NON-REAL EIGENVALUES OF SINGULAR INDEFINITE STURM-LIOUVILLE PROBLEMS

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ABSTRACT. The present paper deals with non-real eigenvalues of singular indefinite Sturm-Liouville boundary value problems with limit-circle type non-oscillation endpoints. The estimate of upper bounds on non-real eigenvalues for the singular indefinite eigenvalue problem associated to the separated self-adjoint boundary conditions with non-principle solutions are obtained.

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1. Introduction

Consider the singular indefinite Sturm-Liouville differential equation

$$-(py')' + qy = \lambda wy \quad \text{in} \quad L^2_{|w|}(a, b) \tag{1.1}$$

associated to a self-adjoint boundary value condition, where $-\infty < a < b < \infty$, the functions p, q, w are real-valued and w changes sign on (a, b). Such a problem is called *indefinite* which is of great importance for its wide applications in physics, such as transport theory and quantum mechanics, and has discrete, real eigenvalues unbounded from both above and below. The most difference between indefinite and right-definite($w > 0$) problem is the non-real spectral points may appear([9, 18]). For a review of early works on indefinite problems, see [1, 8, 12, 13].

The problem on the related estimates of non-real eigenvalues for the indefinite problems was raised in [13] and stressed in [11]. Recently, the problem for the regular case was solved in [2, 16, 17, 20]. For the singular indefinite Sturm-Liouville problem under the condition of $p(x) = 1$, $w(x) = \text{sgn}(x)$, $q \in L^1$ or $q \in L^\infty$ or uniformly locally integrable potentials and singular limit-point endpoints case were studied in

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[3, 4, 5, 6]. In particular, the authors in [5] considered the bounds of non-real eigenvalues for the more generally coefficients of the weight function w and the potential function p, which deviate from the case $w(x) = \mathrm{sgn}(x)$ and $p(x) = 1$, and particularly, the indefinite weight functions w with finitely or infinitely many sign changes within a compact interval, functions $1/p \in L^\eta(\mathbb{R})$ for $\eta \in [1, \infty]$ and uniformly locally integrable potentials q or $q \in L^s(\mathbb{R})$ for $s \in [1, \infty]$ are also well investigated. Very recently, the operators theory and the bounds of non-real eigenvalues of indefinite Laplacian problems and singular indefinite Sturm-Liouville operators with L^p-potentials have been studied in [7, 15], and a priori bounds and the existence of non-real eigenvalues for the singular indefinite Sturm-Liouville eigenvalue problems with limit-circle type non-oscillation endpoints associated to a special self-adjoint boundary condition with principle solutions are obtained in [19]. For the existence of non-real eigenvalues for the general singular eigenvalue problem with limit-circle type non-oscillation endpoints, we can exploit the transformation $z(\cdot, \lambda) = y(\cdot, \lambda)/v$ transforms this singular problem into the regular indefinite problem (cf. [10, Lemma 4.4], [19, Lemma4.2]), then the existence and non-existence of non-real eigenvalues are proved in [10, 16, 19, 20, 21] and references cited therein.

The present paper will focus on the upper bounds for singular indefinite Sturm-Liouville eigenvalue problems with limit-circle type non-oscillation endpoints associated with the self-adjoint boundary conditions given by the non-principal solution (see the below in (2.6)). A priori bounds of non-real eigenvalues for this eigenvalue problem are obtained. The main ingredient of this paper is the equivalence of boundary conditions at the endpoints in Lemma 2.4.

The rest of this paper is organized as follows. Section 2 contains a basic discussion of singular indefinite Sturm-Liouville problem with limit-circle type non-oscillation endpoints and some preliminary results, then the upper bounds on non-real eigenvalues in terms of integrable conditions of coefficients for the singular perturbation of Legendre eigenvalue problem are shown in Section 3, see Theorem 3.1 and 3.2. An example is given in Section 4 to illustrate the results.

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2. Preliminary knowledge and results

In this section, we give some basic knowledge for the singular differential equation (1.1) under the standard conditions that p, q, w are real-valued functions satisfying

$$p > 0, |w| > 0 \text{ a.e. on } (a, b), \ \frac{1}{p}, w, q \in L^1_{\rm loc}(a, b), \int_a^b \left(\left| \frac{1}{p} \right| + |q| + |w| \right) = \infty, \tag{2.1}$$

and w satisfying

$$\Gamma := \operatorname*{ess\,inf}_{x \in (a,b) \setminus [a_1, b_1]} |w(x)| > 0 \tag{2.2}$$

and w has constant sign a.e. on (a, a_1) and (b_1, b), $-\infty < a < a_1 < b_1 < b < \infty$. Throughout this section the functions p, q, w always satisfy (2.1) and (2.2).

We first introduce some concepts (cf.[10, 14, 22]). The endpoint b (or a) is *oscillatory* if every nontrivial real-valued solution has an infinite number of zeros in (c, b) (or (a, c)) for any $c \in (a, b)$, and it is *non-oscillatory* otherwise. For fixed $\lambda \in \mathbb{R}$, a real solution u of (1.1) is called a *principal solution* at b if there exists $c \in (a, b)$ such that

$$u(x) \neq 0, \ x \in (c, b), \ \int_c^b \frac{1}{pu^2} = \infty.$$

A real solution v of (1.1) is called a *non-principal solution* at b if there exists $c \in (a, b)$ such that

$$v(x) \neq 0, \ x \in (c, b), \ \int_c^b \frac{1}{pv^2} < \infty.$$

If u and v are principal and non-principal solutions at b, respectively, then

$$\frac{u(x)}{v(x)} \to 0 \ \text{ as } \ x \to b.$$

We say that the endpoint b (resp. a) is a *limit-circle type endpoint* if all solutions of (1.1) are in $L^2_{|w|}[c, b)$ (resp. $L^2_{|w|}(a, c]$) for some $c \in (a, b)$. It is well known that the limit-circle type endpoint is independent of $\lambda \in \mathbb{R}$. The endpoint b (*or* a) is *limit-circle type non-oscillation* if it is both limit-circle type and non-oscillation.

The main condition in this paper is

$$\begin{aligned} &\gamma \in L^1(a, b), \quad \text{where} \quad \gamma(t) := \sup_{a<x<b} \left| \frac{1}{p(t)} \int_t^x q(s) \mathrm{d}s \right|, \\ &\mathcal{P} \in L^2_{|w|}(a, b), \quad \text{where} \quad \mathcal{P}(x) := \int_c^x \frac{1}{p(t)} \mathrm{d}t \end{aligned} \tag{2.3}$$

for some (and hence for all) $c \in (a,b)$.

Lemma 2.1. (*cf.* [19, Lemma 2.1]) *Assume that* (2.3) *holds. Then* (1.1) *is limit-circle type non-oscillation at endpoints a and b.*

Lemma 2.2. (*cf.* [19, Lemma 2.2]) *Assume that* (2.3) *holds. If $\int_c^b 1/p(x)dx = \infty$, then for every $\lambda \in \mathbb{R}$, there exists a non-principal solution v of* (1.1) *at b such that $pv'(x) \to v_0 \neq 0$ as $x \to b$. The similar conclusion holds at a.*

Set

$$[f,g] := f(pg') - g(pf'),\ \ f,g \in D_{\max},$$

where $D_{\max} = \{f \in L^2_{|w|}(a,b) : f, pf' \in AC_{\text{loc}}(a,b), \frac{1}{|w|}[-(pf')' + qf] \in L^2_{|w|}(a,b)\}$, $AC_{\text{loc}}(a,b)$ denotes the set of all complex-valued functions which are absolutely continuous on all compact subintervals of (a,b).

Let v_a, v_b be the non-principal solutions of (1.1) at a, b for $\lambda = 0$ defined in Lemma 2.2, respectively, and u_a, u_b be the corresponding principal solutions, and satisfying $[u_a, v_a](x) \equiv 1$, $[u_b, v_b](x) \equiv 1$.

Lemma 2.3. *Let v_b be defined as above and* (2.3) *holds. Then for arbitrary $y \in D_{\max}$ and v_b satisfy*

$$y,\ \sqrt{p}y',\ \sqrt{p}v_b' \in L^2(a,b) \ \ \textit{and} \ \ py'(x)v_b(x) \to 0 \ \ \textit{as} \ \ x \to b.$$

The similar conclusion holds at a.

Proof. From (2.2), one sees that there exists N such that $|w(x)| \geq N$ *a.e.* x outside of a compact interval $[a_1, b_1]$ and

$$\begin{aligned}\int_a^b |y(x)|^2 \mathrm{d}x = \int_{a_1}^{b_1} |y(x)|^2 \mathrm{d}x + \int_{(a,b)\setminus[a_1,b_1]} |y(x)|^2 \mathrm{d}x \\ \leq (b_1 - a_1) \sup_{x \in [a_1,b_1]} |y(x)|^2 + \frac{1}{N} \int_{(a,b)\setminus[a_1,b_1]} |y(x)|^2 |w(x)| \mathrm{d}x < \infty,\end{aligned}$$

where the continuity of y implies the boundedness on $[a_1, b_1]$. Then $y \in L^2(a,b)$. Since v_b and y are the solutions of (1.1) for $\lambda = 0$ and λ, respectively, i.e.,

$$-(pv_b')' + qv_b = 0, \ \ -(py')' + qy = \lambda wy.$$

Then

$$\begin{aligned}&-pv_b'(b)v_b(b)+pv_b'(a)v_b(a)+\int_a^b p|v_b'|^2+\int_a^b qv_b^2=0,\\&-py'(b)\overline{y(b)}+py'(a)\overline{y(a)}+\int_a^b p|y'|^2+\int_a^b q|y|^2=\lambda\int_a^b w|y|^2.\end{aligned}$$

This facts with $y, v_b \in D_{\max}$ and (2.2) led to $\sqrt{p}y',\ \sqrt{p}v_b' \in L^2(a,b)$.

Let $Sy=[-(py')'+qy]/|w|$, $y\in D_{\max}$, then

$$\int_c^x v_b|w|Sy=\int_c^x v_b[-(py')'+qy]=-py'(x)v_b(x)+py'(c)v_b(c)+\int_c^x(py'v_b'+qyv_b),$$

and hence $\lim_{x\to b}(py'v_b)$ exist and finite. Suppose that

$$\lim_{x\to b}p(x)|y'(x)||v_b(x)|=\widetilde{\alpha}>0.$$

Then there exists $b_0>0$ such that $|v_b(x)|>0$ and

$$p(x)|y'(x)|\geq\frac{\widetilde{\alpha}}{|v_b(x)|}$$

for $x\in(b_0,b)$. Multiplication with $|v_b'(t)|$ and integration leads to

$$\int_{b_0}^x p(t)|y'(t)||v_b'(t)|\mathrm{d}t\geq\widetilde{\alpha}\int_{b_0}^x\frac{|v_b'(t)|}{|v_b(t)|}\mathrm{d}t\geq\widetilde{\alpha}\left|\int_{b_0}^x\frac{v_b'(t)}{v_b(t)}\mathrm{d}t\right|=\widetilde{\alpha}\left|\ln\frac{v_b(x)}{v_b(b_0)}\right|.\tag{2.4}$$

By $pv_b'(x)\to v_0\neq 0$ as $x\to b$ in Lemma 2.2 and $\int_c^b 1/p(t)\mathrm{d}t=\infty$, we have

$$v_b(x)=v_b(c)+\int_c^x v_b'(s)\mathrm{d}s=v_b(c)+\int_c^x pv_b'(s)\frac{1}{p(s)}\mathrm{d}s\to\infty,\ x\to b.\tag{2.5}$$

This together with (2.4) implies that the left hand side of (2.4) is bounded since $\sqrt{p}y',\ \sqrt{p}v_b'\in L^2(a,b)$ hold while the right hand side grows to ∞, which is a contradiction and hence the assumption $\lim_{x\to b}p(x)|y'(x)||v_b(x)|=\widetilde{\alpha}>0$ was false, thus $\lim_{x\to b}p(x)y'(x)v_b(x)=0$. The proof of Lemma 2.3 is finished. □

In the following we give a special separated self-adjoint boundary conditions of limit-circle type non-oscillation endpoints in the form $[y,v_a](a)=0,\ [y,v_b](b)=0$, and the corresponding eigenvalue problem is

$$\begin{cases}-(py')'+qy=\lambda wy,\\ [y,v_a](a)=0,\ [y,v_b](b)=0.\end{cases}\tag{2.6}$$

A complex number λ is called an eigenvalue of boundary value problem (2.6) if there is a nontrivial solution $y\in D_{\max}$ satisfying the boundary conditions. Such a solution y is called an eigenfunction of λ. Since the sign change of the weight function, the indefinite eigenvalues problem (2.6) is not self-adjoint in a Hilbert space but it can be interpreted as self-adjoint in the Krein space with indefinite inner product. The following Lemma plays an important role in the proof of the main results of this paper, which gives the equivalence between boundary conditions constructed by the non-principal solutions of (1.1) and the eigenfunctions at the end points.

Lemma 2.4. *Assume that* (2.1), (2.2), (2.3) *hold and* $v_b(\cdot)$ *is the non-principal solution of* (1.1) *at* b *for* $\lambda = 0$ *defined as above. Let* y *be an eigenfunction of* (2.6) *corresponding to the eigenvalue* λ. *If* $\int_c^b 1/p(t)\,dt = \infty$, *then* y *is bounded and*

$$[y, v_b](b) = 0 \Leftrightarrow (py')(x)y(x) \to 0 \ \textit{as}\ x \to b.$$

The similar conclusion holds for $x \to a$.

Proof. Since v_b is a non-principal solution at b, one can choose $c \in (a, b)$ such that $v_b(x) \neq 0,\ x \in [c, b)$. From y and v_b are the solutions of (1.1) for λ and $\lambda = 0$, respectively, we have that for $x \in [c, b)$, integrating $(y(x)/v_b(x))' = -[y, v_b](x)/(pv_b^2)(x)$ over the interval $[c, x]$ gives that

$$y(x) = \left(\frac{y(c)}{v_b(c)} - \int_c^x \frac{[y, v_b](t)}{pv_b^2(t)} \mathrm{d}t \right) v_b(x) := H(x)v_b(x). \tag{2.7}$$

Then H is bounded since $[y, v_b](b) = 0$ and v_b is a non-principal solution at b. Moreover,

$$(py')(x) = H(x)(pv_b')(x) - \frac{[y, v_b](x)}{v_b(x)}. \tag{2.8}$$

It follows from $[y, v_b](b) = 0$ and $py'(x)v_b(x) \to 0$ as $x \to b$ in Lemma 2.3 that $pv_b'(x)y(x) \to 0, x \to b$. This together with $pv_b'(x) \to v_0 \neq 0$ as $x \to b$ in Lemma 2.2 implies that $y(x) \to 0, x \to b$, which together with $y(x) = H(x)v_b(x)$ in (2.7) and $v_b(x) \to \infty,\ x \to b$ in (2.5) led to $H(x) \to 0,\ x \to b$. The facts $pv_b'(x) \to v_0 \neq 0,\ v_b(x) \to \infty,\ [y, v_b](x) \to 0$ as $x \to b$ and (2.8) yields that $py'(x) \to 0,\ x \to b$. Hence $py'(x)y(x) \to 0,\ x \to b$.

Conversely, assume that $y \to \infty,\ x \to b$, it follows from $py'(x)y(x) \to 0,\ x \to b$ that $py'(x) \to 0,\ x \to b$. This together with (2.8), $pv_b'(x) \to v_0 \neq 0,\ v_b(x) \to$

∞, $[y, v_b](x) \to 0$ as $x \to b$ implies that $H(x) \to 0,\ x \to b$. Then from the L'Hóspital principle and $H(x), v_b^{-1}(x) \to 0$ as $x \to b$ one sees that

$$\begin{aligned}\lim_{x\to b} y(x) &= \lim_{x\to b} H(x)v_b(x) = \lim_{x\to b} \frac{H(x)}{v_b^{-1}(x)} \\ &= \lim_{x\to b} \frac{[y, v_b](x)(pv_b^2)^{-1}(x)}{v_b^{-2}(x)v_b'(x)} = \lim_{x\to b} \frac{[y, v_b](x)}{p(x)v_b'(x)} = 0\end{aligned}$$

since $pv_b'(x) \to v_0 \neq 0$, $[y, v_b](x) \to 0$ as $x \to b$, which is a contradiction. So that $y(x) \to 0$ as $x \to b$. This together with $pv_b'(x) \to v_0 \neq 0$ as $x \to b$ in Lemma 2.2 that $pv_b'(x)y(x) \to 0$ as $x \to b$. Hence $[y, v_b](b) = 0$ provided that $py'(x)v_b(x) \to 0$ as $x \to b$ in Lemma 2.3. This completes the proof of Lemma 2.4. $\square$

3. The upper bounds of non-real eigenvalues

In this section we give a priori bounds on non-real eigenvalues of the singular indefinite eigenvalue problem

$$\begin{cases} -(py')' + qy = \lambda wy, \\ [y, v_a](a) = 0,\ [y, v_b](b) = 0, \end{cases} \tag{3.1}$$

where v_a and v_b are non-principle solutions of $-(py')' + qy = 0$ at a and b defined as above, respectively, q is real valued and $q \in L^2(a, b)$. One can verify that the main condition (2.3) holds for this problem, and hence all the conclusions in Section 2 hold for (3.1). Additionally, we assume that for some constant Γ_p, $\Gamma_{p,q}^a$, $\Gamma_{p,q}^b$, Γ_q, $\Gamma_{p,q}$, $x \in [\operatorname{ess\,inf} a, \operatorname{ess\,sup} b]$ satisfied

$$\begin{aligned} &\operatorname{ess\,inf} a \le a_1 \le a_0 \le b_1 \le \operatorname{ess\,sup} b,\ b - a > 1,\ L^1(a,b) \ni \widetilde{p} := \int_{a_0}^{x} \frac{1}{p}, \\ &\sqrt{p(x)} \le \Gamma_p,\ \left| \frac{b - x}{\sqrt{p(x)}} \int_a^x q_-(t)\mathrm{d}t \right| \le \Gamma_{p,q}^b,\ \left| \frac{x - a}{\sqrt{p(x)}} \int_x^b q_-(t)\mathrm{d}t \right| \le \Gamma_{p,q}^a, \\ &\Gamma_q := \frac{1}{b - a} \int_a^b q_-(t)\mathrm{d}t,\ \ \Gamma_{p,q} := \frac{\Gamma_{p,q}^b + \Gamma_{p,q}^a}{b - a},\ \ q_- = \max\{0, -q\} \end{aligned} \tag{3.2}$$

in order to estimate the a priori bounds of non-real eigenvalues.

Since $w^2(x) > 0$ a.e. on (a, b), we can choose $\delta > 0$ such that

$$\Delta(\delta) = \{x \in (a, b) :\ w^2(x) < \delta\},\ m(\delta) = \operatorname{mes} \Delta(\delta). \tag{3.3}$$

Theorem 3.1. *Assume that* $q \in L^2(a,b)$, $w \in AC(a,b)$, $w' \in L^2_p(a,b)$ *and* (2.3), (3.2), (3.3) *hold. If* $\displaystyle\int_a^b \frac{dt}{p(t)} = \infty$, *then for any non-real eigenvalue* λ *of* (3.1) *it holds*

$$
\begin{aligned}
&|\operatorname{Im}\lambda| \le \frac{2}{\delta}\sqrt{2W(\Gamma_q + 2\Gamma_{p,q}^2)},\\
&|\lambda| \le \frac{2}{\delta}\left\{\Gamma_w\left(2(\Gamma_q + 2\Gamma_{p,q}^2) + \sqrt{b-a}\|q\|_2\right) + \sqrt{2W(\Gamma_q + 2\Gamma_{p,q}^2)}\right\},
\end{aligned}
\tag{3.4}
$$

where $\Gamma_w = \max\{|w(x)| : x \in (a,b)\}$ *and* $W = \int_a^b p|w'|^2$.

A point at x which the weight function w changes its sign will be called a *turning point* [12]. If w has only one turning point on (a,b), that is there exist a point $\widetilde{a} \in (a,b)$ such that

$$
(x - \widetilde{a})w(x) > 0 \ a.e. \text{ on } (a,b). \tag{3.5}
$$

Since $(x - \widetilde{a})w(x) > 0$ $a.e.$ on (a,b), we can choose $\eta > 0$ such that

$$
\Delta(\eta) = \{x \in (a,b) : \ (x - \widetilde{a})w(x) < \eta\}, \ m(\eta) = \operatorname{mes}\Delta(\eta). \tag{3.6}
$$

Theorem 3.2. *Assume that* $q \in L^2(a,b)$, (2.3), (3.2), (3.5) *and* (3.6) *hold. If* $\displaystyle\int_a^b \frac{dt}{p(t)} = \infty$, *then for any non-real eigenvalue* λ *of* (3.1) *it holds that*

$$
\begin{aligned}
&|\operatorname{Im}\lambda| \le \frac{2}{\eta}\left(\Gamma_q + \Gamma_p^2 + 2\Gamma_{p,q}^2\right),\\
&|\lambda| \le \frac{2}{\eta}\left\{\left(\Gamma_q + 2\Gamma_{p,q}^2\right)(1 + 4(b-a)) + \Gamma_p^2\right\}.
\end{aligned}
\tag{3.7}
$$

In order to prove the Theorem 3.1 and 3.2, we first give the estimates of $\|\sqrt{p}\varphi'\|_2$ and $\|\sqrt{q_-}\varphi\|_2$, where φ is an eigenfunction of (3.1) corresponding to the non-real eigenvalue λ. That is $[\varphi, v_a](a) = 0$, $[\varphi, v_b](b) = 0$ and

$$
-(p\varphi')' + q\varphi = \lambda w\varphi. \tag{3.8}
$$

Since the problem (3.8) is a linear system and φ is continuous, we can choose φ satisfying $\int_a^b |\varphi(x)|^2 \mathrm{d}x = 1$ in the following discussion.

Lemma 3.3. *Assume that* $q \in L^2(a,b)$ *and* $\displaystyle\int_a^b \frac{dt}{p(t)} = \infty$, *let* λ, φ *be defined as above and* (3.2) *holds, then*

$$
\int_a^b p|\varphi'|^2 \le 2\left(\Gamma_q + 2\Gamma_{p,q}^2\right), \quad \int_a^b q_-|\varphi|^2 \le 2\left(\Gamma_q + 2\Gamma_{p,q}^2\right). \tag{3.9}
$$

Proof. It follows from Lemma 2.4 that φ is bounded and satisfies

$$(p\varphi')(x)\varphi(x) \to 0 \quad \text{as} \quad x \to a \text{ or } b. \tag{3.10}$$

Multiplying both sides of (3.10) by $\overline{\varphi}$ and integrating over the interval (a,b), we get

$$\int_a^b p|\varphi'|^2 + \int_a^b q|\varphi|^2 = \lambda \int_a^b w|\varphi|^2.$$

Here (3.10) is used. This together with $\operatorname{Im}\lambda \neq 0$ yields that $\int_a^b w|\varphi|^2 = 0$ and hence

$$\int_a^b p|\varphi'|^2 + \int_a^b q|\varphi|^2 = 0. \tag{3.11}$$

Let

$$\Theta(x) = \int_a^x q_-(t)\mathrm{d}t - (x-a)\Gamma_q,\ \Gamma_q = \frac{1}{b-a}\int_a^b q_-(t)\mathrm{d}t,\ x \in [a,b].$$

Then one can verify that

$$\Theta(a) = 0 = \Theta(b),\ \Theta'(x) = q_-(x) - \Gamma_q,\ \text{a.e. } x \in (a,b). \tag{3.12}$$

Furthermore, the condition $q \in L^2(a,b)$ implies that

$$|\Theta(x)| = \left|\frac{b-x}{b-a}\int_a^x q_- - \frac{x-a}{b-a}\int_x^b q_-\right| \le \frac{\Gamma_{p,q}^b + \Gamma_{p,q}^a}{b-a}\sqrt{p(x)} = \Gamma_{p,q}\sqrt{p(x)},$$

and hence

$$\int_a^b q_-|\varphi|^2 = \int_a^b (\Theta' + \Gamma_q)|\varphi|^2 = \Gamma_q - 2\operatorname{Re}\left(\int_a^b \Theta\varphi'\overline{\varphi}\right),$$

which together with $|\Theta(x)| \le \Gamma_{p,q}\sqrt{p(x)}$ and $\int_a^b |\varphi|^2 = 1$ yields that

$$\int_a^b q_-|\varphi|^2 \le \Gamma_q + 2\Gamma_{p,q}\left(\int_a^b p|\varphi'|^2\right)^{1/2} \le \Gamma_q + \frac{1}{2}\int_a^b p|\varphi'|^2 + 2\Gamma_{p,q}^2. \tag{3.13}$$

It follows from (3.11), (3.13) and $q = q_+ - q_-$ that

$$\int_a^b p|\varphi'|^2 \le 2\left(\Gamma_q + 2\Gamma_{p,q}^2\right),\quad \int_a^b q_-|\varphi|^2 \le 2\left(\Gamma_q + 2\Gamma_{p,q}^2\right),$$

which completes the proof of Lemma 3.3. □

Lemma 3.4. *Assume that* $q \in L^2(a,b)$, $\displaystyle\int_a^b \frac{dt}{p(t)} = \infty$ *and* (3.2), (3.3) *hold. For any* $\varepsilon_0 > 0$, *there exists* $\varepsilon > 0$ *such that* $\int_{\Delta(\delta)} |\varphi(x)|^2 dx < \varepsilon_0$ *if* $0 < \delta < \varepsilon$ *for all* *eigenfunctions* φ *of* (3.1) *corresponding to a non-real eigenvalue and* $\int_a^b |\varphi|^2 = 1$. *Particularly, when* $\varepsilon_0 = 1/2$, ε *can be chosen such that*

$$2m(\varepsilon)\left(1 + 2(\Gamma_q + 2\Gamma_{p,q}^2)\int_{a_1}^{b_1}\frac{1}{p}\right) + 4\left(\Gamma_q + 2\Gamma_{p,q}^2\right)\int_{\Delta(\delta)}\left|\int_{a_0}^{x}\frac{1}{p}\right| dx \le 1/2.$$

The similar conclusion holds for $\Delta(\eta)$.

Proof. Since $\int_a^b |\varphi|^2 = 1$, there must exist $x_0 \in [a_1, b_1]$ such that $|\varphi(x_0)| \le 1$, and hence it follows from Lemma 3.3 that

$$|\varphi(x)| \le |\varphi(x_0)| + \left|\int_{x_0}^{x}\varphi'\right| \le 1 + \left|\int_{x_0}^{x}\frac{1}{p}\right|^{\frac{1}{2}}\left|\int_{x_0}^{x}p|\varphi'|^2\right|^{\frac{1}{2}} \le 1 + \sqrt{2\left(\Gamma_q + 2\Gamma_{p,q}^2\right)}\left|\int_{x_0}^{x}\frac{1}{p}\right|^{\frac{1}{2}},$$

which implies that

$$|\varphi(x)|^2 \le 2 + 4\left(\Gamma_q + 2\Gamma_{p,q}^2\right)\left|\int_{x_0}^{x}\frac{1}{p}\right|.$$

As a result,

$$\begin{aligned}
&\int_{\Delta(\delta)}|\varphi(x)|^2\mathrm{d}x \le 2m(\varepsilon) + 4\left(\Gamma_q + 2\Gamma_{p,q}^2\right)\int_{\Delta(\delta)}\left|\int_{x_0}^{x}\frac{1}{p}\right|\mathrm{d}x\\
&\le 2m(\varepsilon) + 4\left(\Gamma_q + 2\Gamma_{p,q}^2\right)\int_{\Delta(\delta)}\left|\int_{x_0}^{a_0}\frac{1}{p} + \int_{a_0}^{x}\frac{1}{p}\right|\mathrm{d}x\\
&\le 2m(\varepsilon)\left(1 + 2(\Gamma_q + 2\Gamma_{p,q}^2)\int_{a_1}^{b_1}\frac{1}{p}\right) + 4\left(\Gamma_q + 2\Gamma_{p,q}^2\right)\int_{\Delta(\delta)}\left|\int_{a_0}^{x}\frac{1}{p}\right|\mathrm{d}x.
\end{aligned}$$

Since $m(\delta) \to 0$ as $\delta \to 0$ and $\int_{a_0}^{x}\frac{1}{p} = \widetilde{p} \in L^1(a,b)$, one sees that the last term of the above inequality tends to 0 as $\delta \to 0$ by the continuity of the integral $\int_{\Delta(\delta)}\left|\int_{a_0}^{x}\frac{1}{p}\right|\mathrm{d}x$. $\square$

With the aids of the above lemmas we now proof Theorem 3.1 and 3.2.

The proof of Theorem 3.1. Multiplying both sides of (3.8) by $w\overline{\varphi}$ and integrating by parts on (a,b), we have from $[\varphi, v_a](a) = 0 = [\varphi, v_b](b)$ and (3.10)

$$\lambda\int_a^b w^2|\varphi|^2 = \int_a^b (wp|\varphi'|^2 + wq|\varphi|^2) + \int_a^b w'p\varphi'\overline{\varphi}. \tag{3.14}$$

It follows from $\int_a^b \varphi(x)\mathrm{d}x = 1$ and $b - a > 1$ that $\|\varphi\|_\infty \le 1$. This together with (3.9) in Lemma 3.3 and $\Gamma_w = \max\{|w(x)| : x \in (a,b)\}$ yields that

$$\begin{aligned}
&\left|\int_a^b wp|\varphi'|^2 + \int_a^b wq|\varphi|^2\right| \le \Gamma_w\left(\int_a^b p|\varphi'|^2 + \|\varphi\|_\infty^2\int_a^b |q|\right)\\
&\le \Gamma_w\left(2(\Gamma_q + 2\Gamma_{p,q}^2) + \sqrt{b-a}\|q\|_2\right).
\end{aligned} \tag{3.15}$$

It follows from $w' \in L^2(a,b)$, $W = \int_a^b p|w'|^2$, (3.9) and Schwarz inequality that

$$\left|\int_a^b w'p\varphi'\overline{\varphi}\right| \le \|\varphi\|_\infty \left(\int_a^b p|w'|^2\right)^{1/2} \left(\int_a^b p|\varphi'|^2\right)^{1/2} \le \sqrt{2W(\Gamma_q + 2\Gamma_{p,q}^2)}. \tag{3.16}$$

Recall the definition of $\Delta(\delta)$ in (3.3),

$$\int_a^b w^2|\varphi|^2 \ge \delta \int_{(a,b)\setminus\Delta(\delta)} |\varphi|^2 = \delta\left(\int_a^b |\varphi|^2 - \int_{\Delta(\delta)} |\varphi|^2\right) \ge \delta\,(1-1/2) \ge \delta/2. \tag{3.17}$$

Then from (3.14)-(3.17) we get

$$|\lambda|\frac{\delta}{2} \le |\lambda| \int_a^b w^2|\varphi|^2 \le \Gamma_w\left(2(\Gamma_q + 2\Gamma_{p,q}^2) + \sqrt{b-a}\|q\|_2\right) + \sqrt{2W(\Gamma_q + 2\Gamma_{p,q}^2)}. \tag{3.18}$$

Separating the imaginary parts of (3.14) implies

$$\operatorname{Im}\lambda \int_a^b w^2|\varphi|^2 = \operatorname{Im}\left(\int_a^b w'p\varphi'\overline{\varphi}\right),$$

which together with (3.16) and (3.17) implies that

$$|\operatorname{Im}\lambda|\frac{\delta}{2} \le |\operatorname{Im}\lambda| \int_a^b w^2|\varphi|^2 \le \left|\int_a^b w'p\varphi'\overline{\varphi}\right| \le \sqrt{2W(\Gamma_q + 2\Gamma_{p,q}^2)}. \tag{3.19}$$

Hence we get the inequalities in (3.4) by (3.19) and (3.18). □

The proof of Theorem 3.2. Multiplying both sides of (3.8) by $\overline{\varphi}$ and integrating over the interval $[x,b]$, we have

$$p\varphi'(x)\overline{\varphi}(x) + \int_x^b p|\varphi'|^2 + \int_x^b q|\varphi|^2 = \lambda \int_x^b w|\varphi|^2. \tag{3.20}$$

Now, integrating (3.20) over $[\widetilde{a}, b]$, where $\widetilde{a} \in (a,b)$ in (3.2), gives that

$$\begin{aligned}
&|\lambda| \int_{\widetilde{a}}^b (x-\widetilde{a}) w|\varphi|^2 = |\lambda| \int_{\widetilde{a}}^b \int_x^b w|\varphi|^2 \\
&\le \int_{\widetilde{a}}^b p|\varphi'||\overline{\varphi}| + \int_{\widetilde{a}}^b \int_x^b p|\varphi'|^2 + \int_{\widetilde{a}}^b \int_x^b |q||\varphi|^2 \\
&\le \Gamma_p \left(\int_{\widetilde{a}}^b p|\varphi'|^2\right)^{1/2} + (b-\widetilde{a})\left(\int_{\widetilde{a}}^b (p|\varphi'|^2 + |q||\varphi|^2)\right) \\
&\le \frac{1}{2}\Gamma_p^2 + \frac{1}{2}\int_{\widetilde{a}}^b p|\varphi'|^2 + (b-\widetilde{a})\left(\int_{\widetilde{a}}^b (p|\varphi'|^2 + |q||\varphi|^2)\right)
\end{aligned} \tag{3.21}$$

provided by $\sqrt{p(x)} \le \Gamma_p$, $\int_a^b |\varphi(x)|^2 \mathrm{d}x = 1$ and Schwarz inequality. Integrating by parts on the interval $[a, x]$ and $[a, \widetilde{a}]$, similar with the above discussion we have

$$\begin{aligned}
&|\lambda| \int_a^{\widetilde{a}} (x-\widetilde{a}) w|\varphi|^2 = -|\lambda| \int_a^{\widetilde{a}} \int_a^x w|\varphi|^2 \\
&\le \Gamma_p \left(\int_a^{\widetilde{a}} p|\varphi'|^2 \right)^{1/2} + (b-\widetilde{a}) \left(\int_a^{\widetilde{a}} (p|\varphi'|^2 + |q||\varphi|^2) \right) \\
&\le \frac{1}{2}\Gamma_p^2 + \frac{1}{2} \int_a^{\widetilde{a}} p|\varphi'|^2 + (b-\widetilde{a}) \left(\int_a^{\widetilde{a}} (p|\varphi'|^2 + |q||\varphi|^2) \right)
\end{aligned} \tag{3.22}$$

Hence (3.21) and (3.22) yields that

$$\begin{aligned}
&|\lambda| \int_a^b (x-\widetilde{a}) w|\varphi|^2 \le \Gamma_p^2 + \frac{1}{2} \int_a^b p|\varphi'|^2 + (b-\widetilde{a}) \left(\int_a^b (p|\varphi'|^2 + |q||\varphi|^2) \right) \\
&\le \Gamma_p^2 + \frac{1}{2} \int_a^b p|\varphi'|^2 + (b-a) \left(\int_a^b (p|\varphi'|^2 + q|\varphi|^2) + 2\int_a^b q_-|\varphi|^2 \right) \\
&\le \left(\Gamma_q + 2\Gamma_{p,q}^2\right)(1 + 4(b-a)) + \Gamma_p^2
\end{aligned} \tag{3.23}$$

by (3.11), (3.9) and $|q| = q + 2q_-$. Separating the imaginary parts of (3.20) we get

$$\operatorname{Im} \lambda \int_x^b w|\varphi|^2 = \operatorname{Im}\left(p\varphi'(x)\overline{\varphi}(x)\right).$$

With the similar method, we have

$$|\operatorname{Im} \lambda| \int_a^b (x-\widetilde{a}) w|\varphi|^2 \le \Gamma_q + \Gamma_p^2 + 2\Gamma_{p,q}^2. \tag{3.24}$$

According to the definition of $\Delta(\eta)$ in (3.6), it follows that

$$\int_a^b (x-\widetilde{a}) w|\varphi|^2 \ge \eta \int_{(a,b)\setminus\Delta(\eta)} |\varphi|^2 = \eta \left(\int_a^b |\varphi|^2 - \int_{\Delta(\eta)} |\varphi|^2 \right) \ge \eta/2,$$

which together with (3.23) and (3.24) lead to the inequalities in (3.7). □

4. Example

In this section we give an example to state the upper bounds result in Section 3. Consider the singular eigenvalue problem

$$\begin{cases} -\left[(1-x^2)y'\right]' + qy = \lambda x y, \\ [y, v_{-1}](-1) = 0, \ [y, v_1](1) = 0, \end{cases} \tag{4.1}$$

where $p(x) = 1 - x^2$, $w(x) = x$, $x \in (-1, 1)$, v_{-1} and v_1 are non-principle solutions of $-[(1-x^2)y']' + qy = 0$ at -1 and 1, respectively, q is real valued and $q \in L^2(-1, 1)$.

Since the differential expression in (4.1) and the classical Legendre equation [22, Example 8.3.1, p157]

$$-[(1-x^2)y']' = \lambda y,\ x \in (-1,1)$$

have the same first coefficient term $(1-x^2)$, so we call (4.1) as *"the perturbation of Legendre eigenvalue problem "* with indefinite weight x on $(-1,1)$, which arises from models in the transport theory in physics through variable separation. It follows from Cauchy-Schwarz inequality that

$$\left|\int_t^x q(s)\mathrm{d}s\right| \le \sqrt{1\pm t}\left(\int_{-1}^1 |q|^2\right)^{1/2} = \sqrt{1\pm t}\|q\|_2,$$

where $\|\cdot\|_2$ denote the norm of the space $L^2(-1,1)$ and hence

$$\left|\frac{1}{1-t^2}\int_t^x q(s)\mathrm{d}s\right| \le \left(\sqrt{1\pm t}\right)^{-1}\|q\|_2 \in L^1(-1,1).$$

Clearly

$$\int_0^x \frac{1}{1-t^2}\mathrm{d}t = \frac{1}{2}\ln\frac{1+x}{1-x} \in L^2_{|x|}(-1,1),\quad \Gamma = \operatorname*{ess\,inf}_{x\in(-1,1)\setminus[-1/2,1/2]} |x| > 0.$$

Therefore, the main condition (2.2) and (2.3) hold for this problem. Hence all the conclusions in Section 2 hold for the problem (4.1). Furthermore,

$$\begin{aligned}
&\int_0^x \frac{1}{1-t^2}\mathrm{d}t \in L^1(-1,1),\ \sqrt{1-x^2} \le 1,\\
&\left|\frac{1-x}{\sqrt{1-x^2}}\int_{-1}^x q_-(t)\mathrm{d}t\right| \le \sqrt{1-x}\left(\int_{-1}^x |q_-(t)|^2\mathrm{d}t\right)^{1/2} \le \sqrt{1-x}\|q_-\|_2 \le \sqrt{2}\|q_-\|_2,\\
&\left|\frac{x+1}{\sqrt{1-x^2}}\int_x^1 q_-(t)\mathrm{d}t\right| \le \sqrt{1+x}\left(\int_x^1 |q_-(t)|^2\mathrm{d}t\right)^{1/2} \le \sqrt{1+x}\|q_-\|_2 \le \sqrt{2}\|q_-\|_2.
\end{aligned}$$

Then the condition in (3.2) holds.

Since $x^2 > 0$ a.e. on $(-1,1)$, we can choose $\delta > 0$ such that

$$\Delta(\delta) = \{x \in (-1,1) :\ x^2 < \delta\},\ m(\delta) = \text{mes}\,\Delta(\delta).$$

By Theorem 3.1, the upper bound on non-real eigenvalues λ of (4.1) is given by

$$\begin{aligned}
&|\operatorname{Im}\lambda| \le \frac{4}{\sqrt{3}\delta}\sqrt{\|q_-\|_2\left(\sqrt{2}+8\|q_-\|_2\right)},\\
&|\lambda| \le \frac{2}{\delta}\left\{\sqrt{2}\left(\|q_-\|_2+\|q\|_2\right)+8\|q_-\|_2^2+\frac{2}{\sqrt{3}}\sqrt{\|q_-\|_2\left(\sqrt{2}+8\|q_-\|_2\right)}\right\}.
\end{aligned}$$

Since $w(x) = x$ has only one turning point $\widetilde{a} = 0$ on $(-1, 1)$ and satisfied $x^2 > 0$ $a.e.$ on $(-1, 1)$, we can choose $\eta > 0$ such that

$$\Delta(\eta) = \{x \in (-1,1) : \ x^2 < \eta\}, \ m(\eta) = \text{mes } \Delta(\eta).$$

By Theorem 3.2, the upper bound for any non-real eigenvalue λ of (4.1) hold that

$$|\operatorname{Im}\lambda| \leq \frac{2}{\eta}\left(1 + \frac{\|q_-\|_2}{\sqrt{2}} + 4\|q_-\|_2^2\right), \ |\lambda| \leq \frac{2}{\eta}\left\{9\|q_-\|_2\left(\frac{1}{\sqrt{2}} + 4\|q_-\|_2\right) + 1\right\}.$$

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