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QUADRATIC POLYNOMIAL HAPPY FUNCTIONS

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ABSTRACT. Fix a base b and a monic quadratic polynomial function f with nonnegative coefficients. The corresponding quadratic polynomial happy function maps any positive integer to the sum of the images under f of its nonzero digits. We study the behavior of these functions under iteration. Our main result is that for $b \geq 4$, given any fixed point or element of a cycle under iterations of this happy function, there exist arbitrarily long arithmetic sequences of positive integers each of which eventually maps to that number. This extends past results for generalized happy functions in a new direction.

1. Introduction

Fix $b \geq 2$. Recall the definition of generalized happy functions and numbers [5]: For $e \geq 2$ and $a = \sum_{i=0}^{n} a_i b^i$, where $0 \leq a_i \leq b-1$ are integers with $a_n \neq 0$, the happy function $S_{e,b} : \mathbb{Z}^+ \to \mathbb{Z}^+$ is defined by

$$(1) \qquad S_{e,b}(a) = S_{e,b}\left(\sum_{i=0}^{n} a_i b^i\right) = \sum_{i=0}^{n} a_i^e.$$

If $S_{e,b}^k(a) = 1$ for some $k \in \mathbb{Z}^+$, then a is called an *e-power b-happy number.* If $e = 2$, the number a is called simply a *b-happy number,* and if $e = 2$ and $b = 10$, a *happy number.* (See [4] for a survey of research on happy numbers and generalized happy numbers.)

In this work, we consider the happy functions created by replacing the exponentiation function with a monic quadratic polynomial function with nonnegative coefficients. More concisely, fix $b \geq 2$, $s \geq 0$, and $t \geq 0$. We define $S_{\{s,t,b\}} : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ as follows: Set $S_{\{s,t,b\}}(0) = 0$ and for integers $1 \leq a \leq b-1$, set

$$S_{\{s,t,b\}}(a) = a^2 + as + t.$$

For $n > 0$, $0 \leq a_i \leq b-1$, with $a_n \neq 0$, define

$$(2) \qquad S_{\{s,t,b\}}\left(\sum_{i=0}^{n} a_i b^i\right) = \sum_{i=0}^{n} S_{\{s,t,b\}}(a_i).$$

Though $S_{\{s,t,b\}}$ is the focus of this paper, it is certainly possible to generalize further, allowing other polynomial functions in the definition of this sort of function.

Let $U_{\{s,t,b\}}$ denote the set of all fixed points and cycles of $S_{\{s,t,b\}}$,

$$U_{\{s,t,b\}} = \{a \in \mathbb{Z}^+ \mid S_{\{s,t,b\}}^k(a) = a \text{ for some } k \in \mathbb{Z}^+\}.$$

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Note that if $s = t = 0$, then $S_{\{s,t,b\}} = S_{\{0,0,b\}} = S_{2,b}$, the standard 2-power b-happy function, with 1 as a fixed point. On the other hand, if s or t is nonzero, then 1 is not in the image of $S_{\{s,t,b\}}$. For this reason, we broaden the scope of study from numbers mapping to 1 to numbers mapping to any particular number $u \in U_{\{s,t,b\}}$. Using standard terminology, given integers $b \geq 2$, $s \geq 0$, $t \geq 0$, and $u \in U_{\{s,t,b\}}$, a positive integer a is a *u-attracted number (under $S_{\{s,t,b\}}$)* if, for some $k \in \mathbb{Z}^+$, $S^k_{\{s,t,b\}}(a) = u$.

In [6], it is proved that for b even, there exist arbitrarily long finite sequences of consecutive b-happy numbers, and for b odd, there exist arbitrarily long finite 2-consecutive sequences (that is, arithmetic sequences with constant difference 2) of b-happy numbers. In Section 4 of this paper, we prove the following analogous result for $S_{\{s,t,b\}}$.

Theorem 1. *Fix $b \geq 4$, $s \geq 0$, and $t \geq 0$. Let $D = \gcd(2, s, t, b-1)$ and let $u \in U_{\{s,t,b\}}$. There exist arbitrarily long finite sequences of D-consecutive numbers that are u-attracted under $S_{\{s,t,b\}}$.*

Since equation (2) holds for any $n \in \mathbb{Z}_{\geq 0}$, $S_{\{s,t,b\}}$ is what is frequently referred to as a *digit function*. It should be noted, however, that the function $S_{\{s,t,b\}}$ is not what Chase [2] calls a *digit map*, since it does not necessarily satisfy the conditions $f(1) = 1$ and $\gcd(f(b-1), b) = 1$. This means that Chase's adaptation of the methods developed by Pan [7] in proving results on sequences of consecutive happy numbers [3, 6, 9], will not apply to prove parallel results here.

In the following section, we discuss the fixed points and cycles of $S_{\{s,t,b\}}$. In Section 3, we present some basic properties of $S_{\{s,t,b\}}$ and some important results for the proofs in Section 4, where we prove the main theorem (Theorem 1) on D-consecutive u-attracted numbers.

2. Fixed Points and Cycles

In this section, we discuss fixed points and cycles of $S_{\{s,t,b\}}$. Theorem 2, which is adapted from [8, Theorem 1], provides an easily computable bound below which at least one element of each cycle (including fixed points) must lie. This allows us to calculate the fixed points and cycles for any triple $\{s,t,b\}$ of integer values with $b \geq 2$, $s \geq 0$, and $t \geq 0$. In Tables 1 and 2, we provide the fixed points and cycles of $S_{\{s,t,b\}}$ for $b = 4$ and $b = 5$ with small values of s and t.

Theorem 2. *Fix $b \geq 2$, $s \geq 0$, and $t \geq 0$. Fix $M \in \mathbb{Z}^+$ such that for each $m \geq M$,*

$$b^m > (m+1)\left((b-1)^2 + s(b-1) + t\right). \tag{3}$$

Then, for each $a \geq b^M$, $S_{\{s,t,b\}}(a) < a$.

Proof. Let $a = \sum_{i=0}^{n} a_i b^i \geq b^M$, with $a_n \neq 0$. Then $n \geq M$ and $b^n \leq a$. Since, for each $0 \leq j \leq b-1$, $S_{\{s,t,b\}}(j) \leq (b-1)^2 + s(b-1) + t$, $S_{\{s,t,b\}}(a) \leq (n+1)\left((b-1)^2 + s(b-1) + t\right) < b^n$ by assumption. Since $b^n \leq a$, the proof is complete. □

Examining the tables, one can find many patterns indicating easily proved generalities. For example, if $s + t = b - 1$, then b is a fixed point of $S_{\{s,t,b\}}$. Similarly for $b \geq 4$, if $s = 1$ and $t = b - 4$, then $S_{\{s,t,b\}}$ has the cycle $(b+2, 2b)$.

The fact that $U_{\{s,t,b\}}$ is finite is immediate from Theorem 2. Note that Theorem 2 and its proof are easily modified to apply to any digit function, substituting the maximum of the absolute value of the function evaluated at a single digit into the right hand side of inequality (3).

s	t	Fixed Points and Cycles
0	0	1
0	1	10, (2,5,4)
0	2	9
0	3	4, 12
1	0	12, (6,8)
1	1	(3,13,16), (6,10,14,20)
1	2	4, 8
1	3	(14,24)
2	0	8, (9,11,23,21)
2	1	4, (8,9,13,20)
2	2	(10,20)
2	3	47, (12,18,17), (29,30,35)
3	0	4, (8,10,20), (14,28,22,18)
3	1	29,(30,35)
3	2	18, (12,20), (30,38)
3	3	21, 28, 63, (14,34,26,33,20)

TABLE 1. Sample Base 4 Fixed Points and Cycles

s	t	Fixed Points and Cycles
0	0	1, 13, 18, (4,16,10)
0	1	7, 22, (2,5)
0	2	6, 12, 17, 21
0	3	11, 16
1	0	12, 20
1	1	6, 16, (7,10)
1	2	(18,28)
1	3	5, 15, (20,23,38,29,28)
2	0	6, 11, 15, (14,32)
2	1	29, (4,25)
2	2	5, 10, 48, (15,17,27)
2	3	(12,22,38,35,17,29,33,30)
3	0	10, (18,36)
3	1	5, 43, (34,39,45), (16,24,58,35)
3	2	38, (24,60)
3	3	28, (20,31,21,38,41,35)

TABLE 2. Sample Base 5 Fixed Points and Cycles

3. Key Properties

The properties of $S_{\{s,t,b\}}$ presented in this section are used in Section 4 to prove the main results of this paper. Lemma 3 provides some basic information about the behavior of $S_{\{s,t,b\}}$ that is used repeatedly throughout the paper. Theorem 6 provides a very limited form of a one-sided inverse for $S_{\{s,t,b\}}$, one that is sufficient for completing the proof of Lemma 7 in Section 4.

We begin by noting that for some values of s, t, and b, $S_{\{s,t,b\}}(a)$ is even, regardless of the input, a, and for other values of s, t, and b, $S_{\{s,t,b\}}(a)$ always has the same parity as a. To record this information, we define two constants, each valued either 1 or 2:

$$d = d(s,t) = \gcd(2, s+1, t)$$

and

$$D = D(s,t,b) = \gcd(2, s, t, b-1).$$

Lemma 3. *Fix $b \geq 2$, $s \geq 0$, $t \geq 0$, and $a \geq 0$.*

- $S_{\{s,t,b\}}(a) \equiv 0 \pmod{d}$.
- $S_{\{s,t,b\}}(a) \equiv a \pmod{D}$.

Proof. Clearly the lemma holds if $a = 0$ (by the definition of $S_{\{s,t,b\}}$), the first part holds if $d = 1$, and the second holds if $D = 1$. Let $a = \sum_{i=0}^{n} a_i b^i$ with $a_n \neq 0$ and, for each i, $0 \leq a_i \leq b-1$.

If $d = 2$, then s is odd and t is even, and so

$$S_{\{s,t,b\}}(a) \equiv \sum_{i=0}^{n} (a_i^2 + a_i s + t) \equiv \sum_{i=0}^{n} (a_i + a_i) \equiv 0 \pmod{2}.$$

If $D = 2$, then s and t are even and b is odd; hence,

$$S_{\{s,t,b\}}(a) \equiv \sum_{i=0}^{n} (a_i^2 + a_i s + t) \equiv \sum_{i=0}^{n} a_i \equiv \sum_{i=0}^{n} a_i b^i \equiv a \pmod{2}.$$ □

Note that it follows from Lemma 3 that if $d = 2$, then every element of $U_{\{s,t,b\}}$ is even.

The following two results are key in proving the results in Section 4.

Lemma 4. *Let $b \geq 4$, $s \geq 0$, and $t \geq 0$ be given. Then*

$$\gcd(S_{\{s,t,b\}}(1), S_{\{s,t,b\}}(2), S_{\{s,t,b\}}(3)) = d.$$

Further, if $d = 2$ and $b \geq 5$, then

$$\gcd(S_{\{s,t,b\}}(2), 2S_{\{s,t,b\}}(3), S_{\{s,t,b\}}(4)) = 2.$$

Proof. Set $g = \gcd(S_{\{s,t,b\}}(1), S_{\{s,t,b\}}(2), S_{\{s,t,b\}}(3))$. By Lemma 3, $d \mid g$.

Since g is a factor of $S_{\{s,t,b\}}(1)$, $S_{\{s,t,b\}}(2)$, and $S_{\{s,t,b\}}(3)$, it is also a factor of $S_{\{s,t,b\}}(3) - 2S_{\{s,t,b\}}(2) + S_{\{s,t,b\}}(1) = 2$. If $g = 1$, then $d = 1$, as desired. If $g = 2$, then $g \mid S_{\{s,t,b\}}(2)$ implies that t is even, and then $g \mid S_{\{s,t,b\}}(1)$ implies that s is odd. Hence $d = \gcd(2, s+1, t) = 2$, completing the proof of the first statement.

Now assume that $d = 2$ and $b \geq 5$ and let

$$\hat{g} = \gcd(S_{\{s,t,b\}}(2), 2S_{\{s,t,b\}}(3), S_{\{s,t,b\}}(4)).$$

Then $d \mid \hat{g}$ and $\hat{g} \mid (S_{\{s,t,b\}}(4) - 2S_{\{s,t,b\}}(3) + S_{\{s,t,b\}}(2)) = 2$. Since $d = 2$, the result follows. □

We next show that each sufficiently large multiple of d is in the image of $S_{\{s,t,b\}}$. This is a variation of the Frobenius (or Postage Stamp) Problem. We use the following simplification of a result from Brauer's work on partitions [1, Theorem 1] in the proof.

Theorem 5 (Brauer)**.** *Let $a_1, a_2, \ldots, a_k$ be relatively prime positive integers. There exists $C \in \mathbb{Z}^+$ such that for each $n \geq C$, there exist positive integers x_1, x_2, $\ldots x_k$ such that $a_1x_1 + a_2x_2 + \cdots a_kx_k = n$.*

Theorem 6. *Given $b \geq 4$, $s \geq 0$, and $t \geq 0$, there exists a positive integer $N = N(s,t,b)$ such that for each $n \in d\mathbb{Z}^+$ with $n \geq N$, there exists $n' \in d\mathbb{Z}^+$ such that $S_{\{s,t,b\}}(n') = n$.*

Proof. First consider the case in which $d = 1$ or b is even. By Lemma 4,

$$\gcd(S_{\{s,t,b\}}(1)/d, S_{\{s,t,b\}}(2)/d, S_{\{s,t,b\}}(3)/d) = 1.$$

It follows from Theorem 5 that there exists some $N_1 \in \mathbb{Z}^+$ such that for each $n \in d\mathbb{Z}^+$ with $n \geq dN_1$, there exist $n_i \geq 0$ satisfying $n_1 S_{\{s,t,b\}}(1)/d + n_2 S_{\{s,t,b\}}(2)/d + n_3 S_{\{s,t,b\}}(3)/d = n/d$. Thus we have $n_1 S_{\{s,t,b\}}(1) + n_2 S_{\{s,t,b\}}(2) + n_3 S_{\{s,t,b\}}(3) = n$.

Let

$$n' = \sum_{i=1}^{n_1} b^i + \sum_{j=1}^{n_2} 2b^{n_1+j} + \sum_{k=1}^{n_3} 3b^{n_1+n_2+k}.$$

Since $d = 1$ or b is even, $n' \in d\mathbb{Z}^+$. Further,

$$S_{\{s,t,b\}}(n') = \sum_{i=1}^{n_1} S_{\{s,t,b\}}(1) + \sum_{j=1}^{n_2} S_{\{s,t,b\}}(2) + \sum_{k=1}^{n_3} S_{\{s,t,b\}}(3) = n,$$

as desired.

For the case with $d = 2$ and b odd, note that $b \geq 5$. By Lemma 4,

$$\gcd(S_{\{s,t,b\}}(2)/2, S_{\{s,t,b\}}(3), S_{\{s,t,b\}}(4)/2) = 1$$

and so by Theorem 5, there exists some $N_2 \in \mathbb{Z}^+$ such that for each $n \in 2\mathbb{Z}^+$ with $n \geq 2N_2$, there exist $n_i \geq 0$ satisfying $n_1 S_{\{s,t,b\}}(2)/2 + n_2 S_{\{s,t,b\}}(3) + n_3 S_{\{s,t,b\}}(4)/2 = n/2$. Letting

$$n' = \sum_{i=1}^{n_1} 2b^i + \sum_{j=1}^{2n_2} 3b^{n_1+j} + \sum_{k=1}^{n_3} 4b^{n_1+2n_2+k},$$

$n' \equiv 2n_1 + 6n_2 + 4n_3 \equiv 0 \pmod 2$ and

$$S_{\{s,t,b\}}(n') = \sum_{i=1}^{n_1} S_{\{s,t,b\}}(2) + \sum_{j=1}^{2n_2} S_{\{s,t,b\}}(3) + \sum_{k=1}^{n_3} S_{\{s,t,b\}}(4) = n.$$

To complete the proof, set $N = N(s,t,b) = max\{2N_1, 2N_2\}$. □

4. Sequences of u-Attracted Numbers

Fix integers $b \geq 4$, $s \geq 0$, and $t \geq 0$. The goal of this section is to prove Theorem 1. Our approach involves ideas from [6], in particular using the concept of a "good" set, but giving it a different meaning from that used in the original paper. Recall the definitions of $d = d(s,t)$ and $D = D(s,t,b)$ from the beginning of Section 3, and of $N = N(s,t,b)$, from Theorem 6.

For fixed s, t, and b, we say that a finite set $H \subseteq \mathbb{Z}^+$ is *good* if for each $u \in U_{\{s,t,b\}}$, there exist $k \geq 0$ and $n \in d\mathbb{Z}^+$ with $n \geq N = N(s,t,b)$ such that, for each $h \in H$, $S^k_{\{s,t,b\}}(h+n) = u$. Define $I : \mathbb{Z}^+ \to \mathbb{Z}^+$ by $I(x) = x+1$.

Lemma 7. *Fix $b \geq 4$, $s \geq 0$, and $t \geq 0$. Let $F : \mathbb{Z}^+ \to \mathbb{Z}^+$ be the composition of a finite sequence of the functions $S_{\{s,t,b\}}$ and I^d, and let H be a finite subset of $\mathbb{Z}^+$. If $F(H)$ is good, then H is good.*

Proof. Let H and F be given with $F(H)$ good. Clearly, if F is the identity function, then H is good. So we assume that F is of one of the two forms: $F = S_{\{s,t,b\}}F'$ or $F = I^dF'$, where F' is also the composition of a finite sequence of the functions $S_{\{s,t,b\}}$ and I^d. Noting that, by Lemma 3, $F'(H)$ is also contained in a single coset modulo D, we assume by induction that if $F'(H)$ is good, then H is good.

Since $F(H)$ is good, for each $u \in U_{\{s,t,b\}}$, there exist $k \geq 0$ and $n \in d\mathbb{Z}$ with $n \geq N$ such that, for each $h \in H$, $S^k_{\{s,t,b\}}(F(h)+n) = u$.

If $F = I^dF'$, then for each $h \in H$,

$$S^k_{\{s,t,b\}}(F'(h)+(d+n)) = S^k_{\{s,t,b\}}(I^dF'(h)+n) = S^k_{\{s,t,b\}}(F(h)+n) = u.$$

Hence $F'(H)$ is good, implying that H is good.

On the other hand, if $F = S_{\{s,t,b\}}F'$, then since $n \in d\mathbb{Z}$ with $n \geq N$, by Theorem 6, there exists $n' \in d\mathbb{Z}^+$ such that $S_{\{s,t,b\}}(n') = n$. Fix $r \in \mathbb{Z}^+$ such that $b^r > N$ and, for each $h \in H$, $b^r > F'(h)$. Then for each $h \in H$, $S^{k+1}_{\{s,t,b\}}(F'(h)+b^rn') = S^k_{\{s,t,b\}}(S_{\{s,t,b\}}F'(h)+S_{\{s,t,b\}}(n')) = S^k_{\{s,t,b\}}(F(h)+n) = u$. Hence, again, $F'(H)$ is good, implying that H is good. □

The following lemma provides the base case for the induction used in the proof of Theorem 9.

Lemma 8. *Fix $b \geq 4$, $s \geq 0$, and $t \geq 0$. If $H = \{h\}$, then H is good.*

Proof. Let $u \in U_{\{s,t,b\}}$ and fix $v \in U_{\{s,t,b\}}$ such that $S_{\{s,t,b\}}(v) = u$. If $d = 2$, then by Lemma 3, u and v are even, and, replacing H by $S_{\{s,t,b\}}(H)$ and applying Lemma 7 if needed, we may assume that h is even. Fix r such that $b^r > h+N$. Let $n = b^rv - h \in d\mathbb{Z}^+$. Then, $S_{\{s,t,b\}}(h+n) = S_{\{s,t,b\}}(b^rv) = u$. Thus, H is good. □

Now we are ready to prove that any finite set contained in a single coset modulo D is good.

Theorem 9. *Fix $b \geq 4$, $s \geq 0$, and $t \geq 0$. Any finite set $H \subseteq \mathbb{Z}^+$ contained in a single coset modulo D is good.*

Proof. Let $H \subseteq \mathbb{Z}^+$ be a finite set contained in a single coset modulo D. Note that, by Lemma 3, $S_{\{s,t,b\}}(H)$ is also contained in a single coset modulo D, and therefore the same holds for $F(H)$ where F is the composition of a finite sequence of $S_{\{s,t,b\}}$ and I^d.

If $|H| = 1$, then by Lemma 8, H is good. So we assume that $|H| > 1$ and assume by induction that any finite subset of $\mathbb{Z}^+$ of cardinality less than $|H|$ that is contained in a single coset modulo D is good. Let h_1, h_2 be distinct elements of H with $h_1 > h_2$.

If $d = 2$, by Lemma 3, $S_{\{s,t,b\}}(H) \subseteq 2\mathbb{Z}$ and so, by Lemma 7, we can replace H by $S_{\{s,t,b\}}(H)$, if needed, to assume that h_1 and h_2 are both even.

Case 1: $h_1 \equiv h_2 \pmod{b-1}$. Fix $v \in \mathbb{Z}$ such that $h_1 - h_2 = (b-1)v$ and note that if $d = 2$ and b is even, then v is even. Fix r such that $b^r > bv + h_1 + N$. Let $n = \delta_0 b^{r+1} + b^r + v - h_2$, where

$$\delta_0 = \begin{cases} 1 & \text{if } d = 2 \text{ and } 2 \mid (v - h_2) \\ 0 & \text{otherwise.} \end{cases}$$

Then, regardless of the values of b and d, $n \in d\mathbb{Z}^+$. Further,

$$I^n(h_1) = h_1 + \delta_0 b^{r+1} + b^r + v - h_2 = \delta_0 b^{r+1} + b^r + v + (b-1)v = \delta_0 b^{r+1} + b^r + bv$$

and

$$I^n(h_2) = h_2 + \delta_0 b^{r+1} + b^r + v - h_2 = \delta_0 b^{r+1} + b^r + v.$$

Thus, $I^n(h_1)$ and $I^n(h_2)$ have the same nonzero digits. Hence, $S_{\{s,t,b\}} I^n(h_1) = S_{\{s,t,b\}} I^n(h_2)$, and so, by induction, $S_{\{s,t,b\}} I^n(H)$ is good and, by Lemma 7, H is good.

Case 2: $h_1 \not\equiv h_2 \pmod{b-1}$. Fix $w = h_1 - h_2$. If $d = 2$ and b is even, set $\delta_1 = 1$, otherwise set $\delta_1 = 0$.

Fix $r' \in \mathbb{Z}^+$ such that $b^{r'} > h_1 + 1 + N$ and, if t is odd, such that

$$J = J(t, s, w, r', \delta_1) = r't - (1+\delta_1)s + \delta_1 - 1 - S_{\{s,t,b\}}(w - (1+\delta_1)) \tag{4}$$

is even.

If t is even and b is odd (so $\delta_1 = 0$), then either s is odd and $d = 2$ or s is even and $D = 2$. In the first instance, Lemma 3 implies that $S_{\{s,t,b\}}(w - (1+\delta_1))$ is even. In the second, $w = h_1 - h_2 \in 2\mathbb{Z}$, by assumption, and so Lemma 3 implies that $S_{\{s,t,b\}}(w - (1+\delta_1))$ is odd. Thus, if t is even and b is odd, J is even.

Now, if J is even, then fix $0 \le j < b-1$ such that $2j \equiv J \pmod{b-1}$. If J is odd, then b is even. Thus, 2 is invertible modulo $b-1$, and again we can fix $0 \le j < b-1$ such that $2j \equiv J \pmod{b-1}$.

If $j = 0$, set $\delta' = 1$, otherwise set $\delta' = 0$. If $d = 2$ and both b and j are odd, set $\delta_2 = 1$, otherwise set $\delta_2 = 0$.

Set

$$n' = \delta_2 b^{r'+2} + (j+1) b^{r'+\delta'} - h_2 - (1+\delta_1).$$

If $d = 2$ and b is odd, then by the definition of δ_2, n' is even. If $d = 2$ and b is even, then by the definition of δ_1, n' is even. Hence, in any case, $n' \in d\mathbb{Z}$. Further,

$$\begin{aligned} S_{\{s,t,b\}}(h_1 + n') &= S_{\{s,t,b\}}(h_1 + \delta_2 b^{r'+2} + (j+1) b^{r'+\delta'} - h_2 - (1+\delta_1)) \\ &= S_{\{s,t,b\}}(\delta_2 b^{r'+2} + (j+1) b^{r'+\delta'} + w - (1+\delta_1)) \\ &= S_{\{s,t,b\}}(\delta_2) + (j+1)^2 + (j+1)s + t + S_{\{s,t,b\}}(w - (1+\delta_1)) \\ &= S_{\{s,t,b\}}(\delta_2) + j^2 + 2j + 1 + js + s + t + S_{\{s,t,b\}}(w - (1+\delta_1)). \end{aligned}$$

Therefore, using equation (4),

$$\begin{aligned}S_{\{s,t,b\}}(h_1+n') \equiv S_{\{s,t,b\}}(\delta_2) + j^2 + &\\ &\left(r't-(1+\delta_1)s+\delta_1-1-S_{\{s,t,b\}}(w-(1+\delta_1))\right)\\ &+1+js+s+t+S_{\{s,t,b\}}(w-(1+\delta_1))\\ \equiv S_{\{s,t,b\}}(\delta_2)+j^2+js+(r'+1)t-\delta_1 s+\delta_1 \pmod{b-1}.\end{aligned}$$

And

$$\begin{aligned}S_{\{s,t,b\}}(h_2+n') &= S_{\{s,t,b\}}(h_2+\delta_2 b^{r'+2}+(j+1)b^{r'+\delta'}-h_2-(1+\delta_1))\\ &= S_{\{s,t,b\}}(\delta_2 b^{r'+2}+(j+1)b^{r'+\delta'}-(1+\delta_1))\\ &= S_{\{s,t,b\}}\left(\delta_2 b^{r'+2}+jb^{r'+\delta'}+\left(\sum_{i=1}^{r'+\delta'-1}(b-1)b^i\right)+(b-(1+\delta_1))\right)\\ &= S_{\{s,t,b\}}(\delta_2)+j^2+js+(1-\delta')t\\ &\qquad +(r'+\delta'-1)\left((b-1)^2+(b-1)s+t\right)\\ &\qquad +(b-(1+\delta_1))^2+(b-(1+\delta_1))s+t,\end{aligned}$$

implying that

$$\begin{aligned}S_{\{s,t,b\}}(h_2+n') &\equiv S_{\{s,t,b\}}(\delta_2)+j^2+js+(r'+1)t-\delta_1 s+\delta_1\\ &\equiv S_{\{s,t,b\}}(h_1+n') \pmod{b-1}.\end{aligned}$$

Applying Case 1 to the elements $S_{\{s,t,b\}}(h_1+n')$ and $S_{\{s,t,b\}}(h_2+n')$ in the set $S_{\{s,t,b\}}I^{n'}(H)$ we conclude that $S_{\{s,t,b\}}I^{n'}(H)$ is good. Hence, by Lemma 7, H is good. □

We now prove Theorem 1.

Proof of Theorem 1. Let $b \geq 4$, $s \geq 0$, $t \geq 0$, and $u \in U_{\{s,t,b\}}$ be given. Fix an arbitrary $m \in \mathbb{Z}^+$. Let $H = \{D, 2D, \ldots, mD\}$. By Theorem 9, H is good. Hence there exist $k \geq 0$ and $n \in d\mathbb{Z}^+$ such that $n \geq N(s,t,b)$ and, for each $h \in H$, $S^k_{\{s,t,b\}}(h+n) = u$. Thus,

$$D+n, 2D+n, \ldots, mD+n$$

is a sequences of length m, of D-consecutive numbers that are u-attracted under $S_{\{s,t,b\}}$. □

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