DERIVATION MODULE AND THE HILBERT-KUNZ MULTIPLICITY OF THE CO-ORDINATE RING OF A PROJECTIVE MONOMIAL CURVE

OM PRAKASH BHARDWAJ AND INDRANATH SENGUPTA

Abstract. Let $n_0, n_1, \ldots, n_p$ be a sequence of positive integers such that $n_0 < n_1 < \cdots < n_p$ and $\gcd(n_0, n_1, \ldots, n_p) = 1$. Let $S = \langle (0, n_p), (n_0, n_p - n_0), \ldots, (n_{p-1}, n_p - n_{p-1}), (n_p, 0)\rangle$ be an affine semigroup in $\mathbb{N}^2$. The semigroup ring $k[S]$ is the co-ordinate ring of the projective monomial curve in the projective space $\mathbb{P}_k^{p+1}$, which is defined parametrically by

$$x_0 = v^{n_p}, \quad x_1 = u^{n_0} v^{n_p - n_0}, \quad \ldots, \quad x_p = u^{n_{p-1}} v^{n_p - n_{p-1}}, \quad x_{p+1} = u^{n_p}.$$

In this article, we consider the case when $n_0, n_1, \ldots, n_p$ forms an arithmetic sequence, and give an explicit set of minimal generators for the derivation module $\mathrm{Der}_k(k[S])$. Further, we give an explicit formula for the Hilbert-Kunz multiplicity of the co-ordinate ring of a projective monomial curve.

1. Introduction

Let k be a field of characteristic 0, and $(R, \mathfrak{m})$ be a local (graded) k-algebra. Finding an explicit set of minimal generators for the derivation module $\mathrm{Der}_k(R)$ of $(R, \mathfrak{m})$ is an important problem in the literature, where $\mathrm{Der}_k(R)$ denote the R-module of k-derivations of R. Previously, this problem has been studied by many authors, for reference see ([6], [12], [13], [17], [19]).

Let $r \geq 1$, and S be an affine semigroup in $\mathbb{N}^r$ generated by $a_0, a_1, \ldots, a_p$. The semigroup ring $k[S] := \oplus_{s \in S} k\mathbf{t}^s$ of S is a k-subalgebra of the polynomial ring $k[t_1, \ldots, t_r]$, where $t_1, \ldots, t_r$ are indeterminates and $\mathbf{t}^s = \prod_{i=1}^r t_i^{s_i}$ for all $s = (s_1, \ldots, s_r) \in S$. If $r = 1$, then S is a submonoid in $\mathbb{N}$, and the semigroup ring $k[S]$ is isomorphic to a numerical semigroup ring. When S is a numerical semigroup, Kraft in [6], proved that the derivation module $\mathrm{Der}_k(k[S])$ is minimally generated by the set $\{t^{\alpha+1}\frac{\partial}{\partial t} \mid \alpha \in \mathrm{PF}(S) \cup \{0\}\}$, where $\mathrm{PF}(S)$ denotes the set of pseudo-Frobenius elements of S. For $r \geq 2$, if S is an affine semigroup in $\mathbb{N}^r$, then Tamone and Molinelli ([7], [8]), give the structure of k-derivations of $k[S]$ for some special type of semigroups. In [7], they consider the affine semigroup $S = \langle (0, n_p), (n_0, n_p - n_0), \ldots, (n_{p-1}, n_p - n_{p-1}), (n_p, 0)\rangle$ in $\mathbb{N}^2$, where $n_0, n_1, \ldots, n_p$ is a sequence of positive integers such that $n_0 < n_1 < \cdots < n_p$ and $\gcd(n_0, n_1, \ldots, n_p) = 1$. For $i = 1, 2$, let S_1 and S_2 be the natural projections onto first and second component of S. With the assumption on the generators, note that S_1 and S_2 are numerical semigroups. For these type of affine semigroups, when $k[S]$ is Cohen-Macaulay, they give the structure of the derivations of the derivation module $\mathrm{Der}_k(k[S])$ using the set of pseudo-Frobenius elements of S_1 and S_2 (see

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2.6). In this article, we give the explicit generators of the derivation module of the co-ordinate ring of the projective monomial curve defined by an arithmetic sequence using the structures of derivations given by Tamone and Molinelli [7].

Now, we summarize the contents of the paper. In this article, we consider the affine semigroup $S = \langle (0, n_p), (n_0, n_p - n_0), \ldots, (n_{p-1}, n_p - n_{p-1}), (n_p, 0)\rangle$ in $\mathbb{N}^2$, where $n_0, n_1, \ldots, n_p$ is a sequence of positive integers such that $n_0 < n_1 < \cdots < n_p$ and $\gcd(n_0, n_1, \ldots, n_p) = 1$, and $k[S]$ the semigroup algebra associated to S, which is isomorphic to the co-ordinate ring of a projective monomial curve in $\mathbb{P}^{p+1}_k$.

In section 2, we recall some definitions about numerical semigroups and when $k[S]$ is Cohen-Macaulay, we summarize the structure of the generating set of the derivation module of $k[S]$ in Theorem 2.6. In section 3, we consider $p = 1$, i.e the sequence of positive integers n_0, n_1 such that $n_0 < n_1$ and $\gcd(n_0, n_1) = 1$. In Theorem 3.1, we give the explicit set of minimal generators of the derivation module for the co-ordinate ring of the projective monomial curve in $\mathbb{P}^2_k$ defined by the positive integers n_0 and n_1.

In section 4, we consider and the sequence $n_0, n_1, n_2, \ldots, n_p$, which forms a minimal arithmetic sequence and consider the affine semigroup $S = \langle (0, n_p), (n_0, n_p - n_0), (n_1, n_p - n_1), \ldots, (n_{p-1}, n_p - n_{p-1}), (n_p, 0)\rangle$. From [1], we know that $k[S]$ is Cohen-Macaulay. In Proposition 4.1, we prove that $\mu(\mathrm{Der}_k(k[S]_{\mathfrak{m}})) = r + 3$, where r is the Cohen-Macaulay type of $k[S]$ and $\mathfrak{m}$ is the maximal homogeneous ideal of $k[S]$. In Corollary 4.2, for $n_0 = ap + b$, $0 \le b < p$, we write the formula for $\mu(\mathrm{Der}_k(k[S]_{\mathfrak{m}}))$, which is,

$$\mu(\mathrm{Der}_k(k[S]_{\mathfrak{m}})) = \begin{cases} 4 & \text{if} \quad p = 1; \\ p+2 & \text{if} \quad p \ge 2, b = 0; \\ p+3 & \text{if} \quad p \ge 2, b = 1; \\ b+2 & \text{if} \quad p \ge 2, 1 < b < p. \end{cases}$$

In Theorem 4.5, we give an explicit set of minimal generators for $\mathrm{Der}_k(k[S])$. In section 5, we compute the Hilbert-Kunz multiplicity of the semigroup algebra $k[S]$. In Theorem 5.1, we prove that the Hilbert-Kunz multiplicity of $k[S]$ is equal to $1 + \frac{1}{n_p}\left(\sum_{r=1}^{p}(n_r - 1)(n_r - n_{r-1})\right)$, where $n_0 = 0$. It is interesting to note that in the case of semigroup algebras, the computation of Hilbert-Kunz multiplicity is independent of the characteristic of the base field.

2. Preliminaries

Throughout the article, $\mathbb{Z}$ and $\mathbb{N}$ denote the sets of integers and non-negative integers respectively.

Definition 2.1. *Let S be a submonoid of $\mathbb{N}$ such that $\mathbb{N} \setminus S$ is finite, then S is called a numerical semigroup. Equivalently, there exist $n_0, \ldots, n_p \in \mathbb{N}$ such that $\gcd(n_0, \ldots, n_p) = 1$ and*

$$S = \langle n_0, \ldots, n_p\rangle = \left\{ \sum_{i=0}^{p} \lambda_i n_i \mid \lambda_i \in \mathbb{N}, \forall i \right\}.$$

Since $\mathbb{N} \setminus S$ is finite, the largest number in $\mathbb{N} \setminus S$ is called the Frobenius number of S, and it is denoted by $F(S)$.

Definition 2.2. *Let S be a numerical semigroup. For any $s \in S$, if $s = \sum_{i=0}^{p} \lambda_i n_i$ is the unique expression for s in S, then we say s has unique factorization in S. In other words,* *we say s has a unique factorization in S if given any two expressions of s, $s = \sum_{i=0}^{p} \lambda_i n_i$ and $s = \sum_{i=0}^{p} \lambda'_i n_i$, we have $\lambda_i = \lambda'_i$ for all $i \in [0, p]$.*

Given $0 \neq s \in S$, the set of lengths of s in S is defined as

$$\mathcal{L}(s) = \left\{ \sum_{i=0}^{p} \lambda_i \mid s = \sum_{i=0}^{p} \lambda_i n_i, \quad \lambda_i \in \mathbb{N} \right\}.$$

Definition 2.3. *A subset $T \subseteq S$ is called homogeneous if either it is empty or $\mathcal{L}(s)$ is singleton for all $0 \neq s \in T$.*

Definition 2.4. *Let S be a numerical semigroup and a be a non-zero element of S. The set $\mathrm{Ap}(S, a) = \{s \in S \mid s - a \notin S\}$ is called the Apéry set of S with respect to a.*

Definition 2.5. *Let S be a numerical semigroup. An element $f \in \mathbb{Z} \setminus S$ is called a pseudo-Frobenius number if $f + s \in S$ for all $s \in S \setminus \{0\}$. The set of pseudo-Frobenius numbers of S is denoted by $\mathrm{PF}(S)$. Note that $F(S) \in \mathrm{PF}(S)$ and $F(S)$ is the maximum element of $\mathrm{PF}(S)$.*

The cardinality of the set of pseudo-Frobenius elements is known as the type of the numerical semigroup S, which is equal to the Cohen-Macaulay type of the numerical semigroup ring $k[S]$. Let $a_0, a_1, \ldots, a_p \in \mathbb{N}^r$ then

$$S = \langle a_0, a_1, \ldots, a_p \rangle = \left\{ \sum_{i=0}^{p} \lambda_i a_i \mid \lambda_i \in \mathbb{N}, \forall i \right\}$$

is called an *affine semigroup* generated by $a_0, a_1, \ldots, a_p$. For $r = 1$, affine semigroups correspond to numerical semigroups. Let k be a field, the semigroup ring $k[S] := \oplus_{s \in S} k\mathbf{t}^s$ of S is a k-subalgebra of the polynomial ring $k[t_1, \ldots, t_r]$, where $t_1, \ldots, t_r$ are indeterminates and $\mathbf{t}^s = \prod_{i=1}^{r} t_i^{s_i}$, for all $s = (s_1, \ldots, s_r) \in S$. The semigroup ring $k[S] = k[\mathbf{t}^{a_0}, \mathbf{t}^{a_1}, \ldots, \mathbf{t}^{a_p}]$ of S can be represented as a quotient of a polynomial ring using a canonical surjection $\pi : k[x_0, x_1, \ldots, x_p] \to k[S]$, given by $\pi(x_i) = \mathbf{t}^{a_i}$ for all $i = 0, 1, \ldots, p$.

Let $n_0, n_1, \ldots, n_p$ be a sequence of positive integers such that $n_0 < n_1 < \cdots < n_p$. Let $\mathcal{C}$ be a projective monomial curve in the projective space $\mathbb{P}_k^{p+1}$, defined parametrically by

$$x_0 = v^{n_p}, \quad x_1 = u^{n_0} v^{n_p - n_0}, \quad \ldots, \quad x_p = u^{n_{p-1}} v^{n_p - n_{p-1}}, \quad x_{p+1} = u^{n_p}.$$

Let $k[\mathcal{C}]$ denote the co-ordinate ring of $\mathcal{C}$. Then $k[\mathcal{C}] = k[S]$, where $S = \langle (0, n_p), (n_0, n_p - n_0), (n_1, n_p - n_1), \ldots, (n_{p-1}, n_p - n_{p-1}), (n_p, 0) \rangle$ is an affine semigroup in $\mathbb{N}^2$. For such affine semigroup rings, we recall the following theorem from [7], which gives a set of generators of the derivation module $\mathrm{Der}_k(k[S])$.

Theorem 2.6. [7] *Let $S = \langle (0, n_p), (n_0, n_p - n_0), (n_1, n_p - n_1), \ldots, (n_{p-1}, n_p - n_{p-1}), (n_p, 0) \rangle$ be an affine semigroup in $\mathbb{N}^2$, where $n_0 < n_1 < \cdots < n_p$ and $\gcd(n_0, n_1, \ldots, n_p) = 1$. Let S_1 and S_2 be the numerical semigroups corresponding the natural projections to the first and second components of S respectively. If the semigroup ring $k[S]$ is Cohen-Macaulay then the derivation module $\mathrm{Der}_k(k[S])$ is generated by $D_1 \cup \{u\frac{\partial}{\partial u}\} \cup D_2 \cup \{t\frac{\partial}{\partial t}\}$, where D_1 and D_2 are defined below.*

(1) *If $S_2 \neq \mathbb{N}$, then $D_1 = \{t^{\beta} u^{\alpha+1} \frac{\partial}{\partial u} \mid \alpha \in \mathrm{PF}(S_2)\}$, and β is the least positive integer such that the pair (β, α) satisfy*

$$(\beta, \alpha) + (n, n_p - n) \in S \quad \textit{for each} \quad n \in \{0, n_0, \ldots, n_{p-1}\}.$$

15 Mar 2023 04:17:51 PDT
221214-Sengupta Version 2 - Submitted to Rocky Mountain J. Math.

(2) *If $S_2 = \mathbb{N}$, then $D_1 = \{t^{1+cn_p}\frac{\partial}{\partial u}\}$, and c is the least non-negative integer such that the pair $(1+cn_p, -1)$ satisfies*

$$(1+cn_p, -1) + (n, n_p - n) \in S \quad \textit{for each} \quad n \in \{0, n_0, \ldots, n_{p-1}\}.$$

(3) *If $S_1 \neq \mathbb{N}$, then $D_2 = \{t^{\delta+1}u^{\gamma}\frac{\partial}{\partial t} \mid \delta \in \mathrm{PF}(S_1)\}$, and γ is the least positive integer such that the pair (δ, γ) satisfy*

$$(\delta, \gamma) + (n, n_p - n) \in S \quad \textit{for each} \quad n \in \{n_0, \ldots, n_p\}.$$

(4) *If $S_1 = \mathbb{N}$, then $D_2 = \{u^{1+en_p}\frac{\partial}{\partial t}\}$, and e is the least non-negative integer such that the pair $(-1, 1+en_p)$ satisfies*

$$(-1, 1+en_p) + (n, n_p - n) \in S \quad \textit{for each} \quad n \in \{n_0, \ldots, n_p\}.$$

3. Derivations in $\mathbb{P}_k^2$

In this section, we give the explicit set of minimal generators of the derivation module for the co-ordinate ring of a projective monomial curve defined by the positive integers n_0 and n_1, such that $n_0 < n_1$ and $\gcd(n_0, n_1) = 1$.

Proposition 3.1. *Let $S = \langle (0, n_1), (n_0, n_1 - n_0), (n_1, 0)\rangle$ be an affine semigroup in $\mathbb{N}^2$, such that $n_0 < n_1$ and $\gcd(n_0, n_1) = 1$. Then we have the following:*

(1) *If $S_1, S_2 \neq \mathbb{N}$, then the derivation module $\mathrm{Der}_k(k[S])$ is mimimally generated by*

$$\left\{t\frac{\partial}{\partial t}, t^{n_0(n_1-1)-n_1+1}u^{(n_1-1)(n_1-n_0)}\frac{\partial}{\partial t}, u\frac{\partial}{\partial u}, t^{n_0(n_1-1)}u^{(n_1-1)(n_1-n_0)-n_1+1}\frac{\partial}{\partial u}\right\}.$$

(2) *If $S_1 = \mathbb{N}$ and $S_2 \neq \mathbb{N}$, then the derivation module $\mathrm{Der}_k(k[S])$ is mimimally generated by*

$$\left\{t\frac{\partial}{\partial t}, u^{1+(n_1-2)n_1}\frac{\partial}{\partial t}, u\frac{\partial}{\partial u}, t^{n_1-1}u^{(n_1-1)(n_1-2)}\frac{\partial}{\partial u}\right\}.$$

(3) *If $S_1 \neq \mathbb{N}$ and $S_2 = \mathbb{N}$, then the derivation module $\mathrm{Der}_k(k[S])$ is mimimally generated by*

$$\left\{t\frac{\partial}{\partial t}, t^{n_0(n_1-1)-n_1+1}u^{n_1-1}\frac{\partial}{\partial t}, u\frac{\partial}{\partial u}, t^{n_0(n_1-1)}\frac{\partial}{\partial u}\right\}.$$

(4) *If $S_1 = S_2 = \mathbb{N}$, then the derivation module $\mathrm{Der}_k(k[S])$ is mimimally generated by*

$$\left\{t\frac{\partial}{\partial t}, u\frac{\partial}{\partial t}, u\frac{\partial}{\partial u}, t\frac{\partial}{\partial u}\right\}.$$

Proof. Let S_1 and S_2 be the projections of S to the first and the second component of S, then we have $S_1 = \langle n_0, n_1\rangle$ and $S_2 = \langle n_1 - n_0, n_1\rangle$. We will prove each case separately by using Theorem 2.6.

Case 1. Suppose $S_1, S_2 \neq \mathbb{N}$. From [15, Proposition 2.13], we have $\mathrm{PF}(S_1) = \{n_0(n_1 - 1) - n_1\}$ and $\mathrm{PF}(S_2) = \{(n_1-1)(n_1-n_0) - n_1\}$. Let $\beta \in S_1$ be such that $(\beta, (n_1-1)(n_1 - n_0) - n_1) + (0, n_1) = (\beta, (n_1-1)(n_1-n_0)) \in S$. Note that $(n_1-1)(n_1-n_0)$ has only one factorization in S_2. Therefore, the possible factorization of $(\beta, (n_1-1)(n_1-n_0))$ in S will be

$$(\beta, (n_1-1)(n_1-n_0)) = (n_1-1)(n_0, n_1-n_0) + \lambda(n_1, 0), \quad \text{for some} \quad \lambda \geq 0.$$

Therefore, we have $\beta = (n_1 - 1)n_0 + \lambda n_1 \geq n_0(n_1 - 1)$. Now for $\beta = n_0(n_1 - 1)$, we have

$$\begin{aligned}(n_0(n_1 - 1), (n_1 - 1)(n_1 - n_0) - n_1) + (0, n_1) &= (n_0(n_1 - 1), (n_1 - 1)(n_1 - n_0)) \\ &= (n_1 - 1)(n_0, n_1 - n_0)\end{aligned}$$

and

$$\begin{aligned}(n_0(n_1 - 1), (n_1 - 1)(n_1 - n_0) - n_1) + (n_0, n_1 - n_0) &= (n_0 n_1, n_1(n_1 - n_0 - 1)) \\ &= n_0(n_1, 0) + (n_1 - n_0 - 1)(0, n_1).\end{aligned}$$

Now, let $\gamma \in S_2$ be such that $(n_0(n_1 - 1) - n_1, \gamma) + (n_1, 0) = (n_0(n_1 - 1), \gamma) \in S$. Note that $n_0(n_1 - 1)$ has only one factorization in S_1. Therefore, the only possible factorization of $(n_0(n_1 - 1), \gamma)$ in S is

$$(n_0(n_1 - 1), \gamma) = (n_1 - 1)(n_0, n_1 - n_0) + \lambda(0, n_1) \quad \text{for} \quad \text{some} \quad \lambda \geq 0.$$

Hence, we have $\gamma = (n_1 - 1)(n_1 - n_0) + \lambda n_1 \geq (n_1 - 1)(n_1 - n_0)$. Now for $\gamma = (n_1 - 1)(n_1 - n_0)$, we have

$$\begin{aligned}(n_0(n_1 - 1) - n_1, (n_1 - 1)(n_1 - n_0)) + (n_0, n_1 - n_0) &= (n_0 n_1 - n_1, n_1(n_1 - n_0)) \\ &= (n_0 - 1)(n_1, 0) + (n_1 - n_0)(0, n_1);\end{aligned}$$

and

$$\begin{aligned}(n_0(n_1 - 1) - n_1, (n_1 - 1)(n_1 - n_0)) + (n_1, 0) &= (n_0(n_1 - 1), (n_1 - 1)(n_1 - n_0)) \\ &= (n_1 - 1)(n_0, n_1 - n_0).\end{aligned}$$

Case 2. Suppose $S_1 = \mathbb{N}$. Therefore, we must have $n_0 = 1$. Let e be a non-negative integer such that $(-1, 1 + en_1) + (n_1, 0) = (n_1 - 1, 1 + en_1) \in S$. Observe that $n_1 - 1$ has only one factorization in S_1. Therefore the only possible factorization of $(n_1 - 1, 1 + en_1)$ in S is

$$(n_1 - 1, 1 + en_1) = (n_1 - 1)(1, n_1 - 1) + \lambda(0, n_1), \quad \text{for} \quad \text{some} \quad \lambda \geq 0.$$

Therefore, we have $1 + en_1 \geq (n_1 - 1)^2$, which implies that $e \geq n_1 - 2$. Now for $e = n_1 - 2$, we have

$$(-1, 1 + (n_1 - 2)n_1) + (n_1, 0) = (n_1 - 1, (n_1 - 1)^2) = (n_1 - 1)(1, n_1 - 1);$$

and

$$(-1, 1 + (n_1 - 2)n_1) + (1, n_1 - 1) = (0, n_1(n_1 - 1)) = (n_1 - 1)(0, n_1).$$

Case 3. Suppose $S_2 = \mathbb{N}$. Therefore we must have $n_1 - n_0 = 1$. Let c be a non-negative integer such that $(1 + cn_1, -1) + (0, n_1) = (1 + cn_1, n_1 - 1) \in S$. Observe that $n_1 - 1$ has only one factorization in S_2. Therefore the only possible factorization of $(1 + cn_1, n_1 - 1)$ in S is

$$(1 + cn_1, n_1 - 1) = (n_1 - 1)(n_0, 1) + \lambda(n_1, 0), \quad \text{for} \quad \text{some} \quad \lambda \geq 0.$$

Therefore, we have $1 + cn_1 \geq (n_1 - 1)n_0 = n_1(n_0 - 1) + 1$. This implies that $c \geq n_0 - 1$. Now for $c = n_0 - 1$, we have

$$(1 + (n_0 - 1)n_1, -1) + (0, n_1) = (1 + (n_0 - 1)(n_0 + 1), (n_1 - 1)) = (n_1 - 1)(n_0, 1);$$

and

$$(1 + (n_0 - 1)n_1, -1) + (n_0, 1) = (n_0 n_1, 0) = n_0(n_1, 0).$$

Case 4. Suppose $S_1 = \mathbb{N} = S_2$. In this case, the only possibility is $S = \langle (0,2),(1,1),(2,0)\rangle$. From the arguements of cases 2 and 3, it is easy to observe that the derivation module $\mathrm{Der}_k(k[S])$ is minimally generated by $\{t\frac{\partial}{\partial t}, u\frac{\partial}{\partial t}, u\frac{\partial}{\partial u}, t\frac{\partial}{\partial u}\}$. □

Example 3.2. Let $S = \langle (0,3),(1,2),(3,0)\rangle$. Here $S_1 = \mathbb{N}$ and $S_2 \neq \mathbb{N}$. The derivation module $\mathrm{Der}_k(k[S])$ is minimally generated by $\{t\frac{\partial}{\partial t}, u^4\frac{\partial}{\partial t}, u\frac{\partial}{\partial u}, t^2u^2\frac{\partial}{\partial u}\}$.

Example 3.3. Let $S = \langle (0,9),(5,4),(9,0)\rangle$. Here $S_1 \neq \mathbb{N} \neq S_2$. The derivation module $\mathrm{Der}_k(k[S])$ is minimally generated by $\{t\frac{\partial}{\partial t}, t^{32}u^{32}\frac{\partial}{\partial t}, u\frac{\partial}{\partial u}, t^{40}u^{24}\frac{\partial}{\partial u}\}$.

4. Derivations in $\mathbb{P}_k^{p+1}$

For a numerical semigroup S, it is well known that $\mu(\mathrm{Der}_k(k[S]_\mathfrak{m})) = r+1$, where r is the Cohen-Macaulay type of $k[S]$ and $\mathfrak{m}$ is the maximal homogeneous ideal of $k[S]$. Such a nice relation between $\mu(\mathrm{Der}_k(k[S]_\mathfrak{m}))$ and Cohen-Macaulay type of $k[S]$ does not hold in general for the affine semigroups (see [7, Remark 3.6]).

We now assume that $n_0, n_1, \ldots, n_p$ is an arithmetic sequence of positive integers i.e., for a fixed positive integer d, $n_i = n_0 + id$ for $i \in [0,p]$, such that $\gcd(n_0, n_1, \ldots, n_p) = 1$. Also assume that the sequence $n_0, n_1, \ldots, n_p$ forms a minimal generating set for a numerical semigroup, and we say that $n_0, n_1, \ldots, n_p$ is a minimal arithmetic sequence. Now define

$$S = \langle (0,n_p),(n_0,n_p-n_0),(n_1,n_p-n_1),\ldots,(n_{p-1},n_p-n_{p-1}),(n_p,0)\rangle,$$

an affine semigroup in $\mathbb{N}^2$. We will denote the natural projections to the first and second components of S by S_1 and S_2 respectively. These notations will be followed throughout the section.

From the [1, Corollary 3.2], we know that $k[S]$ is Cohen-Macaulay. The following Proposition gives a nice relation between $\mu(\mathrm{Der}_k(k[S]_\mathfrak{m}))$ and Cohen-Macaulay type of $k[S]$.

Proposition 4.1. *Let $S = \langle (0,n_p),(n_0,n_p-n_0),(n_1,n_p-n_1),\ldots,(n_{p-1},n_p-n_{p-1}),(n_p,0)\rangle$ be an affine semigroup in $\mathbb{N}^2$, where $n_0,n_1,\ldots,n_p$ is a minimal arithmetic sequence of positive integers such that $\gcd(n_0,n_1,\ldots,n_p) = 1$. Then $\mu(\mathrm{Der}_k(k[S]_\mathfrak{m})) = r+3$, where r is the Cohen-Macaulay type of $k[S]$ and $\mathfrak{m}$ is the maximal homogeneous ideal of $k[S]$.*

Proof. Since $n_0 < n_2 < \ldots < n_p$ is an arithmetic sequence of positive integers, then for a fixed positive integer d, we have $n_i = n_0 + id$ for $i \in [0,p]$. Since S_1 and S_2 are the numerical semigroups corresponding the natural projections to the first and second components of S respectively, we have $S_1 = \langle n_0, n_1, \ldots, n_p\rangle$ and $S_2 = \langle d, n_p\rangle$. For $i = 1,2$, let r_i be the Cohen-Macaulay type of S_i. By [16, Corollary 4.7], we have $r_1 = r$ and by [15, Proposition 2.13], we get $r_2 = 1$. Therefore by [7, Corollary 3.5], we have

$$\mu(\mathrm{Der}_k(k[S]_\mathfrak{m})) = r_1 + r_2 + 2 = r + 3.$$ □

Corollary 4.2. *With the assumptions of Proposition 4.1 and $n_0 = ap+b$, $0 \le b < p$, we have*

$$\mu(\mathrm{Der}_k(k[S]_\mathfrak{m})) = \begin{cases} 4 & \text{if} \quad p = 1; \\ p+2 & \text{if} \quad p \ge 2, b = 0; \\ p+3 & \text{if} \quad p \ge 2, b = 1; \\ b+2 & \text{if} \quad p \ge 2, 1 < b < p. \end{cases}$$

Proof. Since the case $p = 1$ reduces to the Theorem 3.1, we have $\mu(\mathrm{Der}_k(k[S]_{\mathfrak{m}})) = 4$ if $p = 1$. We now assume that $p \geq 2$. Therefore by [10, Theorem 3.1], the Cohen-macaulay type of S_1 is

$$r_1 = \begin{cases} p-1 & \text{if} \quad b = 0; \\ p & \text{if} \quad b = 1; \\ b-1 & \text{if} \quad 1 < b < p. \end{cases}$$

Now, the result follows from Proposition 4.1. □

Lemma 4.3. *Let S_1 be the natural projection to the first component of S. Then the set $\mathrm{Ap}(S_1, n_p)$ is a homogeneous subset of S_1.*

Proof. Define the map $\phi : k[x_0, \ldots, x_p] \longrightarrow k[t]$ such that $x_i \to t^{n_i}$ for $0 \leq i \leq p$. Then, $k[S_1] \cong \frac{k[x_0,\ldots,x_p]}{\ker(\phi)}$. From [11], we have a minimal generating set (say B) of $\mathrm{Ker}(\phi)$ such that one term of each non-homogeneous element of B is divisible by x_p. Hence, the result follows from [18, Proposition 3.9]. □

Lemma 4.4. *Let S_1 be the natural projection to the first component of S and $s \in \mathrm{Ap}(S_1, n_p)$. If $s = \sum_{i=0}^{p} \lambda_i n_i = \sum_{i=0}^{p} \lambda'_i n_i$ has two expressions in S, then*

$$\sum_{i=0}^{p} \lambda_i (p-i)d = \sum_{i=0}^{p} \lambda'_i (p-i)d.$$

Proof. Since

$$s = \sum_{i=0}^{p} \lambda_i n_i = \sum_{i=0}^{p} \lambda'_i n_i,$$

we get

$$\sum_{i=0}^{p} \lambda_i n_0 + \sum_{i=0}^{p} \lambda_i (id) = \sum_{i=0}^{p} \lambda'_i n_0 + \sum_{i=0}^{p} \lambda'_i (id).$$

By Lemma 4.3, we have $\sum_{i=0}^{p} \lambda_i = \sum_{i=0}^{p} \lambda'_i$. Thus, we have

$$\sum_{i=0}^{p} \lambda_i (pd) - \sum_{i=0}^{p} \lambda_i (id) = \sum_{i=0}^{p} \lambda'_i (pd) - \sum_{i=0}^{p} \lambda'_i (id).$$

Therefore, we get

$$\sum_{i=0}^{p} \lambda_i (p-i)d = \sum_{i=0}^{p} \lambda'_i (p-i)d.$$

□

Theorem 4.5. *Suppose $p \geq 2$. Let $S = \langle (0, n_p), (n_0, n_p - n_0), (n_1, n_p - n_1), \ldots, (n_{p-1}, n_p - n_{p-1}), (n_p, 0) \rangle$ be an affine semigroup in $\mathbb{N}^2$, where $n_0, n_1, \ldots, n_p$ is a minimal arithmetic sequence of positive integers, i.e., for $i \in [1, p]$, $n_i = n_0 + id$ for some positive integer d, and $\gcd(n_0, n_1, \ldots, n_p) = 1$. Write $n_0 = ap + b$, $0 \leq b < p$, then we have the following:*

(1) *If $b = 0$, then the derivation module $\mathrm{Der}_k(k[S])$ is mimimally generated by*

$$\left\{ u\frac{\partial}{\partial u}, t^{an_p+d} u^{(d-1)(n_p-1)} \frac{\partial}{\partial u}, t\frac{\partial}{\partial t}, t^{an_p - n_{p-i}+1} u^{d(n_p-i)} \frac{\partial}{\partial t} \mid 1 \leq i \leq p-1 \right\}.$$

(2) *If* $b = 1$*, then the derivation module* $\mathrm{Der}_k(k[S])$ *is mimimally generated by*

$$\left\{u\frac{\partial}{\partial u}, t^{an_p+d}u^{(d-1)(n_p-1)}\frac{\partial}{\partial u}, t\frac{\partial}{\partial t}, t^{an_p-n_{p-i}+1}u^{d(n_p-i)}\frac{\partial}{\partial t} \mid 1 \leq i \leq p\right\}.$$

(3) *If* $b \neq 0, 1$*, then the derivation module* $\mathrm{Der}_k(k[S])$ *is mimimally generated by*

$$\left\{u\frac{\partial}{\partial u}, t^{(a+1)n_p+d}u^{(d-1)(n_p-1)}\frac{\partial}{\partial u}, t\frac{\partial}{\partial t}, t^{an_p+id+1}u^{d(n_p-i)}\frac{\partial}{\partial t} \mid 1 \leq i \leq b-1\right\}.$$

Proof. Let S_1 and S_2 be the numerical semigroups corresponding the natural projections to the first and second components of S respectively. Then we have $\mathrm{PF}(S_2) = \{(d-1)n_p - d\}$. Also by [10, Theorem 3.1], we can write the following formulas for $\mathrm{PF}(S_1)$.

If $b = 0$, then

$$\begin{aligned}\mathrm{PF}(S_1) &= \{an_0 + \ell d - n_0 \mid (a-1)p + 1 \leq \ell \leq ap - 1\}\\ &= \{(a-1)n_p + id \mid 1 \leq i \leq p-1\}\\ &= \{an_p - n_{p-i} \mid 1 \leq i \leq p-1\}.\end{aligned}$$

If $b = 1$, then

$$\begin{aligned}\mathrm{PF}(S_1) &= \{an_0 + \ell d - n_0 \mid (a-1)p + 1 \leq \ell \leq ap\}\\ &= \{an_p - n_{p-i} \mid 1 \leq i \leq p\}.\end{aligned}$$

If $b \neq 0, 1$, then

$$\begin{aligned}\mathrm{PF}(S_1) &= \{(a+1)n_0 + \ell d - n_0 \mid ap + 1 \leq \ell \leq ap + b - 1\}\\ &= \{an_p + id \mid 1 \leq i \leq b-1\}.\end{aligned}$$

Now set $\beta = \begin{cases} an_p + d & \text{if} \quad b = 0, 1\\ (a+1)n_p + d & \text{if} \quad b \neq 0, 1\end{cases}$, and $\quad \alpha = (d-1)n_p - d$.

Also set, for $i \in I$,

$$\delta_i = \begin{cases} an_p - n_{p-i} & \text{if} \quad b = 0, 1\\ an_p + id & \text{if} \quad b \neq 0, 1\end{cases}, \quad \text{and} \quad \gamma_i = d(n_p - i),$$

where

$$I = \begin{cases} [1, p-1] & \text{if} \quad b = 0;\\ [1, p] & \text{if} \quad b = 1;\\ [1, b-1] & \text{if} \quad b \neq 0, 1.\end{cases}$$

Since $k[S]$ is Cohen-Macaulay, to prove $(\beta, \alpha) + (n, n_p - n) \in S$, for each $n \in \{0, n_0, \ldots, n_{p-1}\}$ and $(\delta_i, \gamma_i) + (n, n_p - n) \in S$, for each $i \in I$, $n \in \{n_0, n_1, \ldots, n_p\}$, is equivalent to prove that $(\beta, \alpha) + (n, n_p - n) \in (S_1 \times S_2) \cap G(S)$, for each $n \in \{0, n_0, \ldots, n_{p-1}\}$ and $(\delta_i, \gamma_i) + (n, n_p - n) \in (S_1 \times S_2) \cap G(S)$, for each $i \in I$, $n \in \{n_0, n_1, \ldots, n_p\}$, where $G(S)$ is the group generated by S in $\mathbb{Z}^2$.

Now, observe that $\beta \in S_1$ and since $\alpha = F(S_2)$, we have $\alpha + n_p - n \in S_2$, for each $n \in \{0, n_0, \ldots, n_{p-1}\}$. Therefore, we have $(\beta, \alpha) + (n, n_p - n) \in S_1 \times S_2$, for each $n \in$

$\{0, n_0, \ldots, n_{p-1}\}$. Also, if $b = 0, 1$, we have

$$\beta + \alpha + n + n_p - n = an_p + d + (d-1)n_p - d + n_p = (a+d)n_p,$$

and if $b \neq 0, 1$, we have

$$\beta + \alpha + n + n_p - n = (a+1)n_p + d + (d-1)n_p - d + n_p = (a+d+1)n_p.$$

Therefore by [2, Lemma 4.1], we have $(\beta, \gamma) + (n, n_p - n) \in G(S)$, for each $n \in \{0, n_0, \ldots, n_{p-1}\}$. Now, since $\delta_i \in \mathrm{PF}(S_1)$, we have $\delta_i + n \in S_1$, for all i, n. Also, we have

$$\begin{aligned}\gamma_i + n_p - n = d(n_p - i) + n_p - n &= (d-1)n_p - d + n_{p-(i-1)} + n_p - n \\ &= F(S_2) + n_{p-(i-1)} + n_p - n.\end{aligned}$$

Therefore, we have $(\delta_i, \gamma_i) + (n, n_p - n) \in (S_1 \times S_2)$, for each $i \in I$, $n \in \{n_0, n_1, \ldots, n_p\}$. Also, if $b = 0, 1$, we have

$$\delta_i + \gamma_i + n + n_p - n = an_p - n_{p-i} + d(n_p - i) + n_p = (a+d)n_p,$$

and if $b \neq 0, 1$, we have

$$\delta_i + \gamma_i + n + n_p - n = an_p + id + d(n_p - i) + n_p = (a+d+1)n_p.$$

Therefore by [2, Lemma 4.1], we have $(\delta_i, \gamma_i) + (n, n_p - n) \in G(S)$ for each $i \in I$, $n \in \{n_0, n_1, \ldots, n_p\}$.

To complete the proof, it is sufficient to prove that β and γ_i's are least positive integers such that $(\beta, \alpha) + (n, n_p - n) \in S$, for each $n \in \{0, n_0, \ldots, n_{p-1}\}$ and $(\delta_i, \gamma_i) + (n, n_p - n) \in S$, for each $i \in I$, $n \in \{n_0, n_1, \ldots, n_p\}$. Suppose $b \neq 0, 1$, we have

$$\alpha + n_p = (d-1)n_p - d + n_p = dn_p - d = d(n_0 + pd) - d = (a+d)pd + (b-1)d.$$

If there exist β such that $(\beta, \alpha) + (0, n_p) \in S$, then we get

$$\begin{aligned}\beta \geq (a+d)n_0 + n_0 + (p-b+1)d &\geq an_0 + dn_0 + n_p - (b-1)d \\ &\geq an_0 + dap + n_p + d \\ &\geq (a+1)n_p + d.\end{aligned}$$

Thus, if $b \neq 0, 1$ then $\beta = (a+1)n_p + d$ is minimal satisfying the required properties. Now, suppose $b \in 0, 1$, then we have

$$\alpha + n_p = (d-1)n_p - d + n_p = (a+d-1)pd + (p-1)d \quad \text{if} \quad b = 0,$$

and

$$\alpha + n_p = (d-1)n_p - d + n_p = (a+d)pd \quad \text{if} \quad b = 1.$$

If there exist β such that $(\beta, \alpha) + (0, n_p) \in S$, then we get

$$\beta \geq (a+d-1)n_0 + n_0 + d \geq an_0 + dn_0 + d \geq a(n_0 + pd) + d \quad \text{if} \quad b = 0,$$

and

$$\beta \geq (a+d)n_0 \geq an_0 + d(ap+1) \geq a(n_0 + pd) + d \quad \text{if} \quad b = 1.$$

Thus, if $b \in \{0, 1\}$ then $\beta = an_p + d$ is minimal satisfying the required properties. Now, since we have $\delta_i \in \mathrm{PF}(S_1)$ for all $i \in I$, then $\delta_i + n_p \in \mathrm{Ap}(S_1, n_p)$ for all $i \in I$. Suppose $b \in \{0, 1\}$, we have

$$\begin{aligned}\delta_i + n_p = an_p - n_{p-i} + n_p &= a(n_0 + pd) - n_0 - (p-i)d + n_0 + pd\\ &= (a+d-1)n_0 + apd + id + n_0 - n_0 d\\ &= (a+d-1)n_0 + id + n_0 - bd\\ &= (a+d-1)n_0 + n_{i-b}.\end{aligned}$$

If there exist γ_i such that $(\delta_i, \gamma_i) + (n_p, 0) \in S$, then by Lemma 4.4, we get

$$\begin{aligned}\gamma_i \geq (a+d-1)pd + pd - (i-b)d &\geq (a+d)pd - id + bd\\ &\geq (a+d)pd - id + n_0 d - apd\\ &\geq d(n_0 + pd) - id.\end{aligned}$$

Thus, if $b \in \{0, 1\}$ then $\gamma_i = d(n_p - i)$ is minimal satisfying the required properties. Now, suppose $b \neq 0, 1$. Therefore, we have

$$\begin{aligned}\delta_i + n_p = an_p + id + n_p &= a(n_0 + pd) + id + n_0 + pd\\ &= (a+d)n_0 + n_0 + id - (n_0 - ap)d + pd\\ &= (a+d)n_0 + n_0 + (p+i-b)d\\ &= (a+d)n_0 + n_{p+i-b}.\end{aligned}$$

If there exist γ_i such that $(\delta_i, \gamma_i) + (n_p, 0) \in S$, then by Lemma 4.4, we get

$$\begin{aligned}\gamma_i \geq (a+d)pd + pd - (p+i-b)d &\geq (a+d)pd - id + bd\\ &\geq (a+d)pd - id + n_0 d - apd\\ &\geq d(n_0 + pd) - id.\end{aligned}$$

Thus, if $b \neq 0, 1$, then also $\gamma_i = d(n_p - i)$ is minimal satisfying the required properties. This completes the proof. □

Example 4.6. Let $S = \langle (0,23), (11,12), (13,10), (15,8), (17,6), (19,4), (21,2), (23,0) \rangle$, then $S_1 = \langle 11, 13, 15, 17, 19, 21, 23 \rangle$ and $S_2 = \langle 2, 23 \rangle$. Here we have $n_0 = 11$ and $p = 6$, therefore we get $a = 1$, $b = 5$. Therefore, we have $\mathrm{PF}(S_1) = \{25, 27, 29, 31\}$ and $\mathrm{PF}(S_2) = \{21\}$. In the notation of proof of Theorem 4.5, we have $\delta_1 = 25$, $\delta_2 = 27$, $\delta_3 = 29$, $\delta_4 = 31$. Note that

$$\begin{gathered}\delta_1 + n_6 = 2 \cdot 11 + 2 \cdot 13 = 3 \cdot 11 + 15,\\ \delta_2 + n_6 = 11 + 3 \cdot 13 = 2 \cdot 11 + 13 + 15 = 3 \cdot 11 + 17,\\ \delta_3 + n_6 = 4 \cdot 13 = 11 + 2 \cdot 13 + 15 = 2 \cdot 11 + 2 \cdot 15 = 2 \cdot 11 + 13 + 17 = 3 \cdot 11 + 19,\\ \delta_4 + n_6 = 3 \cdot 13 + 15 = 11 + 13 + 2 \cdot 15 = 11 + 2 \cdot 13 + 17 = 2 \cdot 11 + 15 + 17 = 2 \cdot 11 + 13 + 19 = 3 \cdot 11 + 21,\end{gathered}$$

are the only factorizations of $\delta_1 + n_6, \delta_2 + n_6, \delta_3 + n_6, \delta_4 + n_6$ respectively. Also note that $\mathcal{L}(\delta_i + n_6) = 4$ for all $i \in [1, 4]$. Therefore the minimal choices for γ_i's such that $(\delta_i, \gamma_i) + (n_d, 0) \in S$ are

$$\begin{gathered}\gamma_1 = 2 \cdot 12 + 2 \cdot 10 \quad \text{or} \quad 3 \cdot 12 + 8,\\ \gamma_2 = 12 + 3 \cdot 10 \quad \text{or} \quad 2 \cdot 12 + 10 + 8 \quad \text{or} \quad 3 \cdot 12 + 6,\\ \gamma_3 = 4 \cdot 10 \quad \text{or} \quad 12 + 2 \cdot 10 + 8 \quad \text{or} \quad 2 \cdot 12 + 2 \cdot 8 \quad \text{or} \quad 2 \cdot 12 + 10 + 6 \quad \text{or} \quad 3 \cdot 12 + 4,\\ \gamma_4 = 3 \cdot 10 + 8 \quad \text{or} \quad 12 + 10 + 2 \cdot 8 \quad \text{or} \quad 12 + 2 \cdot 10 + 6 \quad \text{or} \quad 2 \cdot 12 + 8 + 6 \quad \text{or} \quad 2 \cdot 12 + 10 + 4 \quad \text{or} \quad 3 \cdot 12 + 2.\end{gathered}$$

In each case, we have $\gamma_1 = 44$, $\gamma_2 = 42$, $\gamma_3 = 40$ and $\gamma_4 = 38$. Further, we observe that these γ_i's satisfy the condition $(\delta_i, \gamma_i)+(n, n_6-n) \in S$, for each $i \in [1,4]$, $n \in \{n_0, n_1, \ldots, n_6\}$. Now, since $\alpha = 21$, we have $\alpha + n_6 = 44$. Observe that 48 is the smallest natural number such that $(48, \alpha) + (0, n_6) = (48, 44) \in S$. Also, observe that $\beta = 48$ satisfies the property that $(\beta, \alpha) + (n, n_6 - n) \in S$ for all $n \in \{n_0, n_1, \ldots, n_6\}$. Therefore, the set

$$\left\{u\frac{\partial}{\partial u}, t^{48}u^{22}\frac{\partial}{\partial u}, t\frac{\partial}{\partial t}, t^{26}u^{44}\frac{\partial}{\partial t}, t^{28}u^{42}\frac{\partial}{\partial t}, t^{30}u^{40}\frac{\partial}{\partial t}, t^{32}u^{38}\frac{\partial}{\partial t}\right\}$$

forms a minimal generating set for $\mathrm{Der}_k(k[S])$.

5. Hilbert-Kunz multiplicity

Let R be a d-dimensional graded k-algebra, with homogeneous maximal ideal $\mathfrak{m}$. Let M be a finite R-module and $q = \langle x_1, x_2, \ldots, x_s\rangle$ be a homogeneous $\mathfrak{m}$-primary ideal of R, then the Hilbert-Kunz multiplicity is defined by

$$e_{\mathrm{HK}}(q, M) = \lim_{n\to\infty} \frac{\ell_R(M/q^{[n]}M)}{n^d},$$

where $q^{[n]} = \langle x_1^n, x_2^n, \ldots, x_s^n\rangle$. In general, it is not clear that this quantity is well defined. If $\mathrm{char}(k) = p > 0$, then for $n = p^e$, $\lim_{e\to\infty} \frac{\ell_R(M/q^{[p^e]}M)}{p^{ed}}$ always exists (see [9]). If $q = \mathfrak{m}$, then we denote $e_{\mathrm{HK}}(\mathfrak{m}, R)$ by $e_{\mathrm{HK}}(R)$. In this section, we give an explicit formula for the Hilbert-Kunz multiplicity of the co-ordinate ring of the projective monomial curve defined by $n_1, \ldots, n_p$, such that $n_1 < n_2 < \cdots < n_p$ and $\gcd(n_1, n_2, \ldots, n_p) = 1$.

Theorem 5.1. *Let $S = \langle(0, n_p), (n_1, n_p - n_1), (n_2, n_p - n_2), \ldots, (n_{p-1}, n_p - n_{p-1}), (n_p, 0)\rangle$ be an affine semigroup in $\mathbb{N}^2$, where $n_1 < n_2 < \cdots < n_p$ and $\gcd(n_1, n_2, \ldots, n_p) = 1$. Put $n_0 = 0$. Then the Hilbert-Kunz multiplicity of $k[S]$ is*

$$e_{\mathrm{HK}}\left(k[S]\right) = 1 + \frac{1}{n_p}\left(\sum_{r=1}^{p}(n_r - 1)(n_r - n_{r-1})\right).$$

Proof. Let G be the group generated by S in $\mathbb{Z}^2$. Then G is a free $\mathbb{Z}$-module of rank 2. By [2, Lemma 4.1], G has a basis $\{(0, n_p), (1, -1)\}$. Let $\{(1, 0), (0, 1)\}$ be the canonical basis of $\mathbb{Z}^2$ as $\mathbb{Z}$-module. Then we have

$$(0, n_p) = 0 \cdot (1, 0) + n_p(0, 1) \quad \text{and} \quad (1, -1) = (1, 0) - (0, 1).$$

Therefore, the cardinality of $\frac{\mathbb{Z}^2}{G}$ is finite and equal to the modulus of the determinant of the matrix $\begin{bmatrix} 0 & n_p \\ 1 & -1 \end{bmatrix}$. Therefore, we have $|\frac{\mathbb{Z}^2}{G}| = n_p$.

Let J denote the ideal $\langle x^{n_p}, x^{n_1}y^{n_p-n_1}, \ldots, x^{n_{p-1}}y^{n_p-n_{p-1}}, y^{n_p}\rangle$ in $k[x, y]$. Observe that the radical ideal of J is the maximal homogeneous ideal of $k[x, y]$. Therefore, the length of $\frac{k[x,y]}{J}$ is finite and equal to the $\dim_k \frac{k[x,y]}{J}$ as a k-vector space. Let $\mathcal{B}$ be the basis of $\frac{k[x,y]}{J}$ as k-vector space. Then by [14, Theorem 39.6], observe that $\mathcal{B} = B \cup \bigcup_{r=1}^{p} B_r$, where $B = \{1, y, y^2, \cdots, y^{n_p-1}\}$ and for $r \in [1, p]$,

$$B_r = \left\{x^i y^j \mid 1 \le i \le n_r - 1, \quad n_p - n_r \le j \le n_p - n_{r-1} - 1\right\}.$$

The cardinality of B is n_p and the cardinality of B_r is $(n_r-1)(n_r-n_{r-1})$, for each $r \in [1,p]$. Therefore, the length of $\frac{k[x,y]}{J}$ is

$$\ell_{k[x,y]}\left(\frac{k[x,y]}{J}\right) = n_p + \left(\sum_{r=1}^{p}(n_r-1)(n_r-n_{r-1})\right).$$

Now, the result follows from [4, Corollary 2.3]. □

Corollary 5.2. *Let $S = \langle(0,n_p),(n_0,n_p-n_0),(n_1,n_p-n_1),\ldots,(n_{p-1},n_p-n_{p-1}),(n_p,0)\rangle$, where $n_0,n_1,\ldots,n_p$ is a minimal arithmetic sequence of positive integers, such that $n_0 < n_2 < \ldots < n_p$ and $\gcd(n_0,n_1,\ldots,n_p)=1$. Then the Hilbert-Kunz multiplicity of $k[S]$ is*

$$e_{\mathrm{HK}}\left(k[S]\right) = n_0 + \frac{p(p+1)d^2}{2n_p},$$

where d is the common difference.

Proof. By Theorem 5.1, we have

$$\begin{aligned} e_{\mathrm{HK}}\left(k[S]\right) = 1 + \frac{1}{n_0+pd}\left(\sum_{r=1}^{p}(n_r-1)(n_r-n_{r-1})\right) + (n_0-1)n_0 &= \frac{n_0^2+d(n_1+n_2+\cdots+n_p)}{n_0+pd} \\ &= \frac{2n_0^2+2n_0pd+p(p+1)d^2}{2(n_0+pd)}. \end{aligned}$$

□

Example 5.3. Let $A = k[x^3,x^2y,xy^2,y^3]$ be the co-ordinate ring of the twisted cubic curve in the projective space $\mathbb{P}^3$. The affine semigroup parametrizing this curve is $S = \langle(0,3),(1,2),(2,1),(3,0)\rangle$. Therefore, by Corollary 5.2, $e_{\mathrm{HK}}(A)=2$.

Example 5.4. Let $S = \langle(0,19),(7,12),(10,9),(13,6),(16,3),(19,0)\rangle$. Therefore, in the notation of Corollary 5.2, we have $n_0=7$, $p=4$ and $d=3$. Hence $e_{\mathrm{HK}}(k[S]) = 7+\frac{20\cdot 9}{2\cdot 19} = \frac{223}{19}$.

Example 5.5. Let $A = k[x^4,x^3y,xy^3,y^4]$. Then it will correspond to the semigroup ring of the affine semigroup $S = \langle(0,4),(1,3),(3,1),(4,0)\rangle$. Therefore, by Theorem 5.1, $e_{\mathrm{HK}}(A) = 1+\frac{1}{4}\left(0\cdot 1+2\cdot 2+3\cdot 1\right) = \frac{11}{4}$.

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IIT Gandhinagar, Palaj, Gandhinagar, Gujarat-382355 India
Email address: `om.prakash@iitgn.ac.in`

IIT Gandhinagar, Palaj, Gandhinagar, Gujarat-382355 India
Email address: `indranathsg@iitgn.ac.in`