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# REPRESENTATION OF SOLUTIONS AND ASYMPTOTIC BEHAVIOR FOR NONLOCAL DIFFUSION EQUATIONS DESCRIBING TEMPERED LÉVY FLIGHTS

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ABSTRACT. In the paper, we investigate the Cauchy problem of the time-space nonlocal diffusion equation which describes the tempered Lévy flight. The time derivative is defined in the Caputo sense and the spatial derivative is taken as a generalization of the fractional Laplacian. First, representations and asymptotic behaviors for fundamental solutions of the nonlocal diffusion equation are considered. Then, we use the fundamental solution to obtain the representation formula of solutions of the Cauchy problem. In the last, the quantitative decay rates for solutions are proved by employing the Fourier analysis technique.

## 1. Introduction

Let $n \in \mathbb{N}$, $\beta \in (0,1)$ and $\mathbb{P} = \{x \in \mathbb{R}^n : |x| = 1\}$. $\mathscr{B}(\mathbb{R}^n)$ denotes the $\sigma$-algebra of Borel sets of $\mathbb{R}^n$ and $\mathscr{B}(\mathbb{P})$ means the $\sigma$-algebra of Borel sets included in $\mathbb{P}$. Let $\nu_\beta$ be the rotationally invariant Lévy measure of the $2\beta$-stable distribution and let $\mu_\beta$ be the measure on the unit sphere $\mathbb{P}$ given by

$$\mu_\beta(\Omega) = 2\beta\nu_\beta((1,\infty)\Omega), \;\; \Omega \in \mathscr{B}(\mathbb{P}).$$

Then the following relation holds [36].

$$\nu_\beta(G) = \int_{\mathbb{P}}\int_0^\infty \mathbb{I}_G(sy)s^{-1-2\beta}ds\mu_\beta(dy), \;\; G \in \mathscr{B}(\mathbb{R}^n),$$

where $\mathbb{I}_G$ is defined by

$$\mathbb{I}_G(s) = \begin{cases} 1, & s \in G, \\ 0, & s \notin G. \end{cases}$$

In this paper, we consider the Cauchy problem for the nonlocal diffusion equation

$$\partial_t^\alpha u(t,x) = \triangle^{(\beta,\gamma)}u(t,x) + f(t,x), \;\; t>0, x \in \mathbb{R}^n, \tag{1.1}$$

$$u(0,x) = u_0(x), \;\; x \in \mathbb{R}^n. \tag{1.2}$$

Here $\alpha \in (0,1]$, $\beta \in (0,1)$, $\gamma > 0$, $\partial_t^\alpha$ denotes the Caputo fractional differential operator defined by [19] $\partial_t^1$ being the classical differential operator $d/dt$ and

$$\partial_t^\alpha v(t) = \frac{d}{dt}J_t^{1-\alpha}(v-v(0))(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-\alpha}(v(s)-v(0))ds, \;\; t>0$$

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for $\alpha \in (0,1)$, where $J^a$ is the Riemann-Liouville fractional integral operator of order $a \geq 0$ defined by [19] $J^0$ being the identity operator and

$$J_t^a v(t) = \frac{1}{\Gamma(a)} \int_0^t (t-s)^{a-1} v(s) ds$$

for $a > 0$. Also, $\triangle^{(\beta,\gamma)}$ is a generalization of the fractional Laplacian defined by

$$\triangle^{(\beta,\gamma)} v(x) = \int_{\mathbb{R}^n} \big(v(x+y)+v(x-y)-2v(x)\big) \nu_{\beta,\gamma}(dy), \ \ x \in \mathbb{R}^n, \beta \in (0,1), \gamma \geq 0, \tag{1.3}$$

where $\nu_{\beta,\gamma}$ is a Lévy density given by [31]

$$\nu_{\beta,\gamma}(G) = \int_{\mathbb{P}} \int_0^\infty \mathbb{I}_G(sy) s^{-1-2\beta} e^{-\gamma s} ds \mu_\beta(dy), \ \ G \in \mathscr{B}(\mathbb{R}^n). \tag{1.4}$$

We note that

$$\nu_{\beta,0}(dy) = \nu_\beta(dy) = C|y|^{-n-2\beta} dy \tag{1.5}$$

for some $C > 0$. When $n = 1$,

$$\nu_{\beta,\gamma}(dy) = |y|^{-1-2\beta} e^{-\gamma y} dy. \tag{1.6}$$

Substituting (1.4) to (1.3), we obtain

$$\triangle^{(\beta,\gamma)} v(x) = \int_{\mathbb{P}} \mu_\beta(dy) \int_0^\infty (v(x+sy)+v(x-sy)-2v(x)) s^{-1-2\beta} e^{-\gamma s} ds. \tag{1.7}$$

We remark that $\triangle^{(\beta,0)} = -C(-\triangle)^\beta$ for some $C > 0$, where $(-\triangle)^\beta$ means the fractional Laplacian. Also, $\triangle^{(1,0)}$ denotes the Laplacian. Then the equation (1.1) generalizes the following Caputo-Riesz time-space fractional diffusion equation

$$\partial_t^\alpha u(t,x) = -(-\triangle)^\beta u(t,x) + f(t,x), \ \ t > 0, \, x \in \mathbb{R}^n. \tag{1.8}$$

In particular, if $\alpha = 1$, $\beta = 1$ and $\gamma = 0$, then the equation (1.1) stands for the classical heat equation. In this paper, we study the representation and the asymptotic behavior of solutions of the Cauchy problem (1.1)-(1.2).

As a generalization of the Brownian random walk, the continuous time random walk (CTRW in short) is a stochastic process, which is given by the incorporation of the waiting time probability density function (PDF in short) $\psi(t)$ and the jump length PDF $\omega(x)$ [15, 28]. It is well known that the basic formula of a decoupled CTRW process in the Fourier-Laplace space has the following representation [14, 15]

$$\hat{\tilde{u}}(s,\xi) = \frac{1-\hat{\psi}(s)}{s} \frac{1}{1-\hat{\psi}(s)\tilde{\omega}(\xi)}, \tag{1.9}$$

where $\hat{\tilde{u}}(s,\xi)$ denotes the Fourier-Laplace transform of the PDF $u(t,x)$ of being at position $x$ at time $t$, $\hat{\psi}(s)$ stands for the Laplace transform of $\psi(t)$ and $\tilde{\omega}(\xi)$ means the Fourier transform of $\omega(x)$.

Setting the long-tailed waiting time PDF $\psi(t)$ with the asymptotic behavior

$$\psi(t) \sim t^{-(1+\alpha)}, \tag{1.10}$$

the corresponding Laplace space asymptotics is of the form

$$\hat{\psi}(s) \sim 1 - s^\alpha. \tag{1.11}$$

Also, we take the jump length PDF $\omega(x)$ such that

$$\tilde{\omega}(\xi) = e^{-\zeta_{\beta,\gamma}(\xi)}, \tag{1.12}$$

where $\zeta_{\beta,\gamma}$ is defined by

$$\zeta_{\beta,\gamma}(\xi) = \int_{\mathbb{R}^n} (1-\cos(\xi y))\nu_{\beta,\gamma}(dy). \tag{1.13}$$

In fact, it follows from [22, Lemma 6.9] that the function $\zeta_{\beta,\gamma}$ is negative definite. By the negative definiteness of $\zeta_{\beta,\gamma}$, $e^{-\zeta_{\beta,\gamma}(\xi)}$ is a positive definite function. Then, by the Bochner theorem, $\omega(x)$ becomes a PDF. We have

$$e^{-\zeta_{\beta,\gamma}(\xi)} \to 1-\zeta_{\beta,\gamma}(\xi), \;\; \xi \to 0. \tag{1.14}$$

Combining the estimate (1.14) with (1.9) and (1.11), we obtain

$$\hat{\tilde{u}}(s,\xi) = \frac{s^{\alpha-1}}{s^{\alpha}+\zeta_{\beta,\gamma}(\xi)} \tag{1.15}$$

in the $(s,\xi) \to (0,0)$ diffusion limit. Using the inverse Laplace transform and Lemma 2.2, we obtain (1.1) with $f=0$. Here we note that the CTRW is not the only stochastic process that can lead to the time-space fractional diffusion equation (see [3, 29]).

Studying the corresponding equation instead of the random walk gives various of advantages [28]. When $\alpha \in (0,1)$, $\beta = 1$ and $\gamma = 0$, the equation (1.1) corresponds to a time fractional diffusion equation which models the anomalous diffusion process, whose MSD is finite and jump length PDF follows the Gaussian pdf. In the case of $\alpha = 1$, $\beta = 1$ and $\gamma = 0$, the equation (1.1) means a calssical diffusion equation which describes the Brownian random walk. If $\alpha = 1$, $\beta \in (0,1)$ and $\gamma = 0$, then the equation (1.1) reduces to a space fractional diffusion equation, which captures the Lévy process with the infinite mean square displacement (MSD in short). If $\alpha \in (0,1)$, $\beta \in (0,1)$ and $\gamma = 0$, then the equation (1.1) becomes a time-space fractional diffusion equation, which describes the Lévy flight with the diverging MSD. However, from the view point of physics, velocity of propagation of massive particles should be finite, in other words, the divergence of MSD violates physical principles. In order to develop the stochastic processes with non-diverging MSDs in which very long displacement events can occur, Mantegna and Stanley [26] introduced the truncated Lévy flight which looks like a Lévy process in a short time and behaves like a Brownian random walk in a long time. Koponen [21] proposed the tempered Lévy flight which has the analytical expression of the characteristic function and has the same stochastic property as the truncated Lévy flight. Also, the tempered Lévy flight has shown applicability in many fields such as finance, plasma physics, fluid mechanics and so on (see e.g.[5, 6, 7, 32]).

In [32], the tempered Lévy flight, for which the waiting time has the Poisson PDF and the jump length follows the univariate tempered Lévy distribution, was described by the equation (1.1) of the case: $\alpha = 1$. Cartea and del-Castillo-Negrete [7] used the CTRW model to derive a fractional diffusion equation capturing the tempered Lévy flight, for which the waiting time has the Mittag-Leffler PDF and the jump length follows the univariate tempered Lévy distribution with drift component and Brownian motion component. We remark that the transition from the superdiffusion to the subdiffusion, which is the most important property of the tempered Lévy flight discussed in [7], is due to the tempered Lévy density. On the other hand, there are also cases of characteristic crossover from subdiffusion to normal diffusion which can be described by the tempered waiting time PDF [33, 35, 43]. When drift term and Brownian motion term are ignored, the fractional diffusion equation presented in [7] corresponds to the equation (1.1) in one dimensional space. The equation (1.1) captures the random walk, for which the waiting time PDF has the asymptotic behavior (1.10) and the jump length follows the symmetric multivariate tempered Lévy distribution developed in [31]. It follows from (1.1), (1.6) and (1.13) that the MSD $M_2(t)$ of the tempered Lévy flight for the equation (1.1) in one dimensional space satisfies the following relation:

$$\hat{M}_2(s) = -\frac{\partial^2 \tilde{u}(s,\xi)}{\partial \xi^2}\bigg|_{\xi=0} = s^{-(1+\alpha)}\zeta''_{\beta,\gamma}(0) = Cs^{-(1+\alpha)}\int_{\mathbb{R}} |y|^{1-2\beta}e^{-\gamma y}dy.$$

Then the MSD $M_2(t)$ of the tempered Lévy flight is written as follows:

$$M_2(t) = Ct^{\alpha}\int_{\mathbb{R}} |y|^{1-2\beta}e^{-\gamma y}dy. \tag{1.16}$$

In this paper, we consider the multidimensional nonlocal diffusion equation describing the tempered Lévy flight.

Eidelman and Kochubei [10] obtained the various estimates for fundamental solutions of the time fractional diffusion equation corresponding to the equation (1.1) of the case: $\alpha \in (0,1), \beta = 1, \gamma = 0$. Kochubei [20] considered the representation of solutions and the asymptotic behavior for time nonlocal diffusion equations involving distributed order derivative. Kemppainen, Siljander, Vergara and Zacher [17] proved the optimal decay estimates for solutions of general time nonlocal diffusion equations by employing Fourier analysis method and energy method. In [38], Sin considered the long time behavior for the time nonlocal diffusion equation with the generalized Caputo-type differential operator. In [39], the well-posedness and the long-time behavior of Dirichlet problems for multi-term time-fractional wave equations were established by proving a new property of the multivariate Mittag-Leffler functions.

When $\alpha = 1$, the fundamental solution of the equation (1.1) is the same as the transition density of the corresponding Lévy process. In [4], Blumenthal and Getoor established the following estimate for the transition density for the Lévy process corresponding to the equation (1.1) of the case: $\alpha = 1, \beta \in (0,1)$, $\gamma = 0$.

$$u(t,x) \sim \min\{t^{-\frac{n}{2\beta}}, t|x|^{-n-2\beta}\}. \tag{1.17}$$

Watanabe [44] investigated the asymptotic result for the Lévy process whose Lévy density is of the form $\nu(dsdy) = s^{-1-2\beta}ds\mu(dy)$. In [16, 41], Kaleta and Sztonyk obtained the asymptotic estimates for transition density and its derivatives of the tempered Lévy flight.

Chen, Meerschaert and Nane [8] established the probabilistic representations for solutions of the equation (1.8). Allen, Caffarelli and Vasseur [1] studied the Hölder regularity for the nonlocal diffusion equation with the Caputo fractional derivative and a generalization of the fractional Laplacian. In [25], Mainardi, Pagnini and Saxena expressed the fundamental solutions of the Cauchy problem for the time-space fractional diffusion equation in terms of Fox H functions. In [18], Kemppainen, Siljander and Zacher used the Fox H-function and the Fourier analysis technique to prove the asymptotic behavior results for fundamental solutions of the equation (1.8). By employing the Laplace transform, Cheng, Li and Yamamoto [9] obtained the long-time behavior result for the time-space fractional diffusion-reaction equation including (1.8). Liemert and Kienle [23] established the representation formula of the fundamental solution of the one-dimensional fractional diffusion equation with the tempered Riemann-Liouville derivative in space and the Caputo derivative in time. In [37], the existence of solutions of nonlocal diffusion equations involving the generalized Caputo-type derivative and the generalized fractional Laplacian was studied.

This paper is organized as follows. In Section 2, we introduce necessary concepts and lemmata for obtaining the main results of the paper. In Section 3, with the help of the asymptotic behavior result for the transition density of the tempered Lévy flight investigated in [41], we establish the representation formulas of fundamental solutions and the asymptotic behavior results for the time space nonlocal diffusion equation (1.1). Also, based on the asymptotic results, the MSD of the tempered Lévy flight is estimated. In Section 4, we prove the representation formulas of classical solutions of the Cauchy problem (1.1)-(1.2). In Section 5, we use the Fourier analysis method to obtain the $L^2$-decay estimates for solutions.

## 2. Preliminaries

Throughout this paper, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ will stand for the sets of natural, real and complex numbers respectively. $C>0$ expresses the universal positive constant that can be different at different places. Also, $a\lesssim b$ denotes $a\le Cb$ for some constant $C>0$ and $a\gtrsim b$ means $a\ge Cb$ for some constant $C>0$. In addition, we write $a\sim b$ if $a\lesssim b\lesssim a$.

Let $g\in L^1_{loc}(\mathbb{R})$ be a function such that $\int_0^\infty e^{-s_0t}g(t)ds<\infty$ for some $s_0\in\mathbb{R}$. The Laplace transform $\mathscr{L}$ is defined by [30]

$$\mathscr{L}g(s)=\hat{g}(s)=\int_0^\infty e^{-ts}g(t)dt,\ \ Re(s)\ge s_0.$$

Let $v\in L^1(\mathbb{R}^n)$. The Fourier transform $\mathscr{F}$ is defined by [2]

$$\mathscr{F}v(\xi)=\tilde{v}(\xi)=\int_{\mathbb{R}^n}v(x)e^{-ix\xi}dx,\ \ \xi\in\mathbb{R}^n$$

and $\mathscr{F}^*$ is defined by [2]

$$\mathscr{F}^*v(\xi)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}v(x)e^{ix\xi}dx,\ \ \xi\in\mathbb{R}^n$$

If $v\in L^1(\mathbb{R}^n)$ and $\tilde{v}\in L^1(\mathbb{R}^n)$, then $\mathscr{F}^*\mathscr{F}v=v$, which implies that $\mathscr{F}^*$ is the inverse of the Fourier transform $\mathscr{F}$ on $\{v\in L^1(\mathbb{R}^n)|\tilde{v}\in L^1(\mathbb{R}^n)\}$.

Let $a,b\in\mathbb{C}$ and $Re(a)>0$. The two parameter Mittag-Leffler function $E_{a,b}$ is defined by [19]

$$E_{a,b}(z)=\sum_{j=0}^\infty\frac{z^j}{\Gamma(aj+b)},\ \ z\in\mathbb{C}.$$

**Lemma 2.1.** *Let $k>0,a>0$ and $j\in\mathbb{N}$. The following relations hold.*

$$\frac{d^jE_{a,1}(-kt^a)}{dt^j}=-kt^{a-j}E_{a,a-j+1}(-kt^\alpha),\ \ t>0, \tag{2.1}$$

$$\partial_t^aE_{a,1}(-kt^a)=-kE_{a,1}(-kt^a),\ \ t>0, \tag{2.2}$$

$$E_{a,1}(-t)\sim\frac{1}{1+t},\ \ t\ge 0. \tag{2.3}$$

*Proof.* The proof of the relation (2.1) can be found in [34]. The relation (2.2) was proved in [19]. In [42], we can see the proof the relation (2.3). □

Let $a > -1$ and $b \in \mathbb{C}$. The Wright function $W_{a,b}$ is defined by [19]

$$W_{a,b}(z) = \sum_{j=0}^{\infty} \frac{z^j}{j!\Gamma(aj+b)}, \;\; z \in \mathbb{C}. \tag{2.4}$$

Let $r \in (0,1)$. The functions $F_r$ and $M_r$ are special cases of the Wright function defined by [24]

$$F_r(z) = W_{-r,0}(-z) = \sum_{j=1}^{\infty} \frac{(-z)^j}{j!\Gamma(-rj)}, \;\; z \in \mathbb{C}, \tag{2.5}$$

$$M_r(z) = W_{-r,1-r}(-z) = \sum_{j=0}^{\infty} \frac{(-z)^j}{j!\Gamma(-rj+1-r)}, \;\; z \in \mathbb{C}.$$

The functions $F_r$ and $M_r$ are related through

$$F_r(z) = rzM_r(z), \;\; z \in \mathbb{C}. \tag{2.6}$$

By the relation $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$, the following equality holds.

$$F_r(z) = \frac{1}{\pi}\sum_{j=1}^{\infty} (-1)^{j-1} z^j \frac{\Gamma(rj+1)}{j!} \sin(r\pi j), \;\; z \in \mathbb{C}. \tag{2.7}$$

Also, the following relations hold [24].

$$M_r\left(\frac{t}{r}\right) \approx \frac{1}{\sqrt{2\pi(1-r)}} t^{\frac{r-\frac{1}{2}}{1-r}} e^{-\frac{1-r}{r} t^{\frac{1}{1-r}}}, \;\; t \to +\infty, \tag{2.8}$$

$$\int_0^{\infty} M_r(t)dt = 1, \tag{2.9}$$

$$\frac{dW_{a,b}(t)}{dt} = W_{a,a+b}(t), \;\; t \in \mathbb{R}. \tag{2.10}$$

Substituting (1.4) to (1.13), we obtain the equivalent representation of $\zeta_{\beta,\gamma}$

$$\zeta_{\beta,\gamma}(\xi) = 2\int_{\mathbb{P}} \mu_\beta(dy) \int_0^{\infty} s^{-1-2\beta} e^{-\gamma s}(1-\cos(sy\xi))ds. \tag{2.11}$$

**Lemma 2.2.** [40, Lemma 2.1] *Let $\beta \in (0,1)$ and $\gamma \geq 0$. Then the following relation holds.*

$$\triangle^{(\beta,\gamma)} v(x) = \mathscr{F}^*(-\zeta_{\beta,\gamma}(\xi)\mathscr{F}v(\xi))(x), \, v \in \mathscr{S}(\mathbb{R}^n), \tag{2.12}$$

*where $\mathscr{S}(\mathbb{R}^n)$ stands for the Schwartz space of rapidly decaying smooth functions.*

From now on, we will regard $\triangle^{(\beta,\gamma)}$ as the pseudo-differential operator given by (2.12). In other words, $\triangle^{(\beta,\gamma)}$ makes sense for functions $v \in L^1(\mathbb{R}^n)$ such that $\zeta_{\beta,\gamma}\tilde{v} \in L^1(\mathbb{R}^n)$.

**Lemma 2.3.** [40, Lemma 2.2] *Let $\beta \in (0,1)$, $\gamma \geq 0$ and $\kappa = (1,0,\ldots,0) \in \mathbb{R}^n$. Then $\zeta_{\beta,\gamma}$ is rotationally invariant. That is, $\zeta_{\beta,\gamma}(\xi) = \zeta_{\beta,\gamma}(|\xi|\kappa)$ for $\xi \in \mathbb{R}^n$.*

Let $\zeta_{\beta,\gamma}(\xi) = \rho_{\beta,\gamma}(|\xi|)$ for $\xi \in \mathbb{R}^n$. For $\beta \in (0,1), \gamma > 0$ and $r > 0$, we obtain

$$\rho_{\beta,\gamma}(r) = 2\int_{\mathbb{P}} \mu_\beta(dy) \int_0^{\infty} s^{-1-2\beta} e^{-\gamma s}(1-\cos(sy_1 r))ds \tag{2.13}$$

$$= 2r^{2\beta}\int_{\mathbb{P}} \mu_\beta(dy) \int_0^{\infty} w^{-1-2\beta} e^{-\frac{\gamma w}{r}}(1-\cos(wy_1))dw. \tag{2.14}$$

For convenience of notations, we denote

$$K_1 = \int_{\mathbb{P}} y_1^2 \mu_\beta(dy) \int_0^\infty s^{1-2\beta} e^{-\gamma s} ds,$$

$$K_2 = 2\int_{\mathbb{P}} \mu_\beta(dy) \int_0^\infty w^{-1-2\beta}(1-\cos(wy_1))dw,$$

$$K_3 = \cos(1) \int_{\mathbb{P}} y_1^2 \mu_\beta(dy) \int_{s<\frac{1}{\gamma}} s^{1-2\beta} e^{-\gamma s} ds,$$

$$K_4 = 2\int_{\mathbb{P}} \mu_\beta(dy) \int_0^\infty w^{-1-2\beta} e^{-w}(1-\cos(wy_1))dw.$$

**Lemma 2.4.** *Let $\beta \in (0,1)$ and $\gamma > 0$. Then the following inequalities hold.*

$$\rho_{\beta,\gamma}(r) < \min\{K_1 r^2, K_2 r^{2\beta}\}, \quad r > 0,$$

$$\rho_{\beta,\gamma}(r) > \begin{cases} K_3 r^2, & r < \gamma, \\ K_4 r^{2\beta}, & r \geq \gamma. \end{cases}$$

*Proof.* We can prove the desired result as in the proof of [40, Lemma 2.3]. □

## 3. Fundamental solution of nonlocal diffusion equation

In this section, we consider the fundamental solution of the equation (1.1).

**3.1.** ***Fundamental solution of space nonlocal diffusion equation.*** In this subsection, we discuss the space nonlocal diffusion equation of the form

$$\frac{\partial u(t,x)}{\partial t} = \triangle^{(\beta,\gamma)} u(t,x), \quad t > 0, x \in \mathbb{R}^n, \tag{3.1}$$

which is corresponding to the equation (1.1) when $\alpha = 1, \beta \in (0,1), \gamma > 0$ and $f = 0$. Then the fundamental solution $A_{1,\beta,\gamma}$ of the equation (3.1) is represented by

$$A_{1,\beta,\gamma}(t,x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-\zeta_{\beta,\gamma}(\xi)t} \cos(x\xi) d\xi. \tag{3.2}$$

By Lemma 2.4, we have

$$e^{-\zeta_{\beta,\gamma}(\cdot)t} \in L^1(\mathbb{R}^n), \quad t > 0$$

and thus (3.2) makes sense as a convergent integral. Setting $\gamma = 0$ in (3.2), we obtain the fundamental solution of the equation (1.8). Similar to Lemma 2.3, we can prove that $A_{1,\beta,\gamma}(t,x) = A_{1,\beta,\gamma}(t,|x|\kappa)$ for $t > 0$ and $x \in \mathbb{R}^n$, where $\kappa = (1,0,\ldots,0) \in \mathbb{R}^n$.

Similar to the fundamental solution of the classical heat equation, the fundamental solution $A_{1,\beta,\gamma}(t,x)$ of the space nonlocal diffusion equation (3.1) is exponentially decreasing with respect to the spatial variable $x$.

**Lemma 3.1.** *Let $m_1 \geq n + 2\beta$ and $m_2 \geq n$. Then the following relation holds.*

$$A_{1,\beta,\gamma}(t,x) \lesssim \begin{cases} t^{-\frac{n}{2\beta}} \min\{1, t^{1+\frac{n}{2\beta}} |x|^{-m_1}\}, & t \in (0,1], \\ t^{-\frac{n}{2}} \min\{1, t^{1+\frac{m_2}{2}} |x|^{-m_2-2}\}, & t \in (1,\infty). \end{cases} \tag{3.3}$$

*Proof.* The upper estimate of $A_{1,\beta,\gamma}(t,x)$ is presented in [41, Corollary 11] as follows:

$$A_{1,\beta,\gamma}(t,x) \lesssim \begin{cases} t^{-\frac{n}{2\beta}} \min\{1, t^{1+\frac{n}{2\beta}}|x|^{-2\beta-n}e^{-c_1|x|}\}, & t\in(0,1], \\ t^{-\frac{n}{2}}\left(\min\{1,t^{1+\frac{n}{2}}|x|^{-2\beta-n}e^{-c_2|x|}\} + e^{-\frac{c_3|x|}{\sqrt{t}}\ln\left(1+\frac{c_4|x|}{\sqrt{t}}\right)}\right), & t\in(1,\infty) \end{cases} \tag{3.4}$$

for some $c_1,c_2,c_3,c_4>0$. Meanwhile, if $m>0$, then $e^{-x}\lesssim x^{-m}$ for $x>0$. Then we have

$$e^{-\frac{c_3|x|}{\sqrt{t}}\ln(1+\frac{c_4|x|}{\sqrt{t}})} \lesssim \frac{t^{\frac{m_2}{2}+1}}{|x|^{m_2+2}},\ \ t>0, x\in\mathbb{R}^n\backslash\{0\}. \tag{3.5}$$

Substituting (3.5) to (3.4), we obtain the desired result. □

Since $\zeta_{\beta,\gamma}$ is negative definite, it follows from the Bochner theorem that $A_{1,\beta,\gamma}(t,x)\geq 0$ and the following relation holds.

$$e^{-\zeta_{\beta,\gamma}(\xi)t} = \int_{\mathbb{R}^n} A_{1,\beta,\gamma}(t,x)\cos(x\xi)dx = \tilde{A}_{1,\beta,\gamma}(t,\xi). \tag{3.6}$$

Moreover,

$$\int_{\mathbb{R}^n} A_{1,\beta,\gamma}(t,x)dx = 1,\ \ t>0.$$

**Remark 3.1.** *By using the formula* (3.3)*, we can estimate the MSD of the tempered Lévy flight for the equation (3.1). For $t\in(0,1]$ and $m_1>n+2$ , we obtain*

$$M_2(t) = \int_{\mathbb{R}^n}|x|^2A_{1,\beta,\gamma}(t,x)dx \lesssim \int_0^{t^{\frac{n+2\beta}{2\beta m_1}}} t^{-\frac{n}{2\beta}}r^{n+1}dr + \int_{t^{\frac{n+2\beta}{2\beta m_1}}}^{\infty} t r^{n+1-m_1}dr \lesssim t^{\frac{(n+2\beta)(n+2)}{2\beta m_1}-\frac{n}{2\beta}}.$$

*If $m_1>n+2$, then $\frac{(n+2\beta)(n+2)}{2\beta m_1}-\frac{n}{2\beta}<1$ and when $m_1\to n+2$, $\frac{(n+2\beta)(n+2)}{2\beta m_1}-\frac{n}{2\beta}\to 1$. Also, for $t>1$ and $m_2>n$ , we have*

$$M_2(t) = \int_{\mathbb{R}^n}|x|^2A_{1,\beta,\gamma}(t,x)dx \lesssim \int_0^{t^{\frac{1}{2}}} t^{-\frac{n}{2}}r^{n+1}dr + \int_{t^{\frac{1}{2}}}^{\infty} t^{\frac{m_2-n}{2}}r^{n-1-m_2}dr \lesssim t.$$

**Remark 3.2.** *From* (3.3)*, we can easily obtain*

$$A_{1,\beta,\gamma}(t,0) \lesssim \begin{cases} t^{-\frac{n}{2\beta}}, & t\in(0,1], \\ t^{-\frac{n}{2}}, & t\in(1,\infty). \end{cases}$$

*By* [18, Lemma 3.3]*, the above relation shows a transition from superdiffusion to normal diffusion.*

Now we discuss the $L^p$-estimate of the fundamental solution $A_{1,\beta,\gamma}$.

**Theorem 3.1.** *Let $\beta\in(0,1)$ and $\gamma>0$. Then, for any $p\in[1,\infty]$,*

$$\|A_{1,\beta,\gamma}(t,\cdot)\|_{L^p(\mathbb{R}^n)} \lesssim \begin{cases} t^{-\frac{n}{2\beta}(1-\frac{1}{p})}, & t\in(0,1], \\ t^{-\frac{n}{2}(1-\frac{1}{p})}, & t\in(1,\infty). \end{cases}$$

*Proof.* Setting $m_1=n+2\beta$ and $m_2=n$ in (3.3), from [18, Lemma 3.3] and [18, Lemma 5.1], we obtain the desired result. □

**Lemma 3.2.** *Let $\beta \in (0,1)$, $\gamma_1 > \gamma_2 \geq 0$ and $p \geq 1$. Then the following inequality holds.*

$$\|A_{1,\beta,\gamma_2}(t,\cdot)\|_{L^p(\mathbb{R}^n)} \leq \|A_{1,\beta,\gamma_1}(t,\cdot)\|_{L^p(\mathbb{R}^n)}, \ \ t>0.$$

*Proof.* It follows from Theorem 3.1 that $A_{1,\beta,\gamma_1}(t,\cdot), A_{1,\beta,\gamma_2}(t,\cdot) \in L^p(\mathbb{R}^n)$. Also,

$$\zeta_{\beta,\gamma_2}(\xi) - \zeta_{\beta,\gamma_1}(\xi) = 2\int_{\mathbb{P}} \mu(dy) \int_0^\infty s^{-1-2\beta}(e^{-\gamma_2 s} - e^{-\gamma_1 s})(1-\cos(sy\xi))ds = \int_{\mathbb{R}^n}(1-\cos(\xi y))\nu_{12}(dy),$$

where

$$\nu_{12}(G) = 2\int_{\mathbb{P}} \mu(dy)\int_0^\infty \mathbb{I}_G(sy) s^{-1-2\beta}(e^{-\gamma_2 s} - e^{-\gamma_1 s})ds, \ \ G \in \mathscr{B}(\mathbb{R}^n).$$

By the Lévy-Khintchine formula, for $t>0$, $t(\zeta_{\beta,\gamma_2}(\xi) - \zeta_{\beta,\gamma_1}(\xi))$ is negative definite and the function $e^{-t(\zeta_{\beta,\gamma_2}(\xi) - \zeta_{\beta,\gamma_1}(\xi))}$ is a characteristic function of an infinitely divisible distribution. In particular, the function $V(t,x)$ defined by

$$V(t,x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-t(\zeta_{\beta,\gamma_2}(\xi) - \zeta_{\beta,\gamma_1}(\xi))}\cos(x\xi)d\xi, \ \ t>0, x\in\mathbb{R}^n$$

is also a probability density function. Using the relation

$$e^{-t\zeta_{\beta,\gamma_2}(\xi)} = e^{-t\zeta_{\beta,\gamma_1}(\xi)} e^{-t(\zeta_{\beta,\gamma_2}(\xi) - \zeta_{\beta,\gamma_1}(\xi))}, \ \ t>0$$

and Young's inequality for convolution, for $t>0$, we obtain

$$\|A_{1,\beta,\gamma_2}(t,\cdot)\|_{L^p(\mathbb{R}^n)} \leq \|A_{1,\beta,\gamma_1}(t,\cdot)\|_{L^p(\mathbb{R}^n)}\|V(t,\cdot)\|_{L^1(\mathbb{R}^n)} = \|A_{1,\beta,\gamma_1}(t,\cdot)\|_{L^p(\mathbb{R}^n)}.$$

□

Since $\zeta_{\beta,\gamma}(\cdot)e^{-\zeta_{\beta,\gamma}(\cdot)t} \in L^1(\mathbb{R}^n)$ for $t>0$, we have

$$\frac{\partial A_{1,\beta,\gamma}(t,x)}{\partial t} = -\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\zeta_{\beta,\gamma}(\xi)e^{-\zeta_{\beta,\gamma}(\xi)t}\cos(x\xi)d\xi, \tag{3.7}$$

which implies that $\partial A_{1,\beta,\gamma}(t,x)/\partial t$ is continuous with respect to $t$ and $x$. In particular, for any $t>0$, $\partial A_{1,\beta,\gamma}(t,x)/\partial t$ is uniformly continuous on $\mathbb{R}^n$.

**Lemma 3.3.** *Let $\beta \in (0,1)$ and $\gamma > 0$. Then the following relation holds.*

$$\lim_{t\to 0}\frac{\partial A_{1,\beta,\gamma}(t,x)}{\partial t} < \infty, \ \ x \in \mathbb{R}^n \setminus \{0\}. \tag{3.8}$$

*Proof.* Similar to [16, Theorem 3], we can prove the result. □

**3.2. *Fundamental solution of time-space nonlocal diffusion equation.*** In this subsection, we study the equation (1.1) when $\alpha \in (0,1), \beta \in (0,1)$ and $\gamma > 0$. Let $A_{\alpha,\beta,\gamma}$ and $B_{\alpha,\beta,\gamma}$ denote the fundamental solutions of the equation (1.1) corresponding to the initial and forcing condition.

First of all, we consider the fundamental solution $A_{\alpha,\beta,\gamma}$. Applying the Fourier transform with respect to the space variable $x$ in the equation (1.1) with $f=0$, we obtain

$$\partial_t^\alpha \tilde{u}(t,\xi) = -\zeta_{\beta,\gamma}(\xi)\tilde{u}(t,\xi), \ \ t>0, \xi\in\mathbb{R}^n. \tag{3.9}$$

The solution of the equation (3.9) with the condition $\tilde{u}(0,\xi) = 1$ has the form:

$$\tilde{u}(t,\xi) = E_{\alpha,1}(-\zeta_{\beta,\gamma}(\xi)t^\alpha). \tag{3.10}$$

Taking the Laplace transform, we obtain

$$\int_0^\infty E_{\alpha,1}(-\zeta_{\beta,\gamma}(\xi)t^\alpha)e^{-st}dt = \frac{s^{\alpha-1}}{s^\alpha+\zeta_{\beta,\gamma}(\xi)}. \tag{3.11}$$

On the other hand, it is well known that

$$e^{-\tau s^\alpha} = \int_0^\infty e^{-st}\theta(t,\tau)dt, \quad \tau, s>0, \tag{3.12}$$

where

$$\theta(t,\tau) = \frac{1}{\pi}\sum_{j=1}^\infty (-1)^{j-1}\tau^j t^{-\alpha j-1}\frac{\Gamma(j\alpha+1)}{j!}\sin(j\pi\alpha), \quad t,\tau>0. \tag{3.13}$$

It follows from (2.7) that

$$\theta(t,\tau) = \frac{1}{t}F_\alpha\Big(\frac{\tau}{t^\alpha}\Big) = \frac{1}{t}W_{-\alpha,0}\Big(-\frac{\tau}{t^\alpha}\Big), \quad t,\tau>0. \tag{3.14}$$

Define the function $\phi(t,\tau)$ by

$$\phi(t,\tau) = J_t^{1-\alpha}\theta(t,\tau) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\theta(s,\tau)ds, \quad t,\tau>0. \tag{3.15}$$

Then we have

$$\int_0^\infty\int_0^\infty \phi(t,\tau)e^{-\zeta_{\beta,\gamma}(\xi)\tau}d\tau e^{-st}dt = \int_0^\infty s^{\alpha-1}e^{-\tau s^\alpha}e^{-\zeta_{\beta,\gamma}(\xi)\tau}d\tau = \frac{s^{\alpha-1}}{s^\alpha+\zeta_{\beta,\gamma}(\xi)}.$$

From [24, formula (F.52)], we obtain

$$\phi(t,\tau) = \frac{1}{t^\alpha}M_\alpha\Big(\frac{\tau}{t^\alpha}\Big) = \frac{1}{t^\alpha}W_{-\alpha,1-\alpha}\Big(-\frac{\tau}{t^\alpha}\Big), \quad t,\tau>0. \tag{3.16}$$

It follows from the uniqueness of the Laplace transform and (3.11) that

$$\tilde{u}(t,\xi) = E_{\alpha,1}(-\zeta_{\beta,\gamma}(\xi)t^\alpha) = \int_0^\infty \phi(t,\tau)e^{-\zeta_{\beta,\gamma}(\xi)\tau}d\tau, \quad t>0, \xi\in\mathbb{R}^n. \tag{3.17}$$

From (3.3) and (3.16), for $t>0$ and $x\in\mathbb{R}^n$, we deduce

$$\begin{aligned} u(t,x) &= \frac{1}{(2\pi)^n}\int_0^\infty \phi(t,\tau)\int_{\mathbb{R}^n} e^{-\zeta_{\beta,\gamma}(\xi)\tau}\cos(\xi x)d\xi d\tau = \int_0^\infty \phi(t,\tau)A_{1,\beta,\gamma}(\tau,x)d\tau \\ &= \frac{1}{t^\alpha}\int_0^\infty M_\alpha\Big(\frac{\tau}{t^\alpha}\Big)A_{1,\beta,\gamma}(\tau,x)d\tau = \int_0^\infty M_\alpha(s)A_{1,\beta,\gamma}(st^\alpha,x)ds. \end{aligned}$$

Therefore the fundamental solution of the equation (1.1) corresponding to the initial data is represented by

$$A_{\alpha,\beta,\gamma}(t,x) = \int_0^\infty M_\alpha(s)A_{1,\beta,\gamma}(st^\alpha,x)ds, \quad t>0, x\in\mathbb{R}^n. \tag{3.18}$$

It follows from $A_{1,\beta,\gamma}(s,x)\ge 0$ that $A_{\alpha,\beta,\gamma}(t,x)\ge 0$. By the relation (2.9), we obtain

$$\int_{\mathbb{R}^n} A_{\alpha,\beta,\gamma}(t,x)dx = \int_0^\infty M_\alpha(s)\int_{\mathbb{R}^n} A_{1,\beta,\gamma}(st^\alpha,x)dxds = \int_0^\infty M_\alpha(s)ds = 1, \quad t>0.$$

Also, similar to Lemma 2.3, we can prove that $A_{\alpha,\beta,\gamma}(t,x) = A_{\alpha,\beta,\gamma}(t,|x|\kappa)$ for $t>0$ and $x\in\mathbb{R}^n$, where $\kappa=(1,0,...,0)\in\mathbb{R}^n$.

We consider the asymptotic behavior of the fundamental solution $A_{\alpha,\beta,\gamma}(t,x)$.

**Theorem 3.2.** *Let $\alpha \in (0,1), \beta \in (0,1)$, $\gamma > 0$ $m_1 \geq n+2\beta$, $m_2 \geq n$ and $m_3 \geq 0$. Then the fundamental solution $A_{\alpha,\beta,\gamma}$ of the equation (1.1) satisfies the following relations.*
*If $|x| \leq 1$, then*

$$A_{\alpha,\beta,\gamma}(t,x) \lesssim t^{\alpha}|x|^{-n-2\beta} + |x|^{-n} + 1.$$

*If $|x| > 1$, then*

$$A_{\alpha,\beta,\gamma}(t,x) \lesssim t^{\alpha}|x|^{-m_1} + t^{\alpha+\frac{m_2-n}{2}\alpha}|x|^{-m_2-2} + |x|^{-n}.$$

*In particular, if $|x| > 1$ and $t^{-\alpha}|x|^2 \geq 1$, then*

$$A_{\alpha,\beta,\gamma}(t,x) \lesssim t^{\alpha}|x|^{-m_1} + t^{\alpha+\frac{m_2-n}{2}\alpha}|x|^{-m_2-2} + t^{m_3\alpha}|x|^{-n-2m_3}.$$

*Proof.* By using (3.3) and the asymptotic behavior of $M_\alpha$, we obtain the desired result. In particular, (2.8) and the relation $M_\alpha(s) \to 1/\Gamma(1-\alpha)$ as $s \to 0$ are very crucial.

First, we consider the case of $|x| \leq 1$. It follows from (3.3) that

$$\begin{aligned}
&A_{\alpha,\beta,\gamma}(t,x)\\
&\lesssim \int_0^{t^{-\alpha}|x|^{\frac{2\beta m_1}{n+2\beta}}} M_\alpha(s)st^{\alpha}|x|^{-m_1}ds + \int_{t^{-\alpha}|x|^{\frac{2\beta m_1}{n+2\beta}}}^{t^{-\alpha}} M_\alpha(s)(st^{\alpha})^{-\frac{n}{2\beta}}ds + \int_{t^{-\alpha}}^{\infty} M_\alpha(s)(st^{\alpha})^{-\frac{n}{2}}ds\\
&= t^{\alpha}|x|^{-m_1}\int_0^{t^{-\alpha}|x|^{\frac{2\beta m_1}{n+2\beta}}} M_\alpha(s)sds + \left(t^{-\alpha}|x|^{\frac{2\beta m_1}{n+2\beta}}t^{\alpha}\right)^{-\frac{n}{2\beta}}\int_{t^{-\alpha}|x|^{\frac{2\beta m_1}{n+2\beta}}}^{t^{-\alpha}} M_\alpha(s)ds + (t^{-\alpha}t^{\alpha})^{-\frac{n}{2}}\int_{t^{-\alpha}}^{\infty} M_\alpha(s)ds\\
&\lesssim t^{\alpha}|x|^{-m_1} + |x|^{-\frac{m_1 n}{n+2\beta}} + 1.
\end{aligned}$$

Setting $m_1 = n+2\beta$, we obtain the desired result.

Next, we study the case of $|x| > 1$. By (3.3), we obtain

$$\begin{aligned}
&A_{\alpha,\beta,\gamma}(t,x)\\
&\lesssim \int_0^{t^{-\alpha}} M_\alpha(s)st^{\alpha}|x|^{-m_1}ds + \int_{t^{-\alpha}}^{t^{-\alpha}|x|^2} M_\alpha(s)(st^{\alpha})^{\frac{m_2-n}{2}+1}|x|^{-m_2-2}ds + \int_{t^{-\alpha}|x|^2}^{\infty} M_\alpha(s)(st^{\alpha})^{-\frac{n}{2}}ds\\
&\lesssim t^{\alpha}|x|^{-m_1}\int_0^{t^{-\alpha}} M_\alpha(s)sds + t^{\alpha+\frac{m_2-n}{2}\alpha}|x|^{-m_2-2}\int_{t^{-\alpha}}^{t^{-\alpha}|x|^2} M_\alpha(s)s^{\frac{m_2-n}{2}+1}ds + (t^{-\alpha}|x|^2t^{\alpha})^{-\frac{n}{2}}\int_{t^{-\alpha}|x|^2}^{\infty} M_\alpha(s)ds.
\end{aligned}$$

Theorefore, for $|x| > 1$,

$$A_{\alpha,\beta,\gamma}(t,x) \lesssim t^{\alpha}|x|^{-m_1} + t^{\alpha+\frac{m_2-n}{2}\alpha}|x|^{-m_2-2} + |x|^{-n}.$$

It follows from the asymptotic behavior of $M_\alpha$ that the function

$$r^{m_3}\int_r^{\infty} M_\alpha(s)ds$$

has a maximum value in $[1,\infty)$. Then, we have that for $|x|^2 \geq t^\alpha$,

$$\begin{aligned} A_{\alpha,\beta,\gamma}(t,x) &\lesssim t^\alpha |x|^{-m_1} + t^{\alpha+\frac{m_2-n}{2}\alpha}|x|^{-m_2-2} + |x|^{-n}(t^{-\alpha}|x|^2)^{-m_3}(t^{-\alpha}|x|^2)^{m_3}\int_{t^{-\alpha}|x|^2}^\infty M_\alpha(s)ds \\ &\lesssim t^\alpha |x|^{-m_1} + t^{\alpha+\frac{m_2-n}{2}\alpha}|x|^{-m_2-2} + |x|^{-n}(t^{-\alpha}|x|^2)^{-m_3}\sup_{r\geq 1}\int_r^\infty r^{m_3}M_\alpha(s)ds \\ &\lesssim t^\alpha |x|^{-m_1} + t^{\alpha+\frac{m_2-n}{2}\alpha}|x|^{-m_2-2} + t^{m_3\alpha}|x|^{-n-2m_3}. \end{aligned}$$

□

**Remark 3.3.** *From Theorem 3.2, we can obtain the MSD of the tempered Lévy flight for the equation (1.1) with $\alpha \in (0,1)$. For $t \in (0,1)$, $m_1 > n+2$ and $m_2 > n$, we have*

$$\begin{aligned} M_2(t) &= \int_{\mathbb{R}^n} |x|^2 A_{\alpha,\beta,\gamma}(t,x)dx \\ &\lesssim \int_0^{t^{\frac{\alpha}{2}}} (t^\alpha r^{-n-2\beta} + r^{-n} + 1)r^{n+1}dr + \int_{t^{\frac{\alpha}{2}}}^\infty (t^\alpha r^{-m_1} + t^{\alpha+\frac{m_2-n}{2}\alpha}r^{-m_2-2} + t^{m_3\alpha}r^{-n-2m_3})r^{n+1}dr \\ &\lesssim t^{\alpha(2-\beta)} + t^\alpha + t^{\alpha(\frac{n}{2}+1)} + t^{\alpha(\frac{n-m_1}{2}+2)} + t^\alpha + t^\alpha \\ &\lesssim t^{\alpha(\frac{n-m_1}{2}+2)}. \end{aligned}$$

*If $m_1 > n+2$, then $\alpha\left(\frac{n-m_1}{2}+2\right) < \alpha$ and when $m_1 \to n+2$, $\alpha\left(\frac{n-m_1}{2}+2\right) \to \alpha$.*

*For $t > 1$, $m_1 > n+2$ and $m_2 > n$, we have*

$$\begin{aligned} M_2(t) &\lesssim \int_0^1 (t^\alpha r^{-n-2\beta} + r^{-n} + 1)r^{n+1}dr + \int_1^{t^{\frac{\alpha}{2}}} (t^\alpha r^{-m_1} + t^{\alpha+\frac{m_2-n}{2}\alpha}r^{-m_2-2} + r^{-n})r^{n+1}dr \\ &\quad + \int_{t^{\frac{\alpha}{2}}}^\infty (t^\alpha r^{-m_1} + t^{\alpha+\frac{m_2-n}{2}\alpha}r^{-m_2-2} + t^{m_3\alpha}r^{-n-2m_3})r^{n+1}dr \\ &\lesssim \int_0^1 (t^\alpha r^{-n-2\beta} + r^{-n} + 1)r^{n+1}dr + \int_1^\infty \left(t^\alpha r^{-m_1} + t^{\alpha+\frac{m_2-n}{2}\alpha}r^{-m_2-2}\right) r^{n+1}dr + \int_1^{t^{\frac{\alpha}{2}}} rdr \\ &\quad + \int_{t^{\frac{\alpha}{2}}}^\infty (t^{m_3\alpha}r^{-n-2m_3})r^{n+1}dr \lesssim t^{\alpha+\frac{m_2-n}{2}\alpha}. \end{aligned}$$

*If $m_2 > n$, then $\alpha + \frac{m_2-n}{2}\alpha > \alpha$ and when $m_2 \to n$, $\alpha + \frac{m_2-n}{2}\alpha \to \alpha$.*

**Remark 3.4.** *If $n = 1 < 2\beta$, then it follows from (3.18) and (3.3) that*

$$\begin{aligned} A_{\alpha,\beta,\gamma}(t,0) &= \int_0^\infty M_\alpha(s)A_{1,\beta,\gamma}(st^\alpha,0)ds \lesssim \int_0^{t^{-\alpha}} M_\alpha(s)(st^\alpha)^{-\frac{n}{2\beta}}ds + \int_{t^{-\alpha}}^\infty M_\alpha(s)(st^\alpha)^{-\frac{n}{2}}ds \\ &\lesssim t^{-\frac{\alpha}{2\beta}} + t^{-\frac{\alpha}{2}} \lesssim \begin{cases} t^{-\frac{\alpha}{2\beta}}, & t \in (0,1], \\ t^{-\frac{\alpha}{2}}, & t \in (1,\infty). \end{cases} \end{aligned}$$

*By* [18, Lemma 3.3]*, the above estimate shows a transition from superdiffusive dynamics to subdiffusive dynamics.*

Now we obtain the $L^p$-decay estimate for the fundamental solution $A_{\alpha,\beta,\gamma}(t,x)$.

**Theorem 3.3.** *Let $\alpha \in (0,1)$, $\beta \in (0,1)$ and $\gamma > 0$. Then,*

$$\|A_{\alpha,\beta,\gamma}(t,\cdot)\|_{L^p(\mathbb{R}^n)} \lesssim \begin{cases} t^{-\frac{\alpha n}{2\beta}(1-\frac{1}{p})}, & t \in (0,1], \\ t^{-\frac{\alpha n}{2}(1-\frac{1}{p})}, & t \in (1,\infty) \end{cases} \tag{3.19}$$

*for $p \in [1, \bar{p}(n,\beta))$, where*

$$\bar{p}(n,\beta) := \begin{cases} \dfrac{n}{n-2\beta}, & n > 2\beta, \\ \infty, & \text{otherwise.} \end{cases} \tag{3.20}$$

*If $1 = n < 2\beta$, then the estimate (3.19) holds for all $p \in [1,\infty]$.*

**Remark 3.5.** *By* [18, Lemma 5.1]*, the relation* (3.19) *shows a transition from superdiffusion to subdiffusion.*

*Proof.* If $p \in [1, \bar{p}(n,\beta))$, then the integral

$$\int_0^\infty M_\alpha(s) s^{-\frac{n}{2\beta}(1-\frac{1}{p})} ds$$

is finite. Using Theorem 3.1, for $p \in [1, \bar{p}(n,\beta))$, we have

$$\begin{aligned} \|A_{\alpha,\beta,\gamma}(t,\cdot)\|_{L^p(\mathbb{R}^n)} &\le \int_0^{\frac{1}{t^\alpha}} M_\alpha(s) \|A_{1,\beta,\gamma}(st^\alpha,\cdot)\|_{L^p(\mathbb{R}^n)} ds + \int_{\frac{1}{t^\alpha}}^\infty M_\alpha(s) \|A_{1,\beta,\gamma}(st^\alpha,\cdot)\|_{L^p(\mathbb{R}^n)} ds \\ &\lesssim \int_0^{\frac{1}{t^\alpha}} M_\alpha(s)(st^\alpha)^{-\frac{n}{2\beta}(1-\frac{1}{p})} ds + \int_{\frac{1}{t^\alpha}}^\infty M_\alpha(s)(st^\alpha)^{-\frac{n}{2}(1-\frac{1}{p})} ds \\ &\lesssim t^{-\frac{\alpha n}{2\beta}(1-\frac{1}{p})} \int_0^{\frac{1}{t^\alpha}} M_\alpha(s) s^{-\frac{n}{2\beta}(1-\frac{1}{p})} ds + t^{-\frac{\alpha n}{2}(1-\frac{1}{p})} \int_{\frac{1}{t^\alpha}}^\infty M_\alpha(s) s^{-\frac{n}{2}(1-\frac{1}{p})} ds, \end{aligned}$$

which implies (3.19). □

Next we consider the fundamental solution $B_{\alpha,\beta,\gamma}$ of the equation (1.1) when $\alpha \in (0,1), \beta \in (0,1)$ and $\gamma > 0$. Applying the Fourier transform with respect to the space variable $x$ and the Laplace transform with respect to the time variable $t$ in the equation (1.1) with $f(t,x) = \delta(t)\delta(x)$, we obtain

$$\hat{\tilde{u}}(s,\xi) = \frac{1}{s^\alpha + \zeta_{\beta,\gamma}(\xi)}. \tag{3.21}$$

Inverting the Laplace transform in (3.21), we have

$$\tilde{u}(t,\xi) = t^{\alpha-1} E_{\alpha,\alpha}(-\zeta_{\beta,\gamma}(\xi) t^\alpha). \tag{3.22}$$

Meanwhile,

$$\int_0^\infty \int_0^\infty \theta(t,\tau) e^{-\zeta_{\beta,\gamma}(\xi)\tau} d\tau e^{-st} dt = \int_0^\infty e^{-\tau s^\alpha} e^{-\zeta_{\beta,\gamma}(\xi)\tau} d\tau = \frac{1}{s^\alpha + \zeta_{\beta,\gamma}(\xi)}.$$

It follows from the uniqueness of the Laplace transform that

$$\tilde{u}(t,\xi) = t^{\alpha-1} E_{\alpha,\alpha}(-\zeta_{\beta,\gamma}(\xi) t^\alpha) = \int_0^\infty \theta(t,\tau) e^{-\zeta_{\beta,\gamma}(\xi)\tau} d\tau, \quad t > 0, \xi \in \mathbb{R}^n.$$

From (3.3) and (3.14), for $t > 0$ and $x \in \mathbb{R}^n$, we obtain

$$u(t,x) = \frac{1}{(2\pi)^n}\int_0^\infty \theta(t,\tau)\int_{\mathbb{R}^n} e^{-\zeta(\xi)\tau}\cos(\xi x)d\xi d\tau = \int_0^\infty \theta(t,\tau)A_{1,\beta,\gamma}(\tau,x)d\tau$$
$$= \frac{1}{t}\int_0^\infty F_\alpha\left(\frac{\tau}{t^\alpha}\right)A_{1,\beta,\gamma}(\tau,x)d\tau = t^{\alpha-1}\int_0^\infty F_\alpha(s)A_{1,\beta,\gamma}(st^\alpha,x)ds.$$

Then the fundamental solution $B_{\alpha,\beta,\gamma}(t,x)$ is represented by

$$B_{\alpha,\beta,\gamma}(t,x) = t^{\alpha-1}\int_0^\infty F_\alpha(s)A_{1,\beta,\gamma}(st^\alpha,x)ds \quad t>0, x\in\mathbb{R}^n. \tag{3.23}$$

It follows from $A_{1,\beta,\gamma}(s,x) \geq 0$ that $A_{\alpha,\beta,\gamma}(t,x) \geq 0, t > 0, x \in \mathbb{R}^n$. Also, since $\partial_t^{1-\alpha}\phi(t,\tau) = \theta(t,\tau), t, \tau > 0$, we have

$$\partial_t^{1-\alpha}A_{\alpha,\beta,\gamma}(t,x) = B_{\alpha,\beta,\gamma}(t,x), \quad t>0, x\in\mathbb{R}^n. \tag{3.24}$$

Meanwhile, from (2.1), we obtain

(3.25)

$$\tilde{B}_{\alpha,\beta,\gamma}(t,\xi) = t^{\alpha-1}E_{\alpha,\alpha}(-\zeta_{\beta,\gamma}(\xi)t^\alpha) = -\frac{1}{\zeta_{\beta,\gamma}(\xi)}\frac{\partial E_{\alpha,1}(-\zeta_{\beta,\gamma}(\xi)t^\alpha)}{\partial t} = -\frac{1}{\zeta_{\beta,\gamma}(\xi)}\frac{\partial \tilde{A}_{\alpha,\beta,\gamma}(t,\xi)}{\partial t}.$$

Also, we can easily see that $B_{1,\beta,\gamma} = A_{1,\beta,\gamma}$ when $\alpha = 1$.

Similar to Lemma 2.3, we can use the representation formula (3.23) to prove that $B_{\alpha,\beta,\gamma}(t,x) = B_{\alpha,\beta,\gamma}(t,|x|\kappa)$ for $t > 0$ and $x \in \mathbb{R}^n$, where $\kappa = (1,0,...,0) \in \mathbb{R}^n$.

**Theorem 3.4.** *Let $\alpha \in (0,1), \beta \in (0,1)$, $\gamma > 0$ $m_1 \geq n+2\beta$, $m_2 \geq n$ and $m_3 \geq 0$. Then the fundamental solution $B_{\alpha,\beta,\gamma}$ of the equation (1.1) satisfies the following relations.*

*If $|x| \leq 1$, then*

$$B_{\alpha,\beta,\gamma}(t,x) \lesssim t^{2\alpha-1}|x|^{-n-2\beta} + t^{\alpha-1}|x|^{-n} + t^{\alpha-1}.$$

*If $|x| > 1$, then*

$$B_{\alpha,\beta,\gamma}(t,x) \lesssim t^{2\alpha-1}|x|^{-m_1} + t^{2\alpha+\frac{m_2-n}{2}\alpha-1}|x|^{-m_2-2} + t^{\alpha-1}|x|^{-n}.$$

*In particular, if $|x| > 1$ and $t^{-\alpha}|x|^2 \geq 1$, then*

$$B_{\alpha,\beta,\gamma}(t,x) \lesssim t^{2\alpha-1}|x|^{-m_1} + t^{2\alpha+\frac{m_2-n}{2}\alpha-1}|x|^{-m_2-2} + t^{\alpha+m_3\alpha-1}|x|^{-n-2m_3}.$$

*Proof.* By employing the estimate (3.3) and the asymptotic behavior of $F_\alpha$, as in the proof of Theorem 3.2, we can obtain the desired result. In particular, the boundedness of $F_\alpha(s)/s$ and the relation $F_\alpha(s)/s \to \Gamma(1+\alpha)\sin(\pi\alpha)$ as $s \to 0$ are very important in the proof. □

Now we present the $L^p$-decay estimate for the fundamental solution $B_{\alpha,\beta,\gamma}(t,x)$.

**Theorem 3.5.** *Let $\alpha \in (0,1)$, $\beta \in (0,1)$ and $\gamma > 0$. Then,*

$$\|B_{\alpha,\beta,\gamma}(t,\cdot)\|_{L^p(\mathbb{R}^n)} \lesssim \begin{cases} t^{\alpha-1-\frac{\alpha n}{2\beta}(1-\frac{1}{p})}, & t\in(0,1], \\ t^{\alpha-1-\frac{\alpha n}{2}(1-\frac{1}{p})}, & t\in(1,\infty) \end{cases} \tag{3.26}$$

*for $p \in [1, \bar{p}(n,2\beta))$, where $\bar{p}(n,2\beta)$ is given by (3.20). If $n < 4\beta$, then the estimate (3.26) holds for all $p \in [1,\infty]$.*

*Proof.* Similar to Theorem 3.3, we can prove the desired result. □

## 4. Representation formula of solutions

In this section, we establish a representation formula for solutions of (1.1)-(1.2).

**Definition 4.1.** *We call $u \in C([0,\infty)\times\mathbb{R}^n)$ a classical solution of the Cauchy problem (1.1)-(1.2) if*
*(P1) $\mathscr{F}^*((\zeta_{\beta,\gamma}(\cdot)\tilde{u}(t,\cdot))(x)$ is a continuous function of $x$ for any $t>0$,*
*(P2) for any $x\in\mathbb{R}^n$, $J_t^{1-\alpha}u(t,x)$ is continuously differentiable with respect to $t>0$,*
*(P3) $u(t,x)$ satisfies the equation (1.1) for any $(t,x)\in(0,\infty)\times\mathbb{R}^n$ and the initial condition (1.2) for any $x\in\mathbb{R}^n$.*

For the next theorem, we make the following assumption.
(H):there exists a function $g$ satisfying

$$(1+\zeta_{\beta,\gamma}(\cdot))g(\cdot)\in L^1(\mathbb{R}^n) \tag{4.1}$$

such that

$$|\tilde{f}(t,\xi)|\le g(\xi). \tag{4.2}$$

**Theorem 4.1.** *Let $\alpha\in(0,1], \beta\in(0,1)$ and $\gamma>0$. Let $u_0\in C(\mathbb{R}^n)\cap L^1(\mathbb{R}^n)$ be a function such that $\tilde{u}_0\in L^1(\mathbb{R}^n)$. Let $f\in C([0,\infty)\times\mathbb{R}^n)$ be a function satisfying $f(t,\cdot)\in L^1(\mathbb{R}^n)$ for all $t\ge 0$ and the condition (H). Then the function*

$$u(t,x)=\int_{\mathbb{R}^n}A_{\alpha,\beta,\gamma}(t,x-y)u_0(y)dy+\int_0^t\int_{\mathbb{R}^n}B_{\alpha,\beta,\gamma}(t-s,x-y)f(s,y)dyds \tag{4.3}$$

*is a classical solution of the Cauchy problem (1.1)-(1.2).*

**Remark 4.1.** *In order to guarantee the condition (P1), we give the conditions (H), $u_0\in L^1(\mathbb{R}^n)$ and $\tilde{u}_0\in L^1(\mathbb{R}^n)$ in Theorem 4.1. In fact, unlike the equation (1.1) involving the nonlocal operator $\triangle^{(\beta,\gamma)}$, in the case of the classical heat equation with the local operator $\triangle$, the condition $u_0\in C(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ is sufficient for the convolution of $u_0$ with the fundamental solution to be a classical solution of the homogeneous equation.*

**Remark 4.2.** *The definition of the Caputo fractional derivative where the fractional integral applies to the integer-order derivative is more common than the definition used in the present paper. However, the more common definition needs a stronger regularity for functions. So, in order to prove that the function given by the formula (4.3) satisfies the Cauchy problem (1.1)-(1.2) in Theorem 4.1, we should use the definition of the Caputo fractional derivative where the integer-order derivative applies to the fractional integral. If we employ the more common definition, we can never prove the result in Theorem 4.1.*

*Proof.* **Case of $\alpha=1$:**

First, we prove that the function (4.3) satisfies the condition (P1). Using (3.6) and (4.2), for $t>0$ and $\xi\in\mathbb{R}^n$, we have

$$\begin{aligned}|\zeta_{\beta,\gamma}(\xi)\tilde{u}(t,\xi)| &\le \zeta_{\beta,\gamma}(\xi)e^{-\zeta_{\beta,\gamma}(\xi)t}|\tilde{u}_0(\xi)|+\zeta_{\beta,\gamma}(\xi)\int_0^t e^{-\zeta_{\beta,\gamma}(\xi)(t-s)}|\tilde{f}(s,\xi)|ds\\ &\lesssim \frac{1}{t}|\tilde{u}_0(\xi)|+g(\xi)(1-e^{-\zeta_{\beta,\gamma}(\xi)(t)}).\end{aligned}$$

Then it follows from (4.1) that $\zeta_{\beta,\gamma}(\cdot)\tilde{u}(t,\cdot)\in L^1(\mathbb{R}^n)$ for any $t>0$. By the Riemann-Lebesgue lemma, $\mathscr{F}^*((\zeta_{\beta,\gamma}(\cdot)\tilde{u}(t,\cdot))(x)$ is a continuous function of $x$ for any $t>0$.

Next, we show that the function (4.3) satisfies (P2). Since the function

$$\frac{\partial A_{1,\beta,\gamma}(t,x)}{\partial t}$$

is a bounded continuous function of $x$ for any $t>0$ and $u_0 \in L^1(\mathbb{R}^n)$,

$$\frac{\partial A_{1,\beta,\gamma}(t,x-\cdot)}{\partial t} u_0(\cdot) \in L^1(\mathbb{R}^n),$$

which implies that the function

$$\int_{\mathbb{R}^n} \frac{\partial A_{1,\beta,\gamma}(t,x-y)}{\partial t} u_0(y)dy$$

is a continuous function of $x$ for any $t>0$. Meanwhile, we can easily use the conditions (4.1) and (4.2) to prove the estimate

$$|f(t,y)-f(t,x)| \le C|x-y|^r, \quad t \ge 0 \tag{4.4}$$

for $r \in (0, \min\{1, 2\beta\})$. Let

$$v(t,x) := \int_0^t \int_{\mathbb{R}^n} A_{1,\beta,\gamma}(t-s,x-y) f(s,y)dyds.$$

For $\delta > 0$, we have

$$\begin{aligned}
&\frac{v(t+\delta,x)-v(t,x)}{\delta} \\
&= \frac{1}{\delta}\int_t^{t+\delta}\int_{\mathbb{R}^n} A_{1,\beta,\gamma}(t+\delta-s,x-y)(f(s,y)-f(s,x))dyds \\
&\quad + \frac{1}{\delta}\int_t^{t+\delta} f(s,x)ds \\
&\quad + \frac{1}{\delta}\int_0^t\int_{\mathbb{R}^n} (A_{1,\beta,\gamma}(t+\delta-s,x-y)-A_{1,\beta,\gamma}(t-s,x-y)) f(s,y)dyds.
\end{aligned}$$

We can use (4.4) and (3.3) to prove that the first integral converges to 0 when $\delta \to 0$. From (3.8), we obtain

$$\frac{\partial}{\partial t}\int_0^t\int_{\mathbb{R}^n} A_{1,\beta,\gamma}(t-s,x-y)f(s,y)dyds = f(t,x) + \int_0^t\int_{\mathbb{R}^n} \frac{\partial A_{1,\beta,\gamma}(t-s,x-y)}{\partial t} f(s,y)dyds.$$

Therefore, we have

$$\frac{\partial u(t,x)}{\partial t} = \int_{\mathbb{R}^n} \frac{\partial A_{1,\beta,\gamma}(t,x-y)}{\partial t} u_0(y)dy + f(t,x) + \int_0^t\int_{\mathbb{R}^n} \frac{\partial A_{1,\beta,\gamma}(t-s,x-y)}{\partial t} f(s,y)dyds. \tag{4.5}$$

In the last, we deduce that the function (4.3) satisfies the condition (P3). By the conditions (4.2) and (4.1), for any $t>0$, we obtain

$$\zeta_{\beta,\gamma}(\xi)\int_0^t e^{-\zeta_{\beta,\gamma}(\xi)(t-s)}\tilde{f}(s,\xi)ds \in L^1(\mathbb{R}^n).$$

Then, from the uniqueness of the Fourier transform and (4.5), we deduce

$$\begin{aligned}\triangle^{(\beta,\gamma)}u(t,x) &= \mathscr{F}^*\left(-\zeta_{\beta,\gamma}(\xi)e^{-\zeta_{\beta,\gamma}(\xi)t}\tilde{u}_0(\xi)-\zeta_{\beta,\gamma}(\xi)\int_0^t e^{-\zeta_{\beta,\gamma}(\xi)(t-s)}\tilde{f}(s,\xi)ds\right)\\ &= \int_{\mathbb{R}^n}\frac{\partial A_{1,\beta,\gamma}(t,x-y)}{\partial t}u_0(y)dy+\int_0^t\int_{\mathbb{R}^n}\frac{\partial A_{1,\beta,\gamma}(t-s,x-y)}{\partial t}f(s,y)dyds\\ &= \frac{\partial u(t,x)}{\partial t}-f(t,x).\end{aligned}$$

For any $\varepsilon>0$, there exists a $\delta>0$ such that $|u_0(y)-u_0(x)|<\varepsilon$ for all $x,y\in\mathbb{R}^n$ satisfying the relation $|x-y|<\delta$. Using (3.3), for $0<t\le\delta^{2\beta}$ and $x\in\mathbb{R}^n$, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}A_{1,\beta,\gamma}(t,x-y)u_0(y)dy-u_0(x)\right| = \left|\int_{\mathbb{R}^n}A_{1,\beta,\gamma}(t,x-y)(u_0(y)-u_0(x))dy\right|\\ &\le \int_{|x-y|<\delta}A_{1,\beta,\gamma}(t,x-y)|u_0(y)-u_0(x)|dy+\int_{|x-y|>\delta}A_{1,\beta,\gamma}(t,x-y)|u_0(y)-u_0(x)|dy\\ &\lesssim \varepsilon\int_{|x-y|<\delta}A_{1,\beta,\gamma}(t,x-y)dy+\sup_{y\in\mathbb{R}^n}|u_0(y)|\int_{|x-y|>\delta}t|x-y|^{-n-2\beta}dy,\end{aligned}$$

which implies that $\lim_{t\to 0}|u(t,x)-u_0(x)|=0$ for any $x\in\mathbb{R}^n$.

**Case of $\alpha\in(0,1)$:**

First, we prove that the function (4.3) satisfies the condition (P1). Using (3.17), (3.25), (4.2) and (2.3), for any $t>0$ and $\xi\in\mathbb{R}^n$, we have

$$\begin{aligned}|\zeta_{\beta,\gamma}(\xi)\tilde{u}(t,\xi)| &\le \zeta_{\beta,\gamma}(\xi)E_{\alpha,1}(-\zeta_{\beta,\gamma}(\xi)t^{\alpha})|\tilde{u}_0(\xi)|+\int_0^t\left|\frac{\partial E_{\alpha,1}(-\zeta_{\beta,\gamma}(\xi)(t-s)^{\alpha})}{\partial t}\right||\tilde{f}(s,\xi)|ds\\ &\lesssim \frac{1}{t^{\alpha}}|\tilde{u}_0(\xi)|+g(\xi)(1-E_{\alpha,1}(-\zeta_{\beta,\gamma}(\xi)t^{\alpha})).\end{aligned}$$

Then it follows from (4.1) and Riemann-Lebesgue lemma that $\mathscr{F}^*((\zeta_{\beta,\gamma}(\cdot)\tilde{u}(t,\cdot))(x)$ is a continuous function of $x$ for any $t>0$.

Next, we show that the function (4.3) satisfies the condition (P2). Using the relation (3.16) and the asymptotic behavior of $M_\alpha$, we can deduce

$$\lim_{t\to 0}J_t^{1-\alpha}\phi(t,\tau)=\lim_{t\to 0}\int_0^t(t-s)^{-\alpha}\frac{1}{s^{\alpha}}M_\alpha\left(\frac{\tau}{s^{\alpha}}\right)ds=0,\quad \tau>0.$$

Also, from the formula (3.16) and the asymptotic behavior of the Wright function given by [24, formula (F.3)], we can obtain the asymptotic behavior of $J_t^{1-\alpha}\phi(t,\tau)$ when $\tau\to 0$ or $\tau\to\infty$. Then

$$J_t^{1-\alpha}A_{\alpha,\beta,\gamma}(t,x)=\int_0^\infty J_t^{1-\alpha}\phi(t,\tau)A_{1,\beta,\gamma}(\tau,x)d\tau.$$

Employing (3.14) and (2.10), we have

$$\frac{\partial\theta(t,\tau)}{\partial t}=-\frac{1}{t^2}F_\alpha\left(\frac{\tau}{t^{\alpha}}\right)+\frac{\tau\alpha}{t^{2+\alpha}}W_{-\alpha,-\alpha}\left(-\frac{\tau}{t^{\alpha}}\right). \tag{4.6}$$

By the formulas (2.4), (2.5), (2.6) and the asymptotic behavior of the Wright function (see [24, formula (F.3), p. 238]), we can easily obtain

$$\lim_{\tau\to 0}\frac{\partial\theta(t,\tau)}{\partial t}=0,\ \ t>0, \tag{4.7}$$

$$\lim_{\tau\to \infty}\frac{\partial\theta(t,\tau)}{\partial t}=0,\ \ t>0, \tag{4.8}$$

$$\lim_{t\to 0}\frac{\partial\theta(t,\tau)}{\partial t}=0,\ \ \tau>0, \tag{4.9}$$

$$\lim_{t\to \infty}\frac{\partial\theta(t,\tau)}{\partial t}=0,\ \ \tau>0. \tag{4.10}$$

Then

$$\partial_t^\alpha\phi(t,\tau)=\frac{\partial}{\partial t}J_t^{1-\alpha}\phi(t,\tau)=\frac{\partial}{\partial t}J_t^{2-2\alpha}\theta(t,\tau)=J_t^{2-2\alpha}\frac{\partial\theta(t,\tau)}{\partial t},\ \ t,\tau>0. \tag{4.11}$$

By (4.7), (4.9) and (4.11), $J_t^{1-\alpha}A_{\alpha,\beta,\gamma}(t,x)$ is continuously differentiable with respect to $t>0$. Using (3.3), we deduce

$$\partial_t^\alpha A_{\alpha,\beta,\gamma}(t,x)=\int_0^\infty \partial_t^\alpha\phi(t,\tau)A_{1,\beta,\gamma}(\tau,x)d\tau. \tag{4.12}$$

Then the function $\partial_t^\alpha A_{\alpha,\beta,\gamma}(t,x)$ is a continuous function of $x$. By (4.12) and Theorem 3.1, we have $\partial_t^\alpha A_{\alpha,\beta,\gamma}(t,\cdot)\in L^1(\mathbb{R}^n)$ for $t>0$. Also, it follows from the boundedness of $u_0$ that $\partial_t^\alpha A_{\alpha,\beta,\gamma}(t,x-\cdot)u_0(\cdot)\in L^1(\mathbb{R}^n)$ for $t>0$. From (3.24), we deduce

$$\begin{aligned}&\partial_t^\alpha\int_0^t\int_{\mathbb{R}^n}B_{\alpha,\beta,\gamma}(t-s,x-y)f(s,y)dyds\\&=\frac{\partial}{\partial t}\int_0^t\int_{\mathbb{R}^n}J_t^{1-\alpha}B_{\alpha,\beta,\gamma}(t-s,x-y)f(s,y)dyds\\&=\frac{\partial}{\partial t}\int_0^t\int_{\mathbb{R}^n}A_{\alpha,\beta,\gamma}(t-s,x-y)f(s,y)dyds.\end{aligned}$$

As in the proof of Theorem 4.1, we can use (4.1), (4.2), (3.3) and Lemma 2.4 to obtain the following relation.

$$\frac{\partial}{\partial t}\int_0^t\int_{\mathbb{R}^n}A_{\alpha,\beta,\gamma}(t-s,x-y)f(s,y)dyds=f(t,x)+\int_0^t\int_{\mathbb{R}^n}\frac{\partial A_{\alpha,\beta,\gamma}(t-s,x-y)}{\partial t}f(s,y)dyds.$$

Therefore, we have

$$\partial_t^\alpha u(t,x)=\int_{\mathbb{R}^n}\partial_t^\alpha A_{\alpha,\beta,\gamma}(t,x-y)u_0(y)dy+f(t,x)+\int_0^t\int_{\mathbb{R}^n}\frac{\partial A_{\alpha,\beta,\gamma}(t-s,x-y)}{\partial t}f(s,y)dyds. \tag{4.13}$$

In the last, we deduce that the function (4.3) satisfies the condition (P3). Using the uniqueness of the Fourier transform, from (3.9), (3.10), (3.17), (3.24), (2.2) and (4.13), we deduce

$$
\begin{aligned}
&\triangle^{(\beta,\gamma)}u(t,x)\\
=&\mathscr{F}^*\left(\partial_t^\alpha(E_{\alpha,1}(-\zeta_{\beta,\gamma}(\xi)t^\alpha))\tilde{u}_0(\xi)\right)(x)+\mathscr{F}^*\left(\int_0^t\frac{\partial}{\partial t}E_{\alpha,1}(-\zeta_{\beta,\gamma}(\xi)(t-s)^\alpha)\tilde{f}(s,\xi)ds\right)(x)\\
=&\int_0^\infty\partial_t^\alpha\phi(t,\tau)\mathscr{F}^*\left(e^{-\zeta_{\beta,\gamma}(\xi)\tau^\alpha}\tilde{u}_0(\xi)\right)(x)d\tau+\int_0^t\int_0^\infty\frac{\partial\phi(t-s,\tau)}{\partial t}\mathscr{F}^*\left(e^{-\zeta_{\beta,\gamma}(\xi)\tau^\alpha}\tilde{f}(s,\xi)\right)(x)d\tau ds\\
=&\int_0^\infty\partial_t^\alpha\phi(t,\tau)\int_{\mathbb{R}^n}A_{1,\beta,\gamma}(\tau,x-y)u_0(y)dyd\tau+\int_0^t\int_0^\infty\frac{\partial\phi(t-s,\tau)}{\partial t}\int_{\mathbb{R}^n}A_{1,\beta,\gamma}(\tau,x-y)f(s,y)dyd\tau ds\\
=&\int_{\mathbb{R}^n}\partial_t^\alpha A_{\alpha,\beta,\gamma}(t,x-y)u_0(y)dy+\int_0^t\int_{\mathbb{R}^n}\frac{\partial A_{\alpha,\beta,\gamma}(t-s,x-y)}{\partial t}f(s,y)dyds\\
=&\partial_t^\alpha u(t,x)-f(t,x).
\end{aligned}
$$

As in the case of $\alpha=1$, we can use the asymptotic behavior of $A_{\alpha,\beta,\gamma}(t,x)$ presented in Theorem 3.2 to prove that $\lim\limits_{t\to0}|u(t,x)-u_0(x)|=0$ for any $x\in\mathbb{R}^n$. $\square$

## 5. Decay behavior of solutions

In this section, we consider the asymptotic behavior of solutions of the nonlocal diffusion equation (1.1) with the initial condition (1.2).

**Theorem 5.1.** *Let $\alpha\in(0,1]$, $\beta\in(0,1)$ and $\gamma>0$. Suppose that $u_0\in L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ and $f=0$. Then $u$ represented by (4.3) satisfies the following relations.*
*Case of $\alpha=1$:*

$$
\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)}\lesssim\begin{cases}t^{-\frac{n}{4\beta}}, & t\in(0,1],\\ t^{-\frac{n}{4}}, & t\in(1,\infty).\end{cases}
$$

*Case of $\alpha\in(0,1)$:*
*If $n=4$ or $n=3=4\beta$, then*

$$
\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)}\lesssim t^{-\frac{\alpha}{2}},\quad t\in(0,\infty),
$$

*otherwise,*

$$
\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)}\lesssim\begin{cases}t^{-\frac{\alpha}{2}\min\{1,\frac{n}{2\beta}\}}, & t\in(0,1],\\ t^{-\alpha\min\{1,\frac{n}{4}\}}, & t\in(1,\infty).\end{cases}
$$

*Proof.* Case of $\alpha=1$:
Using Theorem 3.1 and Young's inequality for convolution [12], for $t\in(0,1]$, we obtain

$$
\|u(t,\cdot)\|_{L^2(\mathbb{R}^n}\le\|A_{1,\beta,\gamma}(t,\cdot)\|_{L^2(\mathbb{R}^n)}\|u_0\|_{L^1(\mathbb{R}^n)}\lesssim t^{-\frac{n}{4\beta}}\|u_0\|_{L^1(\mathbb{R}^n)}.
$$

Similar to the case of $t\in(0,1]$, we can prove the result when $t>1$.
Case of $\alpha\in(0,1)$:

First, we consider the case of $n<4\beta$. Using Theorem 3.3 and Young's inequality for convolution,

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n} \leq \|A_{\alpha,\beta,\gamma}(t,\cdot)\|_{L^2(\mathbb{R}^n)}\|u_0\|_{L^1(\mathbb{R}^n)} \lesssim \begin{cases} t^{-\frac{\alpha n}{4\beta}}\|u_0\|_{L^1(\mathbb{R}^n)}, & t\in(0,1], \\ t^{-\frac{\alpha n}{4}}\|u_0\|_{L^1(\mathbb{R}^n)}, & t\in(1,\infty). \end{cases}$$

By Plancherel Theorem and (2.3), for $t>0$, we have

$$\begin{aligned} (2\pi)^n\|u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} &= \|\tilde{u}(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n}|\tilde{A}_{\alpha,\beta,\gamma}(t,\xi)|^2|\tilde{u}_0(\xi)|^2d\xi \\ &= \int_{\mathbb{R}^n}|E_{\alpha,1}(-\zeta_{\beta,\gamma}(\xi)t^\alpha)|^2|\tilde{u}_0(\xi)|^2d\xi \lesssim \int_{\mathbb{R}^n}\frac{|\tilde{u}_0(\xi)|^2}{(1+\zeta_{\beta,\gamma}(\xi)t^\alpha)^2}d\xi. \end{aligned} \tag{5.1}$$

For $n>4$ and $t>0$, from (5.1) and Hardy-Littlewood-Sobolev theorem [13], we deduce

$$\begin{aligned} \|u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} \lesssim & t^{-2\alpha}\int_{\mathbb{R}^n}\frac{|\xi|^4t^{2\alpha}}{(1+|\xi|^2t^\alpha)^2}||\xi|^{-2}\tilde{u}_0(\xi)|^2d\xi + t^{-2\alpha}\int_{\mathbb{R}^n}\frac{|\xi|^{4\beta}t^{2\alpha}}{(1+|\xi|^{2\beta}t^\alpha)^2}||\xi|^{-2\beta}\tilde{u}_0(\xi)|^2d\xi \\ \lesssim & t^{-2\alpha}(\|(-\triangle)^{-1}u_0\|^2_{L^2(\mathbb{R}^n)} + \|(-\triangle)^{-\beta}u_0\|^2_{L^2(\mathbb{R}^n)}) \\ \lesssim & t^{-2\alpha}\left(\|u_0\|^2_{L^{\frac{2n}{n+4}}(\mathbb{R}^n)} + \|u_0\|^2_{L^{\frac{2n}{n+4\beta}}(\mathbb{R}^n)}\right). \end{aligned}$$

For $n>2$ and $t>0$, we obtain

$$\begin{aligned} \|u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} \lesssim & t^{-\alpha}\int_{\mathbb{R}^n}\frac{|\xi|^2t^{\alpha}}{(1+|\xi|^2t^\alpha)^2}||\xi|^{-1}\tilde{u}_0(\xi)|^2d\xi + t^{-\alpha}\int_{\mathbb{R}^n}\frac{|\xi|^{2\beta}t^{\alpha}}{(1+|\xi|^{2\beta}t^\alpha)^2}||\xi|^{-\beta}\tilde{u}_0(\xi)|^2d\xi \\ \lesssim & t^{-\alpha}\int_{\mathbb{R}^n}\frac{||\xi|^{-1}\tilde{u}_0(\xi)|^2}{(|\xi|^{-1}t^{-\frac{\alpha}{2}}+|\xi|t^{\frac{\alpha}{2}})^2}d\xi + t^{-\alpha}\int_{\mathbb{R}^n}\frac{||\xi|^{-\beta}\tilde{u}_0(\xi)|^2}{(|\xi|^{-\beta}t^{-\frac{\alpha}{2}}+|\xi|^{\beta}t^{\frac{\alpha}{2}})^2}d\xi \\ \lesssim & t^{-\alpha}(\|(-\triangle)^{-\frac{1}{2}}u_0\|^2_{L^2(\mathbb{R}^n)} + \|(-\triangle)^{-\frac{\beta}{2}}u_0\|^2_{L^2(\mathbb{R}^n)}) \\ \lesssim & t^{-\alpha}\left(\|u_0\|^2_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} + \|u_0\|^2_{L^{\frac{2n}{n+2\beta}}(\mathbb{R}^n)}\right). \end{aligned}$$

If $n<4$ and $t>0$, then, we estimate

$$\int_{\mathbb{R}^n}\frac{1}{(1+|\xi|^2t^\alpha)^2}|\tilde{u}_0(\xi)|^2d\xi \lesssim \|\tilde{u}_0\|^2_{L^\infty(\mathbb{R}^n)}\int_0^\infty\frac{r^{n-1}}{(1+r^2t^\alpha)^2}dr = t^{-\frac{\alpha n}{2}}\|u_0\|^2_{L^1(\mathbb{R}^n)}\int_0^\infty\frac{w^{n-1}}{(1+w^2)^2}dw.$$

Therefore, for $4\beta<n<4$ and $t>0$, we deduce

$$\begin{aligned} \|u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} \lesssim & \int_{\mathbb{R}^n}\frac{1}{(1+|\xi|^2t^\alpha)^2}|\tilde{u}_0(\xi)|^2d\xi + t^{-2\alpha}\int_{\mathbb{R}^n}\frac{|\xi|^{4\beta}t^{2\alpha}}{(1+|\xi|^{2\beta}t^\alpha)^2}||\xi|^{-2\beta}\tilde{u}_0(\xi)|^2d\xi \\ \lesssim & t^{-\frac{\alpha n}{2}}\|u_0\|^2_{L^1(\mathbb{R}^n)} + t^{-2\alpha}\|(-\triangle)^{-\beta}u_0\|^2_{L^2(\mathbb{R}^n)} \\ \lesssim & t^{-\frac{\alpha n}{2}}\|u_0\|^2_{L^1(\mathbb{R}^n)} + t^{-2\alpha}\|u_0\|^2_{L^{\frac{2n}{n+4\beta}}(\mathbb{R}^n)}. \end{aligned}$$

Also, for $2\beta < n < 4$ and $t > 0$, we obtain

$$\begin{aligned}\|u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} \lesssim &\int_{\mathbb{R}^n} \frac{1}{(1+|\xi|^2t^\alpha)^2}|\tilde{u}_0(\xi)|^2 d\xi + t^{-\alpha}\int_{\mathbb{R}^n} \frac{|\xi|^{2\beta}t^\alpha}{(1+|\xi|^{2\beta}t^\alpha)^2}||\xi|^{-\beta}\tilde{u}_0(\xi)|^2 d\xi \\ \lesssim & t^{-\frac{\alpha n}{2}}\|u_0\|^2_{L^1(\mathbb{R}^n)} + t^{-\alpha}\|(-\triangle)^{-\frac{\beta}{2}}u_0\|^2_{L^2(\mathbb{R}^n)} \\ \lesssim & t^{-\frac{\alpha n}{2}}\|u_0\|^2_{L^1(\mathbb{R}^n)} + t^{-\alpha}\|u_0\|^2_{L^{\frac{2n}{n+2\beta}}(\mathbb{R}^n)}.\end{aligned}$$

Combining the previous estimates, we obtain the desired result.

□

The following result shows that the decay rate in Theorem 5.1 is optimal.

**Lemma 5.1.** *Let $\alpha \in (0,1), \beta \in (0,1)$ and $\gamma > 0$. Suppose that $u_0 \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ with $\tilde{u}_0(0) \neq 0$ and $f = 0$. Then the function $u$ represented by (4.3) satisfies the following relation.*

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \gtrsim t^{-\alpha \min\{1,\frac{n}{4}\}}|\tilde{u}_0(0)|, \ \ t \in (1,\infty).$$

*Proof.* Let $r > 0, x \in \mathbb{R}^n$ and $O_r(x) = \{y \in \mathbb{R}^n | |y-x| \leq r\}$. By (2.3), we have

$$\begin{aligned}(2\pi)^n\|u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} =& \int_{\mathbb{R}^n} |E_{\alpha,1}(-\zeta_{\beta,\gamma}(\xi)t^\alpha)|^2|\tilde{u}_0(\xi)|^2 d\xi \\ \gtrsim & \int_{O_r(0)} \frac{1}{(1+\zeta_{\beta,\gamma}(\xi)t^\alpha)^2}|\tilde{u}_0(\xi)|^2 d\xi \\ \gtrsim & \frac{r^n}{(1+\rho_{\beta,\gamma}(r)t^\alpha)^2}\left(r^{-n}\int_{O_r(0)} |\tilde{u}_0(\xi)|^2 d\xi\right).\end{aligned}$$

It follows from $\tilde{u}_0(0) \neq 0$ and Lebesgue differentiation theorem that there exists a $r_0 > 0$ such that

$$r^{-n}\int_{O_r(0)} |\tilde{u}_0(\xi)|^2 d\xi \gtrsim |\tilde{u}_0(0)|^2, \ \ r \in (0, r_0].$$

Then

$$\|u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} \gtrsim \frac{|\tilde{u}_0(0)|^2 r^n}{(1+\rho_{\beta,\gamma}(r)t^\alpha)^2}.$$

Setting $r = r_0$, we have

$$\|u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} \gtrsim \frac{|\tilde{u}_0(0)|^2 r_0^n}{(1+\rho_{\beta,\gamma}(r_0)t^\alpha)^2} \gtrsim t^{-2\alpha}|\tilde{u}_0(0)|^2, \ \ t \in (1,\infty).$$

Taking $r = \rho_{\beta,\gamma}{}^{-1}\left(\frac{\min\{K_3\gamma^2, K_3 r_0^2\}}{1+t^\alpha}\right)$, from Lemma 2.4, we obtain

$$\|u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} \gtrsim \frac{|\tilde{u}_0(0)|^2\left(\frac{\min\{K_3\gamma^2, K_3 r_0^2\}}{1+t^\alpha}\right)^{\frac{n}{2}}}{(1+\min\{K_3\gamma^2, K_3 r_0^2\}\frac{t^\alpha}{1+t^\alpha})^2} \gtrsim t^{-\frac{n\alpha}{2}}|\tilde{u}_0(0)|^2, \ \ t \in (1,\infty).$$

□

## 6. Conclusion

The tempered Lévy flights have been widely applied in many areas such as plasma physics, finance and turbulent transport. In this paper, the Cauchy problems for time-space nonlocal diffusion equations describing the tempered Lévy flights were investigated. First, we established the asymptotic behavior results of fundamental solutions of the nonlocal diffusion equation. The results show that the tempered Lévy flights exhibit a transition from superdiffusive to subdiffusive dynamics. Based on the asymptotic behavior results, the MSD of the tempered Lévy flight was estimated. Second, the representation formula of solutions of the Cauchy problem was obtaind by using the fundamental solutions. In the last, the $L^2$-decay estimate of solutions was deduced by employing the Fourier analysis method.

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