A REVISIT OF POPOVICIU INEQUALITY

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Abstract. Convex functions and their properties have acquired a large area in the literature. In this paper, we further explore the celebrated Popoviciu inequality, which relates the values of a convex function at three points in its domain. In particular, we present refinements, reverses, new Mercer-Popoviciu inequality, and some discussion of the monotony of quotients related to convex functions.

1. Introduction

Let J be a real interval. A function $f:J\to\mathbb{R}$ is said to be convex if it satisfies the simple inequality

$$f((1-t)x+ty)\le(1-t)f(x)+tf(y), \tag{1.1}$$

for all $x,y\in J$ and $0\le t\le 1$. The above inequality can be extended to n parameters via Jensen's inequality which states

$$f\left(\sum_{i=1}^{n}t_ix_i\right)\le\sum_{i=1}^{n}t_if(x_i), \tag{1.2}$$

where $f:J\to\mathbb{R}$ is convex, $x_i\in J, 0\le t_i\le 1$ and $\sum_{i=1}^{n}t_i=1$. The inequalities (1.1) and (1.2) have received considerable attention in the literature due to their significance in different fields, including mathematical inequalities, mathematical analysis, functional analysis, operator theory, probability, and mathematical physics. We refer the reader to [1, 2, 8, 9, 10, 12, 13, 14, 18, 19, 23, 28, 29] as a sample of references that treated these inequalities with possible applications.

Some inequalities about convex functions have been named after their founders, like the Jensen inequality, Mercer inequality, and Popoviciu inequality. In this paper, we are interested in Popoviciu inequality, which states [22]

$$\frac{2}{3}\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{z+x}{2}\right)\right]\le\frac{f\left(x\right)+f\left(y\right)+f\left(z\right)}{3}+f\left(\frac{x+y+z}{3}\right), \tag{1.3}$$

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where $f : J \to \mathbb{R}$ is convex and $x, y, z \in J$. A recent new extension of Popoviciu's inequality is found in [16]. Popoviciu's inequality can be regarded as a generalization of Hlawka's inequality using convexity as a simple geometry tool. If $f(t) = |t|$, then the Popoviciu inequality reduces to the celebrated Hlawka inequality

$$|x+y| + |y+z| + |z+x| \le |x| + |y| + |z| + |x+y+z|.$$

This seems to have appeared for the first time in a paper by Hornich [11]. Our goal in this paper is to explore Popoviciu inequality further. For this, we need some tools, as follows. The basic inequality (1.1) was refined and reversed in [5] by the forms

$$f((1-t)a+tb) \le (1-t)f(a) + tf(b) - 2r\left(\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right), \tag{1.4}$$

and

$$(1-t)f(a) + tf(b) \le f((1-t)a+tb) + 2R\left(\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right) \tag{1.5}$$

where $r = \min\{t, 1-t\}$, $R = \max\{t, 1-t\}$, and $0 \le t \le 1$. Here, $f : J \to \mathbb{R}$ is convex, $a, b \in J$ and $0 \le t \le 1$. We refer the reader to [6, 7, 17, 24, 25, 26, 27] for further discussion of convex inequalities, with applications in different contexts.

In what follows, we will present some results for differentiable convex functions. In this context, the reader is reminded of the inequality

$$f'(a)(b-a) + f(a) \le f(b) \tag{1.6}$$

is valid for the differentiable convex function $f : J \to \mathbb{R}$ and $a, b \in J$. This observation will be used to obtain a reversed version of Popoviciu inequality.

Further, when discussing inequalities for convex functions, the following observation becomes handy. If f is convex on $[m, M]$, and $t \in [m, M]$, then $t = \frac{M-t}{M-m}m + \frac{t-m}{M-m}M$. This, together with (1.1), implies

$$f(t) \le \frac{M-t}{M-m}f(m) + \frac{t-m}{M-m}f(M);\ m \le t \le M. \tag{1.7}$$

This observation has been used extensively in the literature, as seen in [3].

Popoviciu's inequality can be extended to a weighted version [4, Theorem 2b] in the following. We prove it by the use of (1.7) for convenience to the readers, although we can find a different proof in [21, Theorem 6.2].

Proposition 1.1. *Let $p, q, r > 0$ and $f : J \to \mathbb{R}$ be a convex function. For any three points $x, y, z \in J$, we have*

$$(p+q)f\left(\frac{px+qy}{p+q}\right)+(q+r)f\left(\frac{qy+rz}{q+r}\right)+(r+p)f\left(\frac{rz+px}{r+p}\right)$$
$$\le pf(x)+qf(y)+rf(z)+(p+q+r)f\left(\frac{px+qy+rz}{p+q+r}\right). \tag{1.8}$$

Proof. We may assume $x \le y \le z$ without loss of generality.

(i) For the case $x \le y \le \frac{px+qy+rz}{p+q+r}$, we have

$$\frac{px+qy+rz}{p+q+r} \le \frac{px+rz}{p+r} \le z \quad \text{and} \quad \frac{px+qy+rz}{p+q+r} \le \frac{qy+rz}{q+r} \le z.$$

Letting $m := \frac{px+qy+rz}{p+q+r}$, $M := z$ and $t := \frac{px+rz}{p+r}$ in (1.7), we have

$$f\left(\frac{px+rz}{p+r}\right) \le \frac{p+q+r}{p+r}\cdot\frac{p(z-x)}{p(z-x)+q(z-y)}f\left(\frac{px+qy+rz}{p+q+r}\right)$$
$$+\frac{q}{p+r}\cdot\frac{p(x-y)+r(z-y)}{p(z-x)+q(z-y)}f(z). \tag{1.9}$$

Letting $m := \frac{px+qy+rz}{p+q+r}$, $M := z$ and $t := \frac{qy+rz}{q+r}$ in (1.7), we have

$$f\left(\frac{qy+rz}{q+r}\right) \le \frac{p+q+r}{q+r}\cdot\frac{q(z-y)}{p(z-x)+q(z-y)}f\left(\frac{px+qy+rz}{p+q+r}\right)$$
$$+\frac{p}{q+r}\cdot\frac{q(y-x)+r(z-x)}{p(z-x)+q(z-y)}f(z). \tag{1.10}$$

Taking $(p+r)\times$ (1.9) + $(q+r)\times$ (1.10), we have

$$(p+r)f\left(\frac{px+rz}{p+r}\right)+(q+r)f\left(\frac{qy+rz}{g+r}\right) \le (p+q+r)f\left(\frac{px+qy+ez}{p+q+r}\right)+rf(z) \tag{1.11}$$

after calculations. We also have the following inequality by the convexity of f:

$$(p+q)f\left(\frac{px+qy}{p+q}\right) \le pf(x)+qf(y). \tag{1.12}$$

Adding both sides of (1.11) and (1.12), we get (1.8).

(ii) For the case $x \le \frac{px+qy+rz}{p+q+r} \le y$, we have

$$x \le \frac{px+rz}{p+r} \le \frac{px+qy+rz}{p+q+r} \quad \text{and} \quad x \le \frac{px+qy}{p+q} \le \frac{px+qy+rz}{p+q+r}.$$

Letting $m := x$, $M := \dfrac{px+qy+rz}{p+q+r}$ and $t := \dfrac{px+rz}{p+r}$ in (1.7), we have

$$f\left(\frac{px+rz}{p+r}\right) \le \frac{p+q+r}{p+r} \cdot \frac{r(z-x)}{r(z-x)+q(y-x)} f\left(\frac{px+qy+rz}{p+q+r}\right) + \frac{q}{p+r} \cdot \frac{p(y-x)+r(y-z)}{q(y-x)+r(z-x)} f(x). \tag{1.13}$$

Letting $m := x$, $M := \dfrac{px+qy+rz}{p+q+r}$ and $t := \dfrac{px+qy}{p+q}$ in (1.7), we have

$$f\left(\frac{px+qy}{p+q}\right) \le \frac{p+q+r}{p+q} \cdot \frac{q(y-x)}{r(z-x)+q(y-x)} f\left(\frac{px+qy+rz}{p+q+r}\right) + \frac{r}{p+q} \cdot \frac{p(z-x)+q(z-y)}{q(y-x)+r(z-x)} f(x). \tag{1.14}$$

Taking $(p+r)\times$ (1.13) + $(p+q)\times$ (1.14), we have

$$(p+r)f\left(\frac{px+rz}{p+r}\right) + (p+q)f\left(\frac{px+qy}{p+q}\right) \le (p+q+r)f\left(\frac{px+qy+ez}{p+q+r}\right) + pf(x) \tag{1.15}$$

after calculations. We also have the following inequality by the convexity of f:

$$(q+r)f\left(\frac{qy+rz}{q+r}\right) \le qf(y) + rf(z). \tag{1.16}$$

Adding both sides of (1.15) and (1.16), we get (1.8). This completes the proof.

□

The inequality (1.8) recovers the inequality (1.3) by putting $p = q = r$.

Remark 1.1. *If we replace t by $M+m-t$, in (1.7), we can write*

$$f(M+m-t) \le \frac{t-m}{M-m} f(m) + \frac{M-t}{M-m} f(M);\ m \le t \le M.$$

Assuming $x \le y \le z$, the fact

$$\frac{x+y+z}{3} \le \frac{x+z}{2} \le z,$$

implies

$$f\left(\frac{2y+5z-x}{6}\right) \le \frac{x+z-2y}{2(2z-x-y)} f\left(\frac{x+y+z}{3}\right) + \frac{3(z-x)}{2(2z-x-y)} f(z). \tag{1.17}$$

Further, if we let $m = \frac{x+y+z}{3}$ and $M = z$. If we set $t = \frac{x+z}{2}$, then (1.7) implies

$$f\left(\frac{x+z}{2}\right) \le \frac{3(z-x)}{2(2z-x-y)} f\left(\frac{x+y+z}{3}\right) + \frac{x+z-2y}{2(2z-x-y)} f(z). \tag{1.18}$$

Adding (1.17) and (1.18), we get

$$f\left(\frac{x+z}{2}\right) + f\left(\frac{2y+5z-x}{6}\right) \le f\left(\frac{x+y+z}{3}\right) + f(z). \tag{1.19}$$

Since

$$\frac{x+y+z}{3} \le \frac{y+z}{2} \le z,$$

we get

$$f\left(\frac{2x+5z-y}{6}\right) \le \frac{y+z-2x}{2(2z-x-y)} f\left(\frac{x+y+z}{3}\right) + \frac{3(z-y)}{2(2z-x-y)} f(z)\,. \tag{1.20}$$

On the other hand, if we put $t=\frac{y+z}{2}$, *then* (1.7) *implies*

$$f\left(\frac{y+z}{2}\right) \le \frac{3(z-y)}{2(2z-x-y)} f\left(\frac{x+y+z}{3}\right) + \frac{y+z-2x}{2(2z-x-y)} f(z)\,. \tag{1.21}$$

Adding (1.20) *and* (1.21), *we infer that*

$$f\left(\frac{y+z}{2}\right) + f\left(\frac{2x+5z-y}{6}\right) \le f\left(\frac{x+y+z}{3}\right) + f(z)\,. \tag{1.22}$$

Since

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2},$$

we have, by (1.19) *and* (1.22) *that*

$$\begin{aligned}&\frac{1}{2}\left[f\left(\frac{x+z}{2}\right) + f\left(\frac{2y+5z-x}{6}\right) + f\left(\frac{2x+5z-y}{6}\right) + f\left(\frac{y+z}{2}\right)\right] + 2f\left(\frac{x+y}{2}\right)\\ &\le f\left(\frac{x+y+z}{3}\right) + f(x) + f(y) + f(z)\,.\end{aligned}$$

So, a convex function satisfies this latter inequality. We post the question of whether this inequality is equivalent to the convexity of f.

We will also present variants of the celebrated Mercer inequality that states [15]

$$f(M+m-x) \le f(M) + f(m) - f(x), \tag{1.23}$$

for the convex function $f:[m.M]\to\mathbb{R}$ and $x\in[m,M]$. More precisely, we will present a refinement of this inequality using Popoviciu inequality.

Another useful observation about convex functions is that a convex function is not necessarily monotone. However, it is well known that if $f:J\to\mathbb{R}$ is convex, then for each $y\in J$, the function $x\mapsto \frac{f(y)-f(x)}{y-x}$ is non-decreasing in $x\in J$; see [19, Theorem 1.3.1]. This means that if $x,y,z\in J$ are such that $x<y<z$, then

$$\frac{f(y)-f(x)}{y-x} \le \frac{f(z)-f(x)}{z-x} \le \frac{f(z)-f(y)}{z-y}. \tag{1.24}$$

Interestingly, we will present a refinement and a reverse of this inequality using Popoviciu inequality.

2. Main Results

This section will present refinements, reverses, Popoviciu-Mercer inequality, and further results, with one connection among them all, namely the inequality (1.3).

As we mentioned earlier in the introduction, we aim to explore (1.3) and present variants that complement our understanding of this inequality. Below, we present a reversed version of Popoviciu inequality, where (1.6) is used.

Theorem 2.1. *Let $f : J \to \mathbb{R}$ be a differentiable convex function. Then for any $x, y, z \in J$,*

$$\begin{aligned}&\frac{1}{6}\left[f'(x)(y+z-2x)+f'(y)(x+z-2y)+f'(z)(x+y-2z)\right]\\&+\frac{f(x)+f(y)+f(z)}{3}+f\left(\frac{x+y+z}{3}\right)\le\frac{2}{3}\left[f\left(\frac{y+z}{2}\right)+f\left(\frac{x+z}{2}\right)+f\left(\frac{x+y}{2}\right)\right].\end{aligned}$$

Proof. If we replace a and b by x and $\frac{y+z}{2}$, respectively, in (1.6), we get

$$f'(x)\left(\frac{y+z-2x}{2}\right)+f(x)\le f\left(\frac{y+z}{2}\right). \tag{2.1}$$

If we replace a and b by y and $\frac{x+z}{2}$, respectively, in (1.6), we infer that

$$f'(y)\left(\frac{x+z-2y}{2}\right)+f(y)\le f\left(\frac{x+z}{2}\right). \tag{2.2}$$

If we replace a and b by z and $\frac{x+y}{2}$, respectively, in (1.6), we have

$$f'(z)\left(\frac{x+y-2z}{2}\right)+f(z)\le f\left(\frac{x+y}{2}\right). \tag{2.3}$$

Adding (2.1), (2.2), (2.3), and then multiplying by $\frac{1}{3}$, we obtain

$$\begin{aligned}&\frac{1}{3}\left[f'(x)\left(\frac{y+z-2x}{2}\right)+f'(y)\left(\frac{x+z-2y}{2}\right)+f'(z)\left(\frac{x+y-2z}{2}\right)\right]\\&+\frac{f(x)+f(y)+f(z)}{3}\le\frac{1}{3}\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{x+z}{2}\right)\right].\end{aligned} \tag{2.4}$$

On the other hand, by Jensen's inequality, we have

$$\begin{aligned}f\left(\frac{x+y+z}{3}\right)&=f\left(\frac{1}{3}\cdot\frac{x+y}{2}+\frac{1}{3}\cdot\frac{y+z}{2}+\frac{1}{3}\cdot\frac{x+z}{2}\right)\\&\le\frac{1}{3}\cdot f\left(\frac{x+y}{2}\right)+\frac{1}{3}\cdot f\left(\frac{y+z}{2}\right)+\frac{1}{3}\cdot f\left(\frac{x+z}{2}\right)\\&=\frac{1}{3}\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{x+z}{2}\right)\right].\end{aligned} \tag{2.5}$$

Adding the inequalities (2.4) and (2.5), we infer that

$$\begin{aligned}&\frac{1}{3}\left[f'(x)\left(\frac{y+z-2x}{2}\right)+f'(y)\left(\frac{x+z-2y}{2}\right)+f'(z)\left(\frac{x+y-2z}{2}\right)\right]\\&+\frac{f(x)+f(y)+f(z)}{3}+f\left(\frac{x+y+z}{3}\right)\le\frac{2}{3}\left[f\left(\frac{y+z}{2}\right)+f\left(\frac{x+z}{2}\right)+f\left(\frac{x+y}{2}\right)\right],\end{aligned}$$

as desired. □

At this point, it is evident that the reversing term in Theorem 2.1 is negative, considering the validity of (1.3). This can be stated as follows.

Corollary 2.1. *Let $f: J \to \mathbb{R}$ be a differentiable convex function. Then for any $x, y, z \in J$,*

$$[f'(x)(y+z-2x)+f'(y)(x+z-2y)+f'(z)(x+y-2z)]\le 0.$$

Remark 2.1. *While the proof of Corollary 2.1 is immediate from Theorem 2.1, we pay attention to the following observations about convex functions. Notice that the derivative of a differentiable convex function is increasing. Hence, if $f: J \to \mathbb{R}$ is a differentiable convex function, then (1.6) implies the following for any $a, b \in J$,*

$$(a-b)(f'(b)-f'(a))\le 0.$$

Taking this into consideration, we have

$$\begin{aligned}&[f'(x)(y+z-2x)+f'(y)(x+z-2y)+f'(z)(x+y-2z)]\\&=(y-x)(f'(x)-f'(y))+(z-x)(f'(x)-f'(z))+(y-z)(f'(z)-f'(y))\\&\le 0.\end{aligned}$$

Having presented a reverse of (1.3), we discuss possible refinements now. The following is a refinement of Popoviciu's inequality.

Theorem 2.2. *Let $f: J \to \mathbb{R}$ be a convex function and $x, y, z \in J$ $x \le y \le z$.*

(i) *If $x \le y \le \frac{x+y+z}{3}$, then*

$$\begin{aligned}&\frac{2}{3}\left[f\left(\frac{x+z}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{x+y}{2}\right)\right]\\&\le\frac{f(x)+f(y)+f(z)}{3}+f\left(\frac{x+y+z}{3}\right)\\&-\frac{2}{3}\left(\frac{f(z)+f\left(\frac{x+y+z}{3}\right)}{2}-f\left(\frac{z+\frac{x+y+z}{3}}{2}\right)\right)\left(2-\frac{|y+z-2x|-|x-2y+z|}{2z-x-y}\right)\\&-\frac{1}{3R}\left((1-t)f(x)+tf(y)-f((1-t)x+ty)\right),\end{aligned}$$

where $t \in [0,1]$ is arbitrary, and $R = \max\{t, 1-t\}$.

(ii) *If* $x \le \frac{x+y+z}{3} \le y$, *then*

$$\frac{2}{3}\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{x+z}{2}\right)+f\left(\frac{y+z}{2}\right)\right]$$
$$\le f\left(\frac{x+y+z}{3}\right)+\frac{f(x)+f(y)+f(z)}{3}$$
$$-\frac{2}{3}\left(\frac{f\left(\frac{x+y+z}{3}\right)+f(x)}{2}-f\left(\frac{\frac{x+y+z}{3}+x}{2}\right)\right)\left(2-\frac{|x+y-2z|-|x+z-2y|}{y+z-2x}\right).$$
$$-\frac{1}{3R}\left((1-t)f(y)+tf(z)-f((1-t)y+tz)\right),$$

where $t\in[0,1]$ *is arbitrary, and* $R=\max\{t,1-t\}$.

Proof. If we placed $a=m$, $b=M$, and $t=\frac{t-m}{M-m}$ in (1.4), we get

$$f(t)\le\frac{t-m}{M-m}f(M)+\frac{M-t}{M-m}f(m)-\left(1-\frac{|M+m-2t|}{M-m}\right)\left(\frac{f(M)+f(m)}{2}-f\left(\frac{M+m}{2}\right)\right) \tag{2.6}$$

since

$$2\min\left\{\frac{t-m}{M-m},\frac{M-t}{M-m}\right\}=1-\frac{|M+m-2t|}{M-m}.$$

We may assume that $x\le y\le z$. Then we have two cases to treat.

(i) If $x\le y\le\frac{x+y+z}{3}$, then

$$\frac{x+y+z}{3}\le\frac{x+z}{2}\le z \text{ and } \frac{x+y+z}{3}\le\frac{y+z}{2}\le z.$$

(ii) If $x\le\frac{x+y+z}{3}\le y$, then

$$x\le\frac{x+z}{2}\le\frac{x+y+z}{3} \text{ and } x\le\frac{x+y}{2}\le\frac{x+y+z}{3}.$$

We focus on the first case. Letting $m=\frac{x+y+z}{3}$, $M=z$, and $t=\frac{x+z}{2}$, in (2.6), then

$$f\left(\frac{x+z}{2}\right)\le\frac{x+z-2y}{2(2z-x-y)}f(z)+\frac{3(z-x)}{2(2z-x-y)}f\left(\frac{x+y+z}{3}\right)$$
$$-\left(1-\frac{|y+z-2x|}{2z-x-y}\right)\left(\frac{f(z)+f\left(\frac{x+y+z}{3}\right)}{2}-f\left(\frac{z+\frac{x+y+z}{3}}{2}\right)\right).$$

Putting $m=\frac{x+y+z}{3}$, $M=z$, and $t=\frac{y+z}{2}$, in (2.6), then

$$f\left(\frac{y+z}{2}\right)\le\frac{y+z-2x}{2(2z-x-y)}f(z)+\frac{3(z-y)}{2(2z-x-y)}f\left(\frac{x+y+z}{3}\right)$$
$$-\left(1-\frac{|x-2y+z|}{2z-x-y}\right)\left(\frac{f(z)+f\left(\frac{x+y+z}{3}\right)}{2}-f\left(\frac{z+\frac{x+y+z}{3}}{2}\right)\right).$$

Adding these two inequalities, we obtain

$$f\left(\frac{x+z}{2}\right)+f\left(\frac{y+z}{2}\right)\le\frac{1}{2}f\left(z\right)+\frac{3}{2}f\left(\frac{x+y+z}{3}\right)$$
$$-\left(\frac{f\left(z\right)+f\left(\frac{x+y+z}{3}\right)}{2}-f\left(\frac{z+\frac{x+y+z}{3}}{2}\right)\right)\left(2-\frac{\left|y+z-2x\right|-\left|x-2y+z\right|}{2z-x-y}\right).$$

From the inequality (1.5),

$$f\left(\frac{x+y}{2}\right)\le\frac{f\left(x\right)+f\left(y\right)}{2}-\frac{1}{2R}\left(\left(1-t\right)f\left(x\right)+tf\left(y\right)-f\left(\left(1-t\right)x+ty\right)\right)$$

where $R=\max\left\{t,1-t\right\}$ and $0\le t\le1$. The last two inequalities show

$$\frac{2}{3}\left[f\left(\frac{x+z}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{x+y}{2}\right)\right]$$
$$\le\frac{f\left(x\right)+f\left(y\right)+f\left(z\right)}{3}+f\left(\frac{x+y+z}{3}\right)$$
$$-\frac{2}{3}\left(\frac{f\left(z\right)+f\left(\frac{x+y+z}{3}\right)}{2}-f\left(\frac{z+\frac{x+y+z}{3}}{2}\right)\right)\left(2-\frac{\left|y+z-2x\right|-\left|x-2y+z\right|}{2z-x-y}\right)$$
$$-\frac{1}{3R}\left(\left(1-t\right)f\left(x\right)+tf\left(y\right)-f\left(\left(1-t\right)x+ty\right)\right).$$

Now we consider the second case. Letting $m=x$, $M=\frac{x+y+z}{3}$, and $t=\frac{x+z}{2}$, in (2.6), then

$$f\left(\frac{x+z}{2}\right)\le\frac{3\left(z-x\right)}{2\left(y+z-2x\right)}f\left(\frac{x+y+z}{3}\right)+\frac{2y-x-z}{2\left(y+z-2x\right)}f\left(x\right)$$
$$-\left(1-\frac{\left|x+y-2z\right|}{y+z-2x}\right)\left(\frac{f\left(\frac{x+y+z}{3}\right)+f\left(x\right)}{2}-f\left(\frac{\frac{x+y+z}{3}+x}{2}\right)\right).$$

Letting $m=x$, $M=\frac{x+y+z}{3}$, and $t=\frac{x+y}{2}$, in (2.6), then

$$f\left(\frac{x+y}{2}\right)\le\frac{3\left(y-x\right)}{2\left(y+z-2x\right)}f\left(\frac{x+y+z}{3}\right)+\frac{2z-x-y}{2\left(y+z-2x\right)}f\left(x\right)$$
$$-\left(1-\frac{\left|x+z-2y\right|}{y+z-2x}\right)\left(\frac{f\left(\frac{x+y+z}{3}\right)+f\left(x\right)}{2}-f\left(\frac{\frac{x+y+z}{3}+x}{2}\right)\right).$$

Adding these two inequalities, we reach to

$$f\left(\frac{x+z}{2}\right)+f\left(\frac{x+y}{2}\right)\le\frac{3}{2}f\left(\frac{x+y+z}{3}\right)+\frac{1}{2}f\left(x\right)$$
$$-\left(\frac{f\left(\frac{x+y+z}{3}\right)+f\left(x\right)}{2}-f\left(\frac{\frac{x+y+z}{3}+x}{2}\right)\right)\left(2-\frac{\left|x+y-2z\right|-\left|x+z-2y\right|}{y+z-2x}\right).$$

From the inequality (1.5),

$$f\left(\frac{y+z}{2}\right)\le\frac{f\left(y\right)+f\left(z\right)}{2}-\frac{1}{2R}\left(\left(1-t\right)f\left(y\right)+tf\left(z\right)-f\left(\left(1-t\right)y+tz\right)\right)$$

where $R = \max\{t, 1-t\}$ and $0 \le t \le 1$. The last two inequalities provide

$$\begin{aligned}
&\frac{2}{3}\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{x+z}{2}\right)+f\left(\frac{y+z}{2}\right)\right] \\
&\le f\left(\frac{x+y+z}{3}\right)+\frac{f(x)+f(y)+f(z)}{3} \\
&-\frac{2}{3}\left(\frac{f\left(\frac{x+y+z}{3}\right)+f(x)}{2}-f\left(\frac{\frac{x+y+z}{3}+x}{2}\right)\right)\left(2-\frac{|x+y-2z|-|x+z-2y|}{y+z-2x}\right). \\
&-\frac{1}{3R}\left((1-t)f(y)+tf(z)-f((1-t)y+tz)\right).
\end{aligned}$$

This finalizes the proof. □

In Theorem 2.2, we have presented a refinement of Popoviciu inequality. Also, we presented a reversed version in Theorem 2.1. In the following result, we prove a refinement of the reversed version, as in Theorem 2.2.

Theorem 2.3. *Let $f : J \to \mathbb{R}$ be a convex function and $x, y, z \in J$ with $x \le y \le z$.*

(i) *If $x \le y \le \frac{x+y+z}{3}$, then*

$$\begin{aligned}
&\frac{f(x)+f(y)+f(z)}{3}+f\left(\frac{x+y+z}{3}\right) \\
&\le \frac{2}{3}\left[f\left(\frac{x+z}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{x+y}{2}\right)\right] \\
&+\frac{2}{3}\left(\frac{f(z)+f\left(\frac{x+y+z}{3}\right)}{2}-f\left(\frac{z+\frac{x+y+z}{3}}{2}\right)\right)\left(2+\frac{|y+z-2x|-|x-2y+z|}{2z-x-y}\right) \\
&+\frac{1}{3r}\left((1-t)f(x)+tf(y)-f((1-t)x+ty)\right),
\end{aligned}$$

where $t \in [0,1]$ is arbitrary, and $r = \min\{t, 1-t\}$.

(ii) *If $x \le \frac{x+y+z}{3} \le y$, then*

$$\begin{aligned}
&f\left(\frac{x+y+z}{3}\right)+\frac{f(x)+f(y)+f(z)}{3} \le \frac{2}{3}\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{x+z}{2}\right)+f\left(\frac{y+z}{2}\right)\right] \\
&+\frac{2}{3}\left(\frac{f\left(\frac{x+y+z}{3}\right)+f(x)}{2}-f\left(\frac{\frac{x+y+z}{3}+x}{2}\right)\right)\left(2+\frac{|x+y-2z|-|x+z-2y|}{y+z-2x}\right). \\
&+\frac{1}{3r}\left((1-t)f(y)+tf(z)-f((1-t)y+tz)\right),
\end{aligned}$$

where $t \in [0,1]$ is arbitrary, and $r = \min\{t, 1-t\}$.

Proof. If we put $a = m$, $b = M$, and $t = \frac{t-m}{M-m}$ in (1.5), we obtain

$$\frac{t-m}{M-m}f(M)+\frac{M-t}{M-m}f(m) \le f(t)+\left(1+\frac{|M+m-2t|}{M-m}\right)\left(\frac{f(M)+f(m)}{2}-f\left(\frac{M+m}{2}\right)\right) \tag{2.7}$$

due to

$$2\max\left\{\frac{t-m}{M-m},\frac{M-t}{M-m}\right\}=1+\frac{|M+m-2t|}{M-m}.$$

The inequality (1.4) also indicates

$$\frac{f(a)+f(b)}{2}\le f\left(\frac{a+b}{2}\right)+\frac{1}{2r}\left((1-t)f(a)+tf(b)-f((1-t)a+tb)\right) \tag{2.8}$$

where $r=\min\{t,1-t\}$ and $0\le t\le 1$. Employing (2.7) and (2.8) and operating the same strategy as in the proof of Theorem 2.2, we deduce the desired result. □

Notice that Mercer's inequality (1.23) implies

$$f\left(M+m-\frac{x+y+z}{3}\right)\le f(M)+f(m)-f\left(\frac{x+y+z}{3}\right).$$

In the following, we introduce an interesting refinement via Popoviciu's inequality.

Theorem 2.4. *(Popoviciu-Mercer inequality) Let $f:[m,M]\to\mathbb{R}$ be a convex function. Then for any $x,y,z\in[m,M]$*

$$\begin{aligned}
&f\left(M+m-\frac{x+y+z}{3}\right)\\
&\le\frac{1}{3}\left(f\left(M+m-\frac{x+y}{2}\right)+f\left(M+m-\frac{y+z}{2}\right)+f\left(M+m-\frac{x+z}{2}\right)\right)\\
&\le\frac{1}{2}\left(\frac{f(M+m-x)+f(M+m-y)+f(M+m-z)}{3}+f\left(M+m-\frac{x+y+z}{3}\right)\right)\\
&\le\frac{f(M+m-x)+f(M+m-y)+f(M+m-z)}{3}\\
&\le f(M)+f(m)-\frac{f(x)+f(y)+f(z)}{3}\\
&\le f(M)+f(m)-f\left(\frac{x+y+z}{3}\right).
\end{aligned}$$

Proof. Since $m\le x,y,z\le M$, then

$$m\le M+m-x, M+m-y, M+m-z\le M,$$

$$m\le M+m-\frac{x+y}{2}, M+m-\frac{y+z}{2}, M+m-\frac{x+z}{2}\le M,$$

and

$$m\le M+m-\frac{x+y+z}{3}\le M.$$

Consequently,

$$\begin{aligned}
&\frac{2}{3}\left[f\left(M+m-\frac{x+y}{2}\right)+f\left(M+m-\frac{y+z}{2}\right)+f\left(M+m-\frac{x+z}{2}\right)\right]\\
&\le\frac{f(M+m-x)+f(M+m-y)+f(M+m-z)}{3}+f\left(M+m-\frac{x+y+z}{3}\right).
\end{aligned}$$

On the other hand, by the Jensen inequality, we can write

$$\begin{aligned}
& f\left(M+m-\frac{x+y+z}{3}\right) \\
& =f\left(\frac{M+m-x}{3}+\frac{M+m-y}{3}+\frac{M+m-z}{3}\right) \\
& \leq \frac{f(M+m-x)+f(M+m-y)+f(M+m-z)}{3}.
\end{aligned}$$

So,

$$\begin{aligned}
& 2\left(f\left(M+m-\frac{x+y+z}{3}\right)\right) \\
& \leq \frac{2}{3}\left(f\left(M+m-\frac{x+y}{2}\right)+f\left(M+m-\frac{y+z}{2}\right)+f\left(M+m-\frac{x+z}{2}\right)\right) \\
& \leq \frac{f(M+m-x)+f(M+m-y)+f(M+m-z)}{3}+f\left(M+m-\frac{x+y+z}{3}\right) \\
& \leq 2\left(\frac{f(M+m-x)+f(M+m-y)+f(M+m-z)}{3}\right) \\
& \leq 2\left(f(M)+f(m)-\frac{f(x)+f(y)+f(z)}{3}\right),
\end{aligned}$$

which completes the proof. □

We conclude this paper with a new discussion of the monotony of certain quotients related to convex functions. We know that if f is a convex function on J and $x, y, z \in J$ with $x \leq y \leq z$ then (as in (1.24))

$$\frac{f(y)-f(x)}{y-x} \leq \frac{f(z)-f(x)}{z-x} \leq \frac{f(z)-f(y)}{z-y}.$$

The following result provides refinement and reverse of this inequality.

Theorem 2.5. *Let f be a convex function on J and $x, y, z \in J$ with $x \leq y \leq z$. Then*

$$\begin{aligned}
& \frac{f(y)-f(x)}{y-x}+\frac{1}{y-x}\left(1-\frac{|z+x-2y|}{z-x}\right)\left(\frac{f(z)+f(x)}{2}-f\left(\frac{z+x}{2}\right)\right) \\
& \leq \frac{f(z)-f(x)}{z-x} \\
& \leq \frac{f(z)-f(y)}{z-y}-\frac{1}{z-y}\left(1-\frac{|z+x-2y|}{z-x}\right)\left(\frac{f(z)+f(x)}{2}-f\left(\frac{z+x}{2}\right)\right),
\end{aligned}$$

and

$$\begin{aligned}&\frac{f(z)-f(y)}{z-y}-\frac{1}{z-y}\left(1+\frac{|z+x-2y|}{z-x}\right)\left(\frac{f(z)+f(x)}{2}-f\left(\frac{z+x}{2}\right)\right)\\&\leq\frac{f(z)-f(x)}{z-x}\\&\leq\frac{f(y)-f(x)}{y-x}+\frac{1}{y-x}\left(1+\frac{|z+x-2y|}{z-x}\right)\left(\frac{f(z)+f(x)}{2}-f\left(\frac{z+x}{2}\right)\right).\end{aligned}$$

Proof. By (2.6), we have

$$f(y)\leq\frac{y-x}{z-x}f(z)+\frac{z-y}{z-x}f(x)-\left(1-\frac{|z+x-2y|}{z-x}\right)\left(\frac{f(z)+f(x)}{2}-f\left(\frac{z+x}{2}\right)\right).$$

So,

(2.9)
$$\frac{f(y)-f(x)}{y-x}\leq\frac{f(z)-f(x)}{z-x}-\frac{1}{y-x}\left(1-\frac{|z+x-2y|}{z-x}\right)\left(\frac{f(z)+f(x)}{2}-f\left(\frac{z+x}{2}\right)\right).$$

On the other hand,

(2.10)
$$\frac{f(z)-f(y)}{z-y}\geq\frac{f(z)-f(x)}{z-x}+\frac{1}{z-y}\left(1-\frac{|z+x-2y|}{z-x}\right)\left(\frac{f(z)+f(x)}{2}-f\left(\frac{z+x}{2}\right)\right).$$

By (2.9) and (2.10), we get the desired result. The other inequality can be proved similarly, thanks to (2.7). □

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