(\widetilde{j}, \widetilde{k-j}) = \zeta_3^{0,2}(1, \widetilde{k-1}).$$

Similarly, taking all possible values of (a_1, a_2) in Corollary 4.6, we get the following weighted sum formulas of multiple L-values of level 3.

Corollary 4.14. *For an integer k with $k \geq 3$, we have*

$$\sum_{j=2}^{k-1} 2^j \zeta(j, k-j) = (k+1)\zeta(k),$$

$$\sum_{j=2}^{k-1} 2^j \zeta^{(3)}(\overline{j}, k-j) = 2\zeta^{(3)}(\overline{1}, \widetilde{k-1}) - 2\zeta^{(3)}(\overline{1}, k-1) + 2\zeta^{(3)}(\widetilde{k-1}, \overline{1}) - 2\zeta^{(3)}(\widetilde{k-1}, \widetilde{1}) + (k-1)\zeta^{(3)}(\widetilde{k}) + 2\zeta(k),$$

$$\sum_{j=2}^{k-1} 2^j \zeta^{(3)}(\widetilde{j}, k-j) = 2\zeta^{(3)}(\widetilde{1}, \overline{k-1}) - 2\zeta^{(3)}(\widetilde{1}, k-1) + 2\zeta^{(3)}(\overline{k-1}, \widetilde{1}) - 2\zeta^{(3)}(\overline{k-1}, \overline{1}) + (k-1)\zeta^{(3)}(\overline{k}) + 2\zeta(k),$$

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$$
\begin{aligned}
\sum_{j=2}^{k-1} 2^{j-1}\zeta^{(3)}(j,\overline{k-j}) + \sum_{j=2}^{k-1} 2^{j-1}\zeta^{(3)}(\overline{j},\widetilde{k-j}) &= \zeta^{(3)}(\overline{1},\overline{k-1}) - \zeta^{(3)}(\overline{1},\widetilde{k-1}) \\
&\quad + \zeta^{(3)}(\overline{k-1},\overline{1}) - \zeta^{(3)}(\overline{k-1},\widetilde{1}) + k\zeta^{(3)}(\overline{k}) + \zeta^{(3)}(\widetilde{k}), \\
\sum_{j=2}^{k-1} 2^{j-1}\zeta^{(3)}(j,\widetilde{k-j}) + \sum_{j=2}^{k-1} 2^{j-1}\zeta^{(3)}(\widetilde{j},\overline{k-j}) &= \zeta^{(3)}(\widetilde{1},\widetilde{k-1}) - \zeta^{(3)}(\widetilde{1},\overline{k-1}) \\
&\quad + \zeta^{(3)}(\widetilde{k-1},\widetilde{1}) - \zeta^{(3)}(\widetilde{k-1},\overline{1}) + k\zeta^{(3)}(\widetilde{k}) + \zeta^{(3)}(\overline{k}), \\
\sum_{j=2}^{k-1} 2^{j-1}\zeta^{(3)}(\overline{j},\overline{k-j}) + \sum_{j=2}^{k-1} 2^{j-1}\zeta^{(3)}(\widetilde{j},\widetilde{k-j}) &= \zeta^{(3)}(\overline{1},k-1) - \zeta^{(3)}(\overline{1},\overline{k-1}) \\
&\quad + \zeta^{(3)}(\widetilde{1},k-1) - \zeta^{(3)}(\widetilde{1},\widetilde{k-1}) + (k-1)\zeta(k) + \zeta^{(3)}(\overline{k}) + \zeta^{(3)}(\widetilde{k}).
\end{aligned}
$$

Using Corollary 4.14 and (4.15), we get the following weighted sum formulas of double zeta values of level 3.

Corollary 4.15. *For an integer k with $k \geq 3$, we have*

$$
\begin{aligned}
\sum_{j=2}^{k-1} 2^{j}\zeta_3(\overline{j},\widetilde{k-j}) &= (2\omega+4)\left(\zeta^{(3)}(\overline{1},k-1) - \zeta^{(3)}(\widetilde{1},\widetilde{k-1})\right) \\
&\quad + (2\omega-2)\left(\zeta^{(3)}(\overline{1},\overline{k-1}) - \zeta^{(3)}(\widetilde{1},k-1)\right) \\
&\quad + (4\omega+2)\left(\zeta^{(3)}(\widetilde{1},\overline{k-1}) - \zeta^{(3)}(\overline{1},\widetilde{k-1})\right) \\
&\quad + (3k-3)\zeta(k) + (3k-3)\omega\zeta^{(3)}(\overline{k}) - (3k-3)(\omega+1)\zeta^{(3)}(\widetilde{k}), \\
\sum_{j=2}^{k-1} 2^{j}\zeta_3(\widetilde{j},\overline{k-j}) &= (2\omega+4)\left(\zeta^{(3)}(\widetilde{1},k-1) - \zeta^{(3)}(\overline{1},\overline{k-1})\right) \\
&\quad + (2\omega-2)\left(\zeta^{(3)}(\widetilde{1},\widetilde{k-1}) - \zeta^{(3)}(\overline{1},k-1)\right) \\
&\quad + (4\omega+2)\left(\zeta^{(3)}(\overline{1},\widetilde{k-1}) - \zeta^{(3)}(\widetilde{1},\overline{k-1})\right) \\
&\quad + (3k-3)\zeta(k) + (3k-3)\omega\zeta^{(3)}(\widetilde{k}) - (3k-3)(\omega+1)\zeta^{(3)}(\overline{k}).
\end{aligned}
$$

Acknowledgement

The authors would like to thank the anonymous referee for his/her constructive comments. This work was supported by the Fundamental Research Funds for the Central Universities (grant number 22120210552).

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