DEPTH AND SINGULAR VARIETIES OF EXTERIOR EDGE IDEALS

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ABSTRACT. Edge ideals of finite simple graphs are well-studied over polynomial rings. In this paper, we initiate the study of edge ideals over exterior algebras, specifically focusing on the depth and singular varieties of such ideals. We prove an upper bound on the depth of the edge ideal associated to a general graph and a more refined bound for bipartite graphs, and we show that both are tight. We also compute the depth of several large families of graphs including cycles, complete multipartite graphs, spider graphs, and Ferrers graphs. Finally, we focus on the effect whiskering a graph has on the depth of the associated edge ideal.

1. INTRODUCTION

Let V be a vector space with basis $e_1, \ldots, e_n$ over a field $\Bbbk$, and let $E = \bigwedge_{\Bbbk} \langle e_1, \ldots, e_n \rangle$ be the exterior algebra of V. The standard basis elements $e_{k_1} \wedge \cdots \wedge e_{k_s}$ of E, with $k_1 < \ldots < k_s$, are called monomials in E, and an ideal I of E generated by monomials is called a monomial ideal. Given a finite, simple graph G with vertex set $v_1, \ldots, v_n$, we consider the corresponding edge ideal $I_E(G) = (e_i \wedge e_j \mid \{v_i, v_j\}$ is an edge of $G)$. The corresponding ideals $I_S(G)$ over a polynomials ring S are well-studied, where there are many results linking combinatorial properties of G to algebraic properties of $I_S(G)$. The situation over exterior algebras is the focus of this paper.

Monomial ideals more generally were studied by Aramova, Avramov, and Herzog. It follows from [AAH00, 1.3] that $I_S(G)$ and $I_E(G)$ have the same regularity for any graph G. On the other hand, the depths of $E/I_E(G)$ and $S/I_S(G)$ can be quite different. The aim of this paper is to bound, and in some cases compute exactly, the depth of $E/I_E(G)$ in terms of combinatorial properties of G.

Our main results include a tight general upper bound on depth of edge ideals.

Theorem 4.1. *Let G be a graph on n vertices, none of which are isolated. Then*

$$\operatorname{depth}_S S/I_S(G) \leq n + 2 - 2\sqrt{n} \qquad and \qquad \operatorname{depth}_E E/I_E(G) \leq n + 1 - 2\sqrt{n}.$$

Moreover, when n is a square, both sets of bounds are tight.

The first inequality recovers a result of Hà and Hibi [HH21, 4.2]. In the case of bipartite graphs, we prove a similar upper bound on the depth of edge ideals.

Theorem 4.4. *Let G be a bipartite graph on n vertices, none of which are isolated. Then*

$$\operatorname{depth}_S S/I_S(G) \leq \left\lfloor \frac{n}{2} \right\rfloor \qquad and \qquad \operatorname{depth}_E E/I_E(G) \leq \left\lfloor \frac{n}{2} \right\rfloor - 1.$$

Moreover, both bounds are tight.

We also compute the depths and singular varieties of exterior edge ideals of several large classes of graphs including complete graphs, complete multipartite graphs, cycles, paths, spiders, and star graphs. We show how one can algorithmically compute the depth of any tree, although an exact formula is likely not possible; see Example 3.1. Finally, we investigate the effect of whiskering a graph on the depth of the associated edge ideal, which is distinct from the situation over a polynomial ring; see Corollary 3.18 and Remark 3.19.

The rest of this paper is organized as follows. Section 2 collects the necessary background and notation that we need for the remainder of the paper and includes several propositions that will be useful in our later proofs. In Section 3, we compute the depth and singular varieties of some general families of graphs, such as cycle graphs, complete multipartite graphs, and whiskered graphs. In Section 4, we provide several upper bounds on the depth of edge ideals. In particular, we establish tight upper bounds over E and S of the depth of an edge ideal and a tight upper bound of the projective dimension of an edge ideal over S and a tight upper bound of the complexity of an edge ideal over E.

2. BACKGROUND

2.1. **Depth and Singular Varieties.** Let $E = \bigwedge_{\Bbbk} \langle e_1, \dots, e_n \rangle$ denote an exterior algebra over the field $\Bbbk$ in n variables. We recall that E has a $\Bbbk$-basis consisting of all monomials $e_T = e_{i_1} e_{i_2} \cdots e_{i_r}$ for each subset $T = \{i_1 < i_2 < \cdots < i_r\} \subseteq [n] = \{1, 2, \dots, n\}$. (We omit the wedge products in the remainder of the paper.) There is a natural grading $E = \bigoplus_{r=0}^{n} E_r$, where E_r is the subspace spanned by all monomials e_T with $|T| = r$, and we call elements of E_r homogeneous of degree r. See [HH11] for further details and any unexplained terminology.

Let M be a graded E-module, and take $\ell \in E_1$. We say that ℓ is M**-regular** if

$$(0 :_M \ell) := \{m \in M \mid \ell m = 0\} = \ell M.$$

Otherwise ℓ is M**-singular**. The set of all M-singular elements in E_1 is called the **singular variety** of M, which we denote by $V_E(M)$. (This is called the rank variety of M in [AAH00].) A sequence $\ell_1, \dots, \ell_r$ of linear forms in E is called an M**-regular sequence** if ℓ_i is regular on $M/(\ell_1, \dots, \ell_{i-1})M$ for all $i \leq r$. The **depth** of M, denoted $\operatorname{depth}_E M$, is the maximum length of such a regular sequence. The following result gathers useful results about the singular varieties of modules over E.

Theorem 2.1 ([AAH00, 3.1, 3.2, 4.1]). *Let E be an exterior algebra in n variables over an algebraically closed field $\Bbbk$, and let L, M, and N be finitely generated graded E-modules. Then:*

(a) $V_E(M)$ is a cone in the vector space E_1.
(b) If $I \subseteq E$ is a monomial ideal, then $V_E(E/I)$ is a union of finitely many coordinate subspaces of E_1.
(c) $\operatorname{depth}_E M = \operatorname{codim}_{E_1} V_E(M)$.
(d) If $0 \to L \to M \to N \to 0$ is an exact sequence of E-linear maps, then any one of the sets $V_E(L)$, $V_E(M)$, $V_E(N)$ is contained in the union of the other two.
(e) $V_E(M \otimes_{\Bbbk} N) = V_E(M) \cap V_E(N)$.

As an easy corollary, we prove a depth lemma for modules in a short exact sequence.

Corollary 2.2. *Let E be an exterior algebra over an algebraically closed field $\Bbbk$, and let $0 \to L \to M \to N \to 0$ be an exact sequence of finitely generated graded E-modules. Then the depth of any one of the modules L, M, N is at least the minimum of the depths of the other two, and equality holds if the other two modules have different depths.*

Proof. We only prove the corollary statement for L since the remaining arguments are identical. Suppose without loss of generality that $\operatorname{depth}_E M \leq \operatorname{depth}_E N$. By part (c) of the preceding theorem

$$\dim V_E(N) = n - \operatorname{depth}_E N \leq n - \operatorname{depth}_E M = \dim V_E(M).$$

Part (d) of the preceding theorem yields

$$\dim V_E(L) \leq \max\{\dim V_E(M), \dim V_E(N)\} = \dim V_E(M).$$

Therefore

$$\operatorname{depth}_E M = n - \dim V_E(M) \leq n - \dim V_E(L) = \operatorname{depth}_E L,$$

proving the first claim.

We now prove the second claim. Suppose that $\operatorname{depth}_E M < \operatorname{depth}_E N$. Using the previous claim, we have the two inequalities

$$\operatorname{depth}_E L \geq \min\{\operatorname{depth}_E M, \operatorname{depth}_E N\} = \operatorname{depth}_E M$$

$$\operatorname{depth}_E M \geq \min\{\operatorname{depth}_E L, \operatorname{depth}_E N\} = \operatorname{depth}_E L.$$

Therefore $\operatorname{depth}_E M = \operatorname{depth}_E L$. $\square$

In practice, it is often easier to specify the singular variety of an E-module M as the vanishing set of some collection of polynomials. We let $S = \Bbbk[x_1, \ldots, x_n]$ denote the ring of polynomial functions on E_1, where x_i is the linear functional dual to e_i, and for any ideal $I \subseteq S$, we set $V(I) = \{\ell \in E_1 \mid f(\ell) = 0 \text{ for all } f \in I\}$. The following result allows one to compute singular varieties over an exterior algebra with an added variable.

Proposition 2.3 ([MM22, 4.1, 4.2]). *Let E' be an exterior algebra over an algebraically closed field $\Bbbk$, and let $E = E'\langle e \rangle$ denote an exterior algebra in one more variable. Denote by*

S' and $S = S'[x]$ *the polynomial rings over* $\Bbbk$ *dual to* E' *and* E *respectively, and let* M *be a finitely generated graded* E'-module with $V_{E'}(M) = V(I)$ *for some ideal* $I \subseteq S'$. *Then:*

(a) $V_E(M \otimes_{\Bbbk} \Bbbk\langle e \rangle/(e)) = V(IS)$, *and* $\operatorname{depth}_E(M \otimes_{\Bbbk} \Bbbk\langle e \rangle/(e)) = \operatorname{depth}_{E'} M$.

(b) $V_E(M \otimes_{\Bbbk} \Bbbk\langle e \rangle) = V(x, IS)$, *and* $\operatorname{depth}_E(M \otimes_{\Bbbk} \Bbbk\langle e \rangle) = \operatorname{depth}_{E'} M + 1$.

For an E-module M, the **complexity** of M is

$$\operatorname{cx}_E M = \inf\{c \in \mathbb{Z} \mid \beta_i^E(M) \leq \alpha i^{c-1} \text{ for some } \alpha \in \mathbb{R} \text{ and all } i \geq 1\}.$$

(See Subsection 2.3 for a review of the Betti numbers $\beta_i^E(M)$ in the commutative case.) The complexity of M can be thought of as the size of the minimal free resolution of an E-module, similar to the projective dimension of an S-module. There is an Auslander-Buchsbaum style theorem relating depth and complexity.

Theorem 2.4. [AAH00, 3.2] *If* $\Bbbk$ *is an infinite field and* M *is a finitely generated* E-module, *then each maximal* M-regular sequence has $\operatorname{depth}_E M$ *elements, and*

$$\operatorname{depth}_E M + \operatorname{cx}_E M = n.$$

2.2. **Exterior Edge Ideals.** Let G be a finite, simple graph with vertices $v_1, \ldots, v_n$. The **exterior edge ideal** of G is the ideal of the exterior algebra $E = \bigwedge_{\Bbbk}\langle e_1, \ldots, e_n \rangle$ defined by:

$$I_E(G) = (e_i e_j \mid \{v_i, v_j\} \text{ is an edge of } G).$$

Our aim is to study the depth and singular variety of the quotient ring $E/I_E(G)$ for various families of graphs. In this subsection, we record some general observations about working with quotients by exterior edge ideals that will be useful in the subsequent sections.

Notation. Throughout the remainder of this paper, unless explicitly specified otherwise, G will denote a (finite, simple) graph with vertices $v_1, \ldots, v_n$, $E = \bigwedge_{\Bbbk}\langle e_1, \ldots, e_n \rangle$ denotes the corresponding exterior algebra over a fixed algebraically closed field $\Bbbk$ of characteristic zero, $I = I_E(G) \subseteq E$ is the exterior edge ideal of G, and $S = \Bbbk[x_1, \ldots, x_n]$ denotes the dual polynomial ring.

For simplicity, we denote the depth, complexity, and singular variety of $E/I_E(G)$ by $\operatorname{depth}_E G$, $\operatorname{cx}_E G$, and $V_E(G)$ respectively. We also write $\operatorname{pd}_S G$ and $\operatorname{depth}_S G$ for the projective dimension and depth of $S/I_S(G)$ over S, where $I_S(G) = (x_i x_j \mid \{v_i, v_j\} \text{ is an edge of } G)$ is the more common commutative edge ideal in S.

If v_i and v_j are vertices of G, we write $v_i v_j$ to denote that $\{v_i, v_j\}$ is an edge of G. The **closed neighborhood** of the vertex v_i is the set $N[v_i] = N(v_i) \cup \{v_i\}$ of neighbors of v_i together with v_i. For any unexplained graph-theoretic terminology, we refer the reader to [Wes96].

The following occurs frequently enough as a special case in the arguments of the rest of the paper that it is worth mentioning to avoid future confusion.

Remark 2.5 (The empty graph). When G is the empty graph, there are no vertices or edges of G so that $E/I_E(G) = E = \Bbbk$. Since $E_1 = 0$, there are no linear forms with which to form a regular sequence so that $\operatorname{depth}_E G = 0$ and $V_E(G) = 0 = E_1$.

Proposition 2.6. *Let v_i, v_j be vertices of G, and suppose that $v_i v_j$ is an edge of G. Then:*

(a) $\operatorname{depth}_E G \geq \min\{\operatorname{depth}_E(G \smallsetminus v_i), \operatorname{depth}_E(G \smallsetminus N[v_i])\}$.
(b) $\operatorname{depth}_E G \geq \min\{\operatorname{depth}_E(G \smallsetminus v_i v_j), \operatorname{depth}_E(G \smallsetminus (N[v_i] \cup N[v_j]))\}$.

Proof. If we set $W = N[v_i]$, $G' = G \smallsetminus W$, and $G'' = G \smallsetminus v_i$ and we let E' and E'' denote the corresponding exterior algebras on the vertices of G' and G'' respectively, then we have a short exact sequence of E-modules

$$0 \longrightarrow \frac{E}{I_{E'}(G')E + (e_j \mid v_j \in W)} \xrightarrow{e_i} \frac{E}{I_E(G)} \longrightarrow \frac{E}{I_{E''}(G'')E + (e_i)} \longrightarrow 0. \qquad (2.1)$$

The first and last modules in the above exact sequence are isomorphic to $E'/I_{E'}(G')$ and $E''/I_{E''}(G'')$ respectively so that $\operatorname{depth}_E G \geq \min\{\operatorname{depth}_E G', \operatorname{depth}_E G''\}$ by Corollary 2.2. (Note that $\operatorname{depth}_E E'/I_{E'}(G') = \operatorname{depth}_{E'} E'/I_{E'}(G')$ by Proposition 2.3 so that there is no harm in writing $\operatorname{depth}_E(G')$. The second depth inequality follows similarly from the short exact sequence

$$0 \longrightarrow \frac{E}{I_{E'}(G')E + (e_j \mid v_j \in W)} \xrightarrow{e_i e_j} \frac{E}{I_E(G'')} \longrightarrow \frac{E}{I_E(G)} \longrightarrow 0,$$

where $W = N[v_i] \cup N[v_j]$, $G' = G \smallsetminus W$, and $G'' = G \smallsetminus v_i v_j$. $\qquad \square$

Proposition 2.7. *Suppose G is a disjoint union $G = G' \sqcup G''$ for some subgraphs G', G''. Then*

$$V_E(G) = V_E(G') \cap V_E(G'')$$

and $\operatorname{depth}_E G = \operatorname{depth}_E G' + \operatorname{depth}_E G''$.

Proof. If E' and E'' denote the corresponding exterior algebras on the vertices of G' and G'' respectively, then $E/I_E(G) \cong E'/I_{E'}(G') \otimes_{\Bbbk} E''/I_{E''}(G'')$ so that the statement about singular varieties follows from Theorem 2.1. If S, S', and S'' denote the polynomial rings dual to E, E', and E'' and we write $V_{E'}(G') = V(I')$ and $V_{E''}(G'') = V(I'')$ for some ideals $I' \subseteq S'$ and $I'' \subseteq S''$, then $V_E(G) = V_E(G') \cap V_E(G'') = V(I'S + I''S)$ by Proposition 2.3. Since $I'S$ and $I''S$ are ideals generated by polynomials in disjoint sets of variables, it follows that

$$\operatorname{depth}_E G = \operatorname{ht}(I'S + I''S) = \operatorname{ht} I' + \operatorname{ht} I'' = \operatorname{depth}_E G' + \operatorname{depth}_E G''. \qquad \square$$

Proposition 2.8. *Let v_i and v_j be vertices of G, and set*

$$J = (e_k \mid v_k \in N[v_i] \cap N[v_j]) + (e_a e_b \mid v_a \in N(v_i), v_b \in N(v_j)).$$

Then $(I_E(G) : e_i - e_j) = I_E(G) + (e_i - e_j) + J$.

Proof. Set $L = I_E(G) + (e_i - e_j) + J$. Observe that if v_k is adjacent to both v_i and v_j, then both $e_k e_i$ and $e_k e_j$ are in $I_E(G)$ and so $e_k \in (I_E(G) : e_i - e_j)$. (This includes the possibility $k = i, j$ since, for example, $e_i^2 = 0$.) Similarly, if v_a is adjacent to v_i and v_b is adjacent to v_j, then $e_a e_b e_i, e_a e_b e_j \in I_E(G)$. Thus, $e_a e_b \in (I_E(G) : e_i - e_j)$. Since $(e_i - e_j)^2 = 0$, this shows that $L \subseteq (I_E(G) : e_i - e_j)$.

Now, let $f \in (I_E(G) : e_i - e_j)$. To show that $f \in L$, we may write f as a linear combination of monomials

$$f = e_i e_j \sum_T \lambda_{T,i,j} e_T + e_i \sum_T \lambda_{T,i} e_T + e_j \sum_T \lambda_{T,j} e_T + \sum_T \lambda_T e_T,$$

where $\lambda_{T,i,j}, \lambda_{T,i}, \lambda_{T,j}, \lambda_T \in \mathbb{k}$ and T runs over all subsets of $[n]$ not containing i or j in each sum. Without loss of generality, we may assume that none of the monomials of f belong to L. In particular, since $e_i e_j = (e_i - e_j)e_j \in L$ we can subtract the first sum from f and assume $\lambda_{T,i,j} = 0$ for all T. Then we have

$$(e_i - e_j)f = \sum_T (\lambda_{T,i} + \lambda_{T,j}) e_i e_j e_T + \sum_T \lambda_T e_i e_T - \sum_T \lambda_T e_j e_T,$$

where all monomials appearing on the right are distinct, and so, they must all belong to $I_E(G)$. If $\lambda_T \neq 0$, then $e_i e_T, e_j e_T \in I_E(G)$ so that either $e_T \in I_E(G)$ or there exist $a, b \in T$ such that $e_i e_a, e_j e_b \in I_E(G)$. Either way, we see that $e_T \in L$ contrary to our assumption, so we must have $\lambda_T = 0$ for all T.

Similarly, if $\lambda_{T,i} + \lambda_{T,j} \neq 0$, then $e_i e_j e_T \in I_E(G)$ implies there is an $a \in T$ such that $e_i e_a$ or $e_j e_a$ belongs to $I_E(G)$. Suppose without loss of generality that $e_i e_a \in I_E(G)$. If $\lambda_{T,i} \neq 0$, then $e_i e_T \in L$ contrary to our assumption on f, so we must have $\lambda_{T,i} = 0$ so that $\lambda_{T,j} \neq 0$. But then $e_j e_T$ is a monomial of f divisible by $e_j e_a = e_i e_a - (e_i - e_j)e_a \in L$, which also contradicts our assumption. Hence, we must have $\lambda_{T,i} + \lambda_{T,j} = 0$ for all T so that

$$f = \sum_T \lambda_{T,i}(e_i - e_j)e_T \in L. \qquad \square$$

Corollary 2.9. *With the same notation as in the preceding proposition, if $N[v_i] \cap N[v_j] = \varnothing$ and every vertex in $N(v_i)$ is adjacent to every vertex in $N(v_j)$, then $e_i - e_j$ is regular on $E/I_E(G)$. Furthermore, if $G/(v_i \sim v_j)$ denotes the graph obtained by identifying v_i and v_j, then*

$$\mathrm{depth}_E G \geq \mathrm{depth}_E G/(v_i \sim v_j) + 1.$$

Proof. The extra assumptions imply that $J \subseteq I_E(G) + (e_i - e_j)$ so that $(I_E(G) : e_i - e_j) = I_E(G) + (e_i - e_j)$, which is equivalent to $e_i - e_j$ being regular on $E/I_E(G)$. If we set $H = G/(v_i \sim v_j)$, the second statement follows from the observation that $(I_E(G), e_i - e_j) = (I_E(H), e_i - e_j)$. $\qquad \square$

2.3. **Free Resolutions and Graded Betti Numbers.** For modules that are quotients of the exterior algebra by a monomial ideal, we can detect depth using tools from commutative algebra whose definitions we recall in this subsection.

If I is an ideal generated by homogeneous polynomials in a polynomial ring $S = \Bbbk[x_1, \ldots, x_n]$ over a field, a **free resolution** of the quotient ring $R = S/I$ is an exact sequence of homomorphisms

$$0 \to F_p \xrightarrow{d_p} F_{p-1} \to \cdots \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} R \to 0,$$

where each $F_i \cong S^{r_i}$ is a free S-module (that is, the image of each map in the sequence is the kernel of the preceding map). Concretely, we can represent each map d_i as multiplication by a matrix with entries in S. Every quotient ring R has a free resolution that is **minimal** in the sense the entries of the matrices representing the maps d_i are all homogeneous polynomials of positive degree, and this resolution is unique up to isomorphism.

By imposing a grading on the free modules F_i in the minimal free resolution of R so that the maps d_i preserve degrees, we can write each free module as a direct sum $F_i \cong \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{i,j}}$, where $S(-j)$ denotes the free graded S-module whose d-th graded component is $S(-j)_d = S_{d-j}$. The numbers $\beta_{i,j}^S(R) := \beta_{i,j}$ that record the number of basis elements of F_i of degree j are then called the **graded Betti numbers** of R over S. When I is a monomial ideal, the minimal free resolution of R even admits a finer $\mathbb{Z}^n$-grading, where $\deg x_i = \mathbf{e}_i$ is the i-th standard basis vector of $\mathbb{Z}^n$. In that case, for each $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, we denote by $\beta_{i,\mathbf{a}}^S(R)$ the corresponding **multigraded Betti number** of R over S. In particular, when I is squarefree monomial and $\beta_{i,\mathbf{a}}^S(R) \neq 0$, it follows from the Taylor resolution, for example, that every entry of $\mathbf{a}$ is either 0 or 1. See [Pee11] for further details.

The next result provides the important connection between measuring depth of monomial ideals over an exterior algebra and free resolutions over a polynomial ring.

Theorem 2.10 ([AAH00, 3.1, 3.2, 4.2]). *Let E be an exterior algebra in n variables over an algebraically closed field $\Bbbk$, $J \subseteq E$ be a monomial ideal, and I be the corresponding square-free monomial ideal of the dual polynomial ring S. Denote by Σ the set of non-vanishing multidegrees in the minimal free resolution of S/I. Then*

$$V_E(E/J) = \bigcup_{\mathbf{a} \in \Sigma} V(x_i \mid a_i = 0),$$

$$\operatorname{depth}_E E/J = n - \max\{j \mid \beta_{i,j}^S(S/I) \neq 0\}.$$

Example 2.11. Consider the edge ideal $I_E(C_3) = (e_1 e_2, e_1 e_3, e_2 e_3) \subseteq E = \bigwedge_{\Bbbk}\langle e_1, e_2, e_3 \rangle$, and let $I_S(C_3) = (x_1 x_2, x_1 x_3, x_2 x_3) \subseteq S = \Bbbk[x_1, x_2, x_3]$ denote the corresponding edge ideal in the dual polynomial ring. It can be checked that the minimal free resolution of $R = S/I_S(C_3)$ is

$$0 \longrightarrow S(-1,-1,-1)^2 \xrightarrow{d_2} \begin{matrix} S(-1,-1,0) \\ \oplus \\ S(-1,0,-1) \\ \oplus \\ S(0,-1,-1) \end{matrix} \xrightarrow{d_1} S \longrightarrow R \longrightarrow 0,$$

where

$$d_2 = \begin{pmatrix} x_3 & 0 \\ -x_2 & x_2 \\ 0 & -x_1 \end{pmatrix} \qquad d_1 = \begin{pmatrix} x_1 x_2 & x_1 x_3 & x_2 x_3 \end{pmatrix}$$

so that the only nonzero multidegrees are $\Sigma = \{(1,1,0),(1,0,1),(0,1,1),(1,1,1)\}$ and the only nonzero graded Betti numbers of R are $\beta_{0,0}^S(R) = 1$, $\beta_{1,2}^S(R) = 3$ and $\beta_{2,3}^S(R) = 2$. Hence, the above theorem implies that

$$V_E(C_3) = V(0) = E_1$$

$$\mathrm{depth}_E C_3 = 3 - 3 = 0.$$

We will see that this example is greatly generalized in Theorem 3.7 below.

Corollary 2.12. *Let G be a graph on n vertices, and let E and S be the corresponding exterior algebra and polynomial ring over $\Bbbk$ as above. Then*

$$\mathrm{depth}_E G \leq \mathrm{depth}_S G - 1$$

with equality if $I_S(G)$ has a linear free resolution over S.

Proof. As $I_S(G)$ is generated by quadratic monomials, $\beta_{i,j}^S(S/I_S(G)) = 0$ for $i > 0$ and $j \leq i$. Thus $\beta_{p,j}^S(S/I_S(G)) \neq 0$ for some $j \geq p + 1$, where $p = \mathrm{pd}_S S/I_S(G)$. Thus

$$\mathrm{depth}_E G = n - \max\{j \mid \beta_{i,j}^S(S/I_S(G)) \neq 0\} \leq n - p - 1 = \mathrm{depth}_S G - 1,$$

where the last equality is the Auslander-Buchsbaum Formula [Pee11, 15.3]. When $I_S(G)$ has a linear free resolution, equality follows from Theorem 2.10. □

Remark 2.13. We note that the above inequality is both tight and the difference $\mathrm{depth}_S G - \mathrm{depth}_E G$ can be arbitrarily large. For example, since $I_S(K_n)$ is Cohen-Macaulay of height $n - 1$, it will follow from Corollary 3.10 that

$$\mathrm{depth}_S K_n - 1 = n - 1 - \mathrm{ht}\, I_S(K_n) = 0 = \mathrm{depth}_E K_n.$$

On the other hand, if C_n denotes the cycle graph on n vertices, then $\mathrm{depth}_E C_n = 0$ by Theorem 3.7, yet

$$\mathrm{depth}_S C_n = \begin{cases} \left\lfloor \frac{n}{3} \right\rfloor & \text{if } n \equiv 1 \pmod 3 \\ \left\lfloor \frac{n+2}{3} \right\rfloor & \text{if } n \not\equiv 1 \pmod 3 \end{cases}$$

by [Bou10, 5.0.6]. Thus, the difference $\mathrm{depth}_S G - \mathrm{depth}_E G$ can be arbitrarily large.

3. Some Singular Variety Computations

In this section, we compute the singular varieties and depths of several large classes of graphs, including paths, cycles, spider graphs, star graphs, complete multipartite graphs, and trees. We also investigate the effect whiskering a graph has on its depth.

3.1. Trees and Cycles.

Theorem 3.1. *Let G be a graph with a vertex $w = v_n$ of degree one, and let v be the unique vertex adjacent to w. Then*

$$V_E(G) = (V_E(G \smallsetminus w) \cap V(x_n)) \cup V_E(G \smallsetminus N[v])$$

so that

$$\mathrm{depth}_E G = \min\{\mathrm{depth}_E(G \smallsetminus w) + 1, \mathrm{depth}_E(G \smallsetminus N[v])\}.$$

Proof. Let $\Sigma(G)$ denote the set of multidegrees of basis elements for any free module in the minimal free resolution of $S/I_S(G)$ over S. We may assume the vertices of G have been numbered so that $v_1, \ldots, v_r$ are the neighbors of v different from w, $v = v_{r+1}$, and $w = v_n$. We also set $G' = G \smallsetminus w$, $S' = \Bbbk[x_1, \ldots, x_{n-1}]$, and $S'' = \Bbbk[x_{r+2}, \ldots, x_{n-1}]$. We then have a short exact sequence of graded S-modules

$$0 \longrightarrow S/(I_S(G') : x_{r+1}x_n)(-2) \overset{x_{r+1}x_n}{\longrightarrow} S/I_{S'}(G')S \longrightarrow S/I_S(G) \longrightarrow 0,$$

where $(I_S(G') : x_{r+1}x_n) = I_{S''}(G \smallsetminus N[v])S + (x_1, \ldots, x_r)$. Let $F_\bullet$ and $F'_\bullet$ denote the minimal free resolutions of $S/(I_S(G') : x_{r+1}x_n)$ and $S/I_{S'}(G')S$ respectively. We note that $F'_\bullet$ is obtained from the minimal free resolution of $S'/I_{S'}(G')$ over S' by tensoring with S since S is free over S' so that the set of multidegrees of basis elements for any free module in $F'_\bullet$ is precisely $\Sigma(G')$ (after identifying $\mathbb{Z}^{n-1}$ with $\mathbb{Z}^{n-1} \times \{0\} \subseteq \mathbb{Z}^n$). Additionally, since

$$S/(I_S(G') : x_{r+1}x_n) \cong S''/I_{S''}(G \smallsetminus N[v]) \otimes_\Bbbk \Bbbk[x_1, \ldots, x_{r+1}, x_n]/(x_1, \ldots, x_r),$$

we know that $F_\bullet$ can be obtained as the tensor product over $\Bbbk$ of the minimal free resolution of $S''/I_{S''}(G \smallsetminus N[w])$ over S'' with the Koszul complex on the variables $x_1, \ldots, x_r$. Hence, if $\Delta = \{0, 1\}^r$, we see that the set of multidegrees of basis elements for any free module in $F_\bullet$ is

$$\Sigma(G \smallsetminus N[v]) + \Delta = \{\mathbf{a} + \mathbf{b} \mid \mathbf{a} \in \Sigma(G \smallsetminus N[v]), \mathbf{b} \in \Delta\},$$

where we again identify $\mathbb{Z}^r$ with $\mathbb{Z}^r \times \{0\}^{n-r} \subseteq \mathbb{Z}^n$.

If $\varphi : F_\bullet(-2) \to F'_\bullet$ is any chain map lifting $S/(I_S(G') : x_{r+1}x_n)(-2) \overset{x_{r+1}x_n}{\longrightarrow} S/I_{S'}(G')S$, then as observed in [Bou10, 2.1.1], the mapping cone of φ provides a minimal free resolution of $S/I_S(G)$ since every multidegree $\mathbf{a} = (a_1, \ldots, a_n)$ appearing as a shift in $F'_\bullet$ must have $a_n = 0$. Since the i-th free module in the mapping cone of φ is just $F'_i \oplus F_{i-1}$ for $i > 0$ and S if $i = 0$, it follows that

$$\Sigma(G) = \Sigma(G') \cup (\Sigma(G \smallsetminus N[v]) + \Delta + \mathbf{e}_{r+1} + \mathbf{e}_n).$$

Hence, Theorem 2.10 and Proposition 2.3 imply that

$$V_E(G) = \left(\bigcup_{\mathbf{a} \in \Sigma(G')} V(x_i, x_n \mid a_i = 0) \right) \cup \left(\bigcup_{\mathbf{b} \in \Sigma(G \smallsetminus N[v])} V(x_i \mid b_i = 0) \right)$$

$$= (V_E(G \smallsetminus w) \cap V(x_n)) \cup V_E(G \smallsetminus N[v]),$$

so that

$$\operatorname{depth}_E G = \operatorname{codim}_{E_1} V_E(G)$$
$$= \min\{\operatorname{codim}_{E_1}(V_E(G \smallsetminus w) \cap V(x_n)), \operatorname{codim}_{E_1} V_E(G \smallsetminus N[v])\}$$
$$= \min\{\operatorname{depth}_E(G \smallsetminus w) + 1, \operatorname{depth}_E(G \smallsetminus N[v])\}. \qquad \square$$

As a consequence of the preceding theorem, we recover the following result of McCullough and Mere on the singular varieties of path graphs.

Corollary 3.2 ([MM22, 4.3]). *Let P_n denote the path graph on $n \geq 1$ vertices with edges $v_i v_{i+1}$ for all $i < n$.*

(a) *If $n \equiv 1 \pmod 3$, then $\operatorname{depth}_E P_n = 1$ and*

$$V_E(P_n) = \bigcup_{i \equiv 1 \,(\mathrm{mod}\, 3)} V(x_i).$$

(b) *If $n \not\equiv 1 \pmod 3$, then $\operatorname{depth}_E P_n = 0$ and $V_E(P_n) = E_1$.*

Proof. For the $n \leq 3$, the result is a straightforward computation, which we omit. If $n \geq 4$ and the result holds for $k < n$, then applying the preceding theorem with $w = v_n$ yields $V_E(P_n) = (V_E(P_{n-1}) \cap V(x_n)) \cup V_E(P_{n-3})$. If $n \not\equiv 1 \pmod 3$, then $V_E(P_{n-3}) = E_1$ by Proposition 2.3 and induction so that $V_E(P_n) = E_1$ and $\operatorname{depth}_E P_n = 0$. If $n \equiv 1 \pmod 3$, then

$$V_E(P_{n-3}) = \bigcup_{\substack{i \equiv 1 \,(\mathrm{mod}\, 3) \\ i < n}} V(x_i) \qquad \text{and} \qquad V(P_{n-1}) = E_1$$
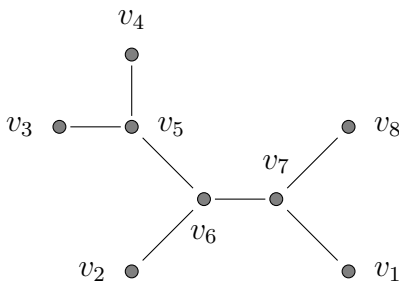
so that

$$V_E(P_n) = V(x_n) \cup \left(\bigcup_{\substack{i \equiv 1 \,(\mathrm{mod}\, 3) \\ i < n}} V(x_i) \right) = \bigcup_{i \equiv 1 \,(\mathrm{mod}\, 3)} V(x_i),$$

whence $\operatorname{depth}_E P_n = 1$. $\qquad \square$

Since every tree has a degree one vertex, we also note that the above theorem can be used to recursively compute the depth or singular variety of any tree.

Example 3.3. Consider the tree G shown below.

The table in Figure 3.1 shows two sequences of subgraphs obtained by deleting one whisker and its closed neighborhood at a time until we obtain a pair of paths for which the depth is known by Corollary 3.2. We then apply Theorem 3.1 to recursively compute the depth of the original tree. In this case, we have strategically chosen the vertices to delete so that the

i	$G_i = G_{i-1} \smallsetminus v_i$	$\operatorname{depth}_E G_i$	$G'_i = G_{i-1} \smallsetminus N[v_i]$	$\operatorname{depth}_E G'_i$
0		1	–	–
1		1		1
2		0		3
3		0		0

FIGURE 3.1. Computing Depth by Trimming Whiskers

graphs G'_i have immediately recognizable depth, but in general, one may obtain a repeatedly branching tree of subgraphs.

The following modification of the depth equality in the theorem will be useful later.

Corollary 3.4. *Let G be a graph with a vertex w of degree one, and let v be the unique vertex adjacent to w. Then*

$$\operatorname{depth}_E G = \min\{\operatorname{depth}_E(G \smallsetminus v), \operatorname{depth}_E(G \smallsetminus N[v])\}.$$

Proof. By Proposition 2.6, we know that

$$\operatorname{depth}_E G \geq \min\{\operatorname{depth}_E(G \smallsetminus v), \operatorname{depth}_E(G \smallsetminus N[v])\}.$$

In the case where $\operatorname{depth}_E(G \smallsetminus v)$ and $\operatorname{depth}_E(G \smallsetminus N[v])$ are different, we get the desired result via Corollary 2.2.

So, suppose $\operatorname{depth}_E(G \smallsetminus v) = \operatorname{depth}_E(G \smallsetminus N[v])$. Applying Proposition 2.6 again to the graph $G \smallsetminus w$, we have

$$\operatorname{depth}_E(G \smallsetminus w) \geq \min\{\operatorname{depth}_E(G \smallsetminus \{v, w\}), \operatorname{depth}_E(G \smallsetminus N[v])\}.$$

Since $G \smallsetminus v = (G \smallsetminus \{v, w\}) \sqcup \{w\}$, we have $\operatorname{depth}_E(G \smallsetminus v) = \operatorname{depth}_E(G \smallsetminus \{v, w\}) + 1$. Since the depths of $G \smallsetminus v$ and $G \smallsetminus N[v]$ were assumed to be equal and $G \smallsetminus \{v, w\}$ has depth one less than that of $G \smallsetminus v$, we get that $\operatorname{depth}_E(G \smallsetminus w) = \operatorname{depth}_E(G \smallsetminus v) - 1$ by Corollary 2.2. Finally, we use Theorem 3.1 to calculate the depth of G:

$$\operatorname{depth}_E G = \min\{\operatorname{depth}_E(G \smallsetminus w) + 1, \operatorname{depth}_E(G \smallsetminus N[v])\}$$
$$= \min\{\operatorname{depth}_E(G \smallsetminus v), \operatorname{depth}_E(G \smallsetminus N[v])\}. \qquad \square$$

A **spider graph** is a tree that has exactly one vertex of degree greater than 2. This vertex will be called the **head**. Since no other vertex can have degree bigger than 2, the spider will have induced paths coming off from the head. We will call these paths the **legs**.

Denote by $SP_{n_1,\ldots,n_k}$ a spider with legs of lengths $1 \leq n_1 \leq n_2 \leq \cdots \leq n_k$, and label the vertices of the leg of length n_r as $v_0 = v_{r,0}, v_{r,1}, \ldots, v_{r,n_r}$ where v_0 is the head of the spider.

Theorem 3.5. *Let $SP_{n_1,\ldots,n_k}$ be a spider graph, and let $E = \bigwedge_{\Bbbk} \langle e_0, e_{r,i} \mid 1 \leq r \leq k, 1 \leq i \leq n_r \rangle$ be the corresponding exterior algebra. Then*

$$V_E(SP_{n_1,\ldots,n_k}) = \left(\bigcap_{n_r \equiv 1 \,(\mathrm{mod}\ 3)} \bigcup_{i \equiv 1 \,(\mathrm{mod}\ 3)} V(x_{r,i}) \right) \cup \left(\bigcap_{n_r \equiv 2 \,(\mathrm{mod}\ 3)} \bigcup_{i \equiv 2 \,(\mathrm{mod}\ 3)} V(x_{r,i}) \right),$$

and $\operatorname{depth}_E SP_{n_1,\ldots,n_k} = \min\{p, q\}$, where p is the number of legs whose length is congruent to 1 mod 3 and q is the number of legs whose length is congruent 2 mod 3.

Proof. Set $G = SP_{n_1,\ldots,n_k}$. If $n_r = 1$ for all r, then $G = K_{1,k}$ is a complete bipartite graph, and the statement of the theorem asserts that $\operatorname{depth}_E G = 0$ and $V_E(G) = E_1$ since the rightmost intersection is an intersection over an empty collection. This case follows from Corollary 3.10, so we may assume that $n_r \geq 2$ for some r.

By deleting the head v_0 of the spider, we obtain an exact sequence of E-modules as (2.1) where $G \smallsetminus v_0 = P_{n_1} \sqcup \cdots \sqcup P_{n_k}$ and $G \smallsetminus N[v_0] = P_{n_1-1} \sqcup \cdots \sqcup P_{n_k-1}$. By combining Proposition

2.7, Proposition 2.3, and Corollary 3.2, we compute the respective singular varieties

$$V_E(G \smallsetminus v_0) = \bigcap_{r=1}^{k} V_E(P_{n_r}) = \bigcap_{n_r \equiv 1 \,(\mathrm{mod}\,3)} V_E(P_{n_r}) = \bigcap_{n_r \equiv 1 \,(\mathrm{mod}\,3)} \left(\bigcup_{i \equiv 1 \,(\mathrm{mod}\,3)} V(x_{r,i}) \right),$$

and

$$V_E(G \smallsetminus N[v_0]) = \bigcap_{r=1}^{k} V_E(P_{n_r - 1}) = \bigcap_{n_r \equiv 2 \,(\mathrm{mod}\,3)} V_E(P_{n_r - 1}) = \bigcap_{n_r \equiv 2 \,(\mathrm{mod}\,3)} \left(\bigcup_{i \equiv 2 \,(\mathrm{mod}\,3)} V(x_{r,i}) \right),$$

where we choose the variables $x_{r,i}$ with $i \equiv 2 \pmod 3$ in the last equality since the r-th component of $G \smallsetminus N[v_0]$ has vertices labeled by $v_{r,2}, v_{r,3}, \ldots, v_{r,n_r}$.

In the former case, we can rewrite $V_E(G \smallsetminus v_0)$ as a union of linear subspaces determined by the vanishing of p variables x_{r,i_r} where for each r with $n_r \equiv 1 \pmod 3$ we choose one $i_r \le n_r$ with $i_r \equiv 1 \pmod 3$. Similarly, we can rewrite $V_E(G \smallsetminus N[v_0])$ as a union of linear subspaces determined by the vanishing of q variables x_{r,i_r} where for each r with $n_r \equiv 2 \pmod 3$ we choose one $i_r \le n_r$ with $i_r \equiv 2 \pmod 3$. By Theorem 2.1, we know that $V_E(G) \subseteq V_E(G \smallsetminus v_0) \cup V_E(G \smallsetminus N[v_0])$. However, we also know that $V_E(G \smallsetminus v_0) \subseteq V_E(G) \cup V_E(G \smallsetminus N[v_0])$, and since it is clear that none of the linear subspaces whose union is $V_E(G \smallsetminus v_0)$ is contained in any of the linear subspaces whose union is $V_E(G \smallsetminus N[v_0])$, we must have $V_E(G \smallsetminus v_0) \subseteq V_E(G)$. An analogous argument shows that $V_E(G \smallsetminus N[v_0]) \subseteq V_E(G)$ as well so that

$$V_E(G) = V_E(G \smallsetminus v_0) \cup V_E(G \smallsetminus N[v_0]).$$

Finally, because every component of $V_E(G \smallsetminus v_0)$ has codimension p and every component of $V_E(G \smallsetminus N[v_0])$ has codimension q, it follows that

$$\mathrm{depth}_E G = \mathrm{codim}_E V_E(G) = \min\{p, q\}. \qquad \square$$

Example 3.6. The graph $SP_{1,2,3,4}$ is shown below.
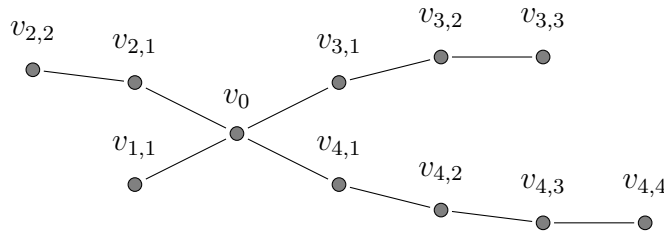
Since $SP_{1,2,3,4}$ has 2 legs of length congruent to 1 mod 3 and 1 leg of length congruent to 2 mod 3, the previous proposition shows that $\mathrm{depth}_E SP_{1,2,3,4} = \min(2, 1) = 1$ and

$$V_E(SP_{1,2,3,4}) = V(x_{1,1}) \cap (V(x_{4,1}) \cup V(x_{4,4})) \cup (V(x_{2,2}))$$
$$= V(x_{1,1}, x_{4,1}) \cup V(x_{1,1}, x_{4,4}) \cup V(x_{2,2}).$$

Having seen how to compute the depth and singular varieties of trees, we now turn our attention to cycles.

Theorem 3.7. *Let C_n denote a cycle graph on $n \geq 3$ vertices. Then $\operatorname{depth}_E C_n = 0$.*

Proof. We assume that the vertices $v_1, \ldots, v_n$ of C_n have been labeled so that the edges are $v_i v_{i+1}$ for $i < n$ and $v_1 v_n$, and we consider cases based on the residue of n mod 3. If $n \equiv 1$ (mod 3) or $n \equiv 2$ (mod 3), applying Proposition 2.6 to the deletion of the vertex v_n yields

$$\operatorname{depth}_E C_n \geq \min\{\operatorname{depth}_E P_{n-1}, \operatorname{depth} P_{n-3}\}.$$

When $n \equiv 1$ (mod 3), Corollary 3.2 yields that $\operatorname{depth}_E P_{n-3} = 1$ and $\operatorname{depth}_E P_{n-1} = 0$. Consequently, $\operatorname{depth}_E C_n = 0$ by Corollary 2.2. When $n \equiv 2$ (mod 3), $\operatorname{depth}_E P_{n-3} = 0$ and $\operatorname{depth}_E P_{n-1} = 1$, so the result follows similarly.

Now, suppose $n \equiv 0$ (mod 3). We may further assume that $n \geq 6$, since the case $n = 3$ is handled by Corollary 3.10. Let G denote the graph obtained from C_n by adding the edge $v_1 v_3$. Applying Proposition 2.6 to the deletion of the edge $v_1 v_3$ yields

$$\operatorname{depth}_E C_n \geq \min\{\operatorname{depth}_E P_{n-5}, \operatorname{depth}_E G\}.$$

Since $\operatorname{depth}_E P_{n-5} = 1$ by Corollary 3.2, we aim to show $\operatorname{depth}_E G = 0$ in order to prove that $\operatorname{depth} C_n = 0$ by Corollary 2.2. Applying Proposition 2.6 again to the deletion of the edge $v_1 v_n$ yields

$$\operatorname{depth}_E G \geq \min\{\operatorname{depth}_E H, \operatorname{depth}_E P_{n-5}\},$$

where $H = G \setminus v_1 v_n$ is the following graph.
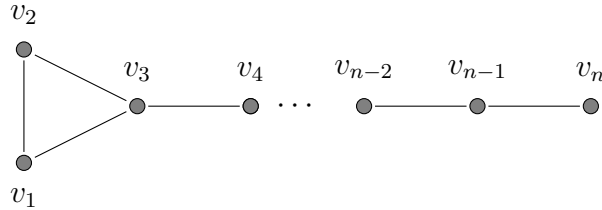
Hence, it further suffices to show that $\operatorname{depth}_E H = 0$. Deleting the vertex v_4 from H yields

$$\operatorname{depth}_E H \geq \min\{\operatorname{depth}_E(P_2 \sqcup P_{n-5}), \operatorname{depth}_E(C_3 \sqcup P_{n-4})\}.$$

By Proposition 2.7, Corollary 3.2, and Corollary 3.10, we see that $\operatorname{depth}_E(P_2 \sqcup P_{n-5}) = 0 + 1 = 1$ and $\operatorname{depth}_E(C_3 \sqcup P_{n-4}) = 0 + 0 = 0$. Hence, $\operatorname{depth}_E H = 0$ by Corollary 2.2 as wanted. $\square$

3.2. Duplicating and Coning.

A vertex of a graph which is connected to every other vertex is called **universal**, and graphs with a universal vertex are called **cones**. Proposition 3.8 states that all cones have depth 0. Special cases of cones include complete graphs, wheel graphs, and windmill graphs.

Proposition 3.8. *If G is a graph with at least two vertices having a universal vertex v, then $\operatorname{depth}_E G = 0$.*

Proof. Let u be any vertex of G different from v, and let H denote the graph obtained by adding a whisker vw to the graph $G \smallsetminus uv$. By Theorem 3.1, we see that

$$\operatorname{depth}_E H = \min\{\operatorname{depth}_E(G \smallsetminus uv) + 1, \operatorname{depth}_E\{u\}\} = 1.$$

We note that G is isomorphic to the graph obtained from H by identifying the vertices u and w. Since $N[w] \cap N[u] = \varnothing$ and every vertex in $N(w) = \{v\}$ is adjacent to every vertex in $N(v)$ by assumption, Corollary 2.9 implies that $\operatorname{depth}_E H \geq \operatorname{depth}_E G + 1$. Since $\operatorname{depth}_E H = 1$, it follows that $\operatorname{depth}_E G = 0$. $\qquad\square$

A vertex v' is called a **duplicate** of another vertex v if $N(v') = N(v)$.

Proposition 3.9. *If G is a graph with a duplicate vertex v and $\operatorname{depth}_E(G \smallsetminus v) = 0$, then $\operatorname{depth}_E G = 0$.*

Proof. If $H = G \smallsetminus v$ and w denotes a duplicate of v in G, we note that $G \smallsetminus N[v] = (H \smallsetminus N[w]) \sqcup \{w\}$. Applying Proposition 2.6 to the deletion of v as well as Proposition 2.7 yields

$$\operatorname{depth}_E G \geq \min\{\operatorname{depth}_E(H \smallsetminus N[w]) + 1, \operatorname{depth}_E H\}.$$

Since $\operatorname{depth}_E H = 0$ by hypothesis, Corollary 2.2 implies that $\operatorname{depth}_E G = \operatorname{depth}_E H = 0$ as desired. $\qquad\square$

A **complete graph** with n vertices is a graph K_n in which every vertex is universal. A **complete r-partite graph** $K_{n_1, \ldots, n_r}$ is a graph whose vertices can be partitioned into r disjoint sets of cardinalities $n_1, \ldots, n_r$ so that no two vertices in the same set are adjacent, and there is an edge between every pair of vertices from different sets. Complete multipartite graphs include complete graphs since $K_{1, \ldots, 1} = K_r$ and **star graphs**, which are trees of the form $K_{1,n}$.

By duplicating a universal vertex, we can combine the previous two results to obtain the following.

Corollary 3.10. *For $K_{n_1, \ldots, n_r} \neq K_1$, we have $\operatorname{depth}_E K_{n_1, \ldots, n_r} = 0$.*

Proof. We prove the claim by induction on the size of n_1, and we note that the base case is handled by Proposition 3.8 since $K_{n_1, \ldots, n_r} \neq K_1$. So assume $n_1 \geq 2$ and that the corollary holds for all complete multipartite graphs $K_{m_1, \ldots, m_s}$ with $m_1 < n_1$. If v is a vertex in the set of size n_1 in the partition of the vertices of $K_{n_1, \ldots, n_r}$, then v is a duplicate vertex since $n_1 \geq 2$ and $K_{n_1, \ldots, n_r} \smallsetminus v = K_{n_1 - 1, n_2, \ldots, n_r}$ has depth 0 by induction, so the result follows from Proposition 3.9. $\qquad\square$

3.3. **Whiskering.** Given a graph G, define $W(G, m)$ to be the graph formed by attaching m new degree 1 vertices to each vertex of G. Then $W(G, m)$ is called a **whiskering** of G and the degree 1 vertices (or their incident edges) are called **whiskers**.

Theorem 3.11. *Let G be a graph on n vertices, and let $m > 0$. Then*

$$\operatorname{depth}_E W(G, m) = m \operatorname{depth}_E W(G, 1).$$

Proof. We will argue by induction on $n \geq 0$. When $n = 0$, both $W(G, m)$ and $W(G, 1)$ are the empty graph, which has depth zero.

Now, suppose that $n \geq 1$ and that the result holds for all graphs with fewer than n vertices. Let v a vertex of G of degree d. Observe that by Corollary 3.4 we have

$$\operatorname{depth}_E W(G, 1) = \min\{\operatorname{depth}_E(W(G, 1) \smallsetminus v), \operatorname{depth}_E(W(G, 1) \smallsetminus N[v])\}$$

$$= \min\{1 + \operatorname{depth}_E(W(G \smallsetminus v, 1)), d + \operatorname{depth}_E(W(G \smallsetminus N[v], 1))\}.$$

Using Corollary 3.4 again and our inductive assumption, we then calculate

$$\operatorname{depth}_E W(G, m) = \min\{\operatorname{depth}_E(W(G, m) \smallsetminus v), \operatorname{depth}_E(W(G, m) \smallsetminus N[v])\}$$

$$= \min\{m + \operatorname{depth}_E W(G \smallsetminus v, m), dm + \operatorname{depth}_E W(G \smallsetminus N[v], m)\}$$

$$= \min\{m + m \operatorname{depth}_E W(G \smallsetminus v, 1), dm + m \operatorname{depth}_E W(G \smallsetminus N[v], 1)\}$$

$$= m \min\{1 + \operatorname{depth}_E W(G \smallsetminus v, 1), d + \operatorname{depth}_E W(G \smallsetminus N[v], 1)\}$$

$$= m \operatorname{depth}_E W(G, 1). \qquad \square$$

Given a graph G, a set of vertices of G is an **independent set** if no two vertices in the set are adjacent. The **independence number** $\alpha(G)$ is the largest number of vertices in any independent subset of G.

Theorem 3.12. *Let G be a graph with n vertices. Then $\operatorname{depth}_E W(G, 1) = n - \alpha(G)$.*

Proof. We will argue by induction on $n \geq 0$. When $n = 0$, $W(G, 1)$ is the empty graph so that $\operatorname{depth}_E W(G, 1) = 0 = 0 - 0 = n - \alpha(G)$.

Now, suppose that $n \geq 1$ and that the result holds for all graphs with fewer than n vertices. Let A be a maximal independent set of G, and let v be a vertex in A with degree d in G. By Corollary 3.4, we have

$$\operatorname{depth}_E W(G, 1) = \min\{\operatorname{depth}_E(W(G, 1) \smallsetminus v), \operatorname{depth}_E(W(G, 1) \smallsetminus N[v])\}$$

$$= \min\{1 + \operatorname{depth}_E W(G \smallsetminus v, 1), d + \operatorname{depth}_E W(G \smallsetminus N[v], 1)\}$$

$$= \min\{1 + n - 1 - \alpha(G \smallsetminus v), d + n - d - 1 - \alpha(G \smallsetminus N[v])\}$$

$$= \min\{n - \alpha(G \smallsetminus v), n - 1 - \alpha(G \smallsetminus N[v])\}$$

$$= n - \max\{\alpha(G \smallsetminus v), 1 + \alpha(G \smallsetminus N[v])\}.$$

If A' is an independent subset of $G \smallsetminus v$, then it is also independent in G because $G \smallsetminus v$ is an induced subgraph of G, so $\alpha(G \smallsetminus v) \leq \alpha(G)$. If A'' is independent in $G \smallsetminus N[v]$, then

$A'' \cup \{v\}$ is independent in G, so $1 + \alpha(G \smallsetminus N[v]) \leq \alpha(G)$. Further, $A \smallsetminus \{v\}$ is independent in $G \smallsetminus N[v]$ so $1 + \alpha(G \smallsetminus N[v]) = \alpha(G)$. Hence

$$\operatorname{depth}_E W(G, 1) = n - \max\{\alpha(G \smallsetminus v), 1 + \alpha(G \smallsetminus N[v])\} = n - \alpha(G). \qquad \square$$

A **vertex cover** of a graph G is a set C of vertices of G such that every edge has one of its ends in G. A vertex cover C is **minimal** if no proper subset of C is a vertex cover. It is easy to see that $C = \{i_1, \ldots, i_r\}$ is a minimal vertex cover of G if and only if the prime ideal $P_C = (x_{i_1}, \ldots, x_{i_r}) \subseteq S$ is a minimal prime of $I_S(G)$ [HH11, 9.1.4]. The **vertex cover number** $\beta(G)$ is the smallest number of vertices in any vertex cover of G. Hence, $\operatorname{ht} I_S(G) = \beta(G)$. As a consequence of [Wes96, 3.1.21], we have the following corollary.

Corollary 3.13. *For any graph G, we have $\operatorname{depth}_E W(G, 1) = \operatorname{ht} I_S(G)$.* $\qquad \square$

Since $W(G, 1)$ is Cohen-Macaulay [Vil90, 2.2], we also recover a special case of a purely commutative result of Biermann and Van Tuyl [BVT13, 4.7] which gives the regularity of whiskered graphs.

Corollary 3.14. *For any graph G, we have $\operatorname{reg} W(G, 1) = \dim S/I_S(G)$.*

Proof. Let n be the number of vertices of G, and let S' denote the polynomial ring of $W(G, 1)$ in $2n$ variables. Applying Theorem 2.10, the previous corollary, and the fact that $W(G, 1)$ is Cohen-Macaulay gives

$$\operatorname{ht} I_S(G) = \operatorname{depth}_E W(G, 1) = 2n - \operatorname{pd}_{S'} W(G, 1) - \operatorname{reg} W(G, 1).$$

The set of vertices of G is a minimal vertex cover, and since $W(G, 1)$ is Cohen-Macaulay, all minimal vertex covers have the same size so that $\operatorname{pd}_{S'} W(G, 1) = \operatorname{ht} I_{S'}(W(G, 1)) = n$. And so, it follows that $\operatorname{reg} W(G, 1) = n - \operatorname{ht} I_S(G) = \dim S/I_S(G)$. $\qquad \square$

Example 3.15. As a consequence of Theorem 3.12, we can compute the depth of every Cohen-Macaulay tree G, as every such tree is of the form $W(G_0, 1)$ for some tree G_0 [Vil90, 2.4, 2.5]. For example, the tree G below is Cohen-Macaulay since it is the whiskering of the tree G_0 on the vertices $v_1, \ldots, v_{10}$, but it is not a spider.
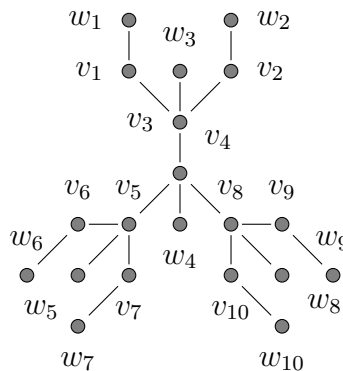
We have $\operatorname{depth}_E G = \beta(G_0) = 3$ since $\{v_3, v_5, v_8\}$ is a vertex cover of G_0 of minimum size.

Corollary 3.16. *Let $n \geq 1$. Then:*

(a) $\operatorname{depth}_E W(P_n, 1) = \left\lfloor \frac{n}{2} \right\rfloor$.

(b) $\operatorname{depth}_E W(C_n, 1) = \left\lceil \frac{n}{2} \right\rceil$ *for $n \geq 3$.*

(c) $\operatorname{depth}_E W(K_n, 1) = n - 1$.

Proof. (a) If P_n has edges $v_i v_{i+1}$ for each $i < n$, then every independent set of P_n is of the of the form $\{v_{i_1}, \ldots, v_{i_t}\}$ where $1 \leq i_1 \leq \cdots \leq i_t \leq n$ and $i_{j+1} \geq i_j + 2$ for all $j < t$. From this, it easily follows that $t \leq \left\lfloor \frac{n+1}{2} \right\rfloor = \left\lceil \frac{n}{2} \right\rceil$. In particular, we see that $\{v_1, v_3, \ldots, v_{2t-1}\}$ is independent for $t = \left\lceil \frac{n}{2} \right\rceil$ so that $\alpha(P_n) = \left\lceil \frac{n}{2} \right\rceil$ and $\operatorname{depth}_E W(P_n, 1) = n - \left\lceil \frac{n}{2} \right\rceil = \left\lfloor \frac{n}{2} \right\rfloor$ by Theorem 3.12.

(b) If A is an independent set of C_n, there is at least one vertex v not in A so that A is independent in $C_n \smallsetminus v = P_{n-1}$. As we already observed that $\alpha(C_n \smallsetminus v) \leq \alpha(C_n)$ in the proof of Theorem 3.12, it follows that $\alpha(C_n) = \alpha(P_{n-1}) = \left\lfloor \frac{n}{2} \right\rfloor$ so that $\operatorname{depth}_E W(C_n, 1) = n - \left\lfloor \frac{n}{2} \right\rfloor = \left\lceil \frac{n}{2} \right\rceil$.

(c) This follows immediately from the fact that every maximal independent set of K_n consists of only one vertex. $\qquad \square$

Example 3.17. The graph $W(K_4, 3)$ is shown below.
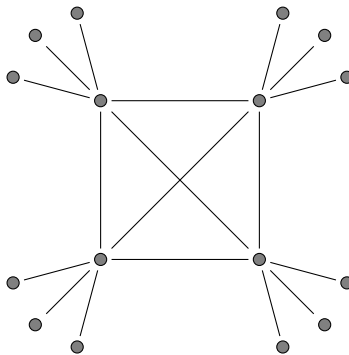
Combining the preceding corollary with Theorem 3.11, we see that $\operatorname{depth}_E W(K_4, 3) = (4 - 1)3 = 9$. The next section will show that this graph has maximal depth among all graphs with 16 vertices (and more generally, that $W(K_m, m - 1)$ has maximal depth among all graphs with m^2 vertices).

We now consider the behavior of the depth of a graph upon iteratively whiskering every vertex at least twice. Given a graph G and integers $m_1, \ldots, m_k > 0$, we define the graph $W(G, m_1, \ldots, m_k)$ recursively by

$$W(G, m_1, \ldots, m_k) := W(W(G, m_1, \ldots, m_{k-1}), m_k).$$

When $m_i = m$ for all i, we also denote this graph by simply $W^k(G, m)$. The following result shows that the depth of these graphs is solely dependent on the number of vertices of the original graph and the number of whiskers added.

Corollary 3.18. *Let G be a graph on n vertices, and let $m_1, m_2, \ldots, m_k > 0$ for some $k \geq 2$. Then*

$$\operatorname{depth}_E W(G, m_1, \ldots, m_k) = n(m_1 + 1)(m_2 + 1) \cdots (m_{k-2} + 1)m_k.$$

In particular, for any $m > 0$ and $k \geq 2$, we have

$$\operatorname{depth}_E W^k(G, m) = nm(m + 1)^{k-2}.$$

Proof. Suppose that $k = 2$. By Theorem 3.11, we may assume that $m_2 = 1$. Hence, it suffices to show that $\operatorname{depth}_E W(G, m_1, 1) = n$. If A is a maximal independent set of $W(G, m_1)$, we can partition A into sets A_v for each vertex v of G, where A_v is the set of vertices in A equal to v or a whisker of v in $W(G, m_1)$. If $v \in A_v$, then $|A_v| = 1$, and otherwise, the maximality of A forces $|A_v| = m_1$. From this, it follows that the set of whiskers of $W(G, m_1)$ is the unique independent set of maximum size. And so, Theorem 3.12 yields $\operatorname{depth}_E W(G, m_1, 1) = n$. The general case follows immediately by a simple induction on k after applying the $k = 2$ case to the graph $G' = W(G, m_1, \ldots, m_{k-2})$, which is easily seen to have $n(m_1 + 1) \cdots (m_{k-2} + 1)$ vertices. $\square$

Remark 3.19. While the preceding corollary shows that for any two graphs G, G' on n vertices, $\operatorname{depth}_E W(G, m_1, m_2) = \operatorname{depth}_E W(G', m_1, m_2)$, it is not true that $\operatorname{depth}_E W(G, m_1) = \operatorname{depth}_E W(G', m_1)$. For example, $\operatorname{depth}_E W(P_4, 1) = 2$ while $\operatorname{depth}_E W(C_4, 1) = 3$ by Proposition 3.16. This is contrary to the case over a polynomial ring where $\operatorname{depth}_S W(G, 1) = n$ for any graph G on n vertices since $W(G, 1)$ is always Cohen-Macaulay [Vil90, 2.2].

4. Bounds on Depth of Edge Ideals

In this section, we prove some upper bounds on the depth of edge ideals. The first main result is a tight upper bound on the depth of arbitrary graphs. Then we prove a more refined bound for bipartite graphs.

Let G be a graph. A subset of pairwise disjoint edges is a **matching**; if the edges in a matching form an induced subgraph of G, it is an **induced matching**. The maximum size of an induced matching of G is the **induced matching number**, denoted $\operatorname{im}(G)$. The maximal size of a minimal vertex cover is denoted $\tau_{\max}(G)$. In particular, for any minimal vertex cover C of a graph G, we have

$$|C| \leq \tau_{\max}(G) = \operatorname{bight} I_S(G) \leq \operatorname{pd}_S S/I_S(G).$$

Next, we give a general upper bound on the depth of exterior and symmetric algebra edge ideals, the first of which recovers [HH21, 4.2].

Theorem 4.1. *Let G be a graph on n vertices, none of which are isolated. Then*

$$\operatorname{depth}_S G \leq n + 2 - 2\sqrt{n}, \qquad\qquad \operatorname{pd}_S G \geq 2\sqrt{n} - 2,$$

$$\operatorname{depth}_E G \leq n + 1 - 2\sqrt{n}, \qquad\qquad \operatorname{cx}_E G \geq 2\sqrt{n} - 1.$$

Moreover, when n is a square, both sets of bounds are tight.

Proof. By [DS13, 4.7], $\operatorname{pd}_S G \geq n - \operatorname{im}(G)$. That $\operatorname{im}(G) \leq n + 2 - 2\sqrt{n}$ was noted first in [Fav88] and follows from [BC79, Theorem 6]. Combining these yields $\operatorname{pd}_S G \geq 2\sqrt{n} - 2$. By the Auslander-Buchsbaum Formula [Pee11, 15.3], $\operatorname{depth}_S G \leq n + 2 - 2\sqrt{n}$. The latter two inequalities follow from Corollary 2.12 and Theorem 2.4. That both bounds are tight follows from the proof of Corollary 4.2. □

Corollary 4.2. *Suppose G is a graph on $n = m^2$ vertices with no isolated vertices. Then* $\operatorname{depth}_E G = n + 1 - 2\sqrt{n}$ *if and only if* $G = W(K_m, m-1)$.

Proof. Suppose $\operatorname{depth}_E G = n + 1 - 2\sqrt{n} = m^2 - 2m + 1$. By Corollary 2.12 and Theorem 4.1, $\operatorname{depth}_S G = m^2 - 2m + 2$, and so, $\operatorname{pd}_S G = 2m - 2$. By [HH21, 3.1], $\tau_{\max}(G) = 2m - 2$ and hence by [HH21, 5.4], G is one of the following graphs:

(1) $P_2 \sqcup P_2$
(2) C_4
(3) $W(K_m, m-1)$.

By Theorem 3.7, $\operatorname{depth}_E C_4 = 0 \neq 1$, so we may disregard C_4. That $\operatorname{depth}_E P_2 \sqcup P_2 = 0 \neq 1$ follows from Proposition 2.7 and Corollary 3.2. Finally, $\operatorname{depth}_E W(K_m, m-1) = (m-1)^2$ by Theorem 3.11 and Corollary 3.16, completing the proof. □

Next, we focus on the depth of bipartite graphs where stronger statements are possible. We first compute the depth of Ferrers graphs. Let $\lambda_1 > \lambda_2 > \cdots > \lambda_m > 0$ be a sequence of positive integers. The associated **Ferrers graph** is the bipartite graph on vertex set $\{v_1, \ldots, v_m\} \sqcup \{w_1, \ldots, w_{\lambda_1}\}$ and edge set $\{\{v_i, w_j\} \mid 1 \leq i \leq m \text{ and } 1 \leq j \leq \lambda_i\}$. Ferrers graphs are exactly those bipartite graphs with the property that, after potentially reordering the vertices, if $\{v_i, w_j\}$ is an edge then so is $\{v_k, w_l\}$ for all $1 \leq k \leq i$ and $1 \leq l \leq j$. Corso and Nagel characterized Ferrers graphs as exactly the bipartite graphs whose symmetric edge ideals have linear free resolutions in [CN09, 4.2]. Using this, we compute the depths of edge ideals of Ferrers graphs.

Proposition 4.3. *Suppose G is a Ferrers graph associated to a sequence of integers $\lambda_1 > \lambda_2 > \cdots > \lambda_m > 0$. Then*

$$\operatorname{depth}_S G = m + \min_j\{\lambda_1 - \lambda_j - j + 1\}$$

and

$$\operatorname{depth}_E G = m + \min_j\{\lambda_1 - \lambda_j - j\}.$$

Proof. Let $n = \lambda_1 + m$ be the number of vertices of G. It follows from [CN09, 2.2] that $\mathrm{pd}_S G = \max_j\{\lambda_j + j - 1\}$. By the Auslander-Buchsbaum Formula, $\mathrm{depth}_S G = n - \mathrm{pd}_S G = m + \min_j\{\lambda_1 - \lambda_j - j + 1\}$. Since $I_S(G)$ has a linear free resolution, it follows from Corollary 2.12 that $\mathrm{depth}_E G = m + \min_j\{\lambda_1 - \lambda_j - j\}$.   □

Finally, we give a tight upper bound on the depth of bipartite graphs.

**Theorem 4.4.** *Let G be a bipartite graph on n vertices none of which are isolated. Then*

$$\mathrm{depth}_S G \leq \left\lfloor \frac{n}{2} \right\rfloor, \qquad\qquad \mathrm{pd}_S G \geq \left\lceil \frac{n}{2} \right\rceil,$$

$$\mathrm{depth}_E G \leq \left\lfloor \frac{n}{2} \right\rfloor - 1, \qquad\qquad \mathrm{cx}_E G \geq \left\lceil \frac{n}{2} \right\rceil + 1.$$

Moreover, both sets of bounds are tight.

Proof. Let G be bipartite on vertex set $V = A \sqcup B$. Then A and B are minimal vertex covers of G. So $\left\lceil \frac{n}{2} \right\rceil \leq \max\{|A|, |B|\} \leq \mathrm{pd}_S G$, and again by the Auslander-Buchsbaum Formula, $\mathrm{depth}_S G = n - \mathrm{pd}_S G \leq n - \left\lceil \frac{n}{2} \right\rceil = \left\lfloor \frac{n}{2} \right\rfloor$. The second two inequalities again follow from Corollary 2.12 and Theorem 2.4. That both bounds are tight follows by considering the Ferrers graph associated to the sequence $\lambda_i = \left\lceil \frac{n}{2} \right\rceil - i + 1$ for $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$ and applying the previous proposition.   □

It seems difficult to completely characterize bipartite graphs G satisfying the equality $\mathrm{depth}_S G = \left\lfloor \frac{n}{2} \right\rfloor$. Indeed, such graphs include all Cohen-Macaulay bipartite graphs, Ferrers graphs with $\lambda_i \leq m - i + 1$ from the previous corollary, and at least some graphs which are neither. There are also non-Ferrers bipartite graphs satisfying $\mathrm{depth}_E G = \left\lfloor \frac{n}{2} \right\rfloor - 1$ such as the following.

Example 4.5. Let G be the following bipartite graph.
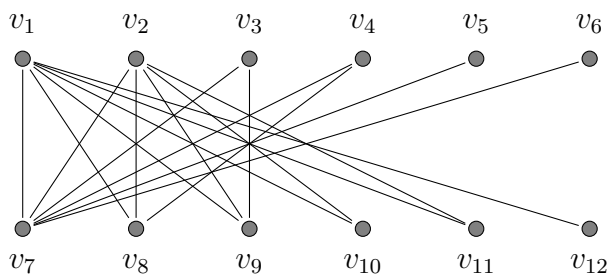
The corresponding edge ideal $I_S(G)$ has the following Betti table.

	0	1	2	3	4	5	6
0:	1	-	-	-	-	-	-
1:	-	17	50	66	47	18	3
2:	-	-	1	3	3	1	-

It follows from Corollary 2.12 that $\operatorname{depth}_S G = 6 = \frac{12}{2}$ and $\operatorname{depth}_E G = 5 = \frac{12}{2} - 1$. Yet, since $\operatorname{reg}_S G = 2$, G is not a Ferrers graph, and since $(x_1, x_2, x_7, x_8, x_9)$ is a minimal prime of height 5, $I_S(G)$ is not Cohen-Macaulay.

Acknowledgements

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