A strong cancellation theorem for modules over $C_\infty \times C_q$

F.E.A. Johnson

Abstract

The module cancellation problem asks whether, given modules X, X' and Y over a ring Λ, the existence of an isomorphism $X \oplus Y \cong X' \oplus Y$ implies that $X \cong X'$. When q is prime we prove a strong cancellation property for certain modules over $\mathbb{Z}[C_\infty \times C_q]$, generalizing, in part, the strong cancellation property for modules over $\mathbb{Z}[C_q]$ established in the paper of R.Wiegand [18].

Keywords: Strong cancellation semigroup, Swan module.

Mathematics Subject Classification (AMS 2020): 13C05; 13C10; 19A13.

Given modules X, X' and Y over a ring Λ, we ask whether the existence of an isomorphism $X \oplus Y \cong X' \oplus Y$ implies that $X \cong X'$. When $\Lambda = \mathbb{Z}[G]$ is an integral group ring this question is central to the homotopy theory of spaces with fundamental group G (cf [7], [8]). When $C_q = \langle x : x^q = 1 \rangle$ is the finite cyclic group of prime order q there is a very strong cancellation property ([18]) for modules over $\mathbb{Z}[C_q]$ which are free of finite rank over $\mathbb{Z}$. In the present paper, writing C_∞ for the infinite cyclic group, we extend these results in part to modules over $\mathbb{Z}[C_\infty \times C_q]$.

Let Λ be a ring and $\mathfrak{C}$ be a class of Λ-modules, closed with respect to isomorphism; we say that $\mathfrak{C}$ is a *cancellation semigroup* when, for Λ-modules C, C', C'',

(i) $C, C' \in \mathfrak{C} \Longrightarrow C \oplus C' \in \mathfrak{C}$;

(ii) if $C, C', C'' \in \mathfrak{C}$ and $C \oplus C' \cong C \oplus C''$ then $C' \cong C''$.

$\mathfrak{C}$ is a *strong cancellation semigroup* when in addition, given a *nonzero* Λ-module S,

(iii) if $C \oplus S \in \mathfrak{C}$ then $C \in \mathfrak{C} \Longrightarrow S \in \mathfrak{C}$.

Let $\Lambda = R[C_q]$ be the group algebra of C_q over the commutative ring R. Let $\epsilon : \Lambda \to R$ be the canonical augmentation homomorphism, put $I = \mathrm{Ker}(\epsilon)$ and denote by $\mathfrak{Q}(R, q)$ the following class of Λ-modules

$$\mathfrak{Q}(R, q) = \{\Lambda^a \oplus I^b \mid a \geq 0,\ b \geq 0,\ a + b > 0\}.$$

Taking $R = \mathbb{Z}[t, t^{-1}]$ to be the ring of Laurent polynomials we identify R with integral group ring $\mathbb{Z}[C_\infty]$ and Λ with $\mathbb{Z}[C_\infty \times C_q]$ so that the elements of $\mathfrak{Q}(R, q)$ are modules over $C_\infty \times C_q$. We will prove:

Main Theorem : $\mathfrak{Q}(\mathbb{Z}[C_\infty], q)$ is a strong cancellation semigroup for any prime q.

We proceed via a number of subsidiary results. When S is a Λ-module we will show:

21 Mar 2023 05:02:02 PDT
221213-Johnson Version 2 - Submitted to Rocky Mountain J. Math.

(I) $\quad S \oplus I \cong_\Lambda I^{(d+1)} \implies S \cong_\Lambda I^{(d)}$.

(II) $\quad \Lambda \oplus S \cong_\Lambda \Lambda^{(c+1)} \oplus I^{(d)} \implies S \cong_\Lambda \Lambda^{(c)} \oplus I^{(d)}$.

(III) $\quad I \oplus S \cong_\Lambda \Lambda^{(c)} \oplus I^{(d+1)} \implies S \cong_\Lambda \Lambda^{(c)} \oplus I^{(d)}$.

The paper is organized as follows; §1- §4 are preliminary; **(I)** is proved in §5; **(II)** and **(III)** are proved in §6 and the Main Theorem is proved in §7. We conclude in §8 with a brief account of the cancellation problem in general.

§1: A recognition criterion for projective modules :

Throughout this paper, without further mention, Λ will denote a commutative algebra which is free of finite rank over a Noetherian ring R. In particular, Λ is itself Noetherian. We first note that:

(1.1) Let P be a finitely generated Λ-module such that $\mathrm{Ext}^1(P, N) = 0$ for all finitely generated Λ-modules N; then P is projective.

Proof : Let $\pi : \Lambda^a \twoheadrightarrow P$ be a surjective Λ homomorphism, put $K = \mathrm{Ker}(\pi)$ and consider the exact sequence $\mathcal{P} = (0 \to K \hookrightarrow \Lambda^a \xrightarrow{\pi} P \to 0)$. As Λ is Noetherian then K is finitely generated so that $\mathrm{Ext}^1(P, K) = 0$. Hence $\mathcal{P}$ splits so that $P \oplus K \cong \Lambda^a$ and P is projective. $\square$

The following are easy consequences of the Noetherian condition on Λ and R.

(1.2) If M, N are finitely generated Λ-modules then $\mathrm{Ext}^k_\Lambda(M, N)$ is a finitely generated R-module for each $k \geq 1$.

(1.3) Let A, B be finitely generated R-modules; if $A \oplus B \cong A$ then $B = 0$.

(1.4) If M, P, Q are finitely generated Λ-modules and $M \oplus P \cong M \oplus Q$ then

$$P \text{ is projective} \iff Q \text{ is projective.}$$

Proof of (1.4) It suffices to prove ($\Longleftarrow$). Let N be a finitely generated Λ-module ; as $M \oplus P \cong M \oplus Q$ then $\mathrm{Ext}^1(M, N) \oplus \mathrm{Ext}^1(P, N) \cong \mathrm{Ext}^1(M, N) \oplus \mathrm{Ext}^1(Q, N)$. As Q is projective then $\mathrm{Ext}^1(Q, N) = 0$; hence $\mathrm{Ext}^1(M, N) \oplus \mathrm{Ext}^1(P, N) \cong \mathrm{Ext}^1(M, N)$. By **(1.2)**, $\mathrm{Ext}^1(M, N)$ and $\mathrm{Ext}^1(P, N)$ are finitely generated R-modules It follows from **(1.3)** that $\mathrm{Ext}^1(P, N) = 0$ so that P is projective by **(1.1)**. $\square$

§2: A cancellation theorem :

We now assume given an algebra Ω which is again free of finite rank over R together with a surjective homomorphism of R-algebras $\eta : \Lambda \to \Omega$. We regard Ω as module over Λ via coinduction by η. As Λ is Noetherian then in the exact sequence

$$0 \to J \xrightarrow{j} \Lambda \xrightarrow{\eta} \Omega \to 0$$

$J = \mathrm{Ker}(\eta)$ is finitely generated over Λ. We assume also that:

21 Mar 2023 05:02:02 PDT
221213-Johnson Version 2 - Submitted to Rocky Mountain J. Math.

$$\textbf{(2.1)} \qquad \mathrm{Hom}_\Lambda(J,\Omega) \;=\; 0.$$

$$\textbf{(2.2)} \qquad X \oplus \Omega^{(n)} \cong_\Lambda \Omega^{(m+n)} \implies X \cong_\Lambda \Omega^{(m)}$$

$$\textbf{(2.3)} \qquad X \oplus J^{(n)} \cong_\Lambda J^{(m+n)} \implies X \cong_\Lambda J^{(m)}$$

With these assumptions we have:

Proposition 2.4 : Let S be a Λ-module for which there exists an isomorphism $f : \Lambda^{(b+c)} \oplus J^{(d)} \xrightarrow{\simeq} S \oplus \Lambda^{(c)}$ for some $c, d \geq 0$. Then for any homomorphism $\alpha : S \to \Omega^{(b)}$ where $b \geq 1$ there exists a commutative diagram of Λ-modules as follows in which the rows are exact, f_- is injective and f_+ is surjective.

$$\textbf{(2.5)} \qquad \left\{ \begin{array}{ccccccc} 0 \to & J^{(b+c+d)} & \stackrel{\widehat{j}}{\to} & \Lambda^{(b+c)} \oplus J^{(d)} & \stackrel{\widehat{\eta}}{\to} & \Omega^{(b+c)} & \to 0 \\ & \downarrow f_- & & \downarrow f & & \downarrow f_+ & \\ 0 \to & S_0 \oplus J^{(c)} & \stackrel{\widehat{i}}{\to} & S \oplus \Lambda^{(c)} & \stackrel{\widehat{\alpha}}{\to} & \mathrm{Im}(\alpha) \oplus \Omega^{(c)} & \to 0 \end{array} \right. .$$

Moreover, f_- and f_+ are uniquely determined by f.

Proof : For any integer $n \geq 1$ we write

$$j^{(n)} = \underbrace{\begin{pmatrix} j & & & \\ & j & & \\ & & \ddots & \\ & & & j \end{pmatrix}}_{n} \quad ; \quad \eta^{(n)} = \underbrace{\begin{pmatrix} \eta & & & \\ & \eta & & \\ & & \ddots & \\ & & & \eta \end{pmatrix}}_{n}.$$

The homomorphisms in the rows are defined by the following matrices.

$$\widehat{j} = \begin{pmatrix} j^{(b+c)} & 0 \\ 0 & \mathrm{Id}_{J^{(d)}} \end{pmatrix} \quad ; \quad \widehat{\eta} = \begin{pmatrix} \eta^{(b+c)} & 0 \\ 0 & 0 \end{pmatrix};$$

$$\widehat{i} = \begin{pmatrix} i & 0 \\ 0 & j^{(c)} \end{pmatrix} \quad ; \quad \widehat{\alpha} = \begin{pmatrix} \alpha & 0 \\ 0 & \eta^{(c)} \end{pmatrix}.$$

As $\mathrm{Hom}_\Lambda(J,\Omega) = 0$ and $\mathrm{Im}(\alpha) \subset \Omega^{(b)}$ then $\widehat{\alpha} \circ f \circ \widehat{j} = 0$. Hence

$$f(\mathrm{Im}(\widehat{j})) \subset \mathrm{Ker}(\widehat{\alpha}) = \mathrm{Im}(\widehat{i})$$

and f restricts to an injective homomorphism $f_- : J^{(b+c+d)} \to S_0 \oplus J^{(c)}$ as indicated. As $\mathrm{Im}(\widehat{j}) = \mathrm{Ker}(\widehat{\eta})$ then $f(\mathrm{Ker}(\widehat{\eta})) \subset \mathrm{Ker}(\widehat{\alpha})$. After making the Noether identifications $\Omega^{(b+c)} \cong (\Lambda^{(b+c)} \oplus J^{(d)})/\mathrm{Ker}(\widehat{\eta})$; $\mathrm{Im}(\alpha) \oplus \Omega^{(c)} \cong (S \oplus \Lambda^{(c)})/\mathrm{Ker}(\widehat{\alpha})$ then f induces a homomorphism $f_+ : \Omega^{(b+c)} \to \mathrm{Im}(\alpha) \oplus \Omega^{(c)}$ as indicated. As $f_+ \circ \widehat{\eta} = \widehat{\alpha} \circ f$ and $\widehat{\alpha} \circ f$ is surjective then f_+ is surjective and uniquely determined by f. $\square$

Continuing the above discussion we now have:

21 Mar 2023 05:02:02 PDT
221213-Johnson Version 2 - Submitted to Rocky Mountain J. Math.

Proposition 2.6 : The following statements are equivalent :

i) f_+ is injective;

ii) f_+ is an isomorphism;

iii) f_- is an isomorphism;

iv) $S_0 \oplus J^{(c)} \cong J^{(b+c+d)}$;

v) $S_0 \cong J^{(b+d)}$;

vi) $\mathrm{Hom}_\Lambda(S_0, \Omega) = 0$;

vii) $\mathrm{Im}(\alpha) \cong \Omega^{(b)}$.

Proof : As f_+ is surjective then i) $\iff$ ii).

ii) $\implies$ vii); If f_+ is an isomorphism then $\mathrm{Im}(\alpha) \oplus \Omega^{(c)} \cong \Omega^{(b+c)}$ so that, by assumption **(2.2)**, $\mathrm{Im}(\alpha) \cong \Omega^{(b)}$.

vii) $\implies$ i); Suppose that $\psi : \mathrm{Im}(\alpha) \xrightarrow{\simeq} \Omega^{(b)}$ is an isomorphism. As f_+ is surjective then $(\psi \oplus \mathrm{Id}_{\Omega^{(c)}}) \circ f_+ : \Omega^{(b+c)} \to \Omega^{(b+c)}$ is surjective. As Ω is Noetherian then $(\psi \oplus \mathrm{Id}_{\Omega^{(c)}}) \circ f_+$ is an isomorphism and hence f_+ is injective.

ii) $\implies$ iii); As f is an isomorphism, if f_+ is also an isomorphism then, by extending the diagram (*) one place to the left by zeroes, it follows from the Five Lemma that f_- is an isomorphism.

iii) $\implies$ iv) is obvious.

iv) $\implies$ v) follows from the underlying hypothesis **(2.3)**.

v) $\implies$ vi) is clear as $\mathrm{Hom}_\Lambda(J, \Omega) = 0$.

It remains only to show that vi) $\implies$ i). Put $g = f^{-1} : S \oplus \Lambda^{(c)} \xrightarrow{\simeq} \Lambda^{(b+c)} \oplus J^{(d)}$ and extend the diagram **(2.5)** as follows:

$$\textbf{(2.7)} \qquad \left\{ \begin{array}{ccccccc} 0 \to & J^{(b+c+d)} & \xrightarrow{\widehat{j}} & \Lambda^{(b+c)} \oplus J^{(d)} & \xrightarrow{\widehat{\eta}} & \Omega^{(b+c)} & \to 0 \\ & \downarrow f_- & & \downarrow f & & \downarrow f_+ & \\ 0 \to & S_0 \oplus J^{(c)} & \xrightarrow{\widehat{i}} & S \oplus \Lambda^{(c)} & \xrightarrow{\widehat{\alpha}} & \mathrm{Im}(\alpha) \oplus \Omega^{(c)} & \to 0 \\ & & & \downarrow g & & & \\ 0 \to & J^{(b+c+d)} & \xrightarrow{\widehat{j}} & \Lambda^{(b+c)} \oplus J^{(d)} & \xrightarrow{\widehat{\eta}} & \Omega^{(b+c)} & \to 0 \end{array} \right.$$

By our assumption **(2.1)**, $\mathrm{Hom}_\Lambda(J, \Omega) = 0$. By hypothesis, $\mathrm{Hom}_\Lambda(S_0, \Omega) = 0$. It follows that $\widehat{\eta} \circ g \circ \widehat{i} = 0$. As in the the proof of **(2.4)**, we may construct homomorphisms g_- and g_+ which extend **(2.7)** to the following commutative diagram:

$$\begin{array}{ccccccc} 0 \to & J^{(b+c+d)} & \xrightarrow{\widehat{j}} & \Lambda^{(b+c)} \oplus J^{(d)} & \xrightarrow{\widehat{\eta}} & \Omega^{(b+c)} & \to 0 \\ & \downarrow f_- & & \downarrow f & & \downarrow f_+ & \\ 0 \to & S_0 \oplus J^{(c)} & \xrightarrow{\widehat{i}} & S \oplus \Lambda^{(c)} & \xrightarrow{\widehat{\alpha}} & \mathrm{Im}(\alpha) \oplus \Omega^{(c)} & \to 0 \\ & \downarrow g_- & & \downarrow g & & \downarrow g_+ & \\ 0 \to & J^{(b+c+d)} & \xrightarrow{\widehat{j}} & \Lambda^{(b+c)} \oplus J^{(d)} & \xrightarrow{\widehat{\eta}} & \Omega^{(b+c)} & \to 0 \end{array}$$

21 Mar 2023 05:02:02 PDT
221213-Johnson Version 2 - Submitted to Rocky Mountain J. Math.

On eliminating the middle row by composition we obtain the commutative diagram:

$$\begin{array}{ccccccc} 0 \to & J^{(b+c+d)} & \stackrel{\widehat{j}}{\to} & \Lambda^{(b+c)} \oplus J^{(d)} & \stackrel{\widehat{\eta}}{\to} & \Omega^{(b+c)} & \to 0 \\ & \downarrow g_- \circ f_- & & \downarrow g \circ f & & \downarrow g_+ \circ f_+ & \\ 0 \to & J^{(b+c+d)} & \stackrel{\widehat{j}}{\to} & \Lambda^{(b+c)} \oplus J^{(d)} & \stackrel{\widehat{\eta}}{\to} & \Omega^{(b+c)} & \to 0 \end{array}$$

As $g \circ f = \mathrm{Id}$ then $\widehat{\eta} = (g_+ \circ f_+) \circ \widehat{\eta}$. As $\widehat{\eta}$ is surjective it follows that $g_+ \circ f_+ = \mathrm{Id}$. Hence f_+ is injective. This completes the proof. □

Maintaining the above notation we now prove:

Theorem 2.8 : If $S \oplus \Lambda^{(c)} \cong \Lambda^{(b+c)} \oplus J^{(d)}$ then S is a module extension

$$0 \to J^{(b+d)} \xrightarrow{i} S \xrightarrow{\alpha} \Omega^{(b)} \to 0.$$

Proof : As $\mathrm{Hom}_\Lambda(J, \Omega) = 0$ then in what follows we may make the identifications

$$\begin{aligned} \mathrm{Hom}_\Lambda(S \oplus \Lambda^{(c)}, \Omega) &= \mathrm{Hom}_\Lambda(S, \Omega) \oplus \mathrm{Hom}_\Lambda(\Lambda^{(c)}, \Omega); \\ \mathrm{Hom}_\Lambda(\Lambda^{(b+c)} \oplus J^{(d)}, \Omega) &= \mathrm{Hom}_\Lambda(\Lambda^{(b+c)}, \Omega). \end{aligned}$$

Moreover $\mathrm{Hom}_\Lambda(S, \Omega) \oplus \mathrm{Hom}_\Lambda(\Lambda^{(c)}, \Omega) \cong \mathrm{Hom}_\Lambda(\Lambda^{b+c}, \Omega) \cong \mathrm{Hom}_\Lambda(\Lambda, \Omega)^{(b+c)}$. As $\mathrm{Hom}_\Lambda(\Lambda, \Omega) \cong \Omega$ then $\mathrm{Hom}_\Lambda(S, \Omega) \oplus \Omega^{(c)} \cong \Omega^{(b+c)}$. By assumption **(2.2)**, $\mathrm{Hom}_\Lambda(S, \Omega) \cong \Omega^{(b)}$. Let $\{\alpha_1, \cdots, \alpha_b\}$ be an Ω-basis for $\mathrm{Hom}_\Lambda(S, \Omega)$ and define

$$\alpha = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_b \end{pmatrix} : S \longrightarrow \Omega^{(b)} \qquad S_0 = \mathrm{Ker}(\alpha)$$

In particular, we have an exact sequence $0 \to S_0 \to S \to \mathrm{Im}(\alpha) \to 0$. Suppose given an isomorphism $f : \Lambda^{(b+c)} \oplus J^{(d)} \xrightarrow{\simeq} S \oplus \Lambda^{(c)} \oplus J^{(d)}$. As in **(2.4)**, we may construct the following diagram with exact rows

$$\left\{ \begin{array}{ccccccc} 0 \to & J^{(b+c+d)} & \stackrel{\widehat{j}}{\to} & \Lambda^{(b+c)} \oplus J^{(d)} & \stackrel{\widehat{\eta}}{\to} & \Omega^{(b+c)} & \to 0 \\ & \downarrow f_- & & \downarrow f & & \downarrow f_+ & \\ 0 \to & S_0 \oplus J^{(c)} & \stackrel{\widehat{i}}{\to} & S \oplus \Lambda^{(c)} & \stackrel{\widehat{\alpha}}{\to} & \mathrm{Im}(\alpha) \oplus \Omega^{(c)} & \to 0 \end{array} \right.$$

in which f_+ is surjective. We claim that f_+ is also injective. Thus let $\{\eta_{b+1}, \ldots \eta_{b+c}\}$ be the canonical basis for $\mathrm{Hom}_\Lambda(\Lambda^{(c)}, \Omega)$ so that $\{\alpha_1, \ldots, \alpha_b, \eta_{b+1}, \ldots, \eta_{b+c}\}$ is an Ω-basis for $\mathrm{Hom}_\Lambda(S, \Omega) \oplus \mathrm{Hom}_\Lambda(\Lambda^{(c)}, \Omega) = \mathrm{Hom}_\Lambda(S \oplus \Lambda^{(c)}, \Omega)$. From the induced isomorphism $f^* : \mathrm{Hom}_\Lambda(S \oplus \Lambda^{(c)}, \Omega) \xrightarrow{\simeq} \mathrm{Hom}_\Lambda(\Lambda^{(b+c)} \oplus J^{(d)}, \Omega)$ we see that

(*) $\{f^*(\alpha_1),\ldots,f^*(\alpha_b),f^*(\eta_{b+1}),\ldots,f^*(\eta_{b+c})\}$ is an Ω-basis for $\mathrm{Hom}_\Lambda(\Lambda^{(b+c)}\oplus J^{(d)},\Omega)$.

If $\{\eta_1,\ldots\eta_{b+c}\}$ is the canonical basis of $\mathrm{Hom}_\Lambda(\Lambda^{(b+c)}\oplus J^{(d)},\Omega)\cong\Omega^{(b+c)}$ there exists an invertible Ω-linear map $T:\mathrm{Hom}_\Lambda(\Lambda^{(b+c)}\oplus J^{(d)},\Omega)\to\mathrm{Hom}_\Lambda(\Lambda^{(b+c)}\oplus J^{(d)},\Omega)$

such that $$T(\eta_i) = \begin{cases} f^*(\alpha_i) = \alpha_i\circ f & 1\le i\le b \\ f^*(\eta_i) = \eta_{i-b}\circ f & b+1\le i\le b+c. \end{cases}$$

On identifying $\mathrm{Hom}_\Lambda(\Lambda^{(b+c)}\oplus J^{(d)},\Omega) = \mathrm{Hom}_\Lambda(S\oplus\Lambda^{(c)}\oplus J^{(d)},\Omega) = \Omega^{(b+c)}$ it is evident that $\mathrm{Im}(T) = \mathrm{Im}(\widehat{\alpha})\oplus\Omega^{(c)}$ and that the following diagram commutes

$$\begin{array}{ccc} \Lambda^{(b+c)}\oplus J^{(d)} & \stackrel{\widehat{\eta}}{\to} & \Omega^{(b+c)} \\ \downarrow f & & \downarrow T \\ S\oplus\Lambda^{(c)} & \stackrel{\widehat{\alpha}}{\to} & \mathrm{Im}(\alpha)\oplus\Omega^{(c)} \end{array}$$

As f_+ is uniqely determined by f it follows that $f_+ = T$. As T is invertible then f_+ is injective as claimed. As f_+ is injective, then regarding S as a module extension $0\to S_0\to S\stackrel{\alpha}{\to}\Omega^{(b)}\to 0$ it follows from **(2.6)** that $S_0\cong J^{(b+d)}$. □

§3 : Matrices with a Smith Normal Form:

Let Λ be a commutative ring. We denote by ${}_mM_n(\Lambda)$ the set of $m\times n$ matrices with coefficients in Λ; when $m = n$ we write ${}_nM_n(\Lambda) = M_n(\Lambda)$, in which case $M_n(\Lambda)$ is a ring with the canonical Λ-basis $\epsilon(i,j)_{1\le i,j\le n}$ given by $\epsilon(i,j)_{r,s} = \delta_{ir}\delta_{js}$. We denote by $E(i,j;\lambda)$, $D(i,-1)$ the elementary invertible matrices

$$E(i,j;\lambda) = I_n+\lambda\epsilon(i,j) \quad (\lambda\in\Lambda;\ \ i\neq j)$$

$$D(i,-1) = I_n-2\epsilon(i,i).$$

Let $E_n(\Lambda)$ (resp. $\Delta_n(\pm1)$) denote the subgroup of $GL_n(\Lambda)$ generated by the matrices $E(i,j;\lambda)$, (resp. $D(i,-1)$). Then $\Delta_n(\pm1)$ normalizes $E_n(\Lambda)$. and we define

(3.1) $$\widetilde{E}_n(\Lambda) = \Delta_n(\pm1)\cdot E_n(\Lambda).$$

A ring homomorphism $\pi:A\to B$ induces a homomorphism of groups

$$\pi_*:E_n(A)\to E_n(B). \qquad :$$

(3.2) If $\pi:A\to B$ is surjective then $\pi_*:\widetilde{E}_n(A)\to\widetilde{E}_n(B)$ is surjective.

Suppose $k\ge1$ and $m\ge0$ are fixed; if $\alpha_1,\ldots,\alpha_k\in\Lambda$ we write

$${}_{k+m}\Delta_k(\alpha_1,\ldots,\alpha_k) = (\alpha_i\delta_{i,j}) \in {}_{m+k}M_k(\Lambda)$$

where $\delta_{i,j}$ is the Kronecker delta, $1\le i\le k+m$ and $1\le j\le k$; that is:

$$
{}_{k}\Delta_{k}(\alpha_1, \dots, \alpha_k) \quad = \quad \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_k \end{pmatrix} \qquad (m = 0);
$$

$$
{}_{k+m}\Delta_{k}(\alpha_1, \dots, \alpha_k) \quad = \quad \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_k \\ \dots & \dots & \dots \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{pmatrix} \qquad (m > 0).
$$

If $X, Y \in {}_{k+m}M_k(\Lambda)$ we write $X \sim Y$ when $X = E_+ Y E_-$ for some $E_+ \in \widetilde{E}_{k+m}(\Lambda)$ and $E_- \in \widetilde{E}_k(\Lambda)$. Evidently '$\sim$' is an equivalence relation; then $X \in {}_{k+m}M_k(\Lambda)$ has a *Smith normal form* when for some $\alpha_1, \dots, \alpha_k \in \Lambda$,

$$
X \sim {}_{k+m}\Delta_k(\alpha_1, \dots, \alpha_k).
$$

We say the commutative ring Ω is *generalized Euclidean* when each $X \in {}_{k+m}M_k(\Omega)$ has a Smith normal form. The argument of H.J.S. Smith ([6], [13]) shows that:

(3.3) If Λ is a Euclidean domain then Λ is generalized Euclidean.

If $S \subset \Lambda$ - $\{0\}$ is a multiplicative submonoid we denote by Λ_S the localization of Λ obtained by inverting each $s \in S$. By clearing fractions we see that:

(3.4) If Λ is a generalized Euclidean domain then so is Λ_S.

If $\mathbb{F}$ is a field, it follows from the division algorithm that the polynomial ring $\mathbb{F}[t]$ is a Euclidean domain. It now follows from **(3.4)** that:

(3.7) The Laurent polynomial ring $\mathbb{F}[t, t^{-1}]$ over the field $\mathbb{F}$ is generalized Euclidean.

§4 : Swan modules and their duals:

For the remainder if this paper we fix a prime q and put $\Lambda = R[C_q]$ where $R = \mathbb{Z}[C_\infty]$. If M is a Λ-module we denote by $M^\bullet$ its Λ-dual $M^\bullet = \mathrm{Hom}_\Lambda(M, \Lambda)$. There is a canonical homomorphism $\natural : M \to M^{\bullet\bullet}$ given by $\natural(x)(f) = f(x)$. If M is free of finite rank over R then $\natural : M \to M^{\bullet\bullet}$ is an isomorphism.

Let $\epsilon : \Lambda \to R$ be the augmentation homomorphism and let $I = \mathrm{Ker}(\epsilon)$ be the augmentation ideal. By a *Swan module of rank k* we mean an extension module X of the form $0 \to I^{(k)} \to X \to R^{(k)} \to 0$. We see from the augmentation sequence $\mathcal{E} = (0 \to I \xrightarrow{i} \Lambda \xrightarrow{\epsilon} R \to 0)$ that Λ is a Swan module of rank 1. On dualizing the augmentation sequence we obtain an exact sequence

$$
0 \to \mathrm{Hom}_\Lambda(R, \Lambda) \xrightarrow{\epsilon^\bullet} \mathrm{Hom}_\Lambda(\Lambda, \Lambda) \xrightarrow{i^\bullet} \mathrm{Hom}_\Lambda(I, \Lambda) \xrightarrow{\delta} \mathrm{Ext}^1_\Lambda(R, \Lambda).
$$

It is straightforward to see that $R^\bullet \cong R$ and $\Lambda^\bullet \cong \Lambda$ and that:

(4,2) $$\mathrm{Hom}_\Lambda(R, I^\bullet) \; = \; 0.$$

From the Eckmann-Shapiro Lemma we see that $\mathrm{Ext}^1_\Lambda(R,\Lambda) \;\cong \mathrm{Ext}^1_R(R,R) \;=\; 0.$ Hence we have a dual exact sequence $\mathcal{E}^\bullet \;=\; (0 \to R \xrightarrow{\epsilon^\bullet} \Lambda \xrightarrow{i^\bullet} I^\bullet \to 0)$. Observe that $I^\bullet$ has a natural ring structure as $\epsilon^\bullet$ imbeds R onto the two-sided ideal $\left(\sum_{k=0}^{q-1} x^k\right)$ in Λ. In the case under consideration it is also true that, as modules, $I^\bullet \cong I$ but this fact plays no part in what follows. By a *dual Swan module of rank k* we shall mean an extension module X of the form

$$0 \to R^{(k)} \longrightarrow X \to (I^\bullet)^{(k)} \to 0.$$

As Λ and R are self dual, then on dualizing the augmentation exact sequence we see that $\Lambda^{(k)}$ is also a dual Swan module of rank k. Taking $\zeta_q \;=\; \exp(2\pi i/q)$ we may compare $\mathcal{E}^\bullet$ with the corresponding exact sequence over $\mathbb{Z}[C_q]$ (cf [11] p.29)

$$\mathbb{E} \;\;=\;\; (0 \to \left(\sum_{k=0}^{q-1} x^k\right) \hookrightarrow \mathbb{Z}[C_q] \twoheadrightarrow \mathbb{Z}(\zeta_q) \to 0).$$

It is straightforward to see that $\mathcal{E}^\bullet \;=\; \mathbb{E} \otimes_\mathbb{Z} \mathbb{Z}[t,t^{-1}]$, from which it follows that:

(4.3) $$I^\bullet \;\cong\; \mathbb{Z}(\zeta_q)[t,t^{-1}].$$

Proposition 4.4: $\quad X \oplus (I^\bullet)^{(n)} \;\cong_\Lambda\; (I^\bullet)^{(m+n)} \;\Longrightarrow\; X \;\cong_\Lambda\; (I^\bullet)^{(m)}$

Proof : Evidently $X \oplus (I^\bullet)^{(n)} \;\cong_{I^\bullet}\; (I^\bullet)^{(m+n)}$ so that, considered purely as a module over $I^\bullet \;=\; \mathbb{Z}(\zeta_q)[t,t^{-1}]$, X is stably free of rank m. As $\mathbb{Z}(\zeta_q)$ is a Dedekind domain it follows from the solution to the Serre Conjecture (cf. [10]. p.189. Cor. 4.12) that X is induced from a projective module P over $\mathbb{Z}(\zeta_q)$. As X is stably free and $\mathbb{Z}(\zeta_q)$ is a retract of $I^\bullet$ then P is also stably free. However as $\mathbb{Z}(\zeta_q)$ satisfies the Eichler condition then every stably free $\mathbb{Z}(\zeta_q)$ module is free (cf. [16], p.176-178) . Thus P is free and X, being induced from a free module, is also free; that is $X \;\cong_{I^\bullet}\; (I^\bullet)^{(m)}$. However, the Λ-module structure on $I^\bullet$ is coinduced from the epimorphism $i^\bullet : \Lambda \xrightarrow{i^\bullet} I^\bullet$. As $X \oplus (I^\bullet)^{(n)} \;\cong_\Lambda\; (I^\bullet)^{(m+n)}$ then the Λ-structure on X is also coinduced from $I^\bullet$ and hence $X \;\cong_\Lambda\; (I^\bullet)^{(m)}$. □

A similar but slightly easier argument (in this case we may appeal to the earlier theorem of Sheshadri [12]) shows that:

Proposition 4.5: $\quad X \oplus R^{(n)} \;\cong_\Lambda\; R^{(m+n)} \;\Longrightarrow\; X \;\cong_\Lambda\; R^{(m)}.$

Statements **(4.2)**, **(4.4)** and **(4.5)** allow us to apply the arguments of §2 on taking

$$\Omega \;=\; I^\bullet \quad ; \quad J = R.$$

As $\mathrm{Ext}^1_\Lambda(I^\bullet, R) \;\cong \mathbb{F}_q[t,t^{-1}]$ we identify

(4.6) $$\mathrm{Ext}^1_\Lambda((I^\bullet)^{(k+m)}, R^{(k)}) \;=\; {}_{k+m}M_k(\mathbb{F}_q[t,t^{-1}]).$$

In the simplest case, a dual Swan module of rank 1 is defined by an extension

$$\mathcal{X} \;=\; (0 \to R \to X \to \; I^\bullet \to 0)$$

21 Mar 2023 05:02:02 PDT
221213-Johnson Version 2 - Submitted to Rocky Mountain J. Math.

and is classified up to congruence by an element $\mathbf{e}_{\mathcal{X}} \in \mathrm{Ext}^1_\Lambda(I^\bullet, R) \cong \mathbb{F}[t, t^{-1}]$. We write $X = X(\alpha)$ where $\alpha \in \mathbb{F}[t, t^{-1}]$ corresponding to $\mathbf{e}_{\mathcal{X}}$ under the isomorphism $\mathrm{Ext}^1_\Lambda(I^\bullet, R) \cong \mathbb{F}[t, t^{-1}]$. Furthermore, we have the following instance of Swan's projectivity criterion ([7], (5.41), p.115):

(4.7) $$X(\alpha) \text{ is projective} \iff \alpha \in \mathbb{F}[t, t^{-1}]^*.$$

Observe that we have a Milnor fibre square

$$(\mathfrak{S}_0) \qquad \begin{tikzcd} \mathbb{Z}[C_q] \arrow[r, "i^\bullet"] \arrow[d, "\epsilon"] & \mathbb{Z}(\zeta_q) \arrow[d, "\nu"] \\ \mathbb{Z} \arrow[r, "\natural"] & \mathbb{F}_q \end{tikzcd} .$$

Applying the functor $- \otimes_{\mathbb{Z}} \mathbb{Z}[t, t^{-1}]$ we obtain another Milnor square

$$(\mathfrak{S}) \qquad \begin{tikzcd} \Lambda \arrow[r, "i^\bullet"] \arrow[d, "\epsilon"] & I^\bullet \arrow[d, "\nu"] \\ \mathbb{Z}[t, t^{-1}] \arrow[r, "\natural"] & \mathbb{F}_q[t, t^{-1}] \end{tikzcd} .$$

We say that X is *locally free of rank* 1 *with respect to* $\mathfrak{S}$ when X is a fibre product

$$\mathfrak{X}(\alpha) = \left\{ \begin{array}{ccc} X & \longrightarrow & I^\bullet \\ \downarrow & & \downarrow \varphi \\ \mathbb{Z}[t, t^{-1}] & \xrightarrow{\natural} & \mathbb{F}_q[t, t^{-1}] \end{array} \right.$$

obtained by glueing via an element $\alpha \in GL_1(\mathbb{F}_q[t, t^{-1}]) = \mathbb{F}_q[t, t^{-1}]^*$. By Milnor's classification ([11] pp. 20-24), any such module is projective over Λ. With respect to the fibre square $\mathfrak{S}$, a locally free module X can equally be described as a projective dual Swan module; that is there is a bijective correspondence of isomorphism classes:

(4.8) $$\left\{ \begin{array}{c} \text{locally free modules} \\ \text{of rank 1 with respect to } \mathfrak{S} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{projective dual Swan} \\ \text{modules of rank 1} \end{array} \right\}$$

By the theorem of Higman ([5], [7] Appendix C), for any commutative integral domain A, the group ring $A[C_\infty]$ has only trivial units; that is $GL_1(A[C_\infty]) \cong A^* \times \langle t \rangle$. As $I^\bullet = \mathbb{Z}(\zeta_q)[C_\infty]$ then $GL_1(I^\bullet) \cong \mathbb{Z}(\zeta_q)^* \times \langle t \rangle$. Likewise, $GL_1(\mathbb{F}_q[t, t^{-1}]) \cong \mathbb{F}_q^* \times \langle t \rangle$. As is well known (cf [2], p87), the canonical homomorphism $\nu : \mathbb{Z}(\zeta_q)^* \to \mathbb{F}_q^*$ is surjective. Hence we see that:

(4.9) $\nu_* : GL_1(I^\bullet) \to GL_1(\mathbb{F}_q[t, t^{-1}])$ is surjective.

Milnor's classification now implies:

(4.10) If X is a projective dual Swan module of rank 1 then $X \cong \Lambda$.

Next consider extensions of the form $\mathcal{X} = (0 \to R^{(k)} \to X \to (I^\bullet)^{(k)} \to 0)$.

Theorem 4.11 : Let X be an extension $\mathcal{X} = (0 \to R^{(k)} \to X \to (I^\bullet)^{(k)} \to 0)$ then X decomposes as a direct sum of dual Swan modules $X \cong X(\alpha_1) \oplus \ldots \oplus X(\alpha_k)$.

Proof : The case $k = 1$ is covered by **(4.7)**. Thus suppose $k \geq 2$. Under the correspondence **(4.6)** $\mathcal{X}$ is classified up to congruence by an extension class $\mathbf{e}_{\mathcal{X}} \in \mathrm{Ext}^1_\Lambda((I^\bullet)^{(k)}, R^{(k)}) = M_k(\mathbb{F}[t, t^{-1}])$. As $\mathbb{F}_q[t, t^{-1}]$ is generalized Euclidean there exist $E_+ \in \widetilde{E}_k(\mathbb{F}_q[t, t^{-1}])$, $E_- \in \widetilde{E}_k(\mathbb{F}_q[t, t^{-1}])$ and $\alpha_1 \ldots, \alpha_k \in \mathbb{F}_q[t, t^{-1}]$ such that

$$E_+ \mathbf{e}_{\mathcal{X}} E_- \; = \; \Delta(\alpha_1, \ldots, \alpha_k).$$

Lifting E_+ to $\widehat{E}_+ \in \widetilde{E}_k(I^\bullet)$ and E_- to $\widehat{E}_- \in \widetilde{E}_k(R)$ we see that X decomposes as

$$X \; = \; X(\alpha_1) \; \oplus \; \ldots \; \oplus \; X(\alpha_k)$$

where, as in **(4.7)**, $X(\alpha_i)$ is classified by $\alpha_i \in \mathbb{F}_q[t, t^{-1}] \cong \mathrm{Ext}^1_\Lambda(I^\bullet, R)$. $\square$

Clearly $\mathbf{e}_{\mathcal{X}}$ is invertible if and only if each $\alpha_i \in \mathbb{F}_q[t, t^{-1}]^*$; hence by (4.7) we have:

Corollary 4.12: Given an extension $\mathcal{X} = (0 \to R^{(k)} \to X \to (I^\bullet)^{(k)} \to 0)$; then the following are equivalent;

i) X is projective; ; ii) $\mathbf{e}_{\mathcal{X}}$ is invertible in $M_k(\mathbb{F}[t, t^{-1}])$; iii) $X \cong \Lambda^{(k)}$.

Finally we consider extensions of the form $\mathcal{S} = (0 \to R^{(k)} \to S \to (I^\bullet)^{(k+m)} \to 0)$ where $m \geq 1$.

Theorem 4.13: Let X be a Λ-module which occurs in an exact sequence

$$\mathcal{X} \; = \; (0 \to R^{(k)} \to X \to (I^\bullet)^{(k+m)} \to 0);$$

then $X \cong X_1 \oplus \ldots \oplus X_k \oplus (I^\bullet)^{(m)}$ where each X_i is a dual Swan module of rank 1.

Proof : By **(4.11)** we may suppose that $m \geq 1$; as above, $\mathcal{X}$ is classified up to congruence by $\mathbf{e}_{\mathcal{X}} \in \mathrm{Ext}^1_\Lambda((I^\bullet)^{(k+m)}, R^{(k)}) = {}_{k+m}M_k(\mathbb{F}_q[t, t^{-1}])$. As $\mathbb{F}_q[t, t^{-1}]$ is generalized Euclidean then

$$E_+ \mathbf{e}_{\mathcal{X}} E_- \; = \; {}_{k+m}\Delta_k(\alpha_1, \ldots, \alpha_k) \; = \; \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_k \\ \cdots & \cdots & \cdots \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{pmatrix}$$

for some $E_+ \in \widetilde{E}_{k+m}(\mathbb{F}_q[t, t^{-1}])$, $E_- \in \widetilde{E}_k(\mathbb{F}_q[t, t^{-1}])$ and $\alpha_1 \ldots, \alpha_k \in \mathbb{F}_q[t, t^{-1}]$ Lifting E_+ to $\widehat{E}_+ \in \widetilde{E}_{k+m}(I^\bullet)$ and E_- to $\widehat{E}_- \in \widetilde{E}_k(R)$ we see that X decomposes as

21 Mar 2023 05:02:02 PDT
221213-Johnson Version 2 - Submitted to Rocky Mountain J. Math.

$$X \;=\; X(\alpha_1) \;\oplus\; \dots \;\oplus\; X(\alpha_k) \;\oplus\; (I^\bullet)^{(m)}$$

where, as above, $X(\alpha_i)$ is the dual Swan module classified by $\alpha_i \in \mathrm{Ext}^1_\Lambda(I^\bullet, R)$. $\square$

§5 : Proof of (I):

Let D be a Dedekind domain. It follows from the theorem of Steinitz ([4], [14]) that:

(5.1) Every stably free D-module is free.

The correspondence $t \mapsto 1$ induces an augmentation homomorphism $D[t, t^{-1}] \to D$ whereby D becomes a module over $D[t, t^{-1}]$. We have 'change of ring' functors

$$\begin{array}{lcl} i_* : \mathcal{M}\mathrm{od}_D \to \mathcal{M}\mathrm{od}_{D[t,t^{-1}]} & ; & i_*(M) \;=\; M \otimes_D D[t, t^{-1}] \\ r_* : \mathcal{M}\mathrm{od}_{D[t,t^{-1}]} \to \mathcal{M}\mathrm{od}_D & ; & r_*(N) \;=\; N \otimes_{D[t,t^{-1}]} D \end{array}$$

under which

(5.2) $\quad r_* \circ i_* \;=\; \mathrm{Id}_{\mathcal{M}\mathrm{od}_D}$.

The following is proved in [10] ((4.12), p.189);

(5.3) If S is a finitely generated projective module over $D[t, t^{-1}]$; then $S \cong i_*(P)$ for some finitely generated projective module P over D.

It follows from **(5.1)**, **(5.2)** and **(5.3)** that:

(5.4) Every stably free $D[t, t^{-1}]$-module is free.

Now put $D \;=\; \mathbb{Z}(\zeta)$ where $\zeta \;=\; \exp(\frac{2\pi i}{q})$ so that $I^\bullet \;=\; R(\zeta) \;=\; D[t, t^{-1}]$. It follows from **(5.4)** that:

(5.5) If S is an $I^\bullet$-module such that $S \oplus I^\bullet \cong_{I^\bullet} (I^\bullet)^{(d+1)}$ then $S \cong_{I^\bullet} (I^\bullet)^{(d)}$.

Now let S be a Λ-module such that $S \oplus I \;\cong_\Lambda\; I^{(d+1)}$. Applying $\mathrm{Hom}(-.\Lambda)$ gives

$$S^\bullet \oplus I^\bullet \;\cong_\Lambda\; (I^\bullet)^{(d+1)}.$$

However $I^\bullet \cdot \Sigma \;=\; 0$ where $\Sigma \;=\; \sum_{r=0}^{q-1} x^r$. Hence $S^\bullet \cdot \Sigma \;=\; 0$, As $\Lambda/\Sigma \cong I^\bullet$ then

$$S^\bullet \oplus I^\bullet \;\cong_{I^\bullet}\; (I^\bullet)^{(d+1)}$$

By **(5.4)**, $S^\bullet \;\cong_{I^\bullet}\; (I^\bullet)^{(d)}$. Thus also $S^\bullet \;\cong_\Lambda\; (I^\bullet)^{(d)}$. On taking Λ-duals then $S \cong_\Lambda I^{(d)}$ and we obtain the following, which is statement **(I)** of the Introduction.

(5.6) If S is a Λ-module such that $S \oplus I \cong_\Lambda I^{(d+1)}$ then $S \cong_\Lambda I^{(d)}$.

§6 : Proof of (II) and (III):

To prove **(II)**, let S be a Λ-module such that $\Lambda \oplus S \;\cong_\Lambda\; \Lambda^{(c+1)} \oplus I^{(d)}$. On taking $\Omega \;=\; R$, $J \;=\; I$ and $b = 1$ in **(2.8)** we see that S occurs as an extension

$$0 \to I^{(c+d)} \to S \xrightarrow{\beta} R^{(c)} \to 0.$$

21 Mar 2023 05:02:02 PDT
221213-Johnson Version 2 - Submitted to Rocky Mountain J. Math.

It follows from **(4.13)** that S decomposes as $S \cong X \oplus I^{(d)}$ where X is a Swan module of rank c. As $\Lambda \oplus S \cong \Lambda^{(c+1)} \oplus I^{(d)}$ then $\Lambda \oplus X \oplus I^{(d)} \cong \Lambda^{(c+1)} \oplus I^{(d)}$. It follows from **(1.4)** that X is projective. Hence $X \cong \Lambda^{(c)}$ by **(4.12)**. Thus $S \cong \Lambda^{(c)} \oplus I^{(d)}$. To summarize we have proved the following which is statement **(II)** of the Introduction:

(6.1) Let S be a Λ-module such that $\Lambda \oplus S \cong_\Lambda \Lambda^{(c+1)} \oplus I^{(d)}$; then

$$S \cong_\Lambda \Lambda^{(c)} \oplus I^{(d)}.$$

The proof of **(III)** is similar to that of **(II)**. Suppose that $f : \Lambda^{(c)} \oplus I^{(d+1)} \xrightarrow{\simeq} I \oplus S$ is an isomorphism with inverse g and put

$$\widehat{\epsilon} = (\underbrace{\epsilon \oplus \ldots \oplus \epsilon}_{c} \oplus 0) : \Lambda^{(c)} \oplus I^{(d+1)} \xrightarrow{\widehat{\epsilon}} R^{(c)};$$

then $\widehat{\epsilon} \circ g : I \oplus S \to R^c$ is surjective. As $\mathrm{Hom}_\Lambda(I, R) = 0$ then $\widehat{\epsilon} \circ g_{|I} \equiv 0$. Putting $\alpha = \widehat{\epsilon} \circ g_{|S}$ and $S_0 = \mathrm{Ker}(\alpha)$ we have an exact sequence

$$0 \to S_0 \to S \to R^{(c)} \to 0.$$

Now consider the following diagram with exact rows

$$\begin{cases} 0 \to & I^{(c+d+1)} & \xrightarrow{\widehat{i}} & \Lambda^{(c)} \oplus I^{(d+1)} & \xrightarrow{\widehat{\epsilon}} & R^{(c)} & \to 0 \\ & \downarrow f_- & & \downarrow f & & \downarrow f_+ & \\ 0 \to & I \oplus S_0 & \xrightarrow{\widehat{j}} & I \oplus S & \xrightarrow{(0,\alpha)} & R^{(c)} & \to 0 \end{cases}$$

As $\mathrm{Hom}_\Lambda(I, R) = 0$ then $(0, \alpha) \circ f \circ i = 0$. Hence there exist homomorphisms $f_+ : R^{(c)} \to R^{(c)}$ and $f_- : I^{(c+d+1)} \to I \oplus S_0$ as indicated. As $(0, \alpha) \circ f$ is surjective then so is f_+. As R is Noetherian then f_+ is an isomorphism. Therefore f_- is also an isomorphism. It follows from **(I)** that $S_0 \cong I^{(c+d)}$ and we have an exact sequence

$$0 \to I^{(c+d)} \longrightarrow S \xrightarrow{\beta} R^{(c)} \to 0$$

By **(4.11)**, S decomposes as $S \cong X \oplus I^{(d)}$ where X is a Swan module of rank c. As $I \oplus S \cong \Lambda^{(c)} \oplus I^{(d+1)}$ then $X \oplus I^{(d+1)} \cong \Lambda^{(c)} \oplus I^{(d+1)}$. It follows from **(1.4)** that X is projective. By (4.12) it follows that $X \cong \Lambda^{(c)}$. Thus $S \cong \Lambda^{(c)} \oplus I^{(d)}$ and we have proved statement **(III)** of the Introduction, namely:

(6.2) Let S be a Λ-module such that $I \oplus S \cong_\Lambda \Lambda^{(c)} \oplus I^{(d+1)}$; then $S \cong_\Lambda \Lambda^{(c)} \oplus I^{(d)}$.

§7 : Proof of Main Theorem :

For integers $a, b \geq 0$ let $Q(a, b) = \Lambda^{(a)} \oplus I^{(b)}$ and define $\mathfrak{Q} = \{Q(a, b) \mid a+b > 0\}$. Clearly $Q(a, b) \oplus Q(c, d) \cong Q(a + c, b + d)$ so that, up to isomorphism, $\mathfrak{Q}$ forms an additive semigroup under '$\oplus$'. To show that $\mathfrak{Q}$ is a cancellation semigroup it suffices to show that the following statement $\mathcal{P}(a, b)$ holds when $a, b \geq$ and $a + b > 0$:

$\mathcal{P}(a, b)$: $\qquad Q(a, b) \oplus S \cong Q(a, b) \oplus Q(c, d) \implies S \cong Q(c, d)$

By **(II)** we see that $\mathcal{P}(1,0)$ holds. Similarly $\mathcal{P}(0,1)$ holds by **(III)**.

Proposition 7.1: $\quad \mathcal{P}(1,0) \wedge \mathcal{P}(a,b) \implies \mathcal{P}(a+1,b).$

Proof: Suppose that $Q(a+1,b) \oplus Q(c,d) \cong Q(a+1,b) \oplus S$. We must show that $Q(c,d) \cong S$. Re-writing we see that $\Lambda \oplus Q(c+a,b+d) \cong \Lambda \oplus Q(a,b) \oplus S$. By $\mathcal{P}(1,0)$ it follows that $Q(c+a,b+d) \cong Q(a,b) \oplus S$ and hence

$$Q(a,b) \oplus Q(c,d) \cong Q(a,b) \oplus S.$$

By $\mathcal{P}(a,b)$ it now follows, as desired, that $Q(c,d) \cong S$. □

Proposition 7.2: $\quad \mathcal{P}(0,1) \wedge \mathcal{P}(a,b) \implies \mathcal{P}(a,b+1).$

Proof : Suppose that $Q(a,b+1) \oplus Q(c,d) \cong Q(a,b+1) \oplus S$. Re-writing we see that $I \oplus Q(c+a,b+d) \cong I \oplus Q(a,b) \oplus S$. By $\mathcal{P}(0,1)$ it follows that $Q(c+a,b+d) \cong Q(a,b) \oplus S$ and hence

$$Q(a,b) \oplus Q(c,d) \cong Q(a,b) \oplus S.$$

The desired conclusion that $Q(c,d) \cong S$ now follows from $\mathcal{P}(a,b)$. □

It follows easily that:

(7.3) $\mathcal{P}(a,b)$ is true for all $a,b \geq 0$ such that $a+b \neq 0$.

However, the truth of the statements $\mathcal{P}(a,b)$ suffices to show that $\mathfrak{Q}$ is a strong cancellation semigroup. To see this, put $\Lambda_{\mathbb{E}} = \Lambda \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{E}$ where $\mathbb{E}$ is the field of fractions of $\mathbb{Z}[t,t^{-1}]$. Then $\Lambda_{\mathbb{E}} = \mathbb{E}[C_q]$ is a semisimple $\mathbb{E}$-algebra. Now suppose that $Q \in \mathfrak{Q}$ and that S is a nonzero Λ-module such that $Q \oplus S \in \mathfrak{Q}$. Write $Q = Q(a,b)$ and $Q \oplus S = Q(e,f)$. By considering the Wedderburn decompositions of $Q \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{E}$ and $S \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{E}$, it follows easily that $a \leq e$ and $b \leq f$. Writing $c = e-a$ and $d = f-b$ we see that $Q(a,b) \oplus S \cong Q(a,b) \oplus Q(c,d)$. It follows from the statement $\mathcal{P}(a,b)$ proved in **(7.3)** that $S \cong Q(c,d)$ and hence $S \in \mathfrak{Q}$. Thus we have shown the following, which is the Main Theorem of the Introduction:

(7.4) $\mathfrak{Q}(\mathbb{Z}[C_\infty],q) = \{\Lambda^{(a)} \oplus I^{(b)} \mid a \geq 0,\ b \geq 0,\ a+b > 0\}$ is a strong cancellation semigroup for any prime q.

§8 : The module cancellation problem in general:

There is a considerable literature on the cancellation problem for modules over group rings. However, the problem considered in the present paper does not seem to have been studied previously. To describe what is known in general, let q be a positive integer and let $\mathfrak{L}(R,q)$ denote the class of modules over $R[C_q]$ which are free of finite positive rank as modules over the commutative ring R. A theorem of Wiegand [18], completed by Swan [17] shows that:

(8.1) $\mathfrak{L}(\mathbb{Z},q)$ is a strong cancellation semigroup $\iff$ q is prime or $q \in \{4,6,8,9,10,14\}$.

The rings $\mathbb{Z}[C_q]$ have Krull dimension 1. In the present paper we replace the coefficient ring $\mathbb{Z}$ by the ring $\mathbb{Z}[C_\infty]$ of Krull dimension 2. In view of Wiegand's result we ask:

Question I: For which integers $q \geq 2$ does $\mathfrak{L}(\mathbb{Z}[C_\infty], q)$ have strong cancellation?

The inclusion $i : C_q \to C_\infty \times C_q$ and projection $\pi : C_\infty \times C_q \to C_q$ induce homomorphisms $i_* : \mathfrak{L}(\mathbb{Z}, q) \to \mathfrak{L}(\mathbb{Z}[C_\infty], C_q)$ and $\pi_* : \mathfrak{L}(\mathbb{Z}[C_\infty], C_q) \to \mathfrak{L}(\mathbb{Z}, q)$ such that $\pi_* \circ i_* = \mathrm{Id}$. Thus $\mathfrak{L}(\mathbb{Z}, q)$ is a retract of $\mathfrak{L}(\mathbb{Z}[C_\infty], q)$. Hence in partial answer to the above general question, Wiegand's conditions are certainly necessary. However to decide whether they are also sufficient would seem to require a complete description of $\mathfrak{L}(\mathbb{Z}[C_\infty], C_q)$ analogous to the well known description of $\mathfrak{L}(\mathbb{Z}, C_q)$ ([3], p.508). To the best of the author's knowledge this has yet to be done.

In the present paper we have restricted attention to the subsemigroup $\mathfrak{Q}(\mathbb{Z}[C_\infty], q)$ of $\mathfrak{L}(\mathbb{Z}[C_\infty], q)$ thereby raising the following:

Question II: For which integers $q \geq 2$ does $\mathfrak{Q}(\mathbb{Z}[C_\infty], q)$ have strong cancellation?

The Main Theorem gives an positive answer to Question II when q is prime. Whether $\mathfrak{Q}(\mathbb{Z}[C_\infty], q)$ has strong cancellation for any non-prime values of q remains an open question. In this connection there are two further considerations; firstly, in contrast to **(8.1)** the author has shown:

(8.2) $\mathfrak{Q}(\mathbb{Z}, q)$ has strong cancellation for all positive integers q.

This is an immediate consequence of (77.6) on p. 248 of [8]. It requires only the additional observations that $\mathbb{Z}[C_q]$ satisfies the Eichler condition and that every projective Swan module over $\mathbb{Z}[C_q]$ is free (cf. the remark on p.279 of [15]). What is nevertheless clear is that the proof of the Main Theorem given above breaks down when q fails to be prime. In particular, when q is not prime there are always projective Swan modules of rank 1 which are not free. This can be shown directly, observing that when q is not prime there are units in $(\mathbb{Z}/q)[C_\infty]$ which do not lift to $\mathbb{Z}(\zeta_q)[C_\infty]$. Alternatively one can appeal to a theorem of Bass and Murthy ([1] Theorem 8.10). Finally, one can ask the corresponding questions for $\mathfrak{Q}(\mathbb{Z}[\underbrace{C_\infty \times \ldots \times C_\infty}_{n}], q)$; in this case the group ring $\mathbb{Z}[\underbrace{C_\infty \times \ldots \times C_\infty}_{n} \times C_q]$ has Krull dimension $n+1$. When $q = 1$, every projective module is free ([10], p.189). More generally, when q is prime the author has shown ([9]) that every projective Swan module is free. However again this conclusion fails when q is not prime.

F.E.A. Johnson

Department of Mathematics,
University College London,
Gower Street,
London WC1E 6BT, U.K.

f.johnson@ucl.ac.uk

21 Mar 2023 05:02:02 PDT
221213-Johnson Version 2 - Submitted to Rocky Mountain J. Math.

REFERENCES

[1] : H. Bass and M.P. Murthy; Grothendieck groups and Picard groups of abelian group rings : Ann. of Math 86 (1967) 16-73.

[2] : B.J. Birch; Cyclotomic fields and Kummer extensions : 'Algebraic Number Theory' (J.W.S. Cassells and A. Fröhlich eds), Academic Press (1967).

[3] : C.W. Curtis and I. Reiner; Representation theory of finite groups and associative algebras; Wiley-Interscience, 1962.

[4] : A. Fröhlich and M.J. Taylor; Algebraic theory of numbers : Cambridge University Press (1991).

[5] : D.G. Higman; The units of group rings : Proc. London. Math. Soc. (2) 46 (1940) 231-248.

[6] : N. Jacobson; Basic Algebra, vol.I. W.H. Freeman (1974).

[7] : F.E.A. Johnson; Syzygies and homotopy theory. Springer Verlag, 2011.

[8] : F.E.A. Johnson; Metacyclic groups and the $D(2)$ problem. World Scientific, 2021.

[9] : F.E.A. Johnson; Swan modules over Laurent polynomials. (in preparation).

[10] : T.Y. Lam; Serre's problem on projective modules. Springer-Verlag (2006).

[11] : J. Milnor; Introduction to Algebraic K-Theory : Annals of Mathematics Studies vol 72, Princeton University Press 1971.

[12] : C.S. Sheshadri; Triviality of vector bundles over the affine space K^2. Proc. Nat. Acad. Sci. USA 44 (1958) 456-458.

[13] : H.J.S. Smith; On systems of linear indeterminate equations and congruences : Phil. Trans. 151 (1861) 293-326.

[14] : E. Steinitz; Rechteckige Systeme und Moduln in algebraischen Zahlkörpen. I : Math. Ann. 71 (1911) 328-354; II : Math. Ann. 72 (1912) 297-345.

[15] : R.G. Swan; Periodic resolutions for finite groups : Ann. of Math. 72 (1960) 267-291.

[16] : R.G. Swan; K-Theory of finite groups and orders. (notes by E.G. Evans). Lecture Notes in Mathematics 149. Springer-Verlag 1970.

[17] : R.G. Swan; Torsion-free cancellation over orders. Illinois Journal of Mathematics 32 (1988), 329-360.

[18] : R. Wiegand; Cancellation of modules over commutative rings of dimension one and two. J, Algebra 88 (1984), 438-459.

21 Mar 2023 05:02:02 PDT
221213-Johnson Version 2 - Submitted to Rocky Mountain J. Math.