ASYMPTOTIC STUDY OF THE THREE-DIMENSIONAL GENERALIZED NAVIER-STOKES EQUATIONS WITH EXPONENTIAL DAMPING

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Abstract. In this paper, we study the long time behavior of solutions of the three dimensional (3D) generalized Navier-Stokes equations with nonlinear exponential damping term $a(e^{b|u|^r}-1)u$ $(a>0, b>0, r\geq 1)$ in a periodic bounded domain. We first study the existence and uniqueness of weak solutions. Then, we investigate the asymptotic behavior of weak solutions via attractors. The difficult issue is that the Cauchy problem could have non-unique solution and then we cannot use directly the classical schemes. To solve this problem, we use a new framework developed by Cheskidov and Lu which called (closed) evolutionary system to obtain various attractors and its properties. Finally, we investigate the determining wavenumbers and this seems to be the first result for a fractional equation.

1. Introduction

We study the 3D generalized Navier-Stokes equations with exponential damping determined by

$$\begin{cases}\partial_t u + \nu(-\Delta)^\alpha u + (u\cdot\nabla)u + a(e^{b|u|^r}-1)u + \nabla p = f,\\ \nabla\cdot u = 0,\end{cases} \tag{1.1}$$

where $u=u(t,x)=(u_1(t,x),u_2(t,x),u_3(t,x))$ and $p=p(t,x)$ denote the fluid velocity vector field and the scalar pressure at the point $(t,x)\in\mathbb{R}^+\times\mathbb{T}$; $(-\Delta)^\alpha$ is α-fractional Laplacian; $f(x,t)$ is the external body force; $\nu>0$ is the constant kinematic viscosity; a is the positive damping coefficient; the exponents b and r are positive constants. It is well-known that the damping term and the fractional power of the Laplacian are very helpful from the mathematical point of view. The damping can be raised as the resistance to the motion and it describes various physical situations such as porous media flow, drag or friction effect, etc (see, e.g., [16, 65] and references therein). Dissipation corresponding to the fractional power of the Laplacian can in principle arise from modeling real physical phenomena. The fractional diffusion operators can model the anomalous diffusion and have now been widely used in turbulence modeling to control the effective range of the non-local dissipation (see, e.g., [5, 15, 33, 35, 39, 56, 57] and references therein). But our

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motivation for studying (1.1) is mainly mathematical and the goal is to understand how the nonlinear exponential damping term affect the asymptotic behavior of the weak solutions. We know that it is flawed because we cannot refer to the many exciting results of the 2D case and there are a lot of differences between the 2D and 3D cases. However, we only consider the 3D case.

The 3D incompressible Navier-Stokes equations were studied for long ago. There are many great results in different issues. In the approach of dynamical systems via attractors, the existence of weak (uniform) global attractors has been established in [27, 29, 45] by using the abstract theory for multivalued semi-flow (processes) and evolutionary systems. Moreover, under some additional assumption, the existence of strong (uniform) global attractors has also been established in [27, 29, 45]. The existence of trajectory attractors were also proved in [54]. On the other hand, the finite number of determining parameters is also interesting issue as we study partial differential equations. As we have known that the finite number of determining modes of the 3D Navier-Stokes equations is not known for lack of regularity (see [18]). Recently, in [25], without making any assumptions regarding regularity properties of solutions or bounds on the global attractor, the existence of a time dependent determining wavenumbers for the forced 3D Navier-Stokes equations defined for each individual solution is investigated (see also [24, 26]). Even though this wavenumber blows up if the solution blows up, its time average is uniformly bounded for all solutions on the weak global attractor. The bound is compared to Kolmogorov's dissipation wavenumber and the Grashof constant.

The 3D incompressible Navier-Stokes equations with polynomial damping has been studied extensively. The existence of weak solutions for this system was established at first in [16]. Then many authors have considered this system for the well-posedness and the long-time behavior of solutions (see, e.g., [40, 41, 42, 48, 58, 60, 61, 62, 63, 71, 75, 76]). In [60, 61, 63], the existence of global attractors, uniform attractors and pullback attractors has established in V and $H^2 \cap V$ by combining asymptotic a priori estimates with Sobolev compactness embedding theorems. The existence of an exponential attractor in V was proved in [62] by using the squeezing property. Specially, the existence of a global attractor in H for weak solutions was proved in [48, 58] by using the abstract theory for multi-valued semi-flow and the upper bound of its fractal dimension by using the methods of ℓ-trajectories.

Recently, the 3D generalized Navier-Stokes equations has been extensively investigated. This system was first studied by J. L. Lions [50] for the existence and uniqueness of weak solutions with $\alpha \in [\frac{5}{4}, \infty)$. In our exponential damping case, we will see that the existence and uniqueness of weak solutions still hold for $\alpha \in [\frac{5}{4}, \infty)$. Moreover, by using the convex integration technique, the existence of non-unique weak solutions with $\alpha < \frac{5}{4}$ was pointed out by T. Luo and E. S. Titi [55]. The global existence and decay of solutions for the 3D generalized Navier-Stokes equations have investigated in [32, 43, 73] (see also in [31, 68, 69, 70] and references therein). The existence of inertial manifolds has studied in [34] for some subcritical case ($\alpha \geq \frac{3}{2}$) on torus. The finite dimensional global attractor and asymptotic determining operators in subcritical case have obtained in [6] as a special case (see also in [74], the MHD equations reduces to the generalized Navier-Stokes equations).

Especially, the 3D incompressible Navier-Stokes equations with exponential damping has been first studied in [10] by J. Benameur. Then, the existence and uniqueness of its strong solutions and the large time decay for some nonlinear exponential damping term have been considered by J. Benameur and et. al. [11, 12, 13].

It is worth noting that so far there are few results studying the properties and the asymptotic behaviour of weak solutions of (1.1). Therefore, analyzing (1.1) seems as an interesting problem. As in the case of the 3D Navier-Stokes system, several difficulties appear and many problems remain open. We still have to cope with the main difficulties such as the absence of results concerning the continuity of weak solutions and the lack of good dissipativity estimates for all weak solutions. The issue how to describe the limit behavior of solutions of evolution equations for which the Cauchy problem can have non-unique solution arouses much interest in recent years (see [19, 20, 22, 46, 54]). In this situation we cannot use directly the classical scheme of construction of a dynamical system in the phase space of initial conditions of the Cauchy problem of a given equation and find a global attractor of this dynamical system. To our knowledge, there are several abstract frameworks for studying dynamical systems without uniqueness such as the abstract theory for multivalued semi-flow (processes). Recently, a new framework work was developed by Cheskidov and Lu in [22, 27, 28, 54] and was called the (closed) evolutionary system. It was first introduced in [27] to study a weak global attractor and a trajectory attractor for the autonomous 3D Navier Stokes equation, and the theory was developed further in [22, 28, 54] to make it applicable to arbitrary autonomous and nonautonomous dissipative partial differential equation without uniqueness. The avantage of this framework lies in a simultaneous use of weak and strong metric and avoid the construction of symbol spaces. The tracking properties of attractors still can be proved which may be the restriction of another frameworks (see [22, 27, 28, 54] for more details).

The main purpose of this paper is to investigate the long time dynamical behavior of the weak solutions of (1.1) via attractors and their properties by using the (closed) evolutionary system (see, e.g., [22, 27, 28, 29, 54]). Then we investigate the determining wavenumbers. The paper is organized as follows. In Section 2, we recall the functional setting and some auxiliary results. In Section 3, we study existence and uniqueness of weak solutions. In Section 4, we prove the existence of various attractors and its properties. In Section 5, we study the determining wavenumbers. Moreover, for completeness, we also summarize the theory of the (closed) evolutionary systems in appendix A and the Littlewood–Paley decomposition for periodic functions in appendix B. In this paper, we sometimes use the symbol C to denote a non-dimensional constant which may change from line to line. We also denote by $A \lesssim B$ an estimate of the form $A \leq CB$ with some positive constant C.

2. Preliminaries

For simplicity, we work on the torus $\mathbb{T} = [-\pi, \pi]^3$ with periodic boundary conditions. Because of the periodic setting and the lack of natural boundary conditions, we can restrict ourselves to deal with initial data and f with vanishing spatial averages; then the solutions will enjoy the same property. This allows us to represent any divergence free velocity vectors u which are periodic and have zero spatial averages as follows

$$u := \sum_{k\in J} u_k\phi_k \text{ with } u_k \in \mathbb{C}^3,\ u_k^* = u_{-k},\ u_k\cdot k = 0\ \ \forall k \in J,$$

where $\phi_k = e^{ik\cdot x}$, $J = \mathbb{Z}^3 \setminus \{0\}$. For $s \in \mathbb{R}$, we define the following spaces

$$V^s := \{u := \sum_{k\in J} u_k\phi_k,\ u_k \in \mathbb{C}^3,\ u_k^* = u_{-k},\ u_k\cdot k = 0,$$
$$\phi_k = e^{ik\cdot x} \text{ and } \sum_{k\in J} |u_k|^2|k|^{2s} < \infty\}.$$

These spaces are also Hilbert spaces with scalar product

$$\langle u, v\rangle_{V^s} = \sum_{k\in J} u_k\cdot v_{-k}|k|^{2s}.$$

For simplicity, we use the notation $\langle\cdot,\cdot\rangle$ denoted the scalar product in V^0 and also the dual pairing of $V^s - V^{-s}$ by $\langle u, v\rangle := \sum_{k\in J} u_k\cdot v_{-k}$. We have the following compact embedding $V^{s+\varepsilon} \hookrightarrow\hookrightarrow V^s$ for any $\varepsilon > 0$. Let $s_1 \le s_2$ and $u \in V^{s_2}$, we have

$$\|u\|_{V^{s_1}} \le \|u\|_{V^{s_2}}. \tag{2.1}$$

Moreover, if $s = \gamma s_1 + (1-\gamma)s_2$, $0 \le \gamma \le 1$, then

$$\|u\|_{V^s} \le \|u\|_{V^{s_1}}^{\gamma}\ \|u\|_{V^{s_2}}^{1-\gamma}. \tag{2.2}$$

Assume that $p \ge 1$. If $0 \le s < \frac{3}{2}$ and $\frac{1}{p} \ge \frac{1}{2} - \frac{s}{3}$, then $V^s \hookrightarrow L^p(\mathbb{T})$ and there exists a constant C depending on s and p such that

$$\|u\|_{L^p(\mathbb{T})} \lesssim \|u\|_{V^s}, \text{ for all } u \in V^s. \tag{2.3}$$

If $s = \frac{3}{2}$, then

$$\|u\|_{L^p(\mathbb{T})} \lesssim \|u\|_{V^s} \text{ for any finite } p \text{ and all } u \in V^s, \tag{2.4}$$

and if $s > \frac{3}{2}$, then

$$\|u\|_{L^\infty(\mathbb{T})} \lesssim \|u\|_{V^s}, \text{ for all } u \in V^s. \tag{2.5}$$

We define the linear operator $\Lambda = (-\Delta)^{\frac{1}{2}}$ as follows

$$\Lambda u = \sum_{k\in J} |k|u_k\phi_k \text{ with } u = \sum_{k\in J} u_k\phi_k,\ \phi_k = e^{ik\cdot x},$$

and its powers Λ^s by

$$\Lambda^s u = \sum_{k\in J} |k|^s u_k\phi_k,$$

hence $(-\Delta)^s = \Lambda^{2s}$. Since Λ^s preserves the divergence free condition $k\cdot u_k = 0$, we infer that Λ^s maps V^α onto $V^{\alpha-s}$. It follows from the construction of Λ^s that

$$\|u\|_{V^s} = \|\Lambda^s u\|_{V^0}. \tag{2.6}$$

In particular, Λ^s maps V^s onto V^0 for all $s > 0$ and so $D(\Lambda^s) = V^s$.

Denote by P_σ the Leray-Helmholtz projection. It is the orthogonal projection from $L^2(\mathbb{T})$ onto V^0 and $P_\sigma\Lambda^s = \Lambda^s P_\sigma$. Setting

$$b(u, v, w) = \int_{\mathbb{T}} \sum_{i,j=1}^{3} u_i \frac{\partial v_j}{\partial x_i} w_j dx.$$

Let $\mathcal{F}$ be the space of formal Fourier series

$$\{u := \sum_{k\in J} \widehat{u}_k \phi_k,\ \widehat{u}_k \in \mathbb{C}^3,\ \phi_k = e^{ik\cdot x}\}.$$

and

$$H^s := \{u \in \mathcal{F} : \|u\|_{H^s}^2 := \sum_{k\in J} |u_k|^2 |k|^{2s} < \infty, \widehat{u^*}_k = \widehat{u}_{-k} \text{ and } \widehat{u}_0 = 0\}.$$

Let $\mathcal{V}$ be the space of divergence free trigonometric polynomials consisting of all $u \in \mathcal{F}$ such that $k \cdot \widehat{u}_k = 0$ for all $k \in J$ and $\widehat{u}_k = 0$ for all but finitely many values of $k \in J$. We see that V^s is the closure of $\mathcal{V}$ in H^s with respect to the $\|\cdot\|_{H^s}$ norm. We need the following lemma, which we quote from [14, 17, 44, 47], to look into the properties of the trilinear form b

Lemma 2.1. *Let $u, v, w \in \mathcal{V}$, it holds that*

(i) $b(u, v, v) = 0$,
(ii) $b(u, v, w) = -b(u, w, v)$,
(iii) $b(u - v, u, u - v) = b(u, u, u - v) - b(v, v, u - v)$

This result may be extended to larger spaces by the density of $\mathcal{V}$ in V^σ for the appropriate values of σ that the trilinear forms are continuous. The following proposition is taken from [38, Proposition 2.5] (see also [9]).

Proposition 2.1. *The trilinear form $b : V^{\sigma_1} \times V^{\sigma_2} \times V^{\sigma_3} \to \mathbb{R}$ is bounded provided that all following conditions hold:*

(i) $\sigma_1 + \sigma_2 + \sigma_3 > \frac{5}{2}$,
(ii) $\sigma_1 + \sigma_2 \geq s$,
(iii) $\sigma_2 + \sigma_3 \geq 1$,
(iv) $\sigma_1 + \sigma_3 \geq 1 - s$,

for some $s \in \{0, 1\}$. If the last three conditions are satisfied and if σ_i is a nonpositive integer for some $i \in \{1, 2, 3\}$, then the condition (i) can be replaced by the nonstrict version of the inequality. The nonstrict inequality is also allowed if for some $s \in \{0, 1\}$,

$$\sigma_1 \geq 0, \quad \sigma_2 \geq s, \quad \sigma_3 \geq 1 - s.$$

We now apply the projection operator P_σ on (1.1). Due to the periodic setting, the weak formulation (1.1) can be rewritten by

$$\partial_t u + \nu \Lambda^{2\alpha} u + B(u, u) + a P_\sigma\big((e^{b|u|^r} - 1)u\big) = P_\sigma f. \tag{2.7}$$

where $B(u, v) := P_\sigma\{(u \cdot \nabla)v\}$. To study (2.7), let us start with a definition of weak solutions for (2.7) with L^2 initial data u_τ.

Definition 2.1. *Let ν, α, a, b be positive and let $r \geq 1$. Given $f \in L^2_{loc}(\mathbb{R}; V^0)$, $u_\tau \in V^0$ and a fixed $T > \tau$. A weak solution to (2.7) on the interval $[\tau, T]$ is a function $u(t, x)$ such that*

$$u \in L^\infty(\tau, T; V^0) \cap L^2(\tau, T; V^\alpha) \cap \mathcal{G}_b^r(\tau, T; L^1(\mathbb{T})) \cap C_w([\tau, T]; V^0),$$

where

$$\mathcal{G}_b^r(\tau, T; L^1(\mathbb{T})) := \big\{u : [\tau, T] \times \mathbb{T} \to \mathbb{R}^3 \text{ measurable, } (e^{b|u|^r} - 1)|u|^2 \in L^1(\tau, T; L^1(\mathbb{T}))\big\}.$$

Moreover, given any $t \in [\tau, T]$ and $v \in V^\gamma \cap L^\infty(\mathbb{T})$, $\gamma > \max\{\frac{5}{2} - \alpha; \alpha\}$, it satisfies $u(\tau) = u_\tau$ and

$$\langle u(t),v\rangle+\nu\int_\tau^t\langle\Lambda^\alpha u(s),\Lambda^\alpha v\rangle ds-\int_\tau^t\langle B(u(s),v),u(s)\rangle ds$$
$$+a\int_\tau^t\langle(e^{b|u(s)|^r}-1)u(s),v\rangle ds=\langle u_\tau,v\rangle+\int_\tau^t\langle f(s),v\rangle ds, \quad (2.8)$$

for a.e. $t\in[\tau,T]$.

Remark 2.1. In the weak formulations above, we see that the trilinear terms are well defined. Indeed, it easily implies that $\gamma>\max\{\frac{5}{2}-\alpha;\alpha\}>1$ and it follows from Proposition 2.1 with $\sigma_1=0,\sigma_2=\gamma,\sigma_3=\alpha$ that

$$|\langle B(u,v),u\rangle|=|b(u,v,u)|\lesssim\|u\|_{V^0}\|v\|_{V^\gamma}\|u\|_{V^\alpha}. \quad (2.9)$$

Lemma 2.2. *If* $r\geq 1$ *and* u *is a weak solution of* (2.7) *determined by Definition 2.1, then*

$$(e^{b|u|^r}-1)u\in L^1(\tau,T;L^1(\mathbb{T})). \quad (2.10)$$

Proof. Indeed, we define

$$\begin{aligned}\Omega:=&[\tau,T]\times\mathbb{T},\\ \Omega_1:=&\{(t,x)\in[\tau,T]\times\mathbb{T};\ 0<|u(t,x)|<1\},\\ \Omega_2:=&\{(t,x)\in[\tau,T]\times\mathbb{T};\ |u(t,x)|\geq 1\}.\end{aligned}$$

We then have

$$\begin{aligned}\int_\Omega(e^{b|u(s)|^r}-1)|u(s)|dxds&=\int_{\Omega_1\cup\Omega_2}(e^{b|u(s)|^r}-1)|u(s)|dxds\\ &=\int_{\Omega_1}(e^{b|u(s)|^r}-1)|u(s)|dxds+\int_{\Omega_2}(e^{b|u(s)|^r}-1)|u(s)|dxds\\ &=\int_{\Omega_1}\frac{e^{b|u(s)|^r}-1}{|u(s)|}|u(s)|^2dxds+\int_{\Omega_2}\frac{1}{|u(s)|}(e^{b|u(s)|^r}-1)|u(s)|^2dxds\\ &\leq M_{br}\int_{\Omega_1}|u(s)|^2dxds+\int_{\Omega_2}(e^{b|u(s)|^r}-1)|u(s)|^2dxds\\ &\qquad\text{where}\quad M_{br}:=\sup_{0<t\leq 1}\frac{e^{bt^r}-1}{t}<\infty\text{ for }r\geq 1,b>0\\ &\leq M_{br}(T-\tau)\|u\|_{L^\infty(\tau,T;V^0)}+\int_\Omega(e^{b|u(s)|^r}-1)|u(s)|^2dxds\\ &\leq M_{br}(T-\tau)\|u\|_{L^\infty(\tau,T;V^0)}+\|(e^{b|u|^r}-1)|u|^2\|_{L^1(\tau,T;L^1(\mathbb{T}))}. \quad (2.11)\end{aligned}$$

This implies the desired result. □

Lemma 2.3. *If* $(e^{b|u|^r}-1)u\in L^1(\tau,T;L^1(\mathbb{T}))$, *then* $u\in\bigcap_{k=1}^{\infty}L^{rk+2}(\tau,T;L^{rk+2}(\mathbb{T}))$.

Proof. We have

$$(e^{b|u|^r}-1)|u|^2=\sum_{k=1}^\infty\frac{b^k}{k!}|u|^{rk+2}. \quad (2.12)$$

This implies that

$$\int_\tau^T\|(e^{b|u(s)|^r}-1)|u(s)|^2\|_{L^1(\mathbb{T})}ds=\sum_{k=1}^\infty\frac{b^k}{k!}\int_\tau^T\|u(s)\|^{rk+2}_{L^{rk+2}(\mathbb{T})}ds. \quad (2.13)$$

We deduce from (2.13) that

$$(e^{b|u|^r}-1)u \in L^1(\tau,T;L^1(\mathbb{T})) \text{ implies } u \in \bigcap_{k=1}^{\infty} L^{rk+2}(\tau,T;L^{rk+2}(\mathbb{T})). \tag{2.14}$$

□

We now recall the following strong continuity result for the velocity (see [14, Lemma 6]).

Lemma 2.4. *Assume that $u \in L^2(\tau,T;V^{s+h})$ and $\frac{du}{dt} \in L^2(\tau,T;V^{s-h})$ for $s \in \mathbb{R}$ and $h > 0$, then $u \in C([\tau,T];V^s)$ and*

$$\frac{d}{dt}\|u(t)\|^2_{V^s} = 2\langle \Lambda^{-h}\frac{du}{dt}(t), \Lambda^h u(t)\rangle_{V^s}. \tag{2.15}$$

We also have the following weak continuity result in time (see [14, Lemma 7])

Lemma 2.5. *Let X and Y be Banach spaces such that $Y \hookrightarrow X$ with a continuous injection. Then*

$$L^\infty(\tau,T;Y) \cap C_w([\tau,T];X) = C_w([\tau,T];Y).$$

In particular, we also have the following important inequalities for the damping (see [8, Lemma 2.2] and [12, Lemma 2.3]).

Lemma 2.6.

(1) Assume that $p \in (1,\infty)$ and $\delta \geq 0$. There exist positive constants c_1 and c_2 such that for all $x,y \in \mathbb{R}^3$,

$$||x|^{p-2}x - |y|^{p-2}y| \leq c_1|x-y|^{1-\delta}(|x|+|y|)^{p-2+\delta},$$

and

$$(|x|^{p-2}x - |y|^{p-2}y)\cdot(x-y) \geq c_2|x-y|^{2+\delta}(|x|+|y|)^{p-2-\delta}.$$

(2) Assume that $b > 0$ and $r > 0$. There exists positive constant c_3 such that for all $x,y \in \mathbb{R}^3$,

$$\big((e^{b|x|^r}-1)x - (e^{b|y|^r}-1)y\big)\cdot(x-y) \geq c_3|x-y|^2\big((e^{b|x|^r}-1)+(e^{b|y|^r}-1)\big).$$

3. Existence and uniqueness of weak solutions

In this section, we will give some results about the existence, uniqueness and regularity of the global weak solutions of system (2.7). Let us first formulate the weak solution existence theorem.

Theorem 3.1. *Assume that ν,α,a,b are positive and $r \geq 1$. Given $u_\tau \in V^0$ and $f \in L^2_{loc}(\mathbb{R};V^0)$. Then, the system (2.7) possesses a global weak solution obeying Definition 2.1 with initial condition u_τ. Furthermore, if $\alpha \geq 1$, the global weak solution then is unique and depends continuity on the initial data.*

Proof. **i) Existence.** The existence of a weak solution of (2.7) is obtained via using the Galerkin approximation method. Therefore, we only outline the main points here.

We define the finite dimensional projectors Π_n in V^0 as

$$\Pi_n u = \sum_{0<|k|\leq n} u_k\phi_k \text{ for } u = \sum_{k\in J} u_k\phi_k \text{ and } \phi_k = e^{ik\cdot x}.$$

Setting $B_n(u,v) := \Pi_n B(u,v)$. We consider the finite dimensional approximation of system (2.7) in the unknowns $u_n = \Pi_n u$. This is the Galerkin approximation for $n = 1, 2, 3, \cdots$

$$\begin{cases} \partial_t u_n = -\nu \Lambda^{2\alpha} u_n - B_n(u_n, u_n) - a\Pi_n P_\sigma\{(e^{b|u_n|^r} - 1)u_n\} + \Pi_n P_\sigma f, \\ u_n(\tau) = \Pi_n u_\tau. \end{cases} \tag{3.1}$$

Obviously, $u_n(\tau)$ strongly converges to u_τ in V^0. We take L^2-scalar product of the first equation with itself u_n; bearing in mind Lemma 2.1, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u_n(t)\|_{V^0}^2 + \nu\|u_n(t)\|_{V^\alpha}^2 + a\|(e^{b|u_n|^r} - 1)|u_n|^2\|_{L^1(\mathbb{T})} &= \int_{\mathbb{T}} f(t)u_n(t)dx \\ &\le \frac{\nu}{2}\|u_n(t)\|_{V^\alpha}^2 + \frac{1}{2\nu}\|f(t)\|_{V^0}^2, \end{aligned} \tag{3.2}$$

where we have used (2.1) and the Cauchy-Schwarz inequality. Therefore,

$$\frac{d}{dt}\|u_n(t)\|_{V^0}^2 + \nu\|u_n(t)\|_{V^\alpha}^2 + 2a\|(e^{b|u_n(t)|^r} - 1)|u_n(t)|^2\|_{L^1(\mathbb{T})} \le \frac{1}{\nu}\|f(t)\|_{V^0}^2. \tag{3.3}$$

For all $t \in [\tau, T]$, we integrate (3.3) in time from τ to t and obtain

$$\begin{aligned} \|u_n(t)\|_{V^0}^2 + \nu\int_\tau^t \|u_n(s)\|_{V^\alpha}^2 ds + 2a\int_\tau^t \|(e^{b|u_n(s)|^r} - 1)|u_n(s)|^2\|_{L^1(\mathbb{T})} ds \\ \le \|u_\tau\|_{V^0}^2 + \frac{1}{\nu}\int_\tau^t \|f(s)\|_{V^0}^2 ds. \end{aligned} \tag{3.4}$$

Since $\|u_\tau\|_{V^0}^2$ and $\int_\tau^t \|f(s)\|_{V^0}^2 ds$ are bounded, it follows from (3.4) and (2.14) that the sequence $\{u_n\}$ is uniformly bounded in $L^\infty(\tau, T; V^0) \cap L^2(\tau, T; V^\alpha) \cap \{\bigcap_{k=1}^{\infty} L^{rk+2}(\tau, T; L^{rk+2}(\mathbb{T}))\}$.

We now consider the first equation of (3.1). We see that the dissipative term $\Lambda^{2\alpha} u_n \in L^2(\tau, T; V^{-\alpha})$ and it follows from (2.9) that $B_n(u_n, u_n) \in L^2(\tau, T; V^{-\gamma})$. Setting $\gamma_0 := \max\{3, 2\alpha, \gamma\}$ and we see that $\gamma \le \gamma_0$. Since $L^1(\mathbb{T}) \hookrightarrow V^{-\gamma_0}$, we deduce that

$$\begin{aligned} \int_\tau^t \|(e^{b|u_n(s)|^r} - 1)|u_n(s)|\|_{V^{-\gamma_0}} ds &\lesssim \int_\tau^t \|(e^{b|u_n(s)|^r} - 1)|u_n(s)|\|_{L^1(\mathbb{T})} ds \\ &\le M_{br}(T - \tau)\|u_n\|_{L^\infty(\tau, T; V^0)} \\ &\quad + \|(e^{b|u_n|^r} - 1)|u_n|^2\|_{L^1(\tau, T; L^1(\mathbb{T}))}, \end{aligned} \tag{3.5}$$

where we have used (2.11). We infer from (3.4) and (3.5) that $(e^{b|u_n|^r} - 1)u_n \in L^1(\tau, T; V^{-\gamma_0})$. Therefore, $\partial_t u_n$ is bounded uniformly in $L^1(\tau, T; V^{-\gamma_0})$. Since

$$V^\alpha \cap \{\bigcap_{k=1}^{\infty} L^{rk+2}(\mathbb{T})\} \hookrightarrow\hookrightarrow V^0 \hookrightarrow V^{-\gamma_0},$$

and

$$V^\alpha \cap \{\bigcap_{k=1}^{\infty} L^{rk+2}(\mathbb{T})\} \hookrightarrow\hookrightarrow V^{\bar\alpha} \hookrightarrow V^{-\gamma_0},$$

for some $\bar\alpha \in (0, \alpha)$ such that $\bar\alpha + \gamma \ge \frac{5}{2}$. We deduce from the Aubin-Lions lemma (see [64]) that $\{u_n\}$ is compact in $L^2(\tau, T; V^0)$ and so we can extract a subsequence, still denoted by u_n, such that

$$u_n \rightharpoonup u \text{ weakly in } L^2(\tau,T;V^\alpha), \tag{3.6}$$

$$u_n \rightharpoonup u \text{ weakly in } L^{rk+2}(\tau,T;L^{rk+2}(\mathbb{T})), \text{ for any positive integer } k, \tag{3.7}$$

$$u_n \rightharpoonup^* u \text{ weakly star in } L^\infty(\tau,T;V^0), \tag{3.8}$$

$$u_n \to u \text{ strongly in } L^2(\tau,T;V^0), \tag{3.9}$$

$$u_n \to u \text{ strongly in } L^2(\tau,T;V^{\bar{\alpha}}). \tag{3.10}$$

We now prove the convergence of nonlinear term. First, we have

$$\int_0^T |b(u_n(t),u_n(t),v) - b(u(t),u(t),v)|dt \le S_n^1 + S_n^2$$

where $v \in V^\gamma$ and the terms S_n^1 and S_n^2 are defined as follows

$$\begin{aligned} S_n^1 &= \int_\tau^T |b(u_n(t),u_n(t)-u(t),v)|dt \\ &= \int_\tau^T |b(u_n(t),v,u_n(t)-u(t))|dt \\ &\text{[By using Lemma 2.1]} \\ &\lesssim \int_\tau^T \|u_n(t)\|_{V^0}\|v\|_{V^\gamma}\|u_n(t)-u(t)\|_{V^{\bar{\alpha}}}dt \\ &\text{[By using Proposition 2.1]} \\ &\lesssim \|v\|_{V^\gamma}\|u_n\|_{L^2(\tau,T;V^0)}\|u_n(t)-u(t)\|_{L^2(\tau,T;V^{\bar{\sigma}})}. \end{aligned}$$

Using (3.10) implies that $\lim_{n\to\infty} S_n^1 = 0$.

$$\begin{aligned} S_n^2 &= \int_\tau^T |b(u_n(t)-u(t),u_n(t),v)|dt \\ &= \int_\tau^T |b(u_n(t)-u(t),v,u_n(t))|dt \\ &\text{[By using Lemma 2.1]} \\ &\lesssim \int_\tau^T \|u_n(t)-u(t)\|_{V^{\bar{\alpha}}}\|v\|_{V^\gamma}\|u_n(t)\|_{V^0}dt \\ &\text{[By using Proposition 2.1]} \\ &\lesssim \|v\|_{V^\gamma}\|u_n\|_{L^2(\tau,T;V^0)}\|u_n(t)-u(t)\|_{L^2(\tau,T;V^{\bar{\sigma}})}. \end{aligned}$$

Using (3.10) implies that $\lim_{n\to\infty} S_n^2 = 0$. Therefore, we have

$$\int_\tau^T b(u_n(t),u_n(t),v)dt \to \int_0^T b(u(t),u(t),v)dt.$$

Using all convergences above, it is classical results to pass to the limit in the variational formulations (2.8) and prove that u is the solution of (2.7) and inherits all the regularity from u_n, i.e.,

$$u \in L^\infty(\tau,T;V^0) \cap L^2(\tau,T;V^\alpha) \cap \mathcal{G}_b^r(\tau,T;L^1(\mathbb{T})),$$

where

$$\mathcal{G}_b^r(\tau,T;L^1(\mathbb{T})) := \left\{u : [\tau,T]\times\mathbb{T} \to \mathbb{R}^3 \text{ measurable}, \ (e^{b|u|^r}-1)|u|^2 \in L^1(\tau,T;L^1(\mathbb{T}))\right\}.$$

We integrate in time the equations for the velocity u and we obtain

$$u(t) = u_\tau + \int_\tau^t [-\nu\Lambda^{2\alpha}u(s) - B(u(s),u(s)) - aP_\sigma\{(e^{b|u(s)|^r} - 1)u(s)\} + P_\sigma f(s)]ds.$$

This implies that $u \in C([\tau,T];V^{-\gamma_0})$. In addition, since $u \in L^\infty(\tau,T;V^0)$, we deduce from Lemma 2.5 that

$$u \in L^\infty(\tau,T;V^0) \cap L^2(\tau,T;V^\alpha) \cap \mathcal{G}_b^r(\tau,T;L^1(\mathbb{T})) \cap C_w([\tau,T];V^0).$$

ii) Uniqueness and continuous dependence on the initial data. Assume that u_1 and u_2 are two weak solutions of (2.7) with initial data $u_{1\tau}$, $u_{2\tau} \in V^0$, respectively. Setting $U = u_1 - u_2$ and then U satisfies

$$\begin{cases} \partial_t U + \nu\Lambda^{2\alpha}U + B(u_1,U) + B(U,u_2) + aP_\sigma\{(e^{b|u_1|^r} - 1)u_1 - (e^{b|u_2|^r} - 1)u_2\} = 0, \\ \nabla\cdot U = 0. \end{cases} \tag{3.11}$$

We first take the L^2-scalar product of the first equation of (3.11) with U and using Lemma 2.1 leads to

$$\frac{1}{2}\frac{d}{dt}\|U\|_{V^0}^2 + \nu\|U\|_{V^\alpha}^2 + b(U,u_2,U) + a\int_{\mathbb{T}}\{(e^{b|u_1|^r} - 1)u_1 - (e^{b|u_2|^r} - 1)u_2\}\cdot U dx = 0. \tag{3.12}$$

Using Lemma 2.6, we get that

$$\int_{\mathbb{T}}\{(e^{b|u_1|^r} - 1)u_1 - (e^{b|u_2|^r} - 1)u_2\}\cdot U dx \geq 0. \tag{3.13}$$

Now, we are going to estimate the nonlinear term $b(U,u_2,U)$.
Case 1. $\alpha = 1$. It follows from (2.14) that we can take $rk \geq 3$ so that $u \in L^{rk+2}(\mathbb{T})$.

Estimation of the nonlinear term now can be done as follows

$$\begin{aligned}
|b(U,u_2,U)| &= |b(U,U,u_2)| \\
&\lesssim \|U\|_{L^{\frac{2(rk+2)}{rk}}(\mathbb{T})}\|\nabla U\|_{L^2(\mathbb{T})}\|u_2\|_{L^{rk+2}(\mathbb{T})} \\
&\quad \text{[By using the Hölder inequality]} \\
&\lesssim \|U\|_{V^1}\|u_2\|_{L^{rk+2}(\mathbb{T})}\|U\|_{L^2(\mathbb{T})}^{\frac{rk-1}{rk+2}}\|U\|_{L^6(\mathbb{T})}^{\frac{3}{rk+2}} \\
&\quad \text{[By using interpolation]} \\
&\lesssim \|U\|_{V^1}^{\frac{rk+5}{rk+2}}\|u_2\|_{L^{rk+2}(\mathbb{T})}\|U\|_{V^0}^{\frac{rk-1}{rk+2}} \\
&\quad \text{[Since } V^0 \hookrightarrow L^2 \text{ and } V^1 \hookrightarrow L^6] \\
&\leq \frac{\nu}{2}\|U\|_{V^1}^2 + C\|u_2\|_{L^{rk+2}(\mathbb{T})}^{\frac{2rk+4}{rk-1}}\|U\|_{V^0}^2 \\
&\quad \text{[By using the Young inequality]} \\
&\leq \frac{\nu}{2}\|U\|_{V^1}^2 + C(1 + \|u_2\|_{L^{rk+2}(\mathbb{T})}^{rk+2})\|U\|_{V^0}^2 \qquad (3.14) \\
&\quad \text{[Since } \frac{2rk+4}{rk-1} \leq rk+2].
\end{aligned}$$

It follows from (3.12), (3.13) and (3.14) that

$$\frac{d}{dt}\|U\|_{V^0}^2 \lesssim (1 + \|u_2\|_{L^{rk+2}(\mathbb{T})}^{rk+2})\|U\|_{V^0}^2. \tag{3.15}$$

Using the Grönwall's inequality, we deduce from (3.15), (2.14) and (3.4) that the weak solution depends continuously on the initial data and is unique.
Case 2. $1 < \alpha < \frac{5}{4}$. This implies that there exists $rk \in (\frac{5-2\alpha}{2\alpha-1}, \frac{5-2\alpha}{\alpha-1})$.

We could estimate the nonlinear term $b(U, u_2, U)$ as

$$\begin{aligned}
|b(U,u_2,U)| &= |b(U,U,u_2)| \\
&\lesssim \|U\|_{L^{\frac{6rk+12}{rk(2\alpha+1)+4(\alpha-1)}}(\mathbb{T})} \|\nabla U\|_{L^{\frac{6}{5-2\alpha}}(\mathbb{T})} \|u_2\|_{L^{rk+2}(\mathbb{T})} \\
&\quad \text{[By using the Hölder inequality]} \\
&\lesssim \|u_2\|_{L^{rk+2}(\mathbb{T})} \|U\|_{V^\alpha} \|U\|_{V^{\frac{rk+5}{rk+2}-\alpha}} \\
&\quad \text{[Since } V^{\alpha-1} \hookrightarrow L^{\frac{6}{5-2\alpha}}(\mathbb{T}) \text{ and } V^{\frac{rk+5}{rk+2}-\alpha} \hookrightarrow L^{\frac{6rk+12}{rk(2\alpha+1)+4(\alpha-1)}}(\mathbb{T})\text{]} \\
&\lesssim \|u_2\|_{L^{rk+2}(\mathbb{T})} \|U\|_{V^0}^{\theta} \|U\|_{V^\alpha}^{2-\theta} \\
&\quad \text{[we have used interpolation inequalities and} \\
&\quad \frac{rk+5}{rk+2} - \alpha = \theta \cdot 0 + (1-\theta)\cdot\alpha \text{ and } 0 \le \theta = 2 - \frac{rk+5}{\alpha(rk+2)} \le 1 \text{]} \\
&\le \frac{\nu}{2}\|U\|_{V^\alpha}^2 + C\|u_2\|_{L^{rk+2}(\mathbb{T})}^{\frac{2}{\theta}} \|U\|_{V^0}^2 \\
&\quad \text{[By using the Young inequality]} \\
&\le \frac{\nu}{2}\|U\|_{V^\alpha}^2 + C\|u_2\|_{L^{rk+2}(\mathbb{T})}^{rk+2} \|U\|_{V^0}^2 \qquad (3.16) \\
&\quad \text{[Since } \frac{2}{\theta} \le rk+2\text{].}
\end{aligned}$$

It follows from (3.12), (3.13) and (3.16) that

$$\frac{d}{dt}\|U\|_{V^0}^2 \lesssim (1 + \|u_2\|_{L^{rk+2}(\mathbb{T})}^{rk+2})\|U\|_{V^0}^2. \qquad (3.17)$$

Using the Grönwall's inequality again, (3.17), (2.14) and (3.4), we obtain clearly the continuous dependence of the weak solution on the initial data, in particular its uniqueness holds provided that $1 < \alpha < \frac{5}{4}$.
Case 3. $\alpha \ge \frac{5}{4}$.

In the hyperdissipative cases ($\alpha \ge \frac{5}{4}$), the nonlinear term $b(U, u_2, U)$ can be estimated by using Proposition 2.1 as follows

$$\begin{aligned}
|b(U,u_2,U)| &\lesssim \|U\|_{V^0}\|u_2\|_{V^\alpha}\|U\|_{V^\alpha} \\
&\quad \text{[By using the Hölder inequality]} \\
&\le \frac{\nu}{2}\|U\|_{V^\alpha} + C\|u_2\|_{V^\alpha}^2\|U\|_{V^0}^2 \qquad (3.18) \\
&\quad \text{[By using the Young inequality].}
\end{aligned}$$

Combining (3.12), (3.13) and (3.18), we get

$$\frac{d}{dt}\|U\|_{V^0}^2 \lesssim \|u_2\|_{V^\alpha}^2\|U\|_{V^0}^2. \qquad (3.19)$$

We also infer from the Grönwall's inequality, (3.4) and (3.19) the uniqueness of the global weak solution of the system (2.7) can be obtained with the less restriction of the damping.

Moreover, using Lemma 2.4 implies that $u \in C([\tau, T]; V^0)$. □

Remark 3.1. We now give some comments on our result.

(i) The existence of the weak solutions of (2.7) still holds with the less regularity of f, i.e., $f \in L^2_{loc}(\mathbb{R}; V^{-\alpha})$. We also see that the weak solution can be extended to the $[\tau, \infty)$ for any $\tau \in \mathbb{R}$. Hence, a weak solution defined globally in times exists for any initial data $u(\tau) \in V^0$.

(ii) This theorem shows us how the strength of nonlinearity and the degree of viscous dissipation can work together to yield the global existence and uniqueness of the weak solution of (2.7). This result is in addition to previous results in [10, 11, 12, 13].

(iii) In case of $r < 1$, since Lemma 2.2 may not be satisfied. This could make our situation much more difficult.

4. Attractors for the generalized Navier-Stokes equations with exponential damping

In this section, we apply the theory of the evolutionary systems established to our generalized Navier-Stokes equations with exponential damping.

Following the ideas in [54, Section 4], [22, Section 8], [29, Section 5-6] and [27], we first define the strong and weak distances as

$$d_s(u_1, u_2) := \|u_1 - u_2\|_{V^0}, \ \forall u_1, u_2 \in V^0,$$

and

$$d_w(u_1, u_2) := \sum_{k \in J} 2^{-|k|} \frac{|u_{1k} - u_{2k}|}{1 + |u_{1k} - u_{2k}|},$$

where u_{ik}, $i = 1, 2$, are Fourier coefficients of u_i, respectively. Note that the weak metric d_w induces the weak topology in any ball in V^0.

Let f_0 be a fixed external force which is translation bounded in $L^2_{loc}(\mathbb{R}; V^0)$, i.e.,

$$\|f_0\|_b^2 := \|f_0\|^2_{L^2_b(\mathbb{R}; V^0)} := \sup_{t \in \mathbb{R}} \int_t^{t+1} \|f_0(s)\|^2_{V^0} ds < \infty.$$

We denote by $L^{2,w}_{loc}(\mathbb{R}; V^0)$ the space $L^2_{loc}(\mathbb{R}; V^0)$ endowed with the local weak convergence topology. Then f_0 is translation compact in $L^{2,w}_{loc}(\mathbb{R}; V^0)$, i.e., the translation family of f_0

$$\Sigma := \{f_0(\cdot + h) : h \in \mathbb{R}\},$$

is precompact in $L^{2,w}_{loc}(\mathbb{R}; V^0)$ (see [21]). Moreover,

$$\|f\|_b^2 \le \|f_0\|_b^2, \ \ \forall f \in \Sigma. \tag{4.1}$$

Let $u(t)$, $t \in [\tau, \infty)$, be a weak solution of (2.7) with the initial data $u(\tau) \in V^0$ and $f \in \Sigma$ guaranteed by Theorem 3.1. Repeating the same arguments in Theorem 3.1 implies that

$$\frac{d}{dt}\|u(t)\|^2_{V^0} + \nu\|u(t)\|^2_{V^\alpha} + 2a\|(e^{b|u(t)|^r} - 1)|u(t)|^2\|_{L^1(\mathbb{T})} \le \frac{1}{\nu}\|f(t)\|^2_{V^0}. \tag{4.2}$$

Thus

$$\frac{d}{dt}\|u(t)\|^2_{V^0} + \nu\|u(t)\|^2_{V^0} \le \frac{1}{\nu}\|f(t)\|^2_{V^0},$$

for t large enough and hence

$$\frac{d}{dt}(\|u(t)\|^2_{V^0} e^{\nu t}) \le \frac{1}{\nu}\|f(t)\|^2_{V^0} e^{\nu t}.$$

Integrating in time from t_0 to t, we receive

$$\|u(t)\|_{V^0}^2 e^{\nu t} - \|u(t_0)\|_{V^0}^2 e^{\nu t_0} \le \frac{1}{\nu}\int_{t_0}^{t} \|f(s)\|_{V^0}^2 e^{\nu s} ds.$$

On the other hand,

$$\begin{aligned}\int_{t_0}^{t} \|f(s)\|_{V^0}^2 e^{\nu s} ds &\le \int_{t-1}^{t} \|f(s)\|_{V^0}^2 e^{\nu s} ds + \int_{t-2}^{t-1} \|f(s)\|_{V^0}^2 e^{\nu s} ds + \cdots \\ &\le \|f\|_b^2 (1 + e^{-\nu} + \cdots) e^{\nu t} \\ &\le \frac{e^{\nu}}{e^{\nu}-1}\|f\|_b^2 e^{\nu t} \\ &\le \frac{e^{\nu}}{e^{\nu}-1}\|f_0\|_b^2 e^{\nu t}.\end{aligned}$$

Therefore

$$\|u(t)\|_{V^0}^2 \le \|u(t_0)\|_{V^0}^2 e^{-\nu(t-t_0)} + \frac{e^{\nu}}{\nu(e^{\nu}-1)}\|f_0\|_b^2, \tag{4.3}$$

for all $t \ge t_0$, t_0 a.e. in $[\tau, \infty)$. Note that we are just looking for estimations and we use the same units of "1" because of simplicity. In fact, units in (4.2) and (4.3) are not dimensionally correct.

It follows from (4.3) that there exists a uniformly (w.r.t. $\tau \in \mathbb{R}$ and $f \in \Sigma$) absorbing ball $B_s(0, R) \subset V^0$, where the radius R depends on ν and $\|f_0\|_b^2$. We denote by X_{cuab} a closed uniformly absorbing ball

$$X_{cuab} := \{u \in V^0 : \|u\|_{V^0} \le R\}. \tag{4.4}$$

Therefore, for any bounded set $B \subset V^0$, there exists a time $\bar{t} \ge 0$ independent of the initial time τ, such that

$$u(t) \in X_{cuab},\ \forall t \ge t_1 := \tau + \bar{t}, \tag{4.5}$$

for every weak solutions u with $f \in \Sigma$ and the initial time $u(\tau) \in B$. Moreover, X_{cuab} is weakly compact in V^0 and metrizable with a metric d_w deducing the weak topology on X_{cuab}.

The following important result holds.

Lemma 4.1. *Let ν, α, a, b be positive and let $r \ge 1$. Assume that u_n is a sequence of weak solutions of* (2.7) *with $f_n \in \Sigma$ satisfying $u_n(t) \in X_{cuab}$ for all $t \ge t_1$. Then*

u_n is bounded in $L^2(t_1, t_2; V^{\alpha})$, $\mathcal{G}_b^r(t_1, t_2; L^1(\mathbb{T}))$ and $L^{\infty}(t_1, t_2; V^0)$,

$\frac{d}{dt}u_n$ is bounded in $L^1(t_1, t_2; V^{-\gamma_0})$,

for all $t_2 \ge t_1$ and $\gamma_0 := \max\{3, 2\alpha\}$. Moreover, there exists a subsequence u_{n_j} converges to some solution u in $C_w([t_1, t_2]; V^0)$, i.e.,

$$\langle u_{n_j}, \psi\rangle \to \langle u, \psi\rangle \text{ uniformly on } [t_1, t_2], \text{ as } n_j \to \infty, \text{ for all } \psi \in V^0.$$

Proof. The proof is a straightforward modification of the results of Theorem 3.1. Therefore, we omit it here (the readers can consult more details in [29, Lemma 5.4], [54, Lemma 5.3], [53, Lemma 3.2] and [59, Lemma 2.1]). □

Let us define the following evolutionary system.

$$\mathcal{E}([\tau,\infty)) := \{u(\cdot) : u(\cdot) \text{ is a weak solution of (2.7) with } f \in \Sigma \text{ on } [\tau,\infty) \text{ and } u(t) \in X_{cuab},\ \forall t \in [\tau,\infty)\}, \tau \in \mathbb{R},$$

$$\mathcal{E}((-\infty,\infty)) := \{u(\cdot) : u(\cdot) \text{ is a weak solution of (2.7) with } f \in \Sigma \text{ on } (-\infty,\infty) \text{ and } u(t) \in X_{cuab},\ \forall t \in (-\infty,\infty)\}.$$

We deduce from the translation identity of (2.7) that all conditions in Definition 5.1 hold for the above evolutionary system.

Since we use the theory of the evolutionary systems, this leads us to check the properties **(A1)**, **(A2)** and **(A3)** for our evolutionary system (see in Appendix A for these properties). Thus, we will use the following condition for the force which is called a normal function and was introduced in [51, 52].

Definition 4.1. *Let $\mathcal{B}$ be a Banach space. A function $g \in L^2_{loc}(\mathbb{R};\mathcal{B})$ is said to be normal in $L^2_{loc}(\mathbb{R};\mathcal{B})$ if for any $\epsilon > 0$, there exists $\delta > 0$, such that*

$$\sup_{t\in\mathbb{R}} \int_t^{t+\delta} \|g(s)\|^2_{\mathcal{B}}\, ds \le \epsilon.$$

It is a classical result that the class of normal functions is a proper closed subspace of the class of translation bounded functions (see [51, 52]). We now prove the following result.

Lemma 4.2. *Let ν, α, a, b be positive and let $r \ge 1$. The evolutionary system $\mathcal{E}$ of* (2.7) *with the force f_0 satisfies* **(A1)** *and* **(A3)**. *Moreover, if f_0 is normal in $L^2_{loc}(\mathbb{R};V^0)$, then the evolutionary system $\mathcal{E}$ of* (2.7) *also satisfies* **(A2)**.

Proof. We first verify that **(A1)** holds. Indeed, we deduce from Definition 2.1, Theorem 3.1 and (4.5) that $\mathcal{E}([0,\infty)) \subset C_w([0,\infty);V^0)$. Let $\{u_n\}_{n=1}^\infty$ be a sequence in $\mathcal{E}([0,\infty))$.

- It follows from Lemma 4.1 that there exists a subsequence, still denoted by $\{u_n\}_{n=1}^\infty$, which converges in $C_w([0,1];V^0)$ to some $u^1 \in C_w([0,1];V^0)$ as $n\to\infty$.
- Passing to a subsequence and dropping a subindex once more, we have that this subsequence converges in $C_w([0,2];V^0)$ to some $u^2 \in C_w([0,2];V^0)$ as $n\to\infty$. Note that $u^1(t) = u^2(t)$ on $[0,1]$.
- Continuing this diagonalization process, we obtain a subsequence $\{u_{n_j}\}$ of $\{u_n\}_{n=1}^\infty$ that converges in $C_w([0,\infty);V^0)$ to some $u \in C_w([0,\infty);V^0)$ as $n_j \to \infty$.

Therefore, **(A1)** holds.

Next, we prove that **(A3)** is valid. Take a sequence $\{u_n\}_{n=1}^\infty \subset \mathcal{E}([0,\infty))$ be such that it is a $d_{C_w([0,T];V^0)}$-Cauchy sequence in $C_w([0,T];V^0)$ for some $T > 0$. Using Lemma 4.1 again and the sequence $\{u_n\}_{n=1}^\infty$ is bounded in $L^2(0,T;V^\alpha)$, we deduce that there exists some $u \in C_w([0,T];V^0)$, such that

$$\int_0^T \|u_n(s) - u(s)\|^2_{V^0} ds \to 0, \text{ as } n \to \infty.$$

In particular, $\|u_n(t)\|_{V^0} \to \|u(t)\|_{V^0}$ as $n\to\infty$ a.e. on $[0,T]$, which means that $\{u_n(t)\}_{n=1}^\infty$ is a d_s-Cauchy sequence a.e. on $[0,T]$. Thus, **(A3)** is valid.

Finally, for any $u \in \mathcal{E}([0,\infty))$ and $t > 0$, using the property of normal functions, we can infer from (4.1) and (4.2) that, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|u(t)\|_{V^0} \le |u(t_0)|_{V^0} + \epsilon,$$

for t_0 a.e. in $(t-\delta, t)$. This implies that **(A2)** holds.

The proof is adapted from [29, Lemma 5.7] and the readers can consult more details in there. □

We now apply the general theory of the evolutionary system which is summarized in Appendix A to get the following results. The following result is a direct consequence of Theorem 5.3, Theorem 5.4, Theorem 5.5 and Lemma 4.2.

Theorem 4.1.

(i) *Assume that ν, α, a, b are positive and $r \ge 1$. Let f_0 be translation bounded in $L^2_{loc}(\mathbb{R}; V^0)$. There exist the weak uniform global attractor $\mathcal{A}_w$ and the weak trajectory attractor $\mathfrak{A}_w$ for (2.7) with the fixed force f_0. The weak uniform global attractor $\mathcal{A}_w$ is the maximal invariant and maximal quasi-invariant set w.r.t. the closure $\bar{\mathcal{E}}$ of the corresponding evolutionary system $\mathcal{E}$ and*

$$\begin{aligned}
\mathcal{A}_w &= \omega_w(X_{cuab}) = \omega_s(X_{cuab}) = \{u(0) : u \in \bar{\mathcal{K}}\},\\
\mathfrak{A}_w &= \Pi_+\bar{\mathcal{K}} = \{u(\cdot)|_{[0,\infty)} : u \in \bar{\mathcal{K}}\},\\
\mathcal{A}_w &= \mathfrak{A}_w(t) = \{u(t) : u \in \mathfrak{A}_w\},\ \forall t \ge 0.
\end{aligned}$$

Moreover, $\mathfrak{A}_w$ satisfies the finite weak uniform tracking property and is weakly equicontinuity on $[0,\infty)$.

(ii) *Furthermore, assume that f_0 is normal in $L^2_{loc}(\mathbb{R}; V^0)$ and every complete trajectory of $\bar{\mathcal{E}}$ is strongly continuous, then the weak global attractor $\mathcal{A}_w$ becomes a strongly compact strong global attractor $\mathcal{A}_s$, and the weak trajectory attractor $\mathfrak{A}_w$ becomes a strongly compact strong trajectory attractor $\mathfrak{A}_s$. Moreover, $\mathfrak{A}_s = \Pi_+\bar{\mathcal{K}}$ satisfies the finite strong uniform tracking property and is strongly equicontinuous on $[0,\infty)$.*

We now give some supplementaries with the better condition of f_0 which is translation compact in $L^2_{loc}(\mathbb{R}; V^0)$, i.e., the closure of the translation family Σ of f_0 in $L^2_{loc}(\mathbb{R}; V^0)$,

$$\bar{\Sigma} := \overline{\{f_0(\cdot + h) : h \in \mathbb{R}\}}^{L^2_{loc}(\mathbb{R};V^0)},$$

is compact in $L^2_{loc}(\mathbb{R}; V^0)$. Following the results in [21, 51, 52], we infer that $L^2_{loc}(\mathbb{R}; V^0)$ is metrizable and the corresponding metric space is complete; and the class of translation compact functions is also a closed subspace of the class of translation bounded functions, but it is a proper subset of the class of normal functions.

The results in Lemma 4.1 are valid for Σ replaced by $\bar{\Sigma}$. We only give the result and omit the proof here since it can be adapted from Theorem 3.1, [29, Lemma 6.1] and the property of the class of translation compact functions as follows

Lemma 4.3. *Let ν, α, a, b be positive and let $r \ge 1$. Assume that u_n is a sequence of weak solutions of (2.7) with $f_n \in \bar{\Sigma}$ satisfying $u_n(t) \in X_{cuab}$ for all $t \ge t_1$. Then*

$$u_n \text{ is bounded in } L^2(t_1, t_2; V^\alpha),\ \mathcal{G}^r_b(t_1, t_2; L^1(\mathbb{T})) \text{ and } L^\infty(t_1, t_2; V^0),$$

$$\frac{d}{dt}u_n \text{ is bounded in } L^1(t_1, t_2; V^{-\gamma_0}),$$

for all $t_2 \geq t_1$ *and* $\gamma_0 := \max\{3, 2\alpha\}$. *Moreover, there exists a subsequence* n_j *such that* $f_{n_j} \in \bar{\Sigma}$ *converges in* $L^2_{loc}(\mathbb{R}; V^0)$ *to some* $f \in \bar{\Sigma}$ *and* u_{n_j} *converges in* $C_w([t_1, t_2]; V^0)$ *to some solution* u *with the force* $f \in \bar{\Sigma}$, *i.e.,*

$$\langle u_{n_j}, \psi\rangle \to \langle u, \psi\rangle \text{ uniformly on } [t_1, t_2], \text{ as } n_j \to \infty, \text{ for all } \psi \in V^0.$$

We now define the following evolutionary system with $\bar{\Sigma}$ as a symbol space. The family of trajectories for this evolutionary system consists of all weak solutions of (2.7) with the force $f \in \bar{\Sigma}$ in X_{cuab} determined by

$$\mathcal{E}_{\bar{\Sigma}}([\tau, \infty)) := \{u(\cdot) : u(\cdot) \text{ is a weak solution on } [\tau, \infty) \text{ with } f \in \bar{\Sigma} \text{ and } u(t) \in X_{cuab}, \ \forall t \in [\tau, \infty)\}, \tau \in \mathbb{R},$$

$$\mathcal{E}_{\bar{\Sigma}}((-\infty, \infty)) := \{u(\cdot) : u(\cdot) \text{ is a weak solution on } (-\infty, \infty) \text{ with } f \in \bar{\Sigma} \text{ and } u(t) \in X_{cuab}, \ \forall t \in (-\infty, \infty)\}.$$

Following step by step as in arguments of [29, Section 6] and [54, Section 4.1] with the straightforward modification of these results, we can prove the following results. Because the proofs are only adapted, we also omit them here.

Lemma 4.4. *Let* ν, α, a, b *be positive and let* $r \geq 1$. *Assume that the external body force* f *belongs to* $\bar{\Sigma}$. *Then, the following results hold for the evolutionary system* $\mathcal{E}_{\bar{\Sigma}}$ *of the family of the 3D generalized Navier-Stokes equations with damping*

(i) *It satisfies* **(B1)**, **(B2)** *and* **(B3)**.
(ii) *It is closed.*
(iii) $\bar{\mathcal{E}}_{\bar{\Sigma}} = \mathcal{E}_{\bar{\Sigma}}$.

Theorem 4.2.

(i) *Assume that* ν, α, a, b *are positive and* $r \geq 1$. *Let* f_0 *be translation compact in* $L^2_{loc}(\mathbb{R}; V^0)$. *Then the weak uniform global attractor* $\mathcal{A}^{\bar{\Sigma}}_w$ *and the weak trajectory attractor* $\mathfrak{A}^{\bar{\Sigma}}_w$ *for* (2.7) *with the external body force* $f \in \bar{\Sigma}$ *exist,* $\mathcal{A}^{\bar{\Sigma}}_w$ *is the maximal invariant and maximal quasi-invariant set w.r.t. the corresponding evolutionary system* $\mathcal{E}_{\bar{\Sigma}}$ *and*

$$\mathcal{A}^{\bar{\Sigma}}_w = \{u(0) : u \in \mathcal{E}_{\bar{\Sigma}}((-\infty, \infty)) = \bigcup_{f \in \bar{\Sigma}} \mathcal{E}_f((-\infty, \infty))\},$$

$$\mathfrak{A}^{\bar{\Sigma}}_w = \Pi_+ \bigcup_{f \in \bar{\Sigma}} \mathcal{E}_f((-\infty, \infty)),$$

$$\mathcal{A}^{\bar{\Sigma}}_w = \mathfrak{A}^{\bar{\Sigma}}_w(t) = \left\{u(t) : u \in \mathfrak{A}^{\bar{\Sigma}}_w\right\}, \ \forall t \geq 0,$$

where $\mathcal{E}_f((-\infty, \infty))$ *is nonempty for any* $f \in \bar{\Sigma}$. *Moreover,* $\mathfrak{A}^{\bar{\Sigma}}_w$ *satisfies the finite weak uniform tracking property and is weakly equicontinuous on* $[0, \infty)$.

(ii) *Furthermore, assume that the external body force* $f \in \bar{\Sigma}$ *and every complete trajectory of the family of the 3D generalized Navier-Stokes equations with damping is strongly continuous, then the weak uniform global attractor* $\mathcal{A}^{\bar{\Sigma}}_w$ *is a strongly compact strong global attractor* $\mathcal{A}^{\bar{\Sigma}}_s$, *and the weak trajectory attractor* $\mathfrak{A}^{\bar{\Sigma}}_w$ *is a strongly compact strong trajectory attractor* $\mathfrak{A}^{\bar{\Sigma}}_s$. *Moreover,* $\mathfrak{A}^{\bar{\Sigma}}_s$ *satisfies the finite strong uniform tracking property and is strongly equicontinuous on* $[0, \infty)$.

We deduce from Lemma 4.4 that $\mathcal{E} \subset \bar{\mathcal{E}} \subset \mathcal{E}_{\bar{\Sigma}}$. We now concern with the question: *Are the attractors $\mathcal{A}_\bullet$, $\mathfrak{A}_\bullet$ and $\mathcal{A}_\bullet^{\bar{\Sigma}}$, $\mathfrak{A}_\bullet^{\bar{\Sigma}}$ in Theorem 4.1 and Theorem 4.2 are identical ?* We may get the negative answer as the weak solutions of (2.7) are not unique. The positive answer is the content of the following results

Theorem 4.3. *Assume that ν, a, b are positive and $\alpha, r \geq 1$. Let f_0 be translation compact in $L^2_{loc}(\mathbb{R}; V^0)$. Let $\mathcal{E}_\Sigma$ be the evolutionary system of (2.7) with the external body force in Σ and $\bar{\mathcal{E}}_\Sigma$ be the closure of $\mathcal{E}_\Sigma$. Let $\mathcal{E}_{\bar{\Sigma}}$ be the evolutionary system of (2.7) with the external body force in $\bar{\Sigma}$. Hence, the following results hold.*

(1) The three weak uniform global attractors $\mathcal{A}_w^\Sigma$, $\bar{\mathcal{A}}_w^\Sigma$ and $\mathcal{A}_w^{\bar{\Sigma}}$ for evolutionary systems $\mathcal{E}_\Sigma$, $\bar{\mathcal{E}}_\Sigma$ and $\mathcal{E}_{\bar{\Sigma}}$, respectively, exist.

(2) $\mathcal{A}_w^\Sigma$, $\bar{\mathcal{A}}_w^\Sigma$ and $\mathcal{A}_w^{\bar{\Sigma}}$ are the maximal invariant and maximal quasi-invariant set with respect to $\bar{\mathcal{E}}_\Sigma$ and satisfy the following

$$\mathcal{A}_w^\Sigma = \bar{\mathcal{A}}_w^\Sigma = \mathcal{A}_w^{\bar{\Sigma}} = \Big\{ u(0) : u \in \mathcal{E}_{\bar{\Sigma}}((-\infty, \infty)) \Big\}.$$

(3) The three weak trajectory attractors $\mathfrak{A}_w^\Sigma$, $\bar{\mathfrak{A}}_w^\Sigma$ and $\mathfrak{A}_w^{\bar{\Sigma}}$ for evolutionary systems $\mathcal{E}_\Sigma$, $\bar{\mathcal{E}}_\Sigma$ and $\mathcal{E}_{\bar{\Sigma}}$, respectively, exist and satisfy the following

$$\mathfrak{A}_w^\Sigma = \bar{\mathfrak{A}}_w^\Sigma = \mathfrak{A}_w^{\bar{\Sigma}} = \Pi_+ \bigcup_{f \in \bar{\Sigma}} \mathcal{E}_f((-\infty, \infty)).$$

Hence, the three weak trajectory attractors satisfy the finite weak uniform tracking property for all the three evolutionary systems and are weakly equicontinuous on $[0, \infty)$.

(4) $\mathcal{A}_w^\Sigma$, $\bar{\mathcal{A}}_w^\Sigma$ and $\mathcal{A}_w^{\bar{\Sigma}}$ are sections of $\mathfrak{A}_w^\Sigma$, $\bar{\mathfrak{A}}_w^\Sigma$ and $\mathfrak{A}_w^{\bar{\Sigma}}$:

$$\begin{aligned} &\mathcal{A}_w^\Sigma = \bar{\mathcal{A}}_w^\Sigma = \mathcal{A}_w^{\bar{\Sigma}} \\ &\quad = \mathfrak{A}_w^\Sigma(t) = \bar{\mathfrak{A}}_w^\Sigma(t) = \mathfrak{A}_w^{\bar{\Sigma}}(t) = \Big\{ u(t) : u \in \mathfrak{A}_w^{\bar{\Sigma}} \Big\}, \ \forall t \geq 0. \end{aligned}$$

(5) The three weak uniform global attractors $\mathcal{A}_w^\Sigma$, $\bar{\mathcal{A}}_w^\Sigma$ and $\mathcal{A}_w^{\bar{\Sigma}}$ for evolutionary systems $\mathcal{E}_\Sigma$, $\bar{\mathcal{E}}_\Sigma$ and $\mathcal{E}_{\bar{\Sigma}}$, respectively, are strongly compact strong uniform global attractors and the three weak trajectory attractors $\mathfrak{A}_w^\Sigma$, $\bar{\mathfrak{A}}_w^\Sigma$ and $\mathfrak{A}_w^{\bar{\Sigma}}$ for evolutionary systems $\mathcal{E}_\Sigma$, $\bar{\mathcal{E}}_\Sigma$ and $\mathcal{E}_{\bar{\Sigma}}$, respectively, are strongly compact strong trajectory attractors. Moreover, the three trajectory attractors satisfy the finite strong uniform tracking property for all the three evolutionary systems and are strongly equicontinuous on $[0, \infty)$.

Remark 4.1. These results also extend and improve the recent results for the Navier-Stokes equations in [22, 27, 29, 54].

5. Determining wavenumbers

In this section we will point out the finite uniform tracking property of attractors via determining wavenumbers. We define the determining wavenumber in the following way:

$$\begin{aligned} \mathcal{N}_u^\alpha(t) := \min\{\lambda_q = 2^q : \lambda_p^{-\alpha+1+\delta} \lambda_q^{-\alpha-\delta} \|u_p\|_{L^\infty(\mathbb{T})} < c_0 \nu, \ \forall p > q \\ \text{and } \lambda_q^{-2\alpha} \sum_{j=0}^{q} \lambda_j \|u_j\|_{L^\infty(\mathbb{T})} < c_0 \nu, \ q \in \mathbb{N}\}, \quad (5.1) \end{aligned}$$

where $0 < \delta < \alpha$ is a fixed (small) parameter, and c_0 is an dimensionless constant that depends only on α and λ_q, $u_p = \Delta_p u$ which is the localized Fourier projection operators (see in Appendix B for more details).

We are now ready to state and prove our main results.

Theorem 5.1. *Assume that ν, a, b are positive, $\alpha \geq \frac{1}{2}$ and $r \geq 1$. Let $u(t)$ and $v(t)$ be two global weak solutions of (2.7) on the weak global attractor $\mathcal{A}$. Let $\mathcal{N}(t) := \max\{\mathcal{N}_u^\alpha(t), \mathcal{N}_v^\alpha(t)\}$ and $Q(t)$ be such that $\mathcal{N}(t) = \lambda_{Q(t)}$. If*

$$u(t)_{\leq Q(t)} = v(t)_{\leq Q(t)}, \ \forall t < 0, \tag{5.2}$$

then

$$u(t) = v(t), \ \forall t \leq 0. \tag{5.3}$$

Proof. Denote $w := u - v$, which satisfies the equation

$$w_t + \nu\Lambda^{2\alpha} w + B(u, w) + B(w, v) + a\big((e^{b|u|^r} - 1)u - (e^{b|v|^r} - 1)v\big) = 0 \tag{5.4}$$

in the sense of distributions. We deduce from (5.2) that $w(t)_{\leq Q(t)} \equiv 0$.

Applying Δ_q to (5.4) yields

$$\partial_t \Delta_q w + \nu\Lambda^{2\alpha}\Delta_q w + \Delta_q(u \cdot \nabla w) + \Delta_q(w \cdot \nabla v) + a\Delta_q\big((e^{b|u|^r} - 1)u - (e^{b|v|^r} - 1)v\big) = 0. \tag{5.5}$$

Dotting (5.5) with $\Delta_q w$, integrating by parts and using $\nabla \cdot u = 0$, we have

$$\frac{1}{2}\frac{d}{dt}\|w_q\|_{L^2(\mathbb{T})}^2 + \nu\|\Lambda^\alpha w_q\|_{L^2(\mathbb{T})}^2 + \int_{\mathbb{T}} \Delta_q(u \cdot \nabla w) w_q dx + \int_{\mathbb{T}} \Delta_q(w \cdot \nabla v) w_q dx + a\int_{\mathbb{T}} \Delta_q\big((e^{b|u|^r} - 1)u - (e^{b|v|^r} - 1)v\big) w_q dx = 0. \tag{5.6}$$

Integrating in time, taking the ℓ^2-norm of the sequence in (5.6), identifying $B^0_{2,2}$ with V^0 and using Lemma 2.6, we deduce that

$$\begin{aligned} \frac{1}{2}\|w(t)\|_{V^0}^2 - \frac{1}{2}\|w(t_0)\|_{V^0}^2 + \nu\int_{t_0}^t \|\Lambda^\alpha w(\tau)\|_{V^0}^2 d\tau &\lesssim \int_{t_0}^t \sum_{q\geq 0} |\int_{\mathbb{T}} \Delta_q(u \cdot \nabla w) w_q dx| d\tau \\ &\quad + \int_{t_0}^t \sum_{q\geq 0} |\int_{\mathbb{T}} \Delta_q(w \cdot \nabla v) w_q dx| d\tau \\ &:= \int_{t_0}^t I d\tau + \int_{t_0}^t J d\tau. \end{aligned} \tag{5.7}$$

Using Bony's paraproduct implies

$$w \cdot \nabla v = \sum_{m=0}^\infty w_{\leq m-2} \cdot \nabla v_m + \sum_{m=0}^\infty w_m \cdot \nabla v_{\leq m-2} + \sum_{m=0}^\infty \widetilde{w}_m \cdot \nabla v_m,$$

where $\widetilde{w}_m = w_{m-1} + w_m + w_{m+1}$. Therefore,

$$\Delta_q(w \cdot \nabla v) = \sum_{m=0}^\infty \Delta_q(w_{\leq m-2} \cdot \nabla v_m) + \sum_{m=0}^\infty \Delta_q(w_m \cdot \nabla v_{\leq m-2}) + \sum_{m=0}^\infty \Delta_q(\widetilde{w}_m \cdot \nabla v_m).$$

We use the triangle inequality and Lemma 5.4 to decompose J as follows

$$\begin{aligned} J \lesssim & \sum_{q\geq 0}\sum_{|q-m|\leq 1} \left|\int_{\mathbb{T}} \Delta_q(w_{\leq m-2}\cdot\nabla v_m)w_q dx\right| \\ & + \sum_{q\geq 0}\sum_{|q-m|\leq 1} \left|\int_{\mathbb{T}} \Delta_q(w_m\cdot\nabla v_{\leq m-2})w_q dx\right| \\ & + \sum_{q\geq 0}\sum_{m\geq q-1} \left|\int_{\mathbb{T}} \Delta_q(\widetilde{w}_m\cdot\nabla v_m)w_q dx\right| \\ & := J_1 + J_2 + J_3. \end{aligned} \tag{5.8}$$

We will estimate the above terms in turn. We adapt the convention that $(Q, m-2]$ is empty if $m-2 \leq Q$. Thus, the first term J_1 can be estimated as follows

$$\begin{aligned} J_1 \leq & \sum_{m\geq Q+2}\sum_{|q-m|\leq 1} \left|\int_{\mathbb{T}^3} \Delta_q(w_{\leq m-2}\cdot\nabla v_m)w_q dx\right| \\ & \quad \text{since } w(t)_{\leq Q(t)} \equiv 0 \text{ and we need } m-2\geq Q \\ \lesssim & \sum_{m\geq Q+2}\sum_{|q-m|\leq 1} \|w_{(Q,m-2]}\|_{L^2(\mathbb{T})}\lambda_m\|v_m\|_{L^\infty(\mathbb{T})}\|w_q\|_{L^2(\mathbb{T})}, \\ & \quad \text{by using Hölder's inequality and Proposition 5.1} \end{aligned} \tag{5.9}$$

It follows from (5.1) that

$$\|v_m\|_{L^\infty(\mathbb{T})} < c_0\nu\lambda_Q^{\alpha+\delta}\lambda_m^{\alpha-1-\delta},\ \forall m > Q, \tag{5.10}$$

and

$$\lambda_Q^{-2\alpha}\|\nabla v_{\leq Q}\|_{L^\infty(\mathbb{T})} \lesssim \lambda_Q^{-2\alpha}\sum_{q=0}^{Q}\lambda_q\|v_q\|_{L^\infty(\mathbb{T})} \lesssim c_0\nu. \tag{5.11}$$

We deduce from (5.9), (5.10) and Young's inequality that

$$\begin{aligned} J_1 \lesssim & c_0\nu\sum_{m\geq Q+2}\sum_{|q-m|\leq 1}\lambda_m^{\alpha-\delta}\lambda_Q^{\alpha+\delta}\|w_q\|_{L^2(\mathbb{T})}\sum_{Q<p\leq m-2}\|w_p\|_{L^2(\mathbb{T})} \\ \lesssim & c_0\nu\sum_{q\geq Q+1}\lambda_q^{\alpha}\|w_q\|_{L^2(\mathbb{T})}\sum_{Q<p\leq q-1}\lambda_q^{-\delta}\lambda_Q^{\alpha+\delta}\|w_p\|_{L^2(\mathbb{T})} \\ & \quad \text{since } Q+1\leq m-1\leq q\leq m+1 \text{ and } \lambda_m^{\alpha-\delta}\lesssim\lambda_q^{\alpha-\delta} \\ \lesssim & c_0\nu\sum_{q\geq Q+1}\lambda_q^{\alpha}\|w_q\|_{L^2(\mathbb{T})}\sum_{Q<p\leq q-1}\lambda_p^{\alpha}\|w_p\|_{L^2(\mathbb{T})}\lambda_p^{-\alpha}\lambda_q^{-\delta}\lambda_Q^{\alpha+\delta} \\ \lesssim & c_0\nu\sum_{q\geq Q+1}\lambda_q^{\alpha}\|w_q\|_{L^2(\mathbb{T})}\sum_{Q<p\leq q-1}\lambda_p^{\alpha}\|w_p\|_{L^2(\mathbb{T})}\lambda_{q-p}^{-\delta} \\ & \quad \text{since } \lambda_p^{-\alpha}\lambda_q^{-\delta}\lambda_Q^{\alpha+\delta}\leq\lambda_p^{-\alpha}\lambda_q^{-\delta}\lambda_p^{\alpha+\delta}=\lambda_{q-p}^{-\delta} \\ \lesssim & c_0\nu\sum_{q\geq Q+1}\lambda_q^{2\alpha}\|w_q\|_{L^2(\mathbb{T})}^2 + c_0\nu\sum_{q\geq Q+1}\Big(\sum_{Q<p\leq q-1}\lambda_p^{\alpha}\|w_p\|_{L^2(\mathbb{T})}\lambda_{q-p}^{-\delta}\Big)^2 \\ & \quad \text{by using Young's inequality} \\ \lesssim & c_0\nu\sum_{q\geq Q+1}\lambda_q^{2\alpha}\|w_q\|_{L^2(\mathbb{T})}^2 + c_0\nu\sum_{q\geq Q+1}\sum_{Q<p\leq q-1}\lambda_p^{2\alpha}\|w_p\|_{L^2(\mathbb{T})}^2\lambda_{q-p}^{-\delta} \end{aligned}$$

by using Cauchy-Schwarz's inequality and $0 < \delta < \alpha$

$$\lesssim c_0\nu \sum_{q\geq Q+1} \lambda_q^{2\alpha} \|w_q\|^2_{L^2(\mathbb{T})} + c_0\nu \sum_{p\geq Q+1} \lambda_p^{2\alpha} \|w_p\|^2_{L^2(\mathbb{T})} \sum_{q\geq p+1} \lambda_{q-p}^{-\delta}$$

$$\lesssim c_0\nu \sum_{q\geq Q+1} \lambda_q^{2\alpha} \|w_q\|^2_{L^2(\mathbb{T})}. \tag{5.12}$$

We now estimate J_2 by the similar strategy. We have

$$\begin{aligned} J_2 &\leq \sum_{m\geq Q+2} \sum_{|q-m|\leq 1} \left|\int_{\mathbb{T}} \Delta_q(w_m \cdot \nabla v_{\leq m-2}) w_q dx\right| \\ &\lesssim \sum_{m\geq Q+2} \sum_{|q-m|\leq 1} \|w_m\|_{L^2(\mathbb{T})} \|\nabla v_{(Q,m-2]}\|_{L^\infty(\mathbb{T})} \|w_q\|_{L^2(\mathbb{T})} \\ &\quad + \sum_{m\geq Q+2} \sum_{|q-m|\leq 1} \|w_m\|_{L^2(\mathbb{T})} \|\nabla v_{\leq Q}\|_{L^\infty(\mathbb{T})} \|w_q\|_{L^2(\mathbb{T})} \\ &:= J_{21} + J_{22}. \end{aligned} \tag{5.13}$$

To estimate J_{21}.

$$J_{21} = \sum_{m\geq Q+2} \sum_{|q-m|\leq 1} \|w_m\|_{L^2(\mathbb{T})} \|\nabla v_{(Q,m-2]}\|_{L^\infty(\mathbb{T})} \|w_q\|_{L^2(\mathbb{T})}$$

$$\lesssim \sum_{m\geq Q+2} \sum_{|q-m|\leq 1} \|w_m\|_{L^2(\mathbb{T})} \|w_q\|_{L^2(\mathbb{T})} \sum_{Q<p\leq m-2} \|\nabla v_p\|_{L^\infty(\mathbb{T})}$$

by using the triangle inequality

$$\lesssim \sum_{m\geq Q+2} \sum_{|q-m|\leq 1} \|w_m\|_{L^2(\mathbb{T})} \|w_q\|_{L^2(\mathbb{T})} \sum_{Q<p\leq m-2} \lambda_p \|v_p\|_{L^\infty(\mathbb{T})}$$

by using Proposition 5.1

$$\lesssim c_0\nu \sum_{q\geq Q+1} \lambda_q^{2\alpha} \|w_q\|^2_{L^2(\mathbb{T})} \sum_{Q<p<q} \lambda_q^{-2\alpha} \lambda_p^{\alpha-\delta} \lambda_Q^{\alpha+\delta}$$

since $Q+1 \leq m-1 \leq q \leq m+1$ and by using (5.10)

$$\lesssim c_0\nu \sum_{q\geq Q+1} \lambda_q^{2\alpha} \|w_q\|^2_{L^2(\mathbb{T})}, \tag{5.14}$$

since $\sum_{Q<p<q} \lambda_q^{-2\alpha} \lambda_p^{\alpha-\delta} \lambda_Q^{\alpha+\delta}$ is bounded.

To estimate J_{22}.

$$J_{22} = \sum_{m\geq Q+2} \sum_{|q-m|\leq 1} \|w_m\|_{L^2(\mathbb{T})} \|\nabla v_{\leq Q}\|_{L^\infty(\mathbb{T})} \|w_q\|_{L^2(\mathbb{T})}$$

$$\lesssim \sum_{q\geq Q+1} \lambda_q^{2\alpha} \|w_q\|^2_{L^2(\mathbb{T})} \lambda_Q^{-2\alpha} \|\nabla v_{\leq Q}\|_{L^\infty(\mathbb{T})}$$

since $Q+1 \leq m-1 \leq q \leq m+1$ and $0 < \lambda_Q \leq \lambda_q$

$$\lesssim c_0\nu \sum_{q\geq Q+1} \lambda_q^{2\alpha} \|w_q\|^2_{L^2(\mathbb{T})}, \tag{5.15}$$

by using (5.11).

We now consider the term J_3. It follows from (5.8) that

$$\begin{aligned}
J_3 &= \sum_{q\geq 0}\sum_{m\geq q-1} |\int_{\mathbb{T}} \Delta_q(\widetilde{w}_m\cdot\nabla v_m)w_q dx| \\
&\lesssim \sum_{q\geq 0}\sum_{m\geq q-1} \int_{\mathbb{T}} |\Delta_q(\widetilde{w}_m\otimes v_m)\nabla w_q| dx \\
&\quad \text{by using integration by parts and divergence free condition} \\
&\lesssim \sum_{m\geq Q+1}\sum_{Q<q\leq m+1} \|\widetilde{w}_m\|_{L^2(\mathbb{T})}\|v_m\|_{L^\infty(\mathbb{T})}\|\nabla w_q\|_{L^2(\mathbb{T})} \\
&\quad \text{since } w(t)_{\leq Q(t)}\equiv 0 \text{ and using Hölder's inequality} \\
&\lesssim \sum_{m\geq Q+1} \|\widetilde{w}_m\|_{L^2(\mathbb{T})}\|v_m\|_{L^\infty(\mathbb{T})}\sum_{Q<q\leq m+1}\lambda_q\|w_q\|_{L^2(\mathbb{T})} \\
&\quad \text{by using Proposition 5.1} \\
&\lesssim c_0\nu\sum_{m\geq Q+1}\lambda_m^{\alpha-1-\delta}\lambda_Q^{\alpha+\delta}\|w_m\|_{L^2(\mathbb{T})}\sum_{Q<q\leq m+1}\lambda_q\|w_q\|_{L^2(\mathbb{T})} \\
&\quad \text{by using (5.1), (5.10) and the triangle inequality} \\
&= c_0\nu\sum_{m\geq Q+1}\lambda_m^{\alpha}\|w_m\|_{L^2(\mathbb{T})}\sum_{Q<q\leq m+1}\lambda_q^{\alpha}\|w_q\|_{L^2(\mathbb{T})}\lambda_Q^{\alpha+\delta}\lambda_m^{-1-\delta}\lambda_q^{1-\alpha} \\
&\lesssim c_0\nu\sum_{m\geq Q+1}\lambda_m^{\alpha}\|w_m\|_{L^2(\mathbb{T})}\sum_{Q<q\leq m+1}\lambda_q^{\alpha}\|w_q\|_{L^2(\mathbb{T})}\lambda_{m-q}^{-(1+\delta)} \\
&\quad \text{since } \lambda_Q^{\alpha+\delta}\lambda_m^{-1-\delta}\lambda_q^{1-\alpha}\leq\lambda_m^{-1-\delta}\lambda_q^{1+\delta}:=\lambda_{m-q}^{-(1+\delta)} \\
&\lesssim c_0\nu\sum_{m\geq Q+1}\lambda_m^{2\alpha}\|w_m\|^2_{L^2(\mathbb{T})} + c_0\nu\sum_{m\geq Q+1}\Big(\sum_{Q<q\leq m+1}\lambda_q^{\alpha}\|w_q\|_{L^2(\mathbb{T})}\lambda_{m-q}^{-(1+\delta)}\Big)^2 \\
&\quad \text{by using Young's inequality} \\
&\lesssim c_0\nu\sum_{m\geq Q+1}\lambda_m^{2\alpha}\|w_m\|^2_{L^2(\mathbb{T})}, \qquad (5.16)
\end{aligned}$$

where we have used $1+\delta>0$ and $Q<q\leq m+1$.

It follows from (5.8), (5.12), (5.13), (5.14), (5.15) and (5.16) that

$$J\lesssim c_0\nu\sum_{q\geq Q+1}\lambda_q^{2\alpha}\|w_q\|^2_{L^2(\mathbb{T})}. \tag{5.17}$$

We now investigate estimation for I. We have

$$I=\sum_{q\geq 0}|\int_{\mathbb{T}}\Delta_q(u\cdot\nabla w)w_q dx|$$

Applying Bony's paraproduct to I yields

$$u\cdot\nabla w=\sum_{m=0}^{\infty}u_{\leq m-2}\cdot\nabla w_m+\sum_{m=0}^{\infty}u_m\cdot\nabla w_{\leq m-2}+\sum_{m=0}^{\infty}u_m\cdot\nabla\widetilde{w}_m,$$

where $\widetilde{w}_m=w_{m-1}+w_m+w_{m+1}$. Therefore,

$$\Delta_q(u\cdot\nabla w)=\sum_{m=0}^{\infty}\Delta_q(u_{\leq m-2}\cdot\nabla w_m)+\sum_{m=0}^{\infty}\Delta_q(u_m\cdot\nabla w_{\leq m-2})+\sum_{m=0}^{\infty}\Delta_q(u_m\cdot\nabla\widetilde{w}_m).$$

Using the triangle inequality and Lemma 5.4, we can decompose I as follows

$$\begin{aligned} I \lesssim &\sum_{q\geq 0}|\sum_{|q-m|\leq 1}\int_{\mathbb{T}}\Delta_q(u_{\leq m-2}\cdot\nabla w_m)w_q dx| \\ &+\sum_{q\geq 0}|\sum_{|q-m|\leq 1}\int_{\mathbb{T}}\Delta_q(u_m\cdot\nabla w_{\leq m-2})w_q dx| \\ &+\sum_{q\geq 0}|\sum_{m\geq q-1}\int_{\mathbb{T}}\Delta_q(u_m\cdot\nabla \widetilde{w}_m)w_q dx| \\ &:= I_1+I_2+I_3. \end{aligned} \tag{5.18}$$

We deduce from (5.50) that

$$\begin{aligned} \Delta_q(u_{\leq m-2}\cdot\nabla w_m) =&[\Delta_q, u_{\leq m-2}\cdot\nabla]w_m + u_{\leq q-2}\cdot\nabla\Delta_q w_m \\ &+(u_{\leq m-2}-u_{\leq q-2})\cdot\nabla\Delta_q w_m. \end{aligned} \tag{5.19}$$

We now can further decompose I_1 as

$$\begin{aligned} I_1 &= \sum_{q\geq 0}|\sum_{|q-m|\leq 1}\int_{\mathbb{T}}\Delta_q(u_{\leq m-2}\cdot\nabla w_m)w_q dx| \\ &\leq \sum_{q\geq 0}|\sum_{|q-m|\leq 1}\int_{\mathbb{T}}[\Delta_q, u_{\leq m-2}\cdot\nabla]w_m w_q dx| \\ &\quad+\sum_{q\geq 0}|\sum_{|q-m|\leq 1}\int_{\mathbb{T}}u_{\leq q-2}\cdot\nabla w_q w_q dx| \\ &\quad+\sum_{q\geq 0}|\sum_{|q-m|\leq 1}\int_{\mathbb{T}}(u_{\leq m-2}-u_{\leq q-2})\cdot\nabla w_q w_q dx| \\ &\quad\text{where we have used } \sum_{|q-m|\leq 1}\Delta_q w_m = w_q \\ &= I_{11}+I_{12}+I_{13}. \end{aligned} \tag{5.20}$$

To estimate I_{11}.

$$\begin{aligned} I_{11} &= \sum_{q\geq 0}|\sum_{|q-m|\leq 1}\int_{\mathbb{T}}[\Delta_q, u_{\leq m-2}\cdot\nabla]w_m w_q dx| \\ &\leq \sum_{q\geq Q+1}\sum_{m\geq Q+1,|q-m|\leq 1}\|[\Delta_q, u_{\leq m-2}\cdot\nabla]w_m\|_{L^2(\mathbb{T})}\|w_q\|_{L^2(\mathbb{T})} \\ &\quad\text{by using Hölder's inequality and } w(t)_{\leq Q(t)}\equiv 0 \\ &\lesssim \sum_{q\geq Q+1}\sum_{m\geq Q+1,|q-m|\leq 1}\|\nabla u_{\leq m-2}\|_{L^\infty(\mathbb{T})}\|w_m\|_{L^2(\mathbb{T})}\|w_q\|_{L^2(\mathbb{T})} \\ &\quad\text{by using (5.51)} \\ &\leq \sum_{q\geq Q+1}\sum_{m\geq Q+1,|q-m|\leq 1}\|\nabla u_{\leq (Q,m-2]}\|_{L^\infty(\mathbb{T})}\|w_m\|_{L^2(\mathbb{T})}\|w_q\|_{L^2(\mathbb{T})} \\ &\quad+\sum_{q\geq Q+1}\sum_{m\geq Q+1,|q-m|\leq 1}\|\nabla u_{\leq Q}\|_{L^\infty(\mathbb{T})}\|w_m\|_{L^2(\mathbb{T})}\|w_q\|_{L^2(\mathbb{T})} \end{aligned}$$

$$\lesssim \sum_{q\geq Q+1} \|w_q\|^2_{L^2(\mathbb{T})} \sum_{Q<p<q} \lambda_p \|u_p\|_{L^\infty(\mathbb{T})}$$

by using Proposition 5.1 and Young's inequality

$$+ c_0\nu\lambda_Q^{2\alpha} \sum_{q\geq Q+1} \|w_q\|^2_{L^2(\mathbb{T})}$$

by using (5.11) and Young's inequality

$$\lesssim c_0\nu \sum_{q\geq Q+1} \|w_q\|^2_{L^2(\mathbb{T})} \sum_{Q<p<q} \lambda_Q^{\alpha+\delta}\lambda_p^{\alpha-\delta}$$

by using (5.10)

$$+ c_0\nu \sum_{q\geq Q+1} \lambda_q^{2\alpha}\|w_q\|^2_{L^2(\mathbb{T})}$$

since $\lambda_Q < \lambda_q$ for all $q \geq Q+1$

$$= c_0\nu \sum_{q\geq Q+1} \lambda_q^{2\alpha}\|w_q\|^2_{L^2(\mathbb{T})} \sum_{Q<p<q} \lambda_Q^{\alpha+\delta}\lambda_p^{\alpha-\delta}\lambda_q^{-2\alpha}$$

$$+ c_0\nu \sum_{q\geq Q+1} \lambda_q^{2\alpha}\|w_q\|^2_{L^2(\mathbb{T})}$$

$$\lesssim c_0\nu \sum_{q\geq Q+1} \lambda_q^{2\alpha}\|w_q\|^2_{L^2(\mathbb{T})} \tag{5.21}$$

since $\lambda_Q < \lambda_p < \lambda_q$ for all $Q < p < q$ and $\delta > 0$.

It follows from integration by parts and $\mathrm{div} u_{\leq m-2} = 0$ that

$$I_{12} = 0. \tag{5.22}$$

To estimate I_{13}.

$$I_{13} = \sum_{q\geq 0} |\sum_{|q-m|\leq 1} \int_{\mathbb{T}} (u_{\leq m-2} - u_{\leq q-2})\cdot \nabla w_q w_q dx|$$

$$\lesssim \sum_{q\geq Q+1} \lambda_q \|u_{(q-4,q]}\|_{L^\infty(\mathbb{T})} \|w_q\|^2_{L^2(\mathbb{T})}$$

by using $w(t)_{\leq Q(t)} \equiv 0$, Young's inequality and Proposition 5.1

$$\leq \sum_{Q+4>q\geq Q+1} \lambda_q \|u_{(q-4,Q]}\|_{L^\infty(\mathbb{T})} \|w_q\|^2_{L^2(\mathbb{T})} + \sum_{q\geq Q+4} \lambda_q \|u_{(Q,q]}\|_{L^\infty(\mathbb{T})} \|w_q\|^2_{L^2(\mathbb{T})}$$

$$\lesssim c_0\nu \sum_{Q+4>q\geq Q+1} \lambda_q^{2\alpha}\|w_q\|^2_{L^2(\mathbb{T})}$$

by using (5.1)

$$+ c_0\nu \sum_{q\geq Q+4} \sum_{Q<p\leq q} \lambda_q \lambda_p^{\alpha-1-\delta}\lambda_Q^{\alpha+\delta}\|w_q\|^2_{L^2(\mathbb{T})}$$

by using the triangle inequality and (5.10)

$$\lesssim c_0\nu \sum_{Q+4>q\geq Q+1} \lambda_q^{2\alpha}\|w_q\|^2_{L^2(\mathbb{T})} + c_0\nu \sum_{q\geq Q+4} \lambda_q^{2\alpha}\|w_q\|^2_{L^2(\mathbb{T})} \sum_{Q<p\leq q} \lambda_q^{1-2\alpha}\lambda_p^{\alpha-1-\delta}\lambda_Q^{\alpha+\delta}$$

$$= c_0\nu \sum_{Q+4>q\geq Q+1} \lambda_q^{2\alpha}\|w_q\|^2_{L^2(\mathbb{T})} + c_0\nu \sum_{q\geq Q+4} \lambda_q^{2\alpha}\|w_q\|^2_{L^2(\mathbb{T})} \sum_{Q<p\leq q} \lambda_{q-p}^{-(2\alpha-1)}\lambda_{p-Q}^{-(\alpha+\delta)}$$

$$\lesssim c_0\nu \sum_{q\geq Q+1} \lambda_q^{2\alpha}\|w_q\|^2_{L^2(\mathbb{T})}, \tag{5.23}$$

where we have used $\alpha \geq \frac{1}{2}$ and $\lambda_Q < \lambda_p \leq \lambda_q$ for all $Q < p \leq q$.

Therefore, we deduce from (5.20), (5.21), (5.22) and (5.23) that

$$I_1 \lesssim c_0\nu \sum_{q\geq Q+1} \lambda_q^{2\alpha}\|w_q\|^2_{L^2(\mathbb{T})}. \tag{5.24}$$

We continue with the estimation of I_2. Since $w(t)_{\leq Q(t)} \equiv 0$, we have

$$\begin{aligned}
I_2 &= \sum_{q\geq 0} \Big| \sum_{|q-m|\leq 1} \int_{\mathbb{T}} \Delta_q(u_m \cdot \nabla w_{\leq m-2}) w_q dx \Big| \\
&= \sum_{q\geq Q+1} \Big| \sum_{m>Q+2, |q-m|\leq 1} \int_{\mathbb{T}} \Delta_q(u_m \cdot \nabla w_{\leq m-2}) w_q dx \Big| \\
&\leq \sum_{q\geq Q+1} \sum_{m>Q+2, |q-m|\leq 1} \|u_m\|_{L^\infty(\mathbb{T})} \|\nabla w_{(Q,m-2]}\|_{L^2(\mathbb{T})} \|w_q\|_{L^2(\mathbb{T})}
\end{aligned}$$

by using Hölder's and Young's inequalities

$$\lesssim c_0\nu \sum_{q\geq Q+1} \sum_{m>Q+2, |q-m|\leq 1} \lambda_Q^{\alpha+\delta} \lambda_m^{\alpha-1-\delta} \|\nabla w_{(Q,m-2]}\|_{L^2(\mathbb{T})} \|w_q\|_{L^2(\mathbb{T})}$$

by using (5.10)

$$\begin{aligned}
&\lesssim c_0\nu \sum_{q\geq Q+1} \lambda_q^{\alpha-1-\delta} \lambda_Q^{\alpha+\delta} \|\nabla w_{(Q,q)}\|_{L^2(\mathbb{T})} \|w_q\|_{L^2(\mathbb{T})} \\
&\lesssim c_0\nu \sum_{q\geq Q+1} \lambda_q^{\alpha-1-\delta} \lambda_Q^{\alpha+\delta} \|w_q\|_{L^2(\mathbb{T})} \sum_{Q<p<q} \lambda_p \|w_p\|_{L^2(\mathbb{T})}
\end{aligned}$$

by using Proposition 5.1 and the triangle inequality

$$\begin{aligned}
&\lesssim c_0\nu \sum_{q\geq Q+1} \lambda_q^{\alpha} \|w_q\|_{L^2(\mathbb{T})} \sum_{Q<p<q} \lambda_p^{\alpha} \|w_p\|_{L^2(\mathbb{T})} \lambda_q^{-1-\delta} \lambda_p^{1-\alpha} \lambda_Q^{\alpha+\delta} \\
&\lesssim c_0\nu \sum_{q\geq Q+1} \lambda_q^{\alpha} \|w_q\|_{L^2(\mathbb{T})} \sum_{Q<p<q} \lambda_p^{\alpha} \|w_p\|_{L^2(\mathbb{T})} \lambda_{q-p}^{-(1+\delta)} \lambda_{p-Q}^{-(\alpha+\delta)} \\
&\lesssim c_0\nu \sum_{q\geq Q+1} \lambda_q^{2\alpha} \|w_q\|^2_{L^2(\mathbb{T})} + c_0\nu \sum_{q\geq Q+1} \Big(\sum_{Q<p<q} \lambda_p^{\alpha} \|w_p\|_{L^2(\mathbb{T})} \lambda_{q-p}^{-(1+\delta)} \lambda_{p-Q}^{-(\alpha+\delta)} \Big)^2
\end{aligned}$$

by using Young's inequalities

$$\lesssim c_0\nu \sum_{q\geq Q+1} \lambda_q^{2\alpha}\|w_q\|^2_{L^2(\mathbb{T})}, \tag{5.25}$$

where we have used $\lambda_Q < \lambda_p \leq \lambda_q$ for all $Q < p \leq q$.

We can estimate I_3 the same as J_3 as follows

$$\begin{aligned}
I_3 &= \sum_{q\geq 0} \Big| \sum_{m\geq q-1} \int_{\mathbb{T}} \Delta_q(u_m \cdot \nabla \widetilde{w}_m) w_q dx \Big| \\
&\lesssim \sum_{q\geq 0} \sum_{m\geq q-1} \int_{\mathbb{T}} |\Delta_q(u_m \otimes \widetilde{w}_m) \nabla w_q| dx
\end{aligned}$$

by using integration by parts and divergence free condition

$$\lesssim \sum_{m\geq Q+1}\sum_{Q<q\leq m+1}\|u_m\|_{L^\infty(\mathbb{T})}\|\widetilde{w}_m\|_{L^2(\mathbb{T})}\|\nabla w_q\|_{L^2(\mathbb{T})}$$

since $w(t)_{\leq Q(t)}\equiv 0$ and using Hölder's inequality

$$\lesssim \sum_{m\geq Q+1}\|\widetilde{w}_m\|_{L^2(\mathbb{T})}\|u_m\|_{L^\infty(\mathbb{T})}\sum_{Q<q\leq m+1}\lambda_q\|w_q\|_{L^2(\mathbb{T})}$$

by using Proposition 5.1

$$\lesssim c_0\nu\sum_{m\geq Q+1}\lambda_m^{\alpha-1-\delta}\lambda_Q^{\alpha+\delta}\|w_m\|_{L^2(\mathbb{T})}\sum_{Q<q\leq m+1}\lambda_q\|w_q\|_{L^2(\mathbb{T})}$$

by using (5.1), (5.10) and the triangle inequality

$$= c_0\nu\sum_{m\geq Q+1}\lambda_m^{\alpha}\|w_m\|_{L^2(\mathbb{T})}\sum_{Q<q\leq m+1}\lambda_q^{\alpha}\|w_q\|_{L^2(\mathbb{T})}\lambda_Q^{\alpha+\delta}\lambda_m^{-1-\delta}\lambda_q^{1-\alpha}$$

$$\lesssim c_0\nu\sum_{m\geq Q+1}\lambda_m^{\alpha}\|w_m\|_{L^2(\mathbb{T})}\sum_{Q<q\leq m+1}\lambda_q^{\alpha}\|w_q\|_{L^2(\mathbb{T})}\lambda_{q-Q}^{-\alpha}\lambda_{m-Q}^{-\delta}$$

since $\lambda_Q^{\alpha+\delta}\lambda_m^{-1-\delta}\lambda_q^{1-\alpha}\lesssim\lambda_Q^{\alpha+\delta}\lambda_m^{-\delta}\lambda_q^{-\alpha}:=\lambda_{q-Q}^{-\alpha}\lambda_{m-Q}^{-\delta}$

$$\lesssim c_0\nu\sum_{m\geq Q+1}\lambda_m^{2\alpha}\|w_m\|^2_{L^2(\mathbb{T})}+c_0\nu\sum_{m\geq Q+1}\Big(\sum_{Q<q\leq m+1}\lambda_q^{\alpha}\|w_q\|_{L^2(\mathbb{T})}\lambda_{q-Q}^{-\alpha}\lambda_{m-Q}^{-\delta}\Big)^2$$

by using Young's inequality

$$\lesssim c_0\nu\sum_{m\geq Q+1}\lambda_m^{2\alpha}\|w_m\|^2_{L^2(\mathbb{T})}, \tag{5.26}$$

where we have used $\lambda_Q\leq\lambda_q$ for all $Q\leq q$ and $\lambda_Q\leq\lambda_m$ for all $Q\leq m$.

We deduce from (5.18), (5.24), (5.25) and (5.26) that

$$I\lesssim c_0\nu\sum_{q\geq Q+1}\lambda_q^{2\alpha}\|w_q\|^2_{L^2(\mathbb{T})}. \tag{5.27}$$

Combining (5.17) and (5.27), we get

$$I+J\lesssim c_0\nu\sum_{q\geq Q+1}\lambda_q^{2\alpha}\|w_q\|^2_{L^2(\mathbb{T})}\leq Cc_0\nu\|\Lambda^\alpha w\|^2_{V^0}. \tag{5.28}$$

It follows from (2.1), (5.7) and (5.28) that if we take $c_0:=\frac{1}{2C}$, we infer

$$\begin{aligned}\|w(t)\|^2_{V^0}&\leq\|w(t_0)\|^2_{V^0}-\nu\int_{t_0}^t\|\Lambda^\alpha w(\tau)\|^2_{V^0}d\tau\\&\leq\|w(t_0)\|^2_{V^0}-\nu\int_{t_0}^t\|w(\tau)\|^2_{V^0}d\tau\end{aligned}$$

for all $t_0\leq t$. Thus

$$\|w(t)\|^2_{V^0}\leq\|w(t_0)\|^2_{V^0}e^{-\nu(t-t_0)} \tag{5.29}$$

for all $t_0\leq t$. Let $t_0\to-\infty$, the proof is completed. □

We see that if we repeat the same arguments in Theorem 5.1, we also obtain the following result

Theorem 5.2. *Assume that ν,a,b are positive, $\alpha\geq\frac{1}{2}$ and $r\geq 1$. Let $u(t)$ and $v(t)$ be two global weak solutions of (2.7) on the weak global attractor $\mathcal{A}$. Let*

$\mathcal{N}(t) := \max\{\mathcal{N}_u^\alpha(t), \mathcal{N}_v^\alpha(t)\}$ *and* $Q(t)$ *be such that* $\mathcal{N}(t) = \lambda_{Q(t)}$*. If*

$$u(t)_{\le Q(t)} = v(t)_{\le Q(t)}, \ \forall t > 0,$$

then

$$\lim_{t\to\infty} \|u(t) - v(t)\|_{V^0} = 0.$$

For simplicity, we will drop the subscript u and superscript α in $\mathcal{N}_u^\alpha$. We still define Q such that $\lambda_Q = \mathcal{N}$.

Lemma 5.1.

(1) If $\lambda_0 \le \mathcal{N} < \infty$*, then*

$$(c_0\nu)^2\mathcal{N}^{4\alpha} \lesssim 16^\alpha \sum_{q=0}^{Q-1} \lambda_q^2 \|u_q\|^2_{L^\infty(\mathbb{T})} + \sup_{p\ge Q} 16^\alpha \lambda_p^{-2\alpha+2+2\delta}\mathcal{N}^{2\alpha-2\delta}\|u_p\|^2_{L^\infty(\mathbb{T})}. \quad (5.30)$$

(2) If $\mathcal{N} = \infty$*, then*

$$\sup_q \lambda_q^{-\alpha+1+\delta}\|u_q\|_{L^\infty(\mathbb{T})} = \infty. \quad (5.31)$$

Proof. If $\lambda_0 \le \mathcal{N} < \infty$, then both conditions in the definition of $\mathcal{N}$ are satisfied for $q = Q$, but one of the conditions is not satisfied for $q = Q-1$, that is,

$$\lambda_p^{-\alpha+1+\delta}\lambda_{Q-1}^{-\alpha-\delta}\|u_p\|_{L^\infty(\mathbb{T})} \ge c_0\nu, \ \text{ for some } p \ge Q, \quad (5.32)$$

or

$$\sum_{q=0}^{Q-1} \lambda_q\|u_q\|_{L^\infty(\mathbb{T})} \ge c_0\nu\lambda_{Q-1}^{2\alpha} := \frac{1}{4^\alpha}c_0\nu\mathcal{N}^{2\alpha}. \quad (5.33)$$

We deduce from (5.32) and $0 < \delta < \alpha$ that

$$(c_0\nu)^2\mathcal{N}^{4\alpha} \le 16^\alpha\lambda_p^{-2\alpha+2+2\delta}\mathcal{N}^{2\alpha-2\delta}\|u_p\|^2_{L^\infty(\mathbb{T})}, \ \text{ for some } p \ge Q. \quad (5.34)$$

It follows from (5.33) that

$$(c_0\nu)^2\mathcal{N}^{4\alpha} \lesssim 16^\alpha \sum_{q=0}^{Q-1} \lambda_q^2\|u_q\|^2_{L^\infty(\mathbb{T})}. \quad (5.35)$$

Combining (5.34) and (5.35), we obtain (5.30).

We now consider the case $\mathcal{N} = \infty$. Then for every $q \in \mathbb{N}$ either

$$\sup_{p>q} \lambda_p^{-\alpha+1+\delta}\lambda_q^{-\alpha-\delta}\|u_p\|_{L^\infty(\mathbb{T})} \ge c_0\nu, \quad (5.36)$$

or

$$\lambda_q^{-2\alpha}\sum_{j=0}^{q} \lambda_j\|u_j\|_{L^\infty(\mathbb{T})} \ge c_0\nu. \quad (5.37)$$

If (5.36) is satisfied, then

$$\limsup_{q\to\infty} \sup_{p>q} \lambda_q^{-\alpha-\delta}\lambda_p^{-\alpha+1+\delta}\|u_p\|_{L^\infty(\mathbb{T})} \ge c_0\nu.$$

This immediately implies that (5.31) holds.

If (5.37) is satisfied, then

$$\limsup_{q\to\infty} \lambda_q^{-2\alpha}\sum_{j=0}^{q} \lambda_j\|u_j\|_{L^\infty(\mathbb{T})} \ge c_0\nu.$$

Using $0 < \delta < \alpha$, we have

$$\begin{aligned}\lambda_q^{-2\alpha} \sum_{j=0}^{q} \lambda_j \|u_j\|_{L^\infty(\mathbb{T})} &= \lambda_q^{-\alpha-\delta} \sum_{j=0}^{q} \lambda_{q-j}^{-\alpha+\delta} \lambda_j^{-\alpha+1+\delta} \|u_j\|_{L^\infty(\mathbb{T})} \\ &\lesssim \lambda_q^{-\alpha-\delta} \sup_{j\le q} \lambda_j^{-\alpha+1+\delta} \|u_j\|_{L^\infty(\mathbb{T})}.\end{aligned}$$

Since $-\alpha - \delta < 0$, we deduce that (5.31) holds. □

Remark 5.1. We have established the determining wavenumbers to estimate the number of determining modes for the 3D generalized Navier-Stokes equations with nonlinear exponential damping term. We see that the determining wavenumber $\mathcal{N}_u^\alpha$ depends on time and may not be bounded. These results also improve and extend the results in [23, 25]. We also see that these results could be extended in the limiting case of no damping. Follow the same arguments in [23, 25], we also might give a bound of the average determining wavenumber in terms of the Kolmogorov dissipation number or Grashof constant. For more details about the determining wavenumbers, we refer readers to [23, 25] and references therein.

Appendix A

In this appendix, for completeness, we briefly recall here the basic definitions and main results on the evolutionary systems which was developed in recent years by Cheskidov and Lu in order to study dynamical systems without uniqueness of solutions. This theory was developed by series of papers of Cheskidov and Lu and all results can be found in [22, 27, 28, 29, 54].

5.1. **Phase space endowed with two metrics.** Assume that a set X is endowed with two metrics $d_s(\cdot,\cdot)$ and $d_w(\cdot,\cdot)$ respectively, satisfying the following conditions:

(1) X is d_w-compact.
(2) If $d_s(u_n, v_n) \to 0$ as $n \to \infty$ for some $u_n, v_n \in X$, then $d_w(u_n, v_n) \to 0$ as $n \to \infty$.

Due to the property (2), $d_w(\cdot,\cdot)$ will be referred to as a weak metric on X. Denote by $\bar{A}^\bullet$ the closure of a set $A \subset X$ in the topology generated by $d_\bullet$. Here (the same below) $\bullet = s$ or w. Note that any strongly compact (d_s-compact) set is weakly compact (d_w-compact), and any weakly closed set is strongly closed.

5.2. **Autonomous case.** Let

$$\mathcal{T} := \{I : I = [\tau, \infty) \subset \mathbb{R}, \text{ or } I = (-\infty, \infty)\},$$

and for each $I \in \mathcal{T}$, let $\mathfrak{F}(I)$ denote the set of all X-valued functions on I. Now we define an evolutionary system $\mathcal{E}$ as follows

Definition 5.1. *A map $\mathcal{E}$ that associates to each $I \in \mathcal{T}$ a subset $\mathcal{E}(I) \subset \mathfrak{F}(I)$ will be called an evolutionary system if the following conditions are satisfied:*

(1) $\mathcal{E}([0,\infty)) \neq \emptyset$.
(2) $\mathcal{E}(I + s) = \{u(\cdot) : u(\cdot + s) \in \mathcal{E}(I)\}$ *for all* $s \in \mathbb{R}$.
(3) $\{u(\cdot)|_{I_2} : u(\cdot) \in \mathcal{E}(I_1)\} \subset \mathcal{E}(I_2)$ *for all pairs* $I_1, I_2 \in \mathcal{T}$*, such that* $I_2 \subset I_1$.
(4) $\mathcal{E}((-\infty,\infty)) = \{u(\cdot) : u(\cdot)|_{[\tau,\infty)} \in \mathcal{E}([\tau,\infty)), \forall \tau \in \mathbb{R}\}$.

We will refer to $\mathcal{E}(I)$ as the set of all trajectories on the time interval I. The set $\mathcal{E}((-\infty,\infty))$ is called the kernel of $\mathcal{E}$ and the trajectories in it are called complete.

Let $C([a,b];X_\bullet)$ be the space of $d_\bullet$-continuous X-valued functions on $[a,b]$ endowed with the metric

$$d_{C([a,b];X_\bullet)}(u,v) := \sup_{t\in[a,b]} d_\bullet(u(t),v(t)).$$

Denote by $C([a,\infty);X_\bullet)$ the space of $d_\bullet$-continuous X-valued functions on $[a,\infty)$ endowed with the metric

$$d_{C([a,\infty);X_\bullet)}(u,v) := \sum_{l\in\mathbb{N}} \frac{1}{2^l} \frac{d_{C([a,a+l];X_\bullet)}(u,v)}{1+d_{C([a,a+l];X_\bullet)}(u,v)}.$$

Note that the convergence in $C([a,\infty);X_\bullet)$ is equivalent to uniform convergence on compact sets.

Let

$$\bar{\mathcal{E}}([\tau,\infty)) := \overline{\mathcal{E}([\tau,\infty))}^{C([\tau,\infty);X_w)},\ \forall\tau\in\mathbb{R},$$

and

$$\bar{\mathcal{E}}((-\infty,\infty)) := \{u(\cdot) : u(\cdot)|_{[\tau,\infty)} \in \bar{\mathcal{E}}([\tau,\infty)), \forall\tau\in\mathbb{R}\}.$$

It can be checked that $\bar{\mathcal{E}}$ is also an evolutionary system and it is called the closure of the evolutionary system $\mathcal{E}$. We add for $\bar{\mathcal{E}}$ the top-script $^-$ to the corresponding notations for $\mathcal{E}$.

Let $\mathcal{K} := \mathcal{E}((-\infty,\infty))$ and $\bar{\mathcal{K}} := \bar{\mathcal{E}}((-\infty,\infty))$, which are called the kernel of $\mathcal{E}$ and $\bar{\mathcal{E}}$, respectively. Let also

$$\Pi_+\mathcal{K} := \{u(\cdot)|_{[0,\infty)} : u\in\mathcal{K}\} \text{ and } \Pi_+\bar{\mathcal{K}} := \{u(\cdot)|_{[0,\infty)} : u\in\bar{\mathcal{K}}\}.$$

We will investigate evolutionary systems $\mathcal{E}$ satisfying the following properties:

(A1) $\mathcal{E}([0,\infty))$ is a precompact set in $C([0,\infty);X_w)$.

(A2) (Energy inequality) Assume that X is a set in some Banach space H satisfying the Radon-Riesz property (see below) with the norm denoted $|\cdot|$, such that $d_s(x,y)=|x-y|$ for $x,y\in X$ and d_w induces the weak topology on X. Assume also that for any $\varepsilon>0$, there exists $\delta>0$, such that for every $u\in\mathcal{E}([0,\infty))$ and $t>0$,

$$|u(t)| \le |u(t_0)| + \varepsilon,$$

for t_0 a.e. in $(t-\delta,t)$.

(A3) (Strong convergence a.e.) Let $u_n\in\mathcal{E}([0,\infty))$ be such that, u_n is $d_{C([0,T];X_w)}$-Cauchy sequence in $C([0,T];X_w)$ for some $T>0$. Then $u_n(t)$ is d_s-Cauchy sequence a.e. in $[0,T]$.

We also recall stronger properties (see [22, 27, 28, 29, 54]) as follows

(B1) $\mathcal{E}([0,\infty))$ is a compact set in $C([0,\infty);X_w)$.

(B2) (Energy inequality) Assume that X is a set in some Banach space H satisfying the Radon-Riesz property (see below) with the norm denoted $|\cdot|$, such that $d_s(x,y)=|x-y|$ for $x,y\in X$ and d_w induces the weak topology on X. Assume also that for any $\varepsilon>0$, there exists $\delta>0$, such that for every $u\in\mathcal{E}([0,\infty))$ and $t>0$,

$$|u(t)| \le |u(t_0)| + \varepsilon,$$

for t_0 a.e. in $(t-\delta,t)$.

(B3) (Strong convergence a.e.) Let $u, u_n \in \mathcal{E}([0,\infty))$ be such that $u_n \to u$ in $C([0,T]; X_w)$ for some $T > 0$. Then $u_n(t) \to u(t)$ strongly a.e. in $[0,T]$.

A Banach $\mathcal{B}$ is said to satisfy the Radon-Riesz property if for any sequence $\{x_n\} \subset \mathcal{B}$,

$$x_n \to x \text{ strongly in } \mathcal{B} \Leftrightarrow \begin{cases} x_n \to x \text{ weakly in } \mathcal{B}, \\ \|x_n\|_{\mathcal{B}} \to \|x\|_{\mathcal{B}}, \end{cases} \quad as\ n \to \infty.$$

In many applications X is bounded closed set in a uniformly convex separable Banach space H. Then the weak topology of H is metrizable on X, and X is compact with respect to such a metric d_w. Moreover, the Radon-Riesz property is automatically satisfied.

If $\mathcal{E}$ satisfies the conditions **(A1)-(A3)**, then $\bar{\mathcal{E}}$ satisfies **(B1)-(B3)** (see [29]).

Let $P(X)$ be the set of all subsets of X. For every $t \geq 0$, define a set-valued map

$$R(t) : P(X) \to P(X),$$
$$R(t)A := \{u(t) : u(0) \in A, u(\cdot) \in \mathcal{E}([0,\infty))\},\ A \subset X.$$

Note that the assumptions on $\mathcal{E}$ implies that $R(t)$ enjoys the following property:

$$R(t+s)A \subset R(t)R(s)A,\ A \subset X,\ t, s \geq 0.$$

Consider an arbitrary evolutionary system $\mathcal{E}$. For a set $A \subset X$ and $r > 0$, denote

$$B_\bullet(A, r) = \{u \in X : d_\bullet(u, A) < r\},$$

where

$$d_\bullet(u, A) := \inf_{x \in A} d_\bullet(u, x),\ \bullet = s, w.$$

Definition 5.2.

(1) *A set $A \subset X$ uniformly attracts a set $B \subset X$ in $d_\bullet$-metric ($\bullet = s, w$) if for any $\varepsilon > 0$, there exists t_0, such that*

$$R(t)B \subset B_\bullet(A, \varepsilon),\ \forall t \geq t_0.$$

(2) *A set $A \subset X$ is a $d_\bullet$-attracting set ($\bullet = s, w$) if it uniformly attracts X in $d_\bullet$-metric.*

Definition 5.3. *A set $\mathcal{A}_\bullet$ is a $d_\bullet$-global attractor ($\bullet = s, w$) if $\mathcal{A}_\bullet$ is a minimal $d_\bullet$-closed $d_\bullet$-attracting set.*

Note that the empty set is never an attracting set. Note also that since X is not strongly compact, the intersection of two d_s-closed d_s-attracting sets might not be d_s-attracting. Nevertheless, the global attractor $\mathcal{A}_\bullet$ is unique if it exists.

Definition 5.4. *The $\omega_\bullet$-limit ($\bullet = s, w$) of a set $A \subset X$ is*

$$\omega_\bullet(A) := \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} R(t)A}^{\bullet}.$$

An equivalent definition of the $\omega_\bullet$-limit set is given by

$$\omega_\bullet(A) = \{x \in X : \text{ there exist sequences } t_n \to \infty \text{ as } n \to \infty \text{ and } x_n \in R(t_n)A, \text{ such that } x_n \to x \text{ in } d_\bullet\text{-metric as } n \to \infty\}.$$

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Definition 5.5. *An evolutionary system $\mathcal{E}$ is asymptotically compact if for any $t_n \to +\infty$ and any $x_n \in R(t_n)X$, the sequence $\{x_n\}$ is relatively strongly compact.*

Theorem 5.3. *Let $\mathcal{E}$ be an evolutionary system satisfying* **(A1)**, **(A2)**, *and* **(A3)**, *and assume that its closure $\bar{\mathcal{E}}$ satisfies $\bar{\mathcal{E}}((-\infty,\infty)) \subset C((-\infty,\infty);X_s)$. Then $\mathcal{E}$ is asymptotically compact.*

Definition 5.6. *Let $\mathcal{E}$ be an evolutionary system. If an map $\mathcal{E}^1$ that associates to each $I \in \mathcal{T}$ a subset $\mathcal{E}^1(I) \subset \mathcal{E}(I)$ is also an evolutionary system, we will call it an evolutionary subsystem of $\mathcal{E}$, and denote by $\mathcal{E}^1 \subset \mathcal{E}$.*

We define the following mapping:

$$\widetilde{R}(t)A := \{u(t) : u(0) \in A, u \in \mathcal{K}\},\ A \subset X, t \in \mathbb{R}.$$

Definition 5.7. *A set $A \subset X$ is positively invariant if*

$$\widetilde{R}(t)A \subset A,\ \forall t \geq 0.$$

A is invariant if

$$\widetilde{R}(t)A = A,\ \forall t \geq 0.$$

A is quasi-invariant if for every $a \in A$ there exists a complete trajectory $u \in \mathcal{K}$ with $u(0) = a$ and $u(t) \in A$ for all $t \in \mathbb{R}$.

We now reconsider the evolutionary systems $\mathcal{E}$ satisfying $\mathcal{E}([0,\infty)) \subset C([0,\infty);X_w)$. Note that $\mathcal{E}([0,\infty))$ may not be closed in $C([0,\infty);X_w)$. Define the family of translation operators $\{T(s)\}_{s\geq 0}$,

$$(T(s)u)(\cdot) := u(\cdot + s)|_{[0,\infty)},\ u \in C([0,\infty);X_w).$$

We consider the dynamics of the translation semigroup $\{T(s)\}_{s\geq 0}$ acting on the phase space $C([0,\infty);X_w)$. Due to the property (3) of the evolutionary system, we see that $T(s)\mathcal{E}([0,\infty)) \subset \mathcal{E}([0,\infty)), \forall s \geq 0$.

Definition 5.8.

(1) *A set $P \subset C([0,\infty);X_w)$ weakly uniformly attracts a set $Q \subset \mathcal{E}([0,\infty))$ if for any $\varepsilon > 0$, there exists t_0, such that*

$$T(t)Q \subset \{v \in C([0,\infty);X_w) : \inf_{u\in P} d_{C([0,\infty);X_w)}(u,v) < \varepsilon\}, \forall t \geq t_0.$$

(2) *A set $P \subset C([0,\infty);X_w)$ is a weak trajectory attracting set for an evolutionary system $\mathcal{E}$ if it weakly uniformly attracts $\mathcal{E}([0,\infty))$.*

Definition 5.9. *A set $\mathfrak{A}_w \subset C([0,\infty);X_w)$ is a weak trajectory attractor for an evolutionary system $\mathcal{E}$ if $\mathfrak{A}_w$ is a minimal weak trajectory attracting set that is*

(i) *Closed in $C([0,\infty);X_w)$.*
(ii) *Invariant: $T(t)\mathfrak{A}_w = \mathfrak{A}_w$, $\forall t \geq 0$.*

Definition 5.10. *A set $P \subset C([0,\infty);X_w)$ satisfies the weak uniform tracking property for an evolutionary system $\mathcal{E}$ if for any $\varepsilon > 0$, there exists t_0, such that for any $t^* > t_0$, every trajectory $u \in \mathcal{E}([0,\infty))$ satisfies*

$$d_{C([t^*,\infty);X_w)}(u(\cdot), v(\cdot - t^*)) < \varepsilon,$$

for some trajectory $v \in P$.

Definition 5.11. *A set $P \subset C([0,\infty); X_w)$ satisfies the finite weak uniform tracking property for an evolutionary system $\mathcal{E}$ if for any $\varepsilon > 0$, there exist t_0 and a finite subset $P^f \subset P$, such that for any $t^* > t_0$, every trajectory $u \in \mathcal{E}([0,\infty))$ satisfies*

$$d_{C([t^*,\infty);X_w)}(u(\cdot), v(\cdot - t^*)) < \varepsilon,$$

for some trajectory $v \in P^f$.

Theorem 5.4. *Let $\mathcal{E}$ be an evolutionary system. Then*

(1) *The weak global attractor $\mathcal{A}_w$ exists, and $\mathcal{A}_w = \omega_w(X)$.*

Furthermore, assume that $\mathcal{E}$ satisfies **(A1)**. *Let $\bar{\mathcal{E}}$ be the closure of $\mathcal{E}$. Then*

(2) *$\mathcal{A}_w = \omega_w(X) = \bar{\omega}_w(X) = \bar{\omega}_s(X) = \bar{\mathcal{A}}_w$.*
(3) *$\mathcal{A}_w$ is the maximal invariant and maximal quasi-invariant set w.r.t. $\bar{\mathcal{E}}$:*

$$\mathcal{A}_w := \{u_0 \in X : u_0 := u(0) \text{ for some } u \in \bar{\mathcal{K}}\}.$$

(4) *The weak trajectory attractor $\mathfrak{A}_w$ exists, it is weakly compact, and $\mathfrak{A}_w = \Pi_+\bar{\mathcal{K}}$. Hence, $\mathfrak{A}_w$ satisfies the finite weak uniform tracking property for $\mathcal{E}$ and is weakly equicontinuous on $[0,\infty)$.*
(5) *$\mathcal{A}_w$ is a section of $\mathfrak{A}_w$:*

$$\mathcal{A}_w = \mathfrak{A}_w(t) := \{u(t) : u \in \mathfrak{A}_w\},\ \forall t \geq 0.$$

Definition 5.12.

(1) *A set $P \subset C([0,\infty); X_w)$ strongly uniformly attracts a set $Q \subset \mathcal{E}([0,\infty))$ if for any $\varepsilon > 0$ and $T > 0$, there exists t_0, such that*

$$T(t)Q \subset \{v \in C([0,\infty); X_w) : \inf_{u\in P} \sup_{\tau\in[0,T]} d_s(u(\tau), v(\tau)) < \varepsilon\}, \forall t \geq t_0.$$

(2) *A set $P \subset C([0,\infty); X_w)$ is a strong trajectory attracting set for an evolutionary system $\mathcal{E}$ if it strongly uniformly attracts $\mathcal{E}([0,\infty))$.*

Note that a strong trajectory attracting set for an evolutionary system $\mathcal{E}$ is a weak trajectory attracting set for $\mathcal{E}$.

Definition 5.13. *A set $\mathfrak{A}_s \subset C([0,\infty); X_w)$ is a strong trajectory attractor for an evolutionary system $\mathcal{E}$ if $\mathfrak{A}_s$ is a minimal strong trajectory attracting set that is*

(1) *Closed in $C([0,\infty); X_w)$.*
(2) *Invariant: $T(t)\mathfrak{A}_s = \mathfrak{A}_s,\ \forall t \geq 0$.*

It is said that $\mathfrak{A}_s$ is strongly compact if it is compact in $C([0,\infty); X_s)$.

Definition 5.14. *A set $P \subset C([0,\infty); X_w)$ satisfies the strong uniform tracking property for an evolutionary system $\mathcal{E}$ if for any $\varepsilon > 0$ and $T > 0$, there exists t_0, such that for any $t^* > t_0$, every trajectory $u \in \mathcal{E}([0,\infty))$ satisfies*

$$d_s(u(t), v(t - t^*)) < \varepsilon,\ \forall t \in [t^*, t^* + T],$$

for some T-time length piece $v \in P_T$. Here $P_T := \{v(\cdot)|_{[0,T]} : v \in P\}$.

Definition 5.15. *A set $P \subset C([0,\infty); X_w)$ satisfies the finite strong uniform tracking property for an evolutionary system $\mathcal{E}$ if for any $\varepsilon > 0$ and $T > 0$, there exist t_0 and a finite subset $P_T^f \subset \mathfrak{A}_s|_{[0,T]}$, such that for any $t^* > t_0$, every trajectory $u \in \mathcal{E}([0,\infty))$ satisfies*

$$d_s(u(t), v(t - t^*)) < \varepsilon,\ \forall t \in [t^*, t^* + T],$$

for some T-time length piece $v \in P_T^f$.

Theorem 5.5. *Let $\mathcal{E}$ be an asymptotically compact evolutionary system. Then*

(1) *The strong global attractor $\mathcal{A}_s$ exists, it is strongly compact, and $\mathcal{A}_s = \mathcal{A}_w$.*

Furthermore, assume that $\mathcal{E}$ satisfies **(A1)**. *Let $\bar{\mathcal{E}}$ be the closure of $\mathcal{E}$. Then*

(2) *The strong trajectory attractor $\mathfrak{A}_s$ exists and $\mathfrak{A}_s = \mathfrak{A}_w = \Pi_+\bar{\mathcal{K}}$, it is strongly compact.*
(3) *$\mathfrak{A}_s$ satisfies the finite strong uniform tracking property for $\mathcal{E}$.*
(4) *$\mathfrak{A}_s = \Pi_+\bar{\mathcal{K}}$ is strongly equicontinuous on $[0,\infty)$, i.e.,*
$$d_s(v(t_1), v(t_2)) \leq \theta(|t_1 - t_2|),\ \forall t_1, t_2 \geq 0, \forall v \in \mathfrak{A}_s,$$
where $\theta(s)$ is a positive function tending to 0 as $s \to 0^+$.

Theorem 5.5 gives us the results that indicate how the dynamics on the global attractor determine the long-time dynamics of all trajectories of an evolutionary system (see [54, Corollary 3.13; Corollary 3.14]). Comparing with Theorem 5.4, Theorem 5.5 implies that the strong compactness of both the strong global attractor and the strong trajectory attractor follow simultaneously once we obtain the asymptotical compactness of an evolutionary system. Moreover, the global attractor is a section of the trajectory attractor and the trajectory attractor consists of the restriction of all the complete trajectories on the global attractor on time semiaxis $[0,\infty)$; the notion of a global attractor stresses the property of attracting trajectories starting from sets in phase space X while the notion of a trajectory attractor emphasizes the uniform tracking property.

The following theorem is an important result for the asymptotical compactness of $\mathcal{E}$.

Theorem 5.6. *An evolutionary system $\mathcal{E}$ is asymptotically compact if and only if its strongly compact strong global attractor $\mathcal{A}_s$ exists*

Corollary 5.1. *Let $\mathcal{E}$ be an evolutionary system satisfying* **(A1)** *and let $\bar{\mathcal{E}}$ be the closure of $\mathcal{E}$. If the strongly compact strong global attractor $\mathcal{A}_s$ for $\mathcal{E}$ exists, then the strongly compact strong trajectory attractor $\mathfrak{A}_s$ for $\mathcal{E}$ exists. Hence*

(1) *$\mathfrak{A}_s = \Pi_+\bar{\mathcal{K}}$ satisfies the finite strong uniform tracking property for $\mathcal{E}$, i.e., for any $\varepsilon > 0$ and $T > 0$, there exist t_0 and a finite subset $P_T^f \subset \mathfrak{A}_s|_{[0,T]}$, such that for any $t^* > t_0$, every trajectory $u \in \mathcal{E}([0,\infty))$ satisfies*
$$d_s(u(t), v(t - t^*)) < \varepsilon,\ \forall t \in [t^*, t^* + T],$$
for some T-time length piece $v \in P_T^f$.
(2) *$\mathfrak{A}_s = \Pi_+\bar{\mathcal{K}}$ is strongly equicontinuous on $[0,\infty)$, i.e.,*
$$d_s(v(t_1), v(t_2)) \leq \theta(|t_1 - t_2|),\ \forall t_1, t_2 \geq 0, \forall v \in \mathfrak{A}_s,$$
where $\theta(s)$ is a positive function tending to 0 as $s \to 0^+$.

5.3. **Nonautonomous case and reducing to autonomous case.** Let Σ be a parameter set and $\{T(h)|h \geq 0\}$ be a family of operators acting on Σ satisfying $T(h)\Sigma = \Sigma$, $\forall h \geq 0$. Any element $\sigma \in \Sigma$ is called (time) symbol and Σ is called (time) symbol space.

Definition 5.16. *A family of maps $\mathcal{E}_\sigma$, $\sigma \in \Sigma$ that for every $\sigma \in \Sigma$ associates to each $I \in \mathcal{T}$ a subset $\mathcal{E}_\sigma(I) \subset \mathfrak{F}(I)$ will be called a nonautonomous evolutionary system if the following conditions are satisfied:*

(1) $\mathcal{E}_\sigma([\tau,\infty)) \neq \emptyset$, $\forall \tau \in \mathbb{R}$.
(2) $\mathcal{E}_\sigma(I+s) = \{u(\cdot) : u(\cdot+s) \in \mathcal{E}_{T(s)\sigma}(I)\}$, $\forall s \geq 0$.
(3) $\{u(\cdot)|_{I_2} : u(\cdot) \in \mathcal{E}_\sigma(I_1)\} \subset \mathcal{E}_\sigma(I_2)$ *for all pairs* $I_1, I_2 \in \mathcal{T}$, *such that* $I_2 \subset I_1$.
(4) $\mathcal{E}_\sigma((-\infty,\infty)) = \{u(\cdot) : u(\cdot)|_{[\tau,\infty)} \in \mathcal{E}_\sigma([\tau,\infty)), \forall \tau \in \mathbb{R}\}$.

Define

$$\mathcal{E}_\Sigma(I) := \bigcup_{\sigma\in\Sigma} \mathcal{E}_\sigma(I),\ \forall I \in \mathcal{T}\backslash\{(-\infty,\infty)\},$$

and

$$\mathcal{E}_\Sigma((-\infty,\infty)) := \{u(\cdot) : u(\cdot)|_{[\tau,\infty)} \in \mathcal{E}_\Sigma([\tau,\infty)), \forall \tau \in \mathbb{R}\}.$$

Therefore, the nonautonomous evolutionary system can be viewed as an (autonomous) evolutionary system in the following way

$$\mathcal{E}(I) := \mathcal{E}_\Sigma(I),\ \forall I \in \mathcal{T}.$$

Consequently, the above notions of invariance, quasi-invariance, and a global attractor for $\mathcal{E}$ can be extended to the nonautonomous evolutionary system $\{\mathcal{E}_\sigma\}_{\sigma\in\Sigma}$. The global attractor in the nonautonomous case will be conventionally called a uniform global attractor (or simply a global attractor). Thus, we will not distinguish between autonomous and nonautonomous evolutionary systems. If it is necessary, we denote an evolutionary system with a symbol space Σ by $\mathcal{E}_\Sigma$ and its global attractor by $\mathcal{A}^\Sigma$, trajectory attractor by $\mathfrak{A}^\Sigma$.

Definition 5.17. *An evolutionary system* $\mathcal{E}_\Sigma$ *is a system with uniqueness if for every* $u_0 \in X$ *and* $\sigma \in \Sigma$, *there is a unique trajectory* $u \in \mathcal{E}_\sigma([0,\infty))$ *such that* $u(0) = u_0$.

Definition 5.18. *An evolutionary system* $\mathcal{E}_\Sigma$ *is (weakly) closed if for any* $\tau \in \mathbb{R}$, $u_n \in \mathcal{E}_{\sigma_n}([\tau,\infty))$, *the convergences* $u_n \to u$ *in* $C([\tau,\infty), X_w)$ *and* $\sigma_n \to \sigma$ *in some topological space* $\mathfrak{T}$ *as* $n \to \infty$ *imply* $u \in \mathcal{E}_\sigma([\tau,\infty))$.

Lemma 5.2. *Let* $\mathfrak{T}$ *be some topological space and* $\Sigma \subset \mathfrak{T}$ *be sequentially compact in itself. Let* $\mathcal{E}_\Sigma$ *be a closed evolutionary system satisfying* **(A1)**. *Then,* $\mathcal{E}_\sigma((-\infty,\infty))$ *is nonempty for any* $\sigma \in \Sigma$, *and*

$$\mathcal{E}_\Sigma((-\infty,\infty)) = \bigcup_{\sigma\in\Sigma} \mathcal{E}_\sigma((-\infty,\infty)),$$

and

$$\mathcal{E}_\Sigma([\tau,\infty)) = \bigcup_{\sigma\in\Sigma} \mathcal{E}_\sigma([\tau,\infty)),$$

is closed in $C([\tau,\infty); X_w)$.

Suppose that $\bar{\Sigma}$ is the sequential closure of Σ in some topological space $\mathfrak{T}$. Let $\mathcal{E}_{\bar{\Sigma}}$ be an evolutionary system with symbol space $\bar{\Sigma}$.

Theorem 5.7. *Let* $\mathcal{E}_\Sigma$ *be an evolutionary system with uniqueness and with symbol space* Σ *satisfying* **(A1)** *and let* $\bar{\mathcal{E}}_\Sigma$ *be the closure of* $\mathcal{E}_\Sigma$. *Let* $\bar{\Sigma}$ *be the sequential closure of* Σ *in some topological space* $\mathfrak{T}$ *and* $\mathcal{E}_{\bar{\Sigma}} \supset \mathcal{E}_\Sigma$ *be a closed evolutionary system with uniqueness and with symbol space* $\bar{\Sigma}$. *Then,* $\mathcal{E}_{\bar{\Sigma}} \subset \bar{\mathcal{E}}_\Sigma$. *Hence,*

(1) The three weak uniform global attractors $\mathcal{A}_w^\Sigma$, $\bar{\mathcal{A}}_w^\Sigma$ *and* $\mathcal{A}_w^{\bar{\Sigma}}$ *for evolutionary systems* $\mathcal{E}_\Sigma$, $\bar{\mathcal{E}}_\Sigma$ *and* $\mathcal{E}_{\bar{\Sigma}}$, *respectively, exist.*

(2) $\mathcal{A}_w^{\Sigma}$, $\bar{\mathcal{A}}_w^{\Sigma}$ and $\mathcal{A}_w^{\bar{\Sigma}}$ *are the maximal invariant and maximal quasi-invariant set with respect to* $\bar{\mathcal{E}}_{\Sigma}$ *and satisfy the following*

$$\mathcal{A}_w^{\Sigma} = \bar{\mathcal{A}}_w^{\Sigma} = \mathcal{A}_w^{\bar{\Sigma}} = \{u_0 : u_0 = u(0) \text{ for some } u \in \bar{\mathcal{E}}_{\Sigma}((-\infty,\infty))\}.$$

(3) *The three weak trajectory attractors* $\mathfrak{A}_w^{\Sigma}$, $\bar{\mathfrak{A}}_w^{\Sigma}$ *and* $\mathfrak{A}_w^{\bar{\Sigma}}$ *for evolutionary systems* $\mathcal{E}_{\Sigma}$, $\bar{\mathcal{E}}_{\Sigma}$ *and* $\mathcal{E}_{\bar{\Sigma}}$, *respectively, exist and satisfy the following*

$$\mathfrak{A}_w^{\Sigma} = \bar{\mathfrak{A}}_w^{\Sigma} = \mathfrak{A}_w^{\bar{\Sigma}} = \Pi_+ \bar{\mathcal{E}}_{\Sigma}((-\infty,\infty)).$$

Hence, the three weak trajectory attractors satisfy the finite weak uniform tracking property for all the three evolutionary systems and are weakly equicontinuous on $[0,\infty)$.

(4) $\mathcal{A}_w^{\Sigma}$, $\bar{\mathcal{A}}_w^{\Sigma}$ *and* $\mathcal{A}_w^{\bar{\Sigma}}$ *are sections of* $\mathfrak{A}_w^{\Sigma}$, $\bar{\mathfrak{A}}_w^{\Sigma}$ *and* $\mathfrak{A}_w^{\bar{\Sigma}}$:

$$\mathcal{A}_w^{\Sigma} = \bar{\mathcal{A}}_w^{\Sigma} = \mathcal{A}_w^{\bar{\Sigma}} = \mathfrak{A}_w^{\Sigma}(t) = \bar{\mathfrak{A}}_w^{\Sigma}(t) = \mathfrak{A}_w^{\bar{\Sigma}}(t), \ \forall t \geq 0.$$

Furthermore, assume that $\bar{\Sigma} \subset \mathfrak{T}$ *is sequentially compact in itself. Then,* $\mathcal{E}_{\bar{\Sigma}} = \bar{\mathcal{E}}_{\Sigma}$. *Hence,*

(5) *The following relationships on kernels hold:*

$$\bar{\mathcal{E}}_{\Sigma}((-\infty,\infty)) = \mathcal{E}_{\bar{\Sigma}}((-\infty,\infty)) = \bigcup_{\sigma \in \bar{\Sigma}} \mathcal{E}_{\sigma}((-\infty,\infty)),$$

and $\mathcal{E}_{\sigma}((-\infty,\infty))$ *is nonempty for any* $\sigma \in \bar{\Sigma}$.

Theorem 5.8. *Assume that all conditions of Theorem 5.7 hold and one of the followings is valid:*

(1) $\bar{\mathcal{E}}_{\Sigma}$ *is asymptotically compact.*
(2) $\mathcal{E}_{\Sigma}$ *satisfies* **(A1)**, **(A2)** *and* **(A3)**, *and* $\bar{\mathcal{E}}_{\Sigma}((-\infty,\infty)) \subset C((-\infty,\infty); X_s)$.
(3) $\bar{\mathcal{E}}_{\Sigma}$ *possesses a strongly compact strong global attractor.*

Then the three weak uniform global attractors in Theorem 5.7 are strongly compact strong uniform global attractors and the three weak trajectory attractors are strongly compact strong trajectory attractors. Moreover, the three trajectory attractors satisfy the finite strong uniform tracking property for all the three evolutionary systems and are strongly equicontinuous on $[0,\infty)$.

Appendix B

In this appendix, we present the Littlewood-Paley decomposition for periodic functions. Our intension here is to provide the techniques for section of determining wavenumbers. The review of the convergence results and properties of partial sums (see, e.g., [30]) allows us to choose the suitable cutoff in the definition of the Littlewood-Paley blocks (or the localized Fourier projections). We choose the square-cutoff defined as follows

$$S_N f(x) := \sum_{|k_j| \leq N, j=1,2,3} \widehat{f}(k) e^{ik\cdot x} = D_N * f, \tag{5.38}$$

where D_N denotes the 3D square Dirichlet kernel

$$D_N := \sum_{|k_j| \leq N, j=1,2,3} e^{ik\cdot x} \ \text{ and } \ \widehat{f}(k) := \frac{1}{(2\pi)^3} \int_{\mathbb{T}} f(x) e^{-ik\cdot x} dx. \tag{5.39}$$

The partial sum defined via the square-cutoff is bounded on any L^p for $1 < p < \infty$ and converges to the original function in L^p. They are stated in the following lemma (see, e.g., [30, 36, 66]).

Lemma 5.3. *The partial sum with the square cutoff $S_N f$ satisfies, for any $f \in L^p(\mathbb{T})$ with $1 < p < \infty$,*

$$\|S_N f\|_{L^p(\mathbb{T})} \leq C_p \|f\|_{L^p(\mathbb{T})},$$

and

$$\|S_N f - f\|_{L^p(\mathbb{T})} \to 0 \text{ as } N \to \infty. \tag{5.40}$$

However, (5.40) *is false for $p = 1$ and for $p = \infty$. In addition, if $f \in L^p(\mathbb{T})$ with $1 < p \leq \infty$, then*

$$S_N f \to f \text{ a.e. as } N \to \infty.$$

For an integer $j \geq 0$, we set A_j to be the 2^j-sized block of 3D integer lattice points,

$$A_j = \{k = (k_1, k_2, k_3) \in \mathbb{Z}^3 : |k_m| \leq 2^j, m = 1, 2, 3\}. \tag{5.41}$$

We define the following localized Fourier projection operators as

$$\Delta_0 f(x) = \sum_{k \in A_0} \widehat{f}(k) e^{ik\cdot x}, \tag{5.42}$$

$$\Delta_j f(x) = \sum_{k \in A_j \setminus A_{j-1}} \widehat{f}(k) e^{ik\cdot x}, j \geq 1, j \in \mathbb{N}. \tag{5.43}$$

For notational convenience, we also write $\Delta_j = 0$ for $j < 0$. With a slight abuse of notation, we set

$$S_j f(x) = \sum_{m=0}^{j} \Delta_m f(x) = \sum_{k \in A_j} \widehat{f}(k) e^{ik\cdot x}. \tag{5.44}$$

In terms of these operators, we can write the Littlewood-Paley decomposition, for any $f \in L^p(\mathbb{T})$ with $1 < p \leq \infty$,

$$f(x) = \sum_{m=0}^{\infty} \Delta_m f(x). \tag{5.45}$$

The following lemma presents useful basic properties of the operators defined above.

Lemma 5.4. *Let $j \geq 0$ be an integer. Let Δ_j and S_j be defined as in* (5.42), (5.43) *and* (5.44). *Then the following properties hold.*

(a) *If $f \in L^p(\mathbb{T})$ with $1 < p \leq \infty$, then*

$$\|\Delta_j f\|_{L^p(\mathbb{T})} \leq C\|f\|_{L^p(\mathbb{T})},$$
$$\|S_j f\|_{L^p(\mathbb{T})} \leq C\|f\|_{L^p(\mathbb{T})},$$

where C's are constants depending on p and d only.

(b) *Let $h \geq 0$ and $j \geq 0$ be integers. Assume $f \in L^p(\mathbb{T})$ with $1 < p \leq \infty$, then*

$$\Delta_h \Delta_j f = 0 \text{ if } h \neq j.$$

(c) *Let $j \geq 0$, $m \geq 0$ and $n \geq 1$ be integers. Assume $f, g \in L^p(\mathbb{T})$ with $1 < p \leq \infty$. Then*

$$\Delta_j(S_{m-n} f \Delta_m g) = 0 \;\; \text{if} \;\; |m-j| \geq n,$$

and

$$\Delta_j(\Delta_m f \widetilde{\Delta}_m g) = 0 \;\; \text{if} \;\; |m-j| \geq n,$$

where

$$\widetilde{\Delta}_m g = \Delta_{m-n+1} g + \Delta_{m-n+2} g + \cdots + \Delta_{m+n-1} g.$$

Proof.

(a) This result follows directly from Lemma 5.3.

(b) We have

$$\Delta_h f(x) = \sum_{\ell \in A_h \setminus A_{h-1}} \widehat{f}(\ell) e^{i\ell \cdot x},$$

where

$$\widehat{f}(\ell) := \frac{1}{(2\pi)^3} \int_{\mathbb{T}} f(x) e^{i\ell \cdot x} dx.$$

Thus,

$$\Delta_j \Delta_h f(x) = \sum_{k \in A_j \setminus A_{j-1}} \widehat{\Delta_h f}(k) e^{ik \cdot x},$$

where

$$\begin{aligned}
\widehat{\Delta_h f}(k) &= \frac{1}{(2\pi)^3} \int_{\mathbb{T}} \Delta_h f(x) e^{ik \cdot x} dx \\
&= \frac{1}{(2\pi)^3} \int_{\mathbb{T}} \sum_{\ell \in A_h \setminus A_{h-1}} \widehat{f}(\ell) e^{i\ell \cdot x} e^{ik \cdot x} dx \\
&= \frac{1}{(2\pi)^3} \sum_{\ell \in A_h \setminus A_{h-1}} \widehat{f}(\ell) \int_{\mathbb{T}} e^{i(\ell+k) \cdot x} dx.
\end{aligned}$$

Since $\int_{\mathbb{T}} e^{i(\ell+k)\cdot x} dx = 0$ if $\ell + k \neq 0$. This implies the proof of (b).

(c) We now prove (c). We have

$$\begin{aligned}
S_{m-n} f(x) &= \sum_{k \in A_{m-n}} \widehat{f}(k) e^{ik \cdot x}, \\
\Delta_m g(x) &= \sum_{\ell \in A_m \setminus A_{m-1}} \widehat{g}(\ell) e^{i\ell \cdot x}.
\end{aligned}$$

Hence

$$S_{m-n} f(x) \Delta_m g(x) = \sum_{k \in A_{m-n}; \ell \in A_m \setminus A_{m-1}} \widehat{f}(k) \widehat{g}(\ell) e^{i(\ell+k)\cdot x},$$

where

$$\begin{aligned}
&k = (k_1, k_2, k_3), \text{ such that } |k_d| \leq 2^{m-n}, d = 1, 2, 3. \\
&\ell = (\ell_1, \ell_2, \ell_3), \text{ such that } 2^{m-1} < |\ell_d| \leq 2^m, d = 1, 2, 3.
\end{aligned}$$

Therefore,

$$2^{m-n} < |k_d + \ell_d| < 2^{m+1}.$$

Thus, $k+\ell \in A_{m+1} \setminus A_{m-n}$. It follows from (b) that if $j \notin (m-n, m+1)$, then

$$\Delta_j(S_{m-n} f \Delta_m g) = 0.$$

This means that $|m-j| \geq n$. By the same manner, we can also prove the remaining equality.

□

We also have the following Bernstein type inequalities for the operators Δ_j (see, e.g., [30, Proposition 2.8]).

Proposition 5.1. *Let $\sigma \geq 0$ and $1 \leq q \leq p \leq \infty$.*

(a) *There exists a constant $C > 0$ such that*

$$\|\Delta_j \Lambda^\sigma f\|_{L^p(\mathbb{T})} \leq C 2^{\sigma j + 3j(\frac{1}{q} - \frac{1}{p})} \|\Delta_j f\|_{L^q(\mathbb{T})}, \tag{5.46}$$

and

$$\|S_j f\|_{L^p(\mathbb{T})} \leq C 2^{3j(\frac{1}{q} - \frac{1}{p})} \|S_j f\|_{L^q(\mathbb{T})}. \tag{5.47}$$

(b) *Let $1 \leq p \leq \infty$. There exists constants $0 < C_1 < C_2$ (depending on p) such that, for any integer $j \geq 0$,*

$$C_1 2^{\sigma j} \|\Delta_j f\|_{L^p(\mathbb{T})} \leq \|\Delta_j \Lambda^\sigma f\|_{L^p(\mathbb{T})} \leq C_2 2^{\sigma j} \|\Delta_j f\|_{L^p(\mathbb{T})}. \tag{5.48}$$

In terms of the operators Δ_j and S_j, we can write a standard product of two periodic functions as a sum of paraproducts, as in the whole space case (see, e.g., [7]) .

$$fg = T_f g + T_g f + R(f,g), \tag{5.49}$$

where

$$T_f g = \sum_{m=0}^{\infty} S_{m-n} f \Delta_m g,$$

$$T_g f = \sum_{m=0}^{\infty} S_{m-n} g \Delta_m f,$$

$$R(f,g) = \sum_{m=0}^{\infty} \sum_{h \geq m-1} \Delta_h f \widetilde{\Delta}_h g,$$

with $\widetilde{\Delta}_h g = \Delta_{h-n+1} g + \Delta_{h-n+2} g + \cdots + \Delta_{h+n-1} g$.

We have simplified the notation by defining

$$u_{\leq Q} := \sum_{m=0}^{Q} u_q, \quad u_{(P,Q]} := \sum_{m=P+1}^{Q} u_q, \quad u_q = \Delta_q u.$$

We will also use the following commutator notation

$$[\Delta_q, u_{\leq m-2} \cdot \nabla] w_m := \Delta_q(u_{\leq m-2} \cdot \nabla w_m) - u_{\leq m-2} \cdot \nabla \Delta_q w_m. \tag{5.50}$$

By using integration by parts, the definition of Δ_q and Young's inequality,

$$\|[\Delta_q, u_{\leq m-2} \cdot \nabla] w_m\|_{L^r(\mathbb{T})} \lesssim \|\nabla u_{\leq m-2}\|_{L^\infty(\mathbb{T})} \|w_m\|_{L^r(\mathbb{T})}, \tag{5.51}$$

for all $r > 1$ (see also [24, 26]).

We now define the Besov type space $B^s_{p,q}(\mathbb{T})$ via the operators Δ_j defined above. Let $\mathcal{S}$ denote the usual Schwarz class and $\mathcal{S}'$ the distributions.

Definition 5.19. *Let $f \in \mathcal{S}'$. The nonhomogeneous Besov space $B^s_{p,q}(\mathbb{T})$ with $1 \le p, q \le \infty$ and $s \in \mathbb{R}$ consists of functions $f \in \mathcal{S}'(\mathbb{T})$ satisfying*

$$\|f\|_{B^s_{p,q}(\mathbb{T})} := \Big\| 2^{js} \|\Delta_j f\|_{L^p(\mathbb{T})} \Big\|_{\ell^q} = \Big[\sum_{j=0}^{\infty} \big(2^{js} \|\Delta_j f\|_{L^p(\mathbb{T})} \big)^q \Big]^{\frac{1}{q}} < \infty.$$

The nonhomogeneous Besov spaces contain Sobolev spaces. Indeed, using the Fourier-Plancherel formula, we find that the Besov space $B^s_{2,2}$ coincides with the Sobolev space V^s (see, e.g., [7, p.99]). Moreover, we have the following embedding: Let $s \in \mathbb{R}$, $1 \le p \le \infty$ and $q_1 \le q_2$, $B^s_{p,q_1}(\mathbb{T}) \subset B^s_{p,q_2}(\mathbb{T})$ (see, e.g., [30, Lemma 2.11]).

We can also define the space-time spaces for periodic functions (see, e.g., [7]).

Definition 5.20. *For $t > 0$, $s \in \mathbb{R}$ and $1 \le p, q, r \le \infty$, the space-time space $\widetilde{L}^r_t B^s_{p,q}$ is defined the norm*

$$\|f\|_{\widetilde{L}^r_t B^s_{p,q}} := \Big\| 2^{js} \|\Delta_j f\|_{L^r_t L^p(\mathbb{T})} \Big\|_{\ell^q}.$$

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