FRACTIONAL MACLAURIN-TYPE INEQUALITIES FOR TWICE-DIFFERENTIABLE FUNCTIONS

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Abstract. In the present paper, an equality is proved for the case of twice-differentiable functions whose second derivatives in absolute value are convex. By using this equality, some Maclaurin-type inequalities are established for the case of the well-known Riemann-Liouville fractional integrals. More precisely, some Maclaurin-type inequalities are obtained by using Hölder and power-mean inequalities. Furthermore, sundry Maclaurin-type inequalities are given by using special cases of obtained theorems.

1. Introduction

Fractional analysis has been investigated by sundry researchers and they have established the fractional derivatives and integrals in different methods with numerous notations. It is well known that the first fractional integral operator is the Riemann-Liouville fractional integral operator. Nowadays, fractional calculus has become one of the famous fields owing to its natural applications in different fields like fluid mechanics, biological modeling, numerical physical science, and so on. Because of significance of fractional calculus, it can be proved the bounds of new inequalities by using not only Hermite-Hadamard type inequalities but also and Simpson, Newton, and Euler-Maclaurin-type inequalities.

Definition 1 (See [12,18]). Let us consider $\mathcal{F} \in L_1[\sigma,\delta]$. The Riemann–Liouville integrals $J^{\alpha}_{\sigma+}\mathcal{F}$ and $J^{\alpha}_{\delta-}\mathcal{F}$ of order $\alpha > 0$ with $\sigma \geq 0$ are described by

$$J^{\alpha}_{\sigma+}\mathcal{F}(x) = \frac{1}{\Gamma(\alpha)} \int_{\sigma}^{x} (x-t)^{\alpha-1} \mathcal{F}(t)dt, \quad x > \sigma$$

and

$$J^{\alpha}_{\delta-}\mathcal{F}(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\delta} (t-x)^{\alpha-1} \mathcal{F}(t)dt, \quad x < \delta,$$

respectively. Here, $\Gamma(\alpha)$ denotes the Gamma function and its described as

$$\Gamma(\alpha) = \int_0^{\infty} e^{-u} u^{\alpha-1} du.$$

The fractional integral reduces to the classical integral for the case of $\alpha = 1$.

Simpson's inequality has Simpson's following rules:

i. Simpson's quadrature formula (Simpson's 1/3 rule) is defined as follows:

$$\int_{\sigma}^{\delta} \mathcal{F}(x)\, dx \approx \frac{\delta-\sigma}{6}\left[\mathcal{F}(\sigma) + 4\mathcal{F}\left(\frac{\sigma+\delta}{2}\right) + \mathcal{F}(\delta)\right]. \tag{1}$$

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ii. Simpson's second formula or Newton-Cotes quadrature formula (Simpson's 3/8 rule (cf. [5])) is described as follows:

$$\int_{\sigma}^{\delta}\mathcal{F}(x)\,dx\approx\frac{\delta-\sigma}{8}\left[\mathcal{F}(\sigma)+3\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+3\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathcal{F}(\delta)\right]. \tag{2}$$

iii. The corresponding dual Simpson's 3/8 formula - the Maclaurin rule based on the Maclaurin formula (cf. [5]) is defined as follows:

$$\int_{\sigma}^{\delta}\mathcal{F}(x)\,dx\approx\frac{\delta-\sigma}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right]. \tag{3}$$

Formulae (1), (2), and (3) provide for the case of any function $\mathcal{F}$ with continuous 4^{th}derivative on $[\sigma,\delta]$.

The well-known Newton-Cotes quadrature featuring the Simpson type with three-point inequality is as follows:

Theorem 1 (See [5]). *Let $\mathcal{F}:[\sigma,\delta]\to\mathbb{R}$ be a four times differentiable and continuous function on (σ,δ), and $\left\|\mathcal{F}^{(4)}\right\|_{\infty}=\sup\limits_{x\in(\sigma,\delta)}\left|\mathcal{F}^{(4)}(x)\right|<\infty$. Then, one has the following inequality*

$$\left|\frac{1}{6}\left[\mathcal{F}(\sigma)+4\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\mathcal{F}(\delta)\right]-\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathcal{F}(x)dx\right|\leq\frac{1}{2880}\left\|\mathcal{F}^{(4)}\right\|_{\infty}(\delta-\sigma)^{4}.$$

Dragomir [8] presented an estimation of remainder for Simpson's quadrature formula to the case of bounded variation functions and applications in theory of special means. In addition, several fractional Simpson type inequalities for the case of function whose second derivatives in absolute value are convex given in [13]. Furthermore, Park [17] proved several estimates of Simpson-like type integral inequalities for the case of functions whose first derivatives in absolute value at certain powers are preinvex. The reader is referred to [1–4, 22] and the references therein for more information about Simpson type inequalities and sundry properties of Riemann–Liouville fractional integrals.

Classical closed type quadrature rules is the Simpson 3/8 rule based on the Simpson 3/8 inequality as follows:

Theorem 2 (See [5]). *If $\mathcal{F}:[\sigma,\delta]\to\mathbb{R}$ is a four times differentiable and continuous function on (σ,δ), and $\left\|\mathcal{F}^{(4)}\right\|_{\infty}=\sup\limits_{x\in(\sigma,\delta)}\left|\mathcal{F}^{(4)}(x)\right|<\infty$, then one has the inequality*

$$\left|\frac{1}{8}\left[\mathcal{F}(\sigma)+3\mathcal{F}\left(\frac{2\sigma+\delta}{3}\right)+3\mathcal{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathcal{F}(\delta)\right]-\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathcal{F}(x)dx\right|\leq\frac{1}{6480}\left\|\mathcal{F}^{(4)}\right\|_{\infty}(\delta-\sigma)^{4}.$$

Simpson's second rule has the rule of three-point Newton-Cotes quadrature, therefore evaluations for the case of three steps quadratic kernel are sometimes called Newton type results in the literature. There has been a growing tendency to investigate such type of inequalities particularly for Newton type inequalities. For example, Erden et al. [16] established several Newton-type inequalities for the case of functions whose the local fractional derivatives in modulus. Noor et al. [19] proved several Newton type integral inequalities for the case of p-harmonic convex functions and some special cases were also investigated as applications. Moreover, several Newton type inequalities based on convexity were presented and some applications for special cases of real functions were also given in [11]. Furthermore, several Newton type inequalities were established with the aid of Riemann-Liouville fractional integrals and several fractional Newton type inequalities for the case of bounded variation functions were also presented in [21]. It can be referred to [9, 14, 15] and the references therein for details and for the unexplained subject about Newton type of inequalities involving convex differentiable functions.

The corresponding dual Simpson's 3/8 formula-the Maclaurin rule based on the Maclaurin inequality is as follows:

Theorem 3 (See [5]). *Suppose $\mathcal{F}:[\sigma,\delta]\to\mathbb{R}$ is a four times differentiable and continuous function on (σ,δ), and $\left\|\mathcal{F}^{(4)}\right\|_\infty=\sup_{x\in(\sigma,\delta)}\left|\mathcal{F}^{(4)}(x)\right|<\infty$. Then, the following inequality holds:*

$$\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right]-\frac{1}{\delta-\sigma}\int_\sigma^\delta\mathcal{F}(x)dx\right|\le\frac{7}{51840}\left\|\mathcal{F}^{(4)}\right\|_\infty(\delta-\sigma)^4.$$

Dedić et al. [6] established a set of inequalities with the help of the Euler-Maclaurin formulae and the results were applied to obtain some error estimates for the case of the Maclaurin quadrature rules. Moreover, a set of inequalities is established by using the Euler-Simpson 3/8 formulae. The results are implemented to obtain some error estimates for the case of the Simpson 3/8 quadrature rules in [7]. For details and for the unexplained subject about these kinds of inequalities, the reader is referred to [5, 10, 20] and the references therein.

The purpose of this paper is to establish Maclaurin-type inequalities for the case of twice-differentiable convex functions by using the Riemann-Liouville fractional integrals. The fundamental definition of fractional calculus and other studies in this field are given in Section 1. We will prove an integral equality in Section 2 that is critical in proving the primary results of this paper. Furthermore, it will be established some Maclaurin-type inequalities for the case of twice-differentiable convex functions by using the Riemann-Liouville fractional integrals. In Section 3, several ideas about Maclaurin type inequalities for the further directions of research will be presented.

2. Main results

Lemma 1. *Let $\mathcal{F}:[\sigma,\delta]\to\mathbb{R}$ be an absolutely continuous function (σ,δ) so that $\mathcal{F}''\in L_1([\sigma,\delta])$. Then, the following equality holds:*

(4)
$$\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right]-\frac{\Gamma(\alpha+1)}{2(\delta-\sigma)^\alpha}\left[J_{\sigma+}^\alpha\mathcal{F}(\delta)+J_{\delta-}^\alpha\mathcal{F}(\sigma)\right]=-\frac{(\delta-\sigma)^2}{2(\alpha+1)}\sum_{i=1}^4 I_i,$$

where

$$\begin{cases} I_1=\int_0^{\frac{1}{6}}\left(t^{\alpha+1}+\alpha t\right)\left[\mathcal{F}''(t\delta+(1-t)\sigma)+\mathcal{F}''(t\sigma+(1-t)\delta)\right]dt,\\ I_2=\int_{\frac{1}{6}}^{\frac{1}{2}}\left(t^{\alpha+1}-\frac{5-3\alpha}{8}t\right)\left[\mathcal{F}''(t\delta+(1-t)\sigma)+\mathcal{F}''(t\sigma+(1-t)\delta)\right]dt,\\ I_3=\int_{\frac{1}{2}}^{\frac{5}{6}}\left(t^{\alpha+1}-\frac{7-\alpha}{8}t\right)\left[\mathcal{F}''(t\delta+(1-t)\sigma)+\mathcal{F}''(t\sigma+(1-t)\delta)\right]dt,\\ I_4=\int_{\frac{5}{6}}^{1}\left(t^{\alpha+1}-t\right)\left[\mathcal{F}''(t\delta+(1-t)\sigma)+\mathcal{F}''(t\sigma+(1-t)\delta)\right]dt. \end{cases}$$

Proof. By using the integration by parts, we can obtain

$$I_1=\int_0^{\frac{1}{6}}\left(t^{\alpha+1}+\alpha t\right)\left[\mathcal{F}''(t\delta+(1-t)\sigma)+\mathcal{F}''(t\sigma+(1-t)\delta)\right]dt \tag{5}$$

$$=\frac{1}{\delta-\sigma}\left(t^{\alpha+1}+\alpha t\right)\left[\mathcal{F}'(t\delta+(1-t)\sigma)-\mathcal{F}'(t\sigma+(1-t)\delta)\right]\Big|_0^{\frac{1}{6}}$$

$$-\frac{1}{\delta-\sigma}\int_{0}^{\frac{1}{6}}\left(\left(\alpha+1\right)t^{\alpha}+\alpha\right)\left[\mathcal{F}'\left(t\delta+\left(1-t\right)\sigma\right)-\mathcal{F}'\left(t\sigma+\left(1-t\right)\delta\right)\right]dt$$

$$=\frac{1}{\delta-\sigma}\left(\frac{1}{6^{\alpha+1}}+\frac{\alpha}{6}\right)\left(\mathcal{F}'\left(\frac{5\sigma+\delta}{6}\right)-\mathcal{F}'\left(\frac{\sigma+5\delta}{6}\right)\right)$$

$$-\frac{1}{\delta-\sigma}\left[\left(\frac{1}{\delta-\sigma}\left(\alpha+1\right)t^{\alpha}+\alpha\right)\left[\mathcal{F}\left(t\delta+\left(1-t\right)\sigma\right)+\mathcal{F}\left(t\sigma+\left(1-t\right)\delta\right)\right]\Big|_{0}^{\frac{1}{6}}\right.$$

$$\left.-\frac{\alpha\left(\alpha+1\right)}{\delta-\sigma}\int_{0}^{\frac{1}{6}}t^{\alpha-1}\left[\mathcal{F}\left(t\delta+\left(1-t\right)\sigma\right)+\mathcal{F}\left(t\sigma+\left(1-t\right)\delta\right)\right]dt\right]$$

$$=\frac{1}{\delta-\sigma}\left(\frac{1}{6^{\alpha+1}}+\frac{\alpha}{6}\right)\left(\mathcal{F}'\left(\frac{5\sigma+\delta}{6}\right)-\mathcal{F}'\left(\frac{\sigma+5\delta}{6}\right)\right)$$

$$-\frac{1}{\left(\delta-\sigma\right)^{2}}\left[\left(\frac{\alpha+1}{6^{\alpha}}+\alpha\right)\left(\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right)-\alpha\left(\mathcal{F}\left(\sigma\right)+\mathcal{F}\left(\delta\right)\right)\right]$$

$$+\frac{\alpha\left(\alpha+1\right)}{\left(\delta-\sigma\right)^{2}}\int_{0}^{\frac{1}{6}}t^{\alpha-1}\left[\mathcal{F}\left(t\delta+\left(1-t\right)\sigma\right)+\mathcal{F}\left(t\sigma+\left(1-t\right)\delta\right)\right]dt.$$

In a similar manner, we get

$$I_{2}=-\frac{1}{\delta-\sigma}\left(\frac{1}{6^{\alpha+1}}+\frac{3\alpha-5}{48}\right)\left(\mathcal{F}'\left(\frac{5\sigma+\delta}{6}\right)-\mathcal{F}'\left(\frac{\sigma+5\delta}{6}\right)\right)$$
$$-\frac{1}{\left(\delta-\sigma\right)^{2}}\left[2\left(\frac{\alpha+1}{2^{\alpha}}+\frac{3\alpha-5}{8}\right)\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right.$$
$$\left.-\left(\frac{\alpha+1}{6^{\alpha}}+\frac{3\alpha-5}{8}\right)\left(\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right)\right]$$
$$+\frac{\alpha\left(\alpha+1\right)}{\left(\delta-\sigma\right)^{2}}\int_{\frac{1}{6}}^{\frac{1}{2}}t^{\alpha-1}\left[\mathcal{F}\left(t\delta+\left(1-t\right)\sigma\right)+\mathcal{F}\left(t\sigma+\left(1-t\right)\delta\right)\right]dt, \tag{6}$$

$$I_{3}=-\frac{1}{\delta-\sigma}\left(\left(\frac{5}{6}\right)^{\alpha+1}+\frac{5\left(\alpha-7\right)}{48}\right)\left(\mathcal{F}'\left(\frac{5\sigma+\delta}{6}\right)-\mathcal{F}'\left(\frac{\sigma+5\delta}{6}\right)\right) \tag{7}$$
$$-\frac{1}{\left(\delta-\sigma\right)^{2}}\left[\left(\left(\alpha+1\right)\left(\frac{5}{6}\right)^{\alpha}+\frac{\alpha-7}{8}\right)\left(\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right)\right.$$

$$-2\left(\frac{\alpha+1}{2^{\alpha}}+\frac{\alpha-7}{8}\right)\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\Bigg]$$

$$+\frac{\alpha(\alpha+1)}{(\delta-\sigma)^{2}}\int\limits_{\frac{1}{2}}^{\frac{5}{6}}t^{\alpha-1}\left[\mathcal{F}(t\delta+(1-t)\sigma)+\mathcal{F}(t\sigma+(1-t)\delta)\right]dt,$$

and

$$I_4=\frac{1}{\delta-\sigma}\left(\left(\frac{5}{6}\right)^{\alpha+1}-\frac{5}{6}\right)\left(\mathcal{F}'\left(\frac{5\sigma+\delta}{6}\right)-\mathcal{F}'\left(\frac{\sigma+5\delta}{6}\right)\right) \tag{8}$$

$$-\frac{1}{(\delta-\sigma)^{2}}\left[\alpha(\mathcal{F}(\sigma)+\mathcal{F}(\delta))-\left((\alpha+1)\left(\frac{5}{6}\right)^{\alpha}-1\right)\left(\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right)\right]$$

$$+\frac{\alpha(\alpha+1)}{(\delta-\sigma)^{2}}\int\limits_{\frac{5}{6}}^{1}t^{\alpha-1}\left[\mathcal{F}(t\delta+(1-t)\sigma)+\mathcal{F}(t\sigma+(1-t)\delta)\right]dt.$$

If we add equalities from (5) to (8), then we readily obtain

$$\sum_{i=1}^{4}I_i=-\frac{(\alpha+1)}{4(\delta-\sigma)^{2}}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right] \tag{9}$$

$$+\frac{\alpha(\alpha+1)}{(\delta-\sigma)^{2}}\left[\int\limits_{0}^{1}t^{\alpha-1}\mathcal{F}(t\delta+(1-t)\sigma)\,dt+\int\limits_{0}^{1}t^{\alpha-1}\mathcal{F}(t\sigma+(1-t)\delta)\,dt\right].$$

If we apply the change of the variable $x=t\delta+(1-t)\sigma$ and $x=t\sigma+(1-t)\delta$ for $t\in[0,1]$ respectively, then equality (9) will be rewritten as follows

$$\sum_{i=1}^{4}I_i=-\frac{(\alpha+1)}{4(\delta-\sigma)^{2}}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right] \tag{10}$$

$$+\frac{\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha+1}}\left[J_{\sigma+}^{\alpha}\mathcal{F}(\delta)+J_{\delta-}^{\alpha}\mathcal{F}(\sigma)\right].$$

Multiplying of (10) by $-\frac{(\delta-\sigma)^2}{2(\alpha+1)}$, the equality (4) is obtained. This ends the proof of Lemma 1. □

Theorem 4. *If the conditions of Lemma 1 satisfy and the function $|\mathcal{F}''|$ is convex on $[\sigma,\delta]$, then we get*

$$\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right]-\frac{\Gamma(\alpha+1)}{2(\delta-\sigma)^{\alpha}}\left[J_{\sigma+}^{\alpha}\mathcal{F}(\delta)+J_{\delta-}^{\alpha}\mathcal{F}(\sigma)\right]\right|$$

$$\leq\frac{(\delta-\sigma)^{2}}{2(\alpha+1)}\left(\Omega_1(\alpha)+\Omega_2(\alpha)+\Omega_3(\alpha)+\Omega_4(\alpha)\right)\left[|\mathcal{F}''(\sigma)|+|\mathcal{F}''(\delta)|\right].$$

Here,

$$\begin{cases} \Omega_1(\alpha) = \int_0^{\frac{1}{6}} \left|t^{\alpha+1} + \alpha t\right| dt = \frac{1}{(\alpha+2)6^{\alpha+2}} + \frac{\alpha}{72}, & \Omega_3(\alpha) = \int_{\frac{1}{2}}^{\frac{5}{6}} \left|t^{\alpha+1} - \frac{7-\alpha}{8}t\right| dt, \\ \Omega_2(\alpha) = \int_{\frac{1}{6}}^{\frac{1}{2}} \left|t^{\alpha+1} - \frac{5-3\alpha}{8}t\right| dt, & \Omega_4(\alpha) = \int_{\frac{5}{6}}^{1} \left|t^{\alpha+1} - t\right| dt = \frac{11}{72} + \frac{1}{\alpha+2}\left(\left(\frac{5}{6}\right)^{\alpha+2} - 1\right). \end{cases}$$

Proof. If we take modulus in Lemma 1. Then, we readily have the following inequality

$$\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right) + 2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right) + 3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right] - \frac{\Gamma(\alpha+1)}{2(\delta-\sigma)^{\alpha}}\left[J_{\sigma+}^{\alpha}\mathcal{F}(\delta) + J_{\delta-}^{\alpha}\mathcal{F}(\sigma)\right]\right| \tag{11}$$

$$\leq \frac{(\delta-\sigma)^2}{2(\alpha+1)}\left[\int_0^{\frac{1}{6}} \left|t^{\alpha+1} + \alpha t\right| \left|\mathcal{F}''(t\delta + (1-t)\sigma) + \mathcal{F}''(t\sigma + (1-t)\delta)\right| dt\right.$$

$$+ \int_{\frac{1}{6}}^{\frac{1}{2}} \left|t^{\alpha+1} - \frac{5-3\alpha}{8}t\right| \left|\mathcal{F}''(t\delta + (1-t)\sigma) + \mathcal{F}''(t\sigma + (1-t)\delta)\right| dt$$

$$+ \int_{\frac{1}{2}}^{\frac{5}{6}} \left|t^{\alpha+1} - \frac{7-\alpha}{8}t\right| \left|\mathcal{F}''(t\delta + (1-t)\sigma) + \mathcal{F}''(t\sigma + (1-t)\delta)\right| dt$$

$$\left.+ \int_{\frac{5}{6}}^{1} \left|t^{\alpha+1} - t\right| \left|\mathcal{F}''(t\delta + (1-t)\sigma) + \mathcal{F}''(t\sigma + (1-t)\delta)\right| dt\right].$$

From the fact that $|\mathcal{F}''|$ is convex, it yields

$$\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right) + 2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right) + 3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right] - \frac{\Gamma(\alpha+1)}{2(\delta-\sigma)^{\alpha}}\left[J_{\sigma+}^{\alpha}\mathcal{F}(\delta) + J_{\delta-}^{\alpha}\mathcal{F}(\sigma)\right]\right|$$

$$\leq \frac{(\delta-\sigma)^2}{2(\alpha+1)}\left[\int_0^{\frac{1}{6}} \left|t^{\alpha+1} + \alpha t\right| \left[t\left|\mathcal{F}''(\delta)\right| + (1-t)\left|\mathcal{F}''(\sigma)\right| + t\left|\mathcal{F}''(\sigma)\right| + (1-t)\left|\mathcal{F}''(\delta)\right|\right] dt\right.$$

$$+ \int_{\frac{1}{6}}^{\frac{1}{2}} \left|t^{\alpha+1} - \frac{5-3\alpha}{8}t\right| \left[t\left|\mathcal{F}''(\delta)\right| + (1-t)\left|\mathcal{F}''(\sigma)\right| + t\left|\mathcal{F}''(\sigma)\right| + (1-t)\left|\mathcal{F}''(\delta)\right|\right] dt$$

$$+ \int_{\frac{1}{2}}^{\frac{5}{6}} \left|t^{\alpha+1} - \frac{7-\alpha}{8}t\right| \left[t\left|\mathcal{F}''(\delta)\right| + (1-t)\left|\mathcal{F}''(\sigma)\right| + t\left|\mathcal{F}''(\sigma)\right| + (1-t)\left|\mathcal{F}''(\delta)\right|\right] dt$$

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$$+\int_{\frac{5}{6}}^{1}\left|t^{\alpha+1}-t\right|\left[t\left|\mathcal{F}''(\delta)\right|+(1-t)\left|\mathcal{F}''(\sigma)\right|+t\left|\mathcal{F}''(\sigma)\right|+(1-t)\left|\mathcal{F}''(\delta)\right|\right]dt\Bigg]$$

$$=\frac{(\delta-\sigma)^2}{2(\alpha+1)}\left[\int_0^{\frac{1}{6}}\left|t^{\alpha+1}+\alpha t\right|dt+\int_{\frac{1}{6}}^{\frac{1}{2}}\left|t^{\alpha+1}-\frac{5-3\alpha}{8}t\right|dt\right.$$

$$\left.+\int_{\frac{1}{2}}^{\frac{5}{6}}\left|t^{\alpha+1}-\frac{7-\alpha}{8}t\right|dt+\int_{\frac{5}{6}}^{1}\left|t^{\alpha+1}-t\right|dt\right]\left[\left|\mathcal{F}''(\sigma)\right|+\left|\mathcal{F}''(\delta)\right|\right].$$

This completes the proof of Theorem 4. □

Corollary 1. *If we choose $\alpha=1$ in Theorem 4, then the following Maclaurin-type inequality holds:*

$$\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right]-\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathcal{F}(t)\,dt\right|\leq\frac{(\delta-\sigma)^2}{64}\left[\left|\mathcal{F}''(\sigma)\right|+\left|\mathcal{F}''(\delta)\right|\right].$$

Theorem 5. *Let us note that the assumptions of Lemma 1 are valid and the function $|\mathcal{F}''|^q$, $q>1$ is convex on $[\sigma,\delta]$. Then, the following inequality*

$$\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right]-\frac{\Gamma(\alpha+1)}{2(\delta-\sigma)^{\alpha}}\left[J_{\sigma+}^{\alpha}\mathcal{F}(\delta)+J_{\delta-}^{\alpha}\mathcal{F}(\sigma)\right]\right|$$

$$\leq\frac{(\delta-\sigma)^2}{2(\alpha+1)}\left[\left(\varphi_1(\alpha,p)+\varphi_4(\alpha,p)\right)\left[\left(\frac{11\left|\mathcal{F}''(\sigma)\right|^q+\left|\mathcal{F}''(\delta)\right|^q}{72}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{F}''(\sigma)\right|^q+11\left|\mathcal{F}''(\delta)\right|^q}{72}\right)^{\frac{1}{q}}\right]\right.$$

$$\left.+\left(\varphi_2(\alpha,p)+\varphi_3(\alpha,p)\right)\left[\left(\frac{2\left|\mathcal{F}''(\sigma)\right|^q+\left|\mathcal{F}''(\delta)\right|^q}{9}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{F}''(\sigma)\right|^q+2\left|\mathcal{F}''(\delta)\right|^q}{9}\right)^{\frac{1}{q}}\right]\right]$$

is valid. Here, $\frac{1}{p}+\frac{1}{q}=1$ and

$$\begin{cases}\varphi_1(\alpha,p)=\left(\int_0^{\frac{1}{6}}\left|t^{\alpha+1}+\alpha t\right|^p dt\right)^{\frac{1}{p}}, & \varphi_3(\alpha,p)=\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|t^{\alpha+1}-\frac{7-\alpha}{8}t\right|^p dt\right)^{\frac{1}{p}},\\ \varphi_2(\alpha,p)=\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|t^{\alpha+1}-\frac{5-3\alpha}{8}t\right|^p dt\right)^{\frac{1}{p}}, & \varphi_4(\alpha,p)=\left(\int_{\frac{5}{6}}^{1}\left|t^{\alpha+1}-t\right|^p dt\right)^{\frac{1}{p}}.\end{cases}$$

Proof. Let us consider Hölder inequality in (11). Then, we can easily have

$$\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right]-\frac{\Gamma(\alpha+1)}{2(\delta-\sigma)^{\alpha}}\left[J_{\sigma+}^{\alpha}\mathcal{F}(\delta)+J_{\delta-}^{\alpha}\mathcal{F}(\sigma)\right]\right|\leq\frac{(\delta-\sigma)^2}{2(\alpha+1)}$$

$$\times\left[\left(\int_0^{\frac{1}{6}}\left|t^{\alpha+1}+\alpha t\right|^p dt\right)^{\frac{1}{p}}\left[\left(\int_0^{\frac{1}{6}}\left|\mathcal{F}''(t\delta+(1-t)\sigma)\right|^q dt\right)^{\frac{1}{q}}+\left(\int_0^{\frac{1}{6}}\left|\mathcal{F}''(t\sigma+(1-t)\delta)\right|^q dt\right)^{\frac{1}{q}}\right]\right.$$

$$+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|t^{\alpha+1}-\frac{5-3\alpha}{8}t\right|^{p}dt\right)^{\frac{1}{p}}\left[\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\mathcal{F}''\left(t\delta+(1-t)\sigma\right)\right|^{q}dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\mathcal{F}''\left(t\sigma+(1-t)\delta\right)\right|^{q}dt\right)^{\frac{1}{q}}\right]$$

$$+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|t^{\alpha+1}-\frac{7-\alpha}{8}t\right|^{p}dt\right)^{\frac{1}{p}}\left[\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\mathcal{F}''\left(t\delta+(1-t)\sigma\right)\right|^{q}dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\mathcal{F}''\left(t\sigma+(1-t)\delta\right)\right|^{q}dt\right)^{\frac{1}{q}}\right]$$

$$+\left(\int_{\frac{5}{6}}^{1}\left|t^{\alpha+1}-t\right|^{p}dt\right)^{\frac{1}{p}}\left[\left(\int_{\frac{5}{6}}^{1}\left|\mathcal{F}''\left(t\delta+(1-t)\sigma\right)\right|^{q}dt\right)^{\frac{1}{q}}+\left(\int_{\frac{5}{6}}^{1}\left|\mathcal{F}''\left(t\sigma+(1-t)\delta\right)\right|^{q}dt\right)^{\frac{1}{q}}\right]\Bigg].$$

Since it is known that $|\mathcal{F}''|^q$ is convex, we obtain

$$\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right]-\frac{\Gamma(\alpha+1)}{2(\delta-\sigma)^{\alpha}}\left[J_{\sigma+}^{\alpha}\mathcal{F}(\delta)+J_{\delta-}^{\alpha}\mathcal{F}(\sigma)\right]\right|$$

$$\leq\frac{(\delta-\sigma)^{2}}{2(\alpha+1)}\left[\left(\int_{0}^{\frac{1}{6}}\left|t^{\alpha+1}+\alpha t\right|^{p}dt\right)^{\frac{1}{p}}\left[\left(\int_{0}^{\frac{1}{6}}t\left|\mathcal{F}''(\delta)\right|^{q}+(1-t)\left|\mathcal{F}''(\sigma)\right|^{q}dt\right)^{\frac{1}{q}}\right.\right.$$

$$\left.+\left(\int_{0}^{\frac{1}{6}}t\left|\mathcal{F}''(\sigma)\right|^{q}+(1-t)\left|\mathcal{F}''(\delta)\right|^{q}dt\right)^{\frac{1}{q}}\right]+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|t^{\alpha+1}-\frac{5-3\alpha}{8}t\right|^{p}dt\right)^{\frac{1}{p}}$$

$$\times\left[\left(\int_{\frac{1}{6}}^{\frac{1}{2}}t\left|\mathcal{F}''(\delta)\right|^{q}+(1-t)\left|\mathcal{F}''(\sigma)\right|^{q}dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}t\left|\mathcal{F}''(\sigma)\right|^{q}+(1-t)\left|\mathcal{F}''(\delta)\right|^{q}dt\right)^{\frac{1}{q}}\right]$$

$$+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|t^{\alpha+1}-\frac{7-\alpha}{8}t\right|^{p}dt\right)^{\frac{1}{p}}\left[\left(\int_{\frac{1}{2}}^{\frac{5}{6}}t\left|\mathcal{F}''(\delta)\right|^{q}+(1-t)\left|\mathcal{F}''(\sigma)\right|^{q}dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}t\left|\mathcal{F}'(\sigma)\right|^{q}+(1-t)\left|\mathcal{F}'(\delta)\right|^{q}dt\right)^{\frac{1}{q}}\right]+\left(\int_{\frac{5}{6}}^{1}\left|t^{\alpha+1}-t\right|^{p}dt\right)^{\frac{1}{p}}$$

$$\left.\times\left[\left(\int_{\frac{5}{6}}^{1}t\left|\mathcal{F}''(\delta)\right|^{q}+(1-t)\left|\mathcal{F}''(\sigma)\right|^{q}dt\right)^{\frac{1}{q}}+\left(\int_{\frac{5}{6}}^{1}t\left|\mathcal{F}''(\sigma)\right|^{q}+(1-t)\left|\mathcal{F}''(\delta)\right|^{q}dt\right)^{\frac{1}{q}}\right]\right]$$

$$= \frac{(\delta-\sigma)^2}{2(\alpha+1)}\left[\left(\left(\int_0^{\frac{1}{6}}\left|t^{\alpha+1}+\alpha t\right|^p dt\right)^{\frac{1}{p}}+\left(\int_{\frac{5}{6}}^{1}\left|t^{\alpha+1}-t\right|^p dt\right)^{\frac{1}{p}}\right)\right.$$

$$\times\left[\left(\frac{11\left|\mathcal{F}''(\sigma)\right|^q+\left|\mathcal{F}''(\delta)\right|^q}{72}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{F}''(\sigma)\right|^q+11\left|\mathcal{F}''(\delta)\right|^q}{72}\right)^{\frac{1}{q}}\right]$$

$$+\left(\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|t^{\alpha+1}-\frac{5-3\alpha}{8}t\right|^p dt\right)^{\frac{1}{p}}+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|t^{\alpha+1}-\frac{7-\alpha}{8}t\right|^p dt\right)^{\frac{1}{p}}\right)$$

$$\left.\times\left[\left(\frac{2\left|\mathcal{F}''(\sigma)\right|^q+\left|\mathcal{F}''(\delta)\right|^q}{9}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{F}''(\sigma)\right|^q+2\left|\mathcal{F}''(\delta)\right|^q}{9}\right)^{\frac{1}{q}}\right]\right].$$

Hence, the proof of Theorem 5 is finished. □

Theorem 6. *Suppose that the assumptions of Lemma 1 hold and the function* $|\mathcal{F}''|^q$, $q \geq 1$ *is convex on* $[\sigma,\delta]$. *Then, the following inequality*

$$\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right]-\frac{\Gamma(\alpha+1)}{2(\delta-\sigma)^\alpha}\left[J_{\sigma+}^{\alpha}\mathcal{F}(\delta)+J_{\delta-}^{\alpha}\mathcal{F}(\sigma)\right]\right|$$

$$\leq \frac{(\delta-\sigma)^2}{2(\alpha+1)}\left[(\Omega_1(\alpha))^{1-\frac{1}{q}}\left[\left(\Omega_5(\alpha)\left|\mathcal{F}''(\delta)\right|^q+(\Omega_1(\alpha)-\Omega_5(\alpha))\left|\mathcal{F}''(\sigma)\right|^q\right)^{\frac{1}{q}}\right.\right.$$

$$\left.+\left(\Omega_5(\alpha)\left|\mathcal{F}''(\sigma)\right|^q+(\Omega_1(\alpha)-\Omega_5(\alpha))\left|\mathcal{F}''(\delta)\right|^q\right)^{\frac{1}{q}}\right]$$

$$+(\Omega_2(\alpha))^{1-\frac{1}{q}}\left[\left(\Omega_6(\alpha)\left|\mathcal{F}''(\delta)\right|^q+(\Omega_2(\alpha)-\Omega_6(\alpha))\left|\mathcal{F}''(\sigma)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\Omega_6(\alpha)\left|\mathcal{F}''(\sigma)\right|^q+(\Omega_2(\alpha)-\Omega_6(\alpha))\left|\mathcal{F}''(\delta)\right|^q\right)^{\frac{1}{q}}\right]$$

$$+(\Omega_3(\alpha))^{1-\frac{1}{q}}\left[\left(\Omega_7(\alpha)\left|\mathcal{F}''(\delta)\right|^q+(\Omega_3(\alpha)-\Omega_7(\alpha))\left|\mathcal{F}''(\sigma)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\Omega_7(\alpha)\left|\mathcal{F}'(\sigma)\right|^q+(\Omega_3(\alpha)-\Omega_7(\alpha))\left|\mathcal{F}'(\delta)\right|^q\right)^{\frac{1}{q}}\right]$$

$$+(\Omega_4(\alpha))^{1-\frac{1}{q}}\left[\left(\Omega_8(\alpha)\left|\mathcal{F}''(\delta)\right|^q+(\Omega_4(\alpha)-\Omega_8(\alpha))\left|\mathcal{F}''(\sigma)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\Omega_8(\alpha)\left|\mathcal{F}''(\sigma)\right|^q+(\Omega_4(\alpha)-\Omega_8(\alpha))\left|\mathcal{F}''(\delta)\right|^q\right)^{\frac{1}{q}}\right]\right]$$

is valid. Here, $\Omega_1(\alpha)$, $\Omega_2(\alpha)$, $\Omega_3(\alpha)$ *and* $\Omega_4(\alpha)$ *are given in Theorem 4 and*

$$\begin{cases} \Omega_5(\alpha) = \int\limits_0^{\frac{1}{6}} \left|t^{\alpha+1} + \alpha t\right| t dt = \frac{1}{(\alpha+3)6^{\alpha+3}} + \frac{\alpha}{648}, & \Omega_7(\alpha) = \int\limits_{\frac{1}{2}}^{\frac{5}{6}} \left|t^{\alpha+1} - \frac{7-\alpha}{8}t\right| t dt, \\ \Omega_6(\alpha) = \int\limits_{\frac{1}{6}}^{\frac{1}{2}} \left|t^{\alpha+1} - \frac{5-3\alpha}{8}t\right| t dt, & \Omega_8(\alpha) = \int\limits_{\frac{5}{6}}^{1} \left|t^{\alpha+1} - t\right| t dt = \frac{91}{648} + \frac{1}{\alpha+3}\left(\left(\frac{5}{6}\right)^{\alpha+3} - 1\right). \end{cases}$$

Proof. If we apply power-mean inequality in (11), then we have

$$\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right) + 2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right) + 3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right] - \frac{\Gamma(\alpha+1)}{2(\delta-\sigma)^{\alpha}}\left[J_{\sigma+}^{\alpha}\mathcal{F}(\delta) + J_{\delta-}^{\alpha}\mathcal{F}(\sigma)\right]\right|$$

$$\leq \frac{(\delta-\sigma)^2}{2(\alpha+1)}\left[\left(\int\limits_0^{\frac{1}{6}} \left|t^{\alpha+1} + \alpha t\right| dt\right)^{1-\frac{1}{q}} \left[\left(\int\limits_0^{\frac{1}{6}} \left|t^{\alpha+1} + \alpha t\right| \left|\mathcal{F}''(t\delta + (1-t)\sigma)\right|^q dt\right)^{\frac{1}{q}}\right.\right.$$

$$\left. + \left(\int\limits_0^{\frac{1}{6}} \left|t^{\alpha+1} + \alpha t\right| \left|\mathcal{F}''(t\sigma + (1-t)\delta)\right|^q dt\right)^{\frac{1}{q}}\right]$$

$$+ \left(\int\limits_{\frac{1}{6}}^{\frac{1}{2}} \left|t^{\alpha+1} - \frac{5-3\alpha}{8}t\right| dt\right)^{1-\frac{1}{q}} \left[\left(\int\limits_{\frac{1}{6}}^{\frac{1}{2}} \left|t^{\alpha+1} - \frac{5-3\alpha}{8}t\right| \left|\mathcal{F}''(t\delta + (1-t)\sigma)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left. + \left(\int\limits_{\frac{1}{6}}^{\frac{1}{2}} \left|t^{\alpha+1} - \frac{5-3\alpha}{8}t\right| \left|\mathcal{F}''(t\sigma + (1-t)\delta)\right|^q dt\right)^{\frac{1}{q}}\right]$$

$$+ \left(\int\limits_{\frac{1}{2}}^{\frac{5}{6}} \left|t^{\alpha+1} - \frac{7-\alpha}{8}t\right| dt\right)^{1-\frac{1}{q}} \left[\left(\int\limits_{\frac{1}{2}}^{\frac{5}{6}} \left|t^{\alpha+1} - \frac{7-\alpha}{8}t\right| \left|\mathcal{F}''(t\delta + (1-t)\sigma)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left. + \left(\int\limits_{\frac{1}{2}}^{\frac{5}{6}} \left|t^{\alpha+1} - \frac{7-\alpha}{8}t\right| \left|\mathcal{F}''(t\sigma + (1-t)\delta)\right|^q dt\right)^{\frac{1}{q}}\right]$$

$$+ \left(\int\limits_{\frac{5}{6}}^{1} \left|t^{\alpha+1} - t\right| dt\right)^{1-\frac{1}{q}} \left[\left(\int\limits_{\frac{5}{6}}^{1} \left|t^{\alpha+1} - t\right| \left|\mathcal{F}''(t\delta + (1-t)\sigma)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_{\frac{5}{6}}^{1}\left|t^{\alpha+1}-t\right|\left|\mathcal{F}''\left(t\sigma+(1-t)\delta\right)\right|^{q}dt\right)^{\frac{1}{q}}\Bigg].$$

From the fact that $|\mathcal{F}''|^q$ is convex, it yields

$$\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right]-\frac{\Gamma(\alpha+1)}{2(\delta-\sigma)^{\alpha}}\left[J_{\sigma+}^{\alpha}\mathcal{F}(\delta)+J_{\delta-}^{\alpha}\mathcal{F}(\sigma)\right]\right|$$

$$\leq\frac{(\delta-\sigma)^{2}}{2(\alpha+1)}\left[\left(\int_{0}^{\frac{1}{6}}\left|t^{\alpha+1}+\alpha t\right|dt\right)^{1-\frac{1}{q}}\left[\left(\int_{0}^{\frac{1}{6}}\left|t^{\alpha+1}+\alpha t\right|\left[t\left|\mathcal{F}''(\delta)\right|^{q}+(1-t)\left|\mathcal{F}''(\sigma)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right.\right.$$

$$\left.+\left(\int_{0}^{\frac{1}{6}}\left|t^{\alpha+1}+\alpha t\right|\left[t\left|\mathcal{F}''(\sigma)\right|^{q}+(1-t)\left|\mathcal{F}''(\delta)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right]$$

$$+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|t^{\alpha+1}-\frac{5-3\alpha}{8}t\right|dt\right)^{1-\frac{1}{q}}\left[\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|t^{\alpha+1}-\frac{5-3\alpha}{8}t\right|\left[t\left|\mathcal{F}''(\delta)\right|^{q}+(1-t)\left|\mathcal{F}''(\sigma)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|t^{\alpha+1}-\frac{5-3\alpha}{8}t\right|\left[t\left|\mathcal{F}''(\sigma)\right|^{q}+(1-t)\left|\mathcal{F}''(\delta)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right]$$

$$+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|t^{\alpha+1}-\frac{7-\alpha}{8}t\right|dt\right)^{1-\frac{1}{q}}\left[\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|t^{\alpha+1}-\frac{7-\alpha}{8}t\right|\left[t\left|\mathcal{F}''(\delta)\right|^{q}+(1-t)\left|\mathcal{F}''(\sigma)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|t^{\alpha+1}-\frac{7-\alpha}{8}t\right|\left[t\left|\mathcal{F}''(\sigma)\right|^{q}+(1-t)\left|\mathcal{F}''(\delta)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right]$$

$$+\left(\int_{\frac{5}{6}}^{1}\left|t^{\alpha+1}-t\right|dt\right)^{1-\frac{1}{q}}\left[\left(\int_{\frac{5}{6}}^{1}\left|t^{\alpha+1}-t\right|\left[t\left|\mathcal{F}''(\delta)\right|^{q}+(1-t)\left|\mathcal{F}''(\sigma)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\int_{\frac{5}{6}}^{1}\left|t^{\alpha+1}-t\right|\left[t\left|\mathcal{F}''(\sigma)\right|^{q}+(1-t)\left|\mathcal{F}''(\delta)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right]\right].$$

Finally, This is the end of proof of Theorem 6. □

Corollary 2. *If we assign $\alpha = 1$ in Theorem 6, then the following Maclaurin-type inequality holds:*

$$\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\delta}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\delta}{6}\right)\right]-\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathcal{F}(t)\,dt\right|$$

$$\leq \frac{(\delta-\sigma)^2}{324}\left[\frac{5^{1-\frac{1}{q}}}{4}\left(\left(\frac{9\left|\mathcal{F}''(\delta)\right|^q+71\left|\mathcal{F}''(\sigma)\right|^q}{16}\right)^{\frac{1}{q}}+\left(\frac{9\left|\mathcal{F}''(\sigma)\right|^q+71\left|\mathcal{F}''(\delta)\right|^q}{16}\right)^{\frac{1}{q}}\right)\right.$$

$$+\frac{71^{1-\frac{1}{q}}}{64}\left(\left(\frac{235\left|\mathcal{F}''(\delta)\right|^q+333\left|\mathcal{F}''(\sigma)\right|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{235\left|\mathcal{F}''(\sigma)\right|^q+333\left|\mathcal{F}''(\delta)\right|^q}{8}\right)^{\frac{1}{q}}\right)$$

$$+\frac{109^{1-\frac{1}{q}}}{64}\left(\left(\frac{539\left|\mathcal{F}''(\delta)\right|^q+333\left|\mathcal{F}''(\sigma)\right|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{539\left|\mathcal{F}''(\sigma)\right|^q+333\left|\mathcal{F}''(\delta)\right|^q}{8}\right)^{\frac{1}{q}}\right)$$

$$\left.+\left(\left(\frac{57\left|\mathcal{F}''(\delta)\right|^q+7\left|\mathcal{F}''(\sigma)\right|^q}{64}\right)^{\frac{1}{q}}+\left(\frac{57\left|\mathcal{F}''(\sigma)\right|^q+7\left|\mathcal{F}''(\delta)\right|^q}{64}\right)^{\frac{1}{q}}\right)\right].$$

3. Conclusion

In the present paper, an equality is proved for the case of the well-known Riemann-Liouville fractional integrals. By using this equality, some Maclaurin-type inequalities are given for the case of twice-differentiable functions whose second derivatives in absolute value are convex. Moreover, several Maclaurin-type inequalities are presented by using special cases of obtained theorems.

In the forthcoming works, the ideas and strategies for our results about corrected Maclaurin-type inequalities by Riemann-Liouville fractional fractional integrals may open new avenues for further research in this area. More precisely, one can obtain Maclaurin-type inequalities for convex functions by using quantum calculus. Furthermore, interested readers can apply these resulting inequalities to different types of fractional integrals such as k-Riemann-Liouville fractional integral, conformable fractional integral, Hadamard fractional integrals, Katugampola fractional integrals, etc.

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