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DOUBLE PHASE OBSTACLE PROBLEMS INVOLVING SET-VALUED CONVECTION AND MIXED BOUNDARY VALUE CONDITIONS: UPPER-BOUND ERROR ESTIMATES

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ABSTRACT. The main purpose of this paper is to study upper-bound error estimates (also known as error bounds) for a class of generalized double phase obstacle problems via regularized gap functions. More precisely, we introduce some regularized gap functions for the double phase obstacle problem in forms introduced by Yamashita and Fukushima, and apply these regularized gap functions to provide the upper-bound error estimates for the double phase obstacle problem.

1. Introduction

In 1976, Auslender [3] introduced a valuable tool called the gap function to formulate variational inequalities by virtue of corresponding optimization problems. A gap function is given by

$$\mathbf{n}(z) = \sup_{v\in C}\langle \pi(z), z-v\rangle,$$

where $z \in C \subset \mathbb{R}^n$, $\pi\colon \mathbb{R}^n \to \mathbb{R}^n$, and $\langle\cdot,\cdot\rangle$ is the scalar product in $\mathbb{R}^n$. The function $\mathbf{n}$ has nonnegative values on C and $\mathbf{n}(z_0) = 0$ if and only if z_0 is a solution to the concerning variational inequality. In general, the gap function $\mathbf{n}$ is not differentiable. This disadvantage was improved by Yamashita and Fukushima [41] with proposing a new gap function which also called the regularized gap function:

$$\mathbf{n}_\theta(z) = \sup_{v\in C}\{\langle \pi(z), z-v\rangle - \theta\|z-v\|^2\},$$

where the regularized parameter $\theta > 0$. The regularized gap function $\mathbf{n}_\theta$ is finite valued and differentiable whenever π is differentiable; see Fukushima [17] for more information. In Ref. [41], Yamashita and Fukushima also provided another gap function based on the Moreau-Yosida regularization involving the regularized gap function $\mathbf{n}_\theta$ as follows:

$$\Psi^{\vartheta}_{\mathbf{n}_\theta}(z) = \inf_{w\in C}\{\mathbf{n}_\theta(w) + \vartheta\|z-w\|^2\},$$

where $\vartheta > 0$. Some error bounds for variational inequalities via the regularized gap functions $\mathbf{n}_\theta$ and $\Psi^{\vartheta}_{\mathbf{n}_\theta}$ were established. Error bound illustrates the upper estimation of the distance between an arbitrary feasible point and the solution set of a certain problem. It was crucial in studying the convergence of iterative methods for solving various classes of variational inequalities. Up to now, the topic on gap functions and error bounds has been important and interesting in optimization theory and nonlinear

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analysis for studying related-optimization problems such as variational inequalities, equilibrium problems and variational-hemivariational inequalities, and so on. For more information on this topic, we refer readers to works, see Refs. [1, 8, 10, 21, 22, 23, 24, 25, 26, 27, 28, 31, 32, 34, 40, 46, 47, 48] and the references therein.

Let Ω be a bounded domain in $\mathbb{R}^N (N \geq 2)$, with Lipschitz boundary $\partial\Omega$ and $\mathbf{S}_0^{1,\mathscr{H}}(\Omega)$ be a subspace of the Sobolev-Musielak-Orlicz space $\mathbf{S}^{1,\mathscr{H}}(\Omega)$ (see Section 2). The double phase operator is given by

$$-\mathrm{div}\left(|\nabla z|^{p-2}\nabla z + \mu(x)|\nabla z|^{q-2}\nabla z\right), \quad z \in \mathbf{S}_0^{1,\mathscr{H}}(\Omega),$$

where $1 < p < q < N$ and $\mu : \overline{\Omega} \to [0,\infty)$. The difference between the (p,q)-differential operator and the double phase operator is that the weight function $\mu : \overline{\Omega} \to [0,\infty)$ can be vanished in Ω.

In the 1980s, Zhikov [50] introduced a class of double phase operators for investigating models of strongly anisotropic materials based on the nonlinear energy functional

$$\phi \mapsto \int_\Omega (|\nabla\phi|^p + \mu(x)|\nabla\phi|^q)\, dx,$$

(also see Refs. [51, 52]). Besides, Zhikov [53] has aslo used double phase operator to describe the models with Lavrentiev's phenomenon. A large number of interesting results for solutions to problems involving this operator has been published up to now, see Refs. [6, 7, 5, 11, 12, 13, 14, 15, 18, 19, 20, 35, 36, 37, 42, 49] and the references therein. Recently, Zeng et al. [43, 44] firstly introduced a double phase implicit obstacle problem involving multivalued operator, and they provided some elegant and effective methods to solve multivalued elliptic problems with double phase differential operators. These works open a new and challenging research direction concerning double phase problems with implicit obstacle constraints, and more and more scholars are attracted to the development on both theoretical and application aspects of double phase obstacle problems. More recently, in order to overcome the challenging and difficulty that nonlinear convection term leads to the invalidity of variational methods, Zeng et al. [45] applied Kakutani-Ky Fan fixed point theorem for multivalued operators along with the theory of nonsmooth analysis and variational methods for pseudomonotone operators to develop a very essential and new framework for investigating double phase problems with implicit obstacle effect and nonlinear convection terms, and obtained the sharpest results concerning existence and compactness to weak solutions. Very recently, based on the ideas in Refs. [39] investigated upper-bound error estimates for a class of double phase obstacle problems by using some regularized gap functions. To the best of my knowledge, such error estimates are the first ones for obstacle problems with the double phase operator.

Motivated by the aforementioned works, this paper represents a continuation of Ref. [39]. First, we consider a class of double phase obstacle problems with set-valued convection and mixed boundary value conditions. Then, several new regularized gap functions for the double phase obstacle problems are introduced. Finally, we provide the upper-bound error estimates for such double phase obstacle problems in terms of regularized gap functions based on the properties of double phase operators and the theory of set-valued analysis.

The paper is organized as follows. Some notations, definitions and related properties of function spaces, set-valued mappings and nonsmooth analysis are recalled in Section 2. In Section 3, we introduce a class of double phase obstacle problems with set-valued convection and mixed boundary value conditions and recall an existence theorem to the double phase obstacle problem under consideration. Section 4 provides some regularized gap functions for such double phase obstacle problem under some suitable conditions. Finally, several error bounds of our problem are discussed in Section 5 via regularized gap functions.

2. Preliminaries

We first recall those elements which will be used throughout the paper. For more details, we refer to Refs. [14, 16, 29, 30].

Let Ω be a bounded domain in $\mathbb{R}^N, N \geq 2$, with Lipschitz boundary $\Gamma := \partial\Omega$. The boundary Γ is divided into two mutually disjoint parts Γ_a and Γ_b with Γ_a having positive Lebesgue measure. Let $r \in [1,\infty)$ and any subset U of $\overline{\Omega}$. We denote the usual Lebesgue spaces $L^r(U) := L^r(U;\mathbb{R})$ and $L^r(\Omega;\mathbb{R}^N)$ equipped with the norm $\|\cdot\|_{r,U}$ given by

$$\|z\|_{r,U} := \left(\int_U |z|^r \, dx\right)^{\frac{1}{r}}.$$

Let $\mathbf{S}^{1,r}(\Omega)$ stand for the Sobolev space endowed with the norm $\|\cdot\|_{1,r,\Omega}$ defined by

$$\|z\|_{1,r,\Omega} := \|z\|_{r,\Omega} + \|\nabla z\|_{r,\Omega} \text{ for all } z \in \mathbf{S}^{1,r}(\Omega).$$

Throughout the paper the symbol $\xrightarrow{\mathbf{w}}$ (resp., $\longrightarrow$) stands for the weak (resp., strong) convergence. By $r' > 1$, we denote the conjugate of $r \in (1,\infty)$, i.e., $\frac{1}{r} + \frac{1}{r'} = 1$. Moreover, we denote by p^* the critical exponent to p given by

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p \geq N. \end{cases} \tag{1}$$

Since Γ_a has positive measure, it follows from Korn's inequality that the function space

$$\mathbf{S}_0^{1,r}(\Omega) := \{z \in \mathbf{S}^{1,r}(\Omega) : z = \mathbf{0} \text{ for a. a. } x \in \Gamma_a\}$$

equipped with the norm $\|\nabla \cdot\|_{p,\Omega}$, is a reflexive Banach space. In what follows, let $\lambda_p > 0$ be the smallest constant such that

$$\|z\|_{p,\Omega}^p \leq \lambda_p \|\nabla z\|_{p,\Omega}^p \tag{2}$$

for all $z \in \mathbf{S}_0^{1,r}(\Omega)$. We now revisit the well-known inequality (see Simon [38, formula (2.2)])

$$\left(|x|^{k-2}x - |y|^{k-2}y\right) \cdot (x-y) \geq a(k)|x-y|^k \tag{3}$$

for $k \geq 2$ and for all $x, y \in \mathbb{R}^N$, where $a(k)$ is a positive constant depending on k.

Throughout the paper, we assume that the function $\mu : \overline{\Omega} \to [0,\infty)$ and exponents p,q satisfy the following conditions (see [16, Proposition 2.18]):

$$0 \le \mu(\cdot) \in L^\infty(\Omega) \ \text{ and } \ 1 < p < N, \quad q < q < p^*. \tag{4}$$

Let $\mathbb{R}_+ := [0,\infty)$ and the modular function $\mathscr{H} : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ be defined by

$$\mathscr{H}(x,s) = s^p + \mu(x)s^q \quad \text{for all } (x,s) \in \Omega \times \mathbb{R}_+.$$

The Musielak-Orlicz space $L^{\mathscr{H}}(\Omega)$ is given by

$$L^{\mathscr{H}}(\Omega) = \left\{ z \mid z : \Omega \to \mathbb{R} \text{ is measurable and } \zeta_{\mathscr{H}}(z) := \int_\Omega \mathscr{H}(x,|z|)\,dx < +\infty \right\}.$$

The space $L^{\mathscr{H}}(\Omega)$ equipped with the Luxemburg norm

$$\|z\|_{\mathscr{H}} = \inf\left\{ \tau > 0 \mid \zeta_{\mathscr{H}}\left(\frac{z}{\tau}\right) \le 1 \right\}$$

is uniformly convex and so it is a reflexive Banach space. Furthermore, we introduce the seminormed function space $L^q_\mu(\Omega)$

$$L^q_\mu(\Omega) = \left\{ z \mid z : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega \mu(x)|z|^q\,dx < +\infty \right\}$$

endowed with the seminorm

$$\|z\|_{q,\mu,\Omega} = \left(\int_\Omega \mu(x)|z|^q\,dx \right)^{\frac{1}{q}}.$$

We know that the embeddings

$$L^q(\Omega) \hookrightarrow L^{\mathscr{H}}(\Omega) \hookrightarrow L^p(\Omega) \cap L^q_\mu(\Omega)$$

are continuous and

$$\min\left\{ \|z\|^p_{\mathscr{H}}, \|z\|^q_{\mathscr{H}} \right\} \le \|z\|^p_{p,\Omega} + \|z\|^q_{q,\mu,\Omega} \le \max\left\{ \|z\|^p_{\mathscr{H}}, \|z\|^q_{\mathscr{H}} \right\} \tag{5}$$

for all $z \in L^{\mathscr{H}}(\Omega)$ (see Colasuonno-Squassina [14, Proposition 2.15 (i), (iv) and (v)]).

The corresponding Sobolev-Musielak-Orlicz space $\mathbf{S}^{1,\mathscr{H}}(\Omega)$ is defined by

$$\mathbf{S}^{1,\mathscr{H}}(\Omega) = \left\{ z \in L^{\mathscr{H}}(\Omega) \mid |\nabla z| \in L^{\mathscr{H}}(\Omega) \right\}.$$

The space $\mathbf{S}^{1,\mathscr{H}}(\Omega)$ is equipped with the norm

$$\|z\|_{1,\mathscr{H}} = \|\nabla z\|_{\mathscr{H}} + \|z\|_{\mathscr{H}},$$

where $\|\nabla z\|_{\mathscr{H}} = \||\nabla z|\|_{\mathscr{H}}$.

Given $A : \mathbf{S}^{1,\mathscr{H}}(\Omega) \to \mathbf{S}^{1,\mathscr{H}}(\Omega)^*$ is an operator defined by

$$\langle A(z), v\rangle_{\mathscr{H}} := \int_\Omega \left(|\nabla z|^{p-2}\nabla z + \mu(x)|\nabla z|^{q-2}\nabla z \right) \cdot \nabla v\,dx, \tag{6}$$

for $z,v \in \mathbf{S}^{1,\mathscr{H}}(\Omega)$, where $\langle\cdot,\cdot\rangle_{\mathscr{H}}$ denotes the duality pairing between $\mathbf{S}^{1,\mathscr{H}}(\Omega)$ and its dual space $\mathbf{S}^{1,\mathscr{H}}(\Omega)^*$. Some properties of the operator A defined by (6) are proposed in the following proposition:

Proposition 2.1. (see Ref. [30]) *The operator A defined by* (6) *is bounded, continuous, monotone.*

Now, we recall some notion and properties concerning set-valued mappings and nonsmooth analysis.

Definition 2.2. (see Ref. [2]) Let Z and X be two Hausdorff topological spaces, $C \subset Z$ be a nonempty set and $\mathscr{M} : Z \rightrightarrows X$ be a set-valued mapping. Then $\mathscr{M}$ is said to be

(a) convex (resp., closed, bounded) valued, if $\mathscr{M}$ is convex (resp., closed, bounded) for each $z \in Z$;

(b) upper semicontinuous at $z_0 \in Z$, if for each open set $U \subset X$ of $\mathscr{M}(z_0)$, there is a neighborhood $N(z_0)$ of z_0 such that $\mathscr{M}(N(z_0)) := \cup_{v \in N(z_0)} \mathscr{M}(v) \subset U$. If it holds for each $z \in C$, then $\mathscr{M}$ is called to be upper semicontinuous on C.

Lemma 2.3. (see Ref. [9]) *Let Y be a Banach space and D be a nonempty subset of another Banach space. Assume that $F : D \rightrightarrows Y$ is a set-valued mapping with nonempty, weakly compact, convex values. Then F is strongly-weakly upper semicontinuous if and only if, for each sequence $\{z_k\} \subset D$ which converges to $z_0 \in D$ and for each sequence $\{\zeta_k\} \subset F(z_k)$, there exists $\zeta_0 \in F(z_0)$ such that $\zeta_k \xrightarrow{\mathbf{w}} \zeta_0$ up to a subsequence.*

Definition 2.4. (see Ref. [33]) Let E be a real Banach space. A function $g : E \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is said to be

(a) proper, if $g \not\equiv +\infty$;

(b) convex, if $g(tz + (1-t)v) \le tg(z) + (1-t)g(v)$ for all $z, v \in E$ and $t \in [0,1]$;

(c) lower semicontinuous at $z_0 \in E$, if for any sequence $\{z_n\} \subset E$ such that $z_n \to z_0$, it holds $g(z_0) \le \liminf g(z_n)$;

(d) upper semicontinuous at $z_0 \in E$, if for any sequence $\{z_n\} \subset E$ such that $z_n \to z_0$, it holds $\limsup g(z_n) \le g(z_0)$;

(e) lower (resp. upper) semicontinuous on E, if g is lower (resp. upper) semicontinuous at every $z_0 \in E$;

(f) continuous on E if, it is both lower and upper semicontinuous on E.

Definition 2.5. (see Ref. [33]) Let E be a real Banach space with its topological dual E^* and $g : E \to \overline{\mathbb{R}}$ be a proper, convex and lower semicontinuous function. The convex subdifferential $\partial_c g : E \rightrightarrows E^*$ of g is defined by

$$\partial_c g(z) = \left\{ w^* \in E^* \mid \langle w^*, v - z \rangle_E \le g(v) - g(z) \ \text{ for all } \ v \in E \right\} \ \text{ for all } \ z \in E.$$

An element $w^* \in \partial_c g(z)$ is called a subgradient of g at $z \in E$.

3. Double phase obstacle problem

In this section, we consider a class of double phase obstacle problems involving set-valued convection and mixed boundary value conditions. This class of problems is a special case of double phase obstacle problems investigated by Zeng et al. [45].

Let Ω be a bounded domain in $\mathbb{R}^N, N \ge 2$, with Lipschitz boundary $\Gamma := \partial\Omega$. The boundary Γ is divided into two mutually disjoint parts Γ_a and Γ_b with Γ_a having positive Lebesgue measure and Γ_b

can be empty. We introduce the following double phase obstacle problem:

$$
(7)\quad \begin{cases}
-\mathrm{div}\big(|\nabla z|^{p-2}\nabla z+\mu(x)|\nabla z|^{q-2}\nabla z\big) & \\
\qquad +|z|^{p-2}z+\mu(x)|z|^{q-2}z\in \mathscr{M}(x,z)+h(x,z,\nabla z) & \text{in } \Omega,\\
z=\mathbf{0} & \text{on } \Gamma_a,\\
-\dfrac{\partial z}{\partial \nu_a}\in \partial_c g(x,z) & \text{on } \Gamma_b,\\
z(x)\le \Psi(x) & \text{in } \Omega,
\end{cases}
$$

where $1<p<q<N$ and

$$
\frac{\partial z}{\partial \nu_a}:=\big(|\nabla z|^{p-2}\nabla z+\mu(x)|\nabla z|^{q-2}\nabla z\big)\cdot \nu,
$$

with ν being the unit normal vector on Γ, $\mu:\overline{\Omega}\to[0,\infty)$ satisfies the condition (4), $\mathscr{M}\colon\Omega\times\mathbb{R}\rightrightarrows\mathbb{R}$ is a set-valued mapping, $g\colon\Gamma_b\times\mathbb{R}\to\mathbb{R}$ is a convex function with respect to the second argument, $\partial_c g(x,z)$ is the convex subdifferential of $z\mapsto g(x,z)$, $h\colon\Omega\times\mathbb{R}\times\mathbb{R}^N$ is a nonlinear convection function and $\Psi\colon\Omega\to\mathbb{R}$ is a given obstacle.

Remark 1. The double phase obstacle problem (7) combines an obstacle effect along with mixed boundary conditions on Γ_a and Γ_b (with the convex subdifferential $\partial_c g$) and the appearance of set-valued mapping $\mathscr{M}$ and the nonlinear convection function h. The problem (7) is a special case of double phase obstacle problems considered in Zeng et al. [45].

Next, we make the following assumptions on the data of the problem (7).

<u>$\mathbf{A}(h)$:</u> $h\colon\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is a Carathéodory function such that

(i) there exist $a_h,b_h\ge 0$ and a function $\alpha_h\in L^{\frac{q_1}{q_1-1}}(\Omega)_+$ satisfying

$$
|h(x,s,\xi)|\le a_h|\xi|^{\frac{p(q_1-1)}{q_1}}+b_h|s|^{q_1-1}+\alpha_h(x)
$$

for a. a. $x\in\Omega$, for all $\xi\in\mathbb{R}^N$ and for all $s\in\mathbb{R}$, where $1<q_1<p^*$ and p^* is the critical exponents to p in the domain considered in (1);

(ii) there exist $c_h,d_h\ge 0$, $\theta_1,\theta_2\in[1,p]$ and a function $\beta_h\in L^1(\Omega)_+$ such that

$$
h(x,s,\xi)s\le c_h|\xi|^{\theta_1}+d_h|s|^{\theta_2}+\beta_h(x)
$$

for a. a. $x\in\Omega$, for all $\xi\in\mathbb{R}^N$ and for all $s\in\mathbb{R}$;

(iii) there exist $e_h,f_h\ge 0$ such that

$$
(h(x,s,\xi)-h(x,t,\xi))(s-t)\le e_h|s-t|^p,
$$
$$
|h(x,s,\xi_1)-h(x,s,\xi_2)|\le f_h|\xi_1-\xi_2|^{p-1}
$$

for a. a. $x\in\Omega$, for all $\xi_1,\xi_2\in\mathbb{R}^N$ and for all $s,t\in\mathbb{R}$.

<u>$\mathbf{A}(\mathscr{M})$:</u> $\mathscr{M}\colon\Omega\times\mathbb{R}\rightrightarrows\mathbb{R}$ satisfies the following conditions:

(i) $\mathscr{M}(x,s)$ is a nonempty, closed, bounded and convex set in $\mathbb{R}$ for a. a. $x\in\Omega$ and all $s\in\mathbb{R}$;

(ii) $x\mapsto\mathscr{M}(x,s)$ is measurable in Ω for all $s\in\mathbb{R}$;

(iii) $s \mapsto \mathscr{M}(x,s)$ is upper semicontinuous for a. a. $x \in \Omega$;

(iv) there exist $\theta_3 \in [1,p]$, $\alpha_{\mathscr{M}} \in L^{p'}(\Omega)_+$ and $\beta_{\mathscr{M}} > 0$ such that

$$|\mathscr{M}(x,s)| \le \alpha_{\mathscr{M}}(x) + \beta_{\mathscr{M}}|s|^{\theta_3 - 1}$$

for a. a. $x \in \Omega$ and for all $s \in \mathbb{R}$.

$\underline{\mathbf{A}(g)}$: $g\colon \Gamma_b \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

(i) $x \mapsto g(x,s)$ is measurable on Γ_b for all $s \in \mathbb{R}$ such that $x \mapsto g(x,0)$ belongs to $L^1(\Gamma_b)$;

(ii) for a. a. $x \in \Gamma_b$, $s \mapsto g(x,s)$ is convex and lower semicontinuous.

$\underline{\mathbf{A}(\Psi)}$: $\Psi\colon \Omega \to [0,+\infty)$ is measurable in Ω;

$\underline{\mathbf{A}(0)}$: The following inequalities hold:

$$\max\left\{e_h, f_h\lambda_p^{\frac{1}{p}}\right\} < a(p) \quad \text{and} \quad \max\{c_h\chi(\theta_1), d_h\chi(\theta_2) + \beta_{\mathscr{M}}\chi(\theta_3)\} < 1,$$

where $a(p) > 0$ is given in (3), λ_p is given in (2) and $\chi\colon [1,p] \to \{1,0\}$ is defined by

$$\chi(\theta) = \begin{cases} 1 & \text{if } \theta = p, \\ 0 & \text{otherwise.} \end{cases}$$

The weak solutions for the problem (7) are given in the following sense.

Definition 3.1. A function $z \in \mathbf{S}^{1,\mathscr{H}}(\Omega)$ is said to be a weak solution of the problem (7) if $z \in \mathscr{P}$ and there exists a function $\zeta \in \mathbf{N}_{\mathscr{M}}^{p'}(z)$ such that

$$\begin{aligned} &\int_\Omega \left(|\nabla z|^{p-2}\nabla z + \mu(x)|\nabla z|^{q-2}\nabla z\right) \cdot \nabla(v-z)\,dx \\ &+ \int_\Omega \left(|z|^{p-2}z + \mu(x)|z|^{q-2}z\right)(v-z)\,dx \\ &+ \int_{\Gamma_b} g(x,v)\,d\Gamma - \int_{\Gamma_b} g(x,z)\,d\Gamma \\ \ge &\int_\Omega \zeta(x)\,(v-z)\,dx + \int_\Omega h(x,z,\nabla z)\,(v-z)\,dx \quad \text{for all } v \in \mathscr{P}, \end{aligned}$$

where the set-valued operator $\mathbf{N}_{\mathscr{M}}^{p'}\colon \mathbf{S}^{1,\mathscr{H}}(\Omega) \rightrightarrows L^{p'}(\Omega)$ is defined by

$$\mathbf{N}_{\mathscr{M}}^{p'}(z) = \left\{\zeta \in L^{p'}(\Omega) : \zeta(x) \in \mathscr{M}(x,z(x)),\ \text{for a. a. } x \in \Omega\right\},\ z \in \mathbf{S}^{1,\mathscr{H}}(\Omega) \tag{8}$$

and

$$\mathscr{P} = \left\{z \in \mathbf{S}_0^{1,\mathscr{H}}(\Omega) : z \le \Psi \text{ in } \Omega\right\}. \tag{9}$$

The operator $\mathbf{N}_{\mathscr{M}}^{p'}$ is known as the set-valued Nemytskij operator associated with the set-valued function $\mathscr{M}$. The following properties of $\mathbf{N}_{\mathscr{M}}^{p'}$ are deduced from Ref. [45, Lemma 1.1].

Lemma 3.2. *Assume that* $\mathbf{A}(\mathscr{M})$ *is satisfied. Then, the following hold:*

(i) $\mathbf{N}^{p'}_{\mathscr{M}}$ *is well-defined and for each* $z \in L^p(\Omega)$*, the set* $\mathbf{N}^{p'}_{\mathscr{M}}(z)$ *is bounded, closed and convex in* $L^{p'}(\Omega)$*;*

(ii) $\mathbf{N}^{p'}_{\mathscr{M}}$ *is strongly-weakly upper semicontinuous.*

Remark 2. Under the assumption $\mathbf{A}(\Psi)$, the set $\mathscr{P}$ defined by (9) is a nonempty, closed and convex subset of $\mathbf{S}^{1,\mathscr{H}}(\Omega)$ (see Ref. [42, page 8]).

The existence result for the problem (7) is provided in the following lemma which is followed from Ref. [45, Theorem 3.9].

Lemma 3.3. *Let* $p \geq 2$*. Assume that* $\mathbf{A}(h), \mathbf{A}(\mathscr{M}), \mathbf{A}(g), \mathbf{A}(\Psi)$ *and* $\mathbf{A}(0)$ *hold. Then the set of solutions for the problem* (7) *is nonempty and weakly compact in* $\mathbf{S}^{1,\mathscr{H}}(\Omega)$*.*

4. Regularized gap functions

Based on the ideas of Yamashita and Fukushima [41] and Tam [39], we shall investigate the regularized gap functions for the problem (7). We now propose the definition of a gap function for the problem (7).

Definition 4.1. A real-valued function $\mathfrak{F}\colon \mathbf{S}^{1,\mathscr{H}}(\Omega) \to \mathbb{R}$ is said to be a gap function for the problem (7), if it satisfies the following properties:

(a) $\mathfrak{F}(z) \geq 0$ for all $z \in \mathscr{P}$.

(b) $z^* \in \mathscr{P}$ is such that $\mathfrak{F}(z^*) = 0$ if and only if z^* is a weak solution to the problem (7).

Let $\omega > 0$ be a fixed parameter. We consider the following functions $\mathscr{Q}_{\mu,\omega} : \mathscr{P} \times L^{p'}(\Omega) \to \mathbb{R}$ and $\Upsilon_{\mu,\omega}\colon \mathscr{P} \to \mathbb{R}$ defined by

$$\begin{aligned}\mathscr{Q}_{\mu,\omega}(z,\zeta) = \sup_{v\in\mathscr{P}} \Bigg(&\int_\Omega \left(|\nabla z|^{p-2}\nabla z + \mu(x)|\nabla z|^{q-2}\nabla z\right)\cdot\nabla(z-v)\,dx + \int_\Omega \zeta(x)\,(v-z)\,dx \\ &+ \int_\Omega \left(|z|^{p-2}z + \mu(x)|z|^{q-2}z\right)(z-v)\,dx - \int_{\Gamma_b} g(x,v)\,d\Gamma + \int_{\Gamma_b} g(x,z)\,d\Gamma \\ &+ \int_\Omega h(x,z,\nabla z)\,(v-z)\,dx - \frac{\omega}{p}\|z-v\|^p_{p,\Omega}\Bigg) \end{aligned}\tag{10}$$

for all $z \in \mathscr{P}$ and $\zeta \in L^{p'}(\Omega)$, and

$$\begin{aligned}&\Upsilon_{\mu,\omega}(z) \\ &= \inf_{\zeta\in\mathbf{N}^{p'}_{\mathscr{M}}(z)} \sup_{v\in\mathscr{P}} \Bigg(\int_\Omega \left(|\nabla z|^{p-2}\nabla z + \mu(x)|\nabla z|^{q-2}\nabla z\right)\cdot\nabla(z-v)\,dx + \int_\Omega \zeta(x)\,(v-z)\,dx \\ &\quad + \int_\Omega \left(|z|^{p-2}z + \mu(x)|z|^{q-2}z\right)(z-v)\,dx - \int_{\Gamma_b} g(x,v)\,d\Gamma + \int_{\Gamma_b} g(x,z)\,d\Gamma \\ &\quad + \int_\Omega h(x,z,\nabla z)\,(v-z)\,dx - \frac{\omega}{p}\|z-v\|^p_{p,\Omega}\Bigg), \end{aligned}\tag{11}$$

for all $z \in \mathbf{S}^{1,\mathscr{H}}(\Omega)$.

It follows from (10) and (11) that $\Upsilon_{\mu,\omega}(z) = \inf_{\zeta \in \mathbf{N}^{p'}_{\mathscr{M}}(z)} \mathscr{Q}_{\mu,\omega}(z,\zeta)$.

Remark 3. Assume that the assumption $\mathbf{A}(\mathscr{M})$ holds. It is easy to see that the function $\mathscr{Q}_{\mu,\omega}$ is convex and continuous in the second component. Moreover, by Lemma 3.2(i), for each $z \in \mathscr{P}$, $\mathbf{N}^{p'}_{\mathscr{M}}(z)$ is a bounded, closed and convex set. Thus, it follows from an elementary result for convex minimization that for each $z \in \mathscr{P}$, there exists $\zeta_z \in \mathbf{N}^{p'}_{\mathscr{M}}(z)$ such that $\Upsilon_{\mu,\omega}(z) = \mathscr{Q}_{\mu,\omega}(z,\zeta_z)$ (see Ref. [4, Theorem 3.3.12]).

In what follows, the function $\Upsilon_{\mu,\omega}$ defined by (11) is called to be a regularized gap function for the problem (7), where $\omega > 0$ is a regularized parameter.

Theorem 4.2. *Suppose the hypotheses of Lemma 3.3. Then, for any $\omega > 0$, the function $\Upsilon_{\mu,\omega}$ is a gap function for the problem* (7).

Proof: (a) Clearly, for each $\omega > 0$ fixed, $z \in \mathscr{P}$ and $\zeta \in \mathbf{N}^{p'}_{\mathscr{M}}(z)$, it follows from the definition of $\mathscr{Q}_{\mu,\omega}$ in (10) that

$$\begin{aligned}\mathscr{Q}_{\mu,\omega}(z,\zeta) \geq & \int_\Omega \left(|\nabla z|^{p-2}\nabla z + \mu(x)|\nabla z|^{q-2}\nabla z\right)\cdot\nabla(z-z)\,dx + \int_\Omega \zeta(x)\,(z-z)\,dx \\ & + \int_\Omega \left(|z|^{p-2}z + \mu(x)|z|^{q-2}z\right)(z-z)\,dx - \int_{\Gamma_b} g(x,z)\,d\Gamma + \int_{\Gamma_b} g(x,z)\,d\Gamma \\ & + \int_\Omega h(x,z,\nabla z)\,(z-z)\,dx - \frac{\omega}{p}\|z-z\|^p_{p,\Omega} \\ =& 0.\end{aligned}$$

By the arbitrariness of $\zeta \in \mathbf{N}^{p'}_{\mathscr{M}}(z)$, we conclude that

$$\Upsilon_{\mu,\omega}(z) = \inf_{\zeta\in\mathbf{N}^{p'}_{\mathscr{M}}(z)} \mathscr{Q}_{\mu,\omega}(z,\zeta) \geq 0,\ \forall z \in \mathscr{P}.$$

(b) Assume that $z^* \in \mathscr{P}$ satisfies $\Upsilon_{\mu,\omega}(z^*) = 0$, that is,

$$\inf_{\zeta\in\mathbf{N}^{p'}_{\mathscr{M}}(z^*)} \mathscr{Q}_{\mu,\omega}(z^*,\zeta) = 0.$$

Thanks to Remark 3, we conclude that there exists $\zeta^* \in \mathbf{N}^{p'}_{\mathscr{M}}(z^*)$ such that

$$\begin{aligned}0 =& \mathscr{Q}_{\mu,\omega}(z^*,\zeta^*) \\ =& \sup_{v\in\mathscr{P}}\Bigg(\int_\Omega \left(|\nabla z^*|^{p-2}\nabla z^* + \mu(x)|\nabla z^*|^{q-2}\nabla z^*\right)\cdot\nabla(z^*-v)\,dx + \int_\Omega \zeta^*(x)\,(v-z^*)\,dx \\ & + \int_\Omega \left(|z^*|^{p-2}z^* + \mu(x)|z^*|^{q-2}z^*\right)(z^*-v)\,dx - \int_{\Gamma_b} g(x,v)\,d\Gamma + \int_{\Gamma_b} g(x,z^*)\,d\Gamma \\ & + \int_\Omega h(x,z^*,\nabla z^*)\,(v-z^*)\,dx - \frac{\omega}{p}\|z^*-v\|^p_{p,\Omega}\Bigg).\end{aligned}$$

This means

$$\begin{aligned}
&\frac{\omega}{p}\|z^*-v\|^p_{p,\Omega}\\
&\geq \int_\Omega \left(|\nabla z^*|^{p-2}\nabla z^* + \mu(x)|\nabla z^*|^{q-2}\nabla z^*\right)\cdot\nabla(z^*-v)\,dx + \int_\Omega \zeta^*(x)\,(v-z^*)\,dx\\
&\qquad + \int_\Omega \left(|z^*|^{p-2}z^* + \mu(x)|z^*|^{q-2}z^*\right)(z^*-v)\,dx - \int_{\Gamma_b} g(x,v)\,d\Gamma + \int_{\Gamma_b} g(x,z^*)\,d\Gamma\\
&\qquad + \int_\Omega h(x,z^*,\nabla z^*)\,(v-z^*)\,dx,\ \forall v\in\mathscr{P}.
\end{aligned}$$

For any $u\in\mathscr{P}$, $\sigma\in(0,1)$, since $\mathscr{P}$ is a convex set, we have $v_\sigma := (1-\sigma)z^*+\sigma u\in \mathscr{P}$. Hence, we insert v_σ into the above inequality to obtain

$$\begin{aligned}
&\int_\Omega \left(|\nabla z^*|^{p-2}\nabla z^* + \mu(x)|\nabla z^*|^{q-2}\nabla z^*\right)\cdot\nabla(z^*-v_\sigma)\,dx + \int_\Omega \zeta^*(x)\,(v_\sigma-z^*)\,dx\\
&+ \int_\Omega \left(|z^*|^{p-2}z^* + \mu(x)|z^*|^{q-2}z^*\right)(z^*-v_\sigma)\,dx - \int_{\Gamma_b} g(x,v_\sigma)\,d\Gamma + \int_{\Gamma_b} g(x,z^*)\,d\Gamma\\
&+ \int_\Omega h(x,z^*,\nabla z^*)\,(v_\sigma-z^*)\,dx\\
&\leq \frac{\omega}{p}\|z^*-v_\sigma\|^p_{p,\Omega} = \frac{\omega\sigma^p}{p}\|z^*-u\|^p_{p,\Omega},\ \forall u\in\mathscr{P}. \qquad (12)
\end{aligned}$$

Using the convexity of $v\mapsto g(x,v)$, one has

$$\begin{aligned}
&\sigma\Bigg(\int_\Omega \left(|\nabla z^*|^{p-2}\nabla z^* + \mu(x)|\nabla z^*|^{q-2}\nabla z^*\right)\cdot\nabla(z^*-u)\,dx + \int_\Omega \zeta^*(x)\,(u-z^*)\,dx\\
&\qquad + \int_\Omega \left(|z^*|^{p-2}z^* + \mu(x)|z^*|^{q-2}z^*\right)(z^*-u)\,dx - \int_{\Gamma_b} g(x,u)\,d\Gamma + \int_{\Gamma_b} g(x,z^*)\,d\Gamma\\
&\qquad + \int_\Omega h(x,z^*,\nabla z^*)\,(u-z^*)\,dx\Bigg)\\
&\leq \int_\Omega \left(|\nabla z^*|^{p-2}\nabla z^* + \mu(x)|\nabla z^*|^{q-2}\nabla z^*\right)\cdot\nabla(z^*-v_\sigma)\,dx + \int_\Omega \zeta^*(x)\,(v_\sigma-z^*)\,dx\\
&\qquad + \int_\Omega \left(|z^*|^{p-2}z^* + \mu(x)|z^*|^{q-2}z^*\right)(z^*-v_\sigma)\,dx - \int_{\Gamma_b} g(x,v_\sigma)\,d\Gamma + \int_{\Gamma_b} g(x,z^*)\,d\Gamma\\
&\qquad + \int_\Omega h(x,z^*,\nabla z^*)\,(v_\sigma-z^*)\,dx,\ \forall u\in\mathscr{P}. \qquad (13)
\end{aligned}$$

Combining (12) and (13), we have

$$\begin{aligned}
&\int_\Omega \left(|\nabla z^*|^{p-2}\nabla z^* + \mu(x)|\nabla z^*|^{q-2}\nabla z^*\right)\cdot\nabla(z^*-u)\,dx + \int_\Omega \zeta^*(x)\,(u-z^*)\,dx\\
&+ \int_\Omega \left(|z^*|^{p-2}z^* + \mu(x)|z^*|^{q-2}z^*\right)(z^*-u)\,dx - \int_{\Gamma_b} g(x,u)\,d\Gamma + \int_{\Gamma_b} g(x,z^*)\,d\Gamma\\
&+ \int_\Omega h(x,z^*,\nabla z^*)\,(u-z^*)\,dx\\
&\leq \frac{\omega\sigma^{p-1}}{p}\|z^*-u\|^p_{p,\Omega}
\end{aligned}$$

for all $u \in \mathscr{P}$. Letting $\sigma \to 0^+$ for the above inequality, it gives

$$\begin{aligned}
&\int_\Omega \left(|\nabla z^*|^{p-2}\nabla z^* + \mu(x)|\nabla z^*|^{q-2}\nabla z^*\right)\cdot\nabla(u-z^*)\,dx + \int_{\Gamma_b} g(x,u)\,d\Gamma\\
&-\int_{\Gamma_b} g(x,z^*)\,d\Gamma + \int_\Omega \left(|z^*|^{p-2}z^* + \mu(x)|z^*|^{q-2}z^*\right)(u-z^*)\,dx\\
\geq &\int_\Omega \zeta^*(x)\,(u-z^*)\,dx + \int_\Omega h(x,z^*,\nabla z^*)\,(u-z^*)\,dx
\end{aligned}$$

for all $u \in \mathscr{P}$. Thus, z^* is a solution to the problem (7).

Conversely, suppose that $z^* \in \mathbf{S}^{1,\mathscr{H}}(\Omega)$ is a weak solution of the problem (7), i.e., $z^* \in \mathscr{P}$ and there exists $\zeta^* \in \mathbf{N}^{p'}_{\mathscr{M}}(z^*)$ such that

$$\begin{aligned}
&\int_\Omega \left(|\nabla z^*|^{p-2}\nabla z^* + \mu(x)|\nabla z^*|^{q-2}\nabla z^*\right)\cdot\nabla(v-z^*)\,dx + \int_{\Gamma_b} g(x,v)\,d\Gamma\\
&-\int_{\Gamma_b} g(x,z^*)\,d\Gamma + \int_\Omega \left(|z^*|^{p-2}z^* + \mu(x)|z^*|^{q-2}z^*\right)(v-z^*)\,dx\\
\geq &\int_\Omega \zeta^*(x)\,(v-z^*)\,dx + \int_\Omega h(x,z^*,\nabla z^*)\,(v-z^*)\,dx,\ \ \forall v \in \mathscr{P}.
\end{aligned}$$

Since $v \in \mathscr{P}$ is arbitrary, we have

$$\begin{aligned}
&\mathscr{Q}_{\mu,\omega}(z^*,\zeta^*)\\
&= \sup_{v\in\mathscr{P}}\Bigg(\int_\Omega \left(|\nabla z^*|^{p-2}\nabla z^* + \mu(x)|\nabla z^*|^{q-2}\nabla z^*\right)\cdot\nabla(z^*-v)\,dx + \int_\Omega \zeta^*(x)\,(v-z^*)\,dx\\
&\qquad + \int_\Omega \left(|z^*|^{p-2}z^* + \mu(x)|z^*|^{q-2}z^*\right)(z^*-v)\,dx - \int_{\Gamma_b} g(x,v)\,d\Gamma + \int_{\Gamma_b} g(x,z^*)\,d\Gamma\\
&\qquad + \int_\Omega h(x,z^*,\nabla z^*)\,(v-z^*)\,dx - \frac{\omega}{p}\|z^*-v\|^p_{p,\Omega}\Bigg)\\
&\leq 0.
\end{aligned}$$

Hence, for any $\zeta \in \mathbf{N}^{p'}_{\mathscr{M}}(z^*)$,

$$\Upsilon_{\mu,\omega}(z^*) = \inf_{\zeta\in\mathbf{N}^{p'}_{\mathscr{M}}(z^*)} \mathscr{Q}_{\mu,\omega}(z^*,\zeta) \leq 0.$$

Since $\Upsilon_{\mu,\omega}(z) \geq 0$ for all $z \in \mathscr{P}$, then $\Upsilon_{\mu,\omega}(z^*) = 0$. The proof is complete. ∎

We shall prove that the regularized gap function $\Upsilon_{\mu,\omega}$ is lower semicontinuous.

Lemma 4.3. *Assume that the hypotheses of Lemma 3.3 are satisfied. Then for each $\omega > 0$, the gap function $\Upsilon_{\mu,\omega}$ is lower semicontinuous.*

Proof: Taking $\ell \in \mathbb{R}$ and a sequence $\{z_k\} \subset \mathbf{S}^{1,\mathscr{H}}(\Omega)$ satisfying $\Upsilon_{\mu,\omega}(z_k) \leq \ell$ for all $k \in \mathbb{N}$, and $z_k \to z_0$ in $\mathbf{S}^{1,\mathscr{H}}(\Omega)$. We show that $\Upsilon_{\mu,\omega}(z_0) \leq \ell$. Indeed, we have

$$\Upsilon_{\mu,\omega}(z_k) = \inf_{\zeta\in\mathbf{N}^{p'}_{\mathscr{M}}(z_k)} \mathscr{Q}_{\mu,\omega}(z_k,\zeta) \leq \ell,$$

for all $k \in \mathbb{N}$. It follows from Remark 3 that for each $k \in \mathbb{N}$, there exists $\zeta_k \in \mathbf{N}^{p'}_{\mathscr{M}}(z_k)$ such that

$$\Upsilon_{\mu,\omega}(z_k) = \mathscr{Q}_{\mu,\omega}(z_k,\zeta_k) = \sup_{v\in\mathscr{P}} \widetilde{\mathscr{Q}}_{\mu,\omega}(z_k,v,\zeta_k),$$

where the function $\widetilde{\mathscr{Q}}_{\mu,\omega}\colon \mathbf{S}^{1,\mathscr{H}}(\Omega)\times\mathbf{S}^{1,\mathscr{H}}(\Omega)\times L^{p'}(\Omega)\to\mathbb{R}$ is defined by

$$\begin{aligned}
&\widetilde{\mathscr{Q}}_{\mu,\omega}(z,v,\zeta)\\
&= \int_\Omega \left(|\nabla z|^{p-2}\nabla z+\mu(x)|\nabla z|^{q-2}\nabla z\right)\cdot\nabla(z-v)\,dx + \int_\Omega \zeta(x)\,(v-z)\,dx\\
&\quad + \int_\Omega \left(|z|^{p-2}z+\mu(x)|z|^{q-2}z\right)(z-v)\,dx - \int_{\Gamma_b} g(x,v)\,d\Gamma + \int_{\Gamma_b} g(x,z)\,d\Gamma\\
&\quad + \int_\Omega h(x,z,\nabla z)\,(v-z)\,dx - \frac{\omega}{p}\|z-v\|^p_{p,\Omega}.
\end{aligned}$$

Then, for all $v\in\mathscr{P}$,

$$\widetilde{\mathscr{Q}}_{\mu,\omega}(z_k,v,\zeta_k) \le \ell, \tag{14}$$

By Lemma 4.3, the operator $\mathbf{N}^{p'}_{\mathscr{M}}$ is strongly-weakly upper semicontinuous with nonempty, weakly compact, convex values. Then using Lemma 2.3, there exists $\zeta_0 \in \mathbf{N}^{p'}_{\mathscr{M}}(z_0)$ such that, passing to a subsequence if necessary,

$$\zeta_k \xrightarrow{\mathrm{w}} \zeta_0 \text{ in } L^{p'}(\Omega).$$

Recall that the operator $A\colon \mathbf{S}^{1,\mathscr{H}}(\Omega)\to\mathbf{S}^{1,\mathscr{H}}(\Omega)^*$ defined by

$$\langle A(z),v\rangle_{\mathscr{H}} := \int_\Omega \left(|\nabla z|^{p-2}\nabla z+\mu(x)|\nabla z|^{q-2}\nabla z\right)\cdot\nabla v\,dx,$$

for $z,v\in\mathbf{S}^{1,\mathscr{H}}(\Omega)$, is continuous (see Proposition 2.1). Then the function $(z,v)\mapsto\langle A(z),v\rangle_{\mathscr{H}}$ is continuous on $\mathbf{S}^{1,\mathscr{H}}(\Omega)\times\mathbf{S}^{1,\mathscr{H}}(\Omega)^*$. Furthermore, the function $z\mapsto g(x,z)$ is lower semicontinuos for a. a. $x\in\Gamma_b$ and functions $z\mapsto h(x,z,\nabla z)$ and $z\mapsto\|z\|_{p,\Omega}$ are continuous for a. a. $x\in\Omega$. Passing to the lower limit as $k\to\infty$ to the inequality (14) and using the compactness of the embedding of $\mathbf{S}^{1,\mathscr{H}}(\Omega)$ to $L^p(\Omega)$, we have

$$\begin{aligned}
\ell &\geq \liminf_{k\to\infty} \widetilde{\mathscr{Q}}_{\mu,\omega}(z_k, v, \zeta_k) \\
&\geq \liminf_{k\to\infty} \int_\Omega \left(|\nabla z_k|^{p-2}\nabla z_k + \mu(x)|\nabla z_k|^{q-2}\nabla z_k\right)\cdot\nabla(z_k - v)\,dx \\
&\quad + \liminf_{k\to\infty} \int_\Omega \left(|z_k|^{p-2}z_k + \mu(x)|z_k|^{q-2}z_k\right)(z_k - v)\,dx \\
&\quad + \liminf_{k\to\infty} \int_\Omega \zeta_k(x)\,(v - z_k)\,dx - \int_{\Gamma_b} g(x,v)\,d\Gamma + \liminf_{k\to\infty}\int_{\Gamma_b} g(x,z_k)\,d\Gamma \\
&\quad + \liminf_{k\to\infty} \int_\Omega h(x,z_k,\nabla z_k)\,(v - z)\,dx - \limsup_{k\to\infty} \frac{\omega}{p}\|z_k - v\|_{p,\Omega}^p \\
&\geq \int_\Omega \left(|\nabla z_0|^{p-2}\nabla z_0 + \mu(x)|\nabla z_0|^{q-2}\nabla z_0\right)\cdot\nabla(z_0 - v)\,dx + \int_\Omega \zeta_0(x)\,(v - z_0)\,dx \\
&\qquad + \int_\Omega \left(|z_0|^{p-2}z_0 + \mu(x)|z_0|^{q-2}z_0\right)(z_0 - v)\,dx - \int_{\Gamma_b} g(x,v)\,d\Gamma + \int_{\Gamma_b} g(x,z_0)\,d\Gamma \\
&\qquad + \int_\Omega h(x,z_0,\nabla z_0)\,(v - z_0)\,dx - \frac{\omega}{p}\|z_0 - v\|_{p,\Omega}^p \\
&= \widetilde{\mathscr{Q}}_{\mu,\omega}(z_0, v, \zeta_0),\ \text{for all } v \in \mathscr{P}.
\end{aligned}$$

Hence, $\sup_{v\in\mathscr{P}} \widetilde{\mathscr{Q}}_{\mu,\omega}(z_0, v, \zeta_0) \leq \ell$. Therefore,

$$\Upsilon_{\mu,\omega}(z_0) = \inf_{\zeta \in \mathbf{N}^{p'}_{\mathscr{M}}(z_0)} \sup_{v\in\mathscr{P}} \widetilde{\mathscr{Q}}_{\mu,\omega}(z_0, v, \zeta) \leq \ell,$$

i.e., the level set $\{z \in \mathbf{S}^{1,\mathscr{H}}(\Omega) \mid \Upsilon_{\mu,\omega}(z) \leq \ell\}$ for any $\ell \in \mathbb{R}$ is closed. Hence, $\Upsilon_{\mu,\omega}$ is lower semicontinuous. ∎

Let ω, $\delta > 0$ be two parameters. Based on Moreau-Yosida regularization of the function $\Upsilon_{\mu,\omega}$, we consider the following function $\Theta^\delta_{\Upsilon_{\mu,\omega}} : \mathbf{S}^{1,\mathscr{H}}(\Omega) \to \mathbb{R}$ defined by

$$\Theta^\delta_{\Upsilon_{\mu,\omega}}(z) = \inf_{w\in\mathscr{P}} \left\{\Upsilon_{\mu,\omega}(w) + \delta\|z - w\|^p_{1,\mathscr{H}}\right\}, \tag{15}$$

for all $z \in \mathscr{P}$.

We now verify that $\Theta^\delta_{\Upsilon_{\mu,\omega}}$ is a gap function for the problem (7). Then, we call $\Theta^\delta_{\Upsilon_{\mu,\omega}}$ to be the Moreau-Yosida regularized gap function for the problem (7).

Theorem 4.4. *Assume that the hypotheses of Lemma 3.3 are satisfied. Then, for all $\omega, \delta > 0$, the function $\Theta^\delta_{\Upsilon_{\mu,\omega}}$ is the gap function for the problem (7).*

Proof: (a) For any $\omega, \delta > 0$ and $w \in \mathscr{P}$, recall that $\Upsilon_{\mu,\omega}$ is a gap function for the problem (7), hence $\Upsilon_{\mu,\omega}(w) \geq 0$. It follows from the definition of $\Theta^\delta_{\Upsilon_{\mu,\omega}}$ in (15) that $\Theta^\delta_{\Upsilon_{\mu,\omega}}(z) \geq 0$, for all $z \in \mathscr{P}$.

(b) Suppose that $z^* \in \mathbf{S}^{1,\mathscr{H}}(\Omega)$ is a weak solution of the problem (7). Thanks to Theorem 4.2, we obtain that $z^* \in \mathscr{P}$ and $\Upsilon_{\mu,\omega}(z^*) = 0$. Moreover, the inequality

$$\Theta^{\delta}_{\Upsilon_{\mu,\omega}}(z^*) = \inf_{w\in\mathscr{P}} \left\{ \Upsilon_{\mu,\omega}(w) + \delta\|z^* - w\|^p_{1,\mathscr{H}} \right\} \le \Upsilon_{\mu,\omega}(z^*) + \delta\|z^* - z^*\|^p_{1,\mathscr{H}} = 0$$

and the fact $\Theta^{\delta}_{\Upsilon_{\mu,\omega}}(z^*) \ge 0$ implies that $\Theta^{\delta}_{\Upsilon_{\mu,\omega}}(z^*) = 0$.

Conversely, let $z^* \in \mathscr{P}$ be such that $\Theta^{\delta}_{\Upsilon_{\mu,\omega}}(z^*) = 0$, i.e.,

$$\inf_{w\in\mathscr{P}} \left\{ \Upsilon_{\mu,\omega}(w) + \delta\|z^* - w\|^p_{1,\mathscr{H}} \right\} = 0.$$

Hence, there exists a minimizing sequence $\{w_k\}$ in $\mathscr{P}$ such that

$$0 \le \Upsilon_{\mu,\omega}(w_k) + \delta\|z^* - w_k\|^p_{1,\mathscr{H}} < \frac{1}{k}. \tag{16}$$

It is obvious that $\Upsilon_{\mu,\omega}(w_k) \to 0$ and $\|z^* - w_k\|^p_{1,\mathscr{H}} \to 0$, as $k \to \infty$. This implies that the sequence $\{w_k\}$ converges to z^* in $\mathbf{S}^{1,\mathscr{H}}(\Omega)$, as $k \to \infty$. Combining the nonnegativity and lower semicontinuity of $\Upsilon_{\mu,\omega}$ (see Lemma 4.3), one has

$$0 \le \Upsilon_{\mu,\omega}(z^*) \le \liminf_{k\to+\infty} \Upsilon_{\mu,\omega}(w_k) = 0,$$

i.e., $\Upsilon_{\mu,\omega}(z^*) = 0$. Since $\Upsilon_{\mu,\omega}$ is a gap function for the problem (7), we get that z^* is a weak solution to the problem (7). The proof is complete. ∎

5. Upper-bound error estimates

In this section, we shall provide some error bounds for the problem (7) associated with the regularized gap function $\Upsilon_{\mu,\omega}$ and the Moreau-Yosida regularized gap function $\Theta^{\delta}_{\Upsilon_{\mu,\omega}}$, accordingly.

To obtain error bounds for the problem (7), we introduce the following assumption:

$\mathbf{A}(\mathscr{M}^*)$: For the set-valued mapping $\mathscr{M} : \Omega \times \mathbb{R} \rightrightarrows \mathbb{R}$, there is a constant $c_{\mathscr{M}} > 0$ such that

$$(\zeta_1 - \zeta_2)(s_1 - s_2) \le c_{\mathscr{M}} |s_1 - s_2|^p,$$

for all $\zeta_i \in \mathscr{M}(x, s_i)$, $s_i \in \mathbb{R}$, $i = 1,2$ and for a. a. $x \in \Omega$.

Theorem 5.1. *Let $z^* \in \mathbf{S}^{1,\mathscr{H}}(\Omega)$ be a weak solution of the problem (7). Assume that all assumptions of Lemma 3.3 and the hypothesis $\mathbf{A}(\mathscr{M}^*)$ hold. Assume further that $\omega > 0$ satisfies*

$$\min\left\{ a(p) - f_h \lambda_p^{\frac{1}{p}}, a(p) - e_h - c_{\mathscr{M}} - \frac{\omega}{p}, a(q) \right\} > 0.$$

Then, for each $z \in \mathscr{P}$, we have

$$\|z - z^*\|_{1,\mathscr{H}} \le \max\left\{ \mathscr{E}^{\frac{1}{p}}(z), \mathscr{E}^{\frac{1}{q}}(z) \right\}, \tag{17}$$

where

$$\mathscr{E}(z) = \frac{\Upsilon_{\mu,\omega}(z)}{\min\left\{ a(p) - f_h \lambda_p^{\frac{1}{p}}, a(p) - e_h - c_{\mathscr{M}} - \frac{\omega}{p}, a(q) \right\}}.$$

Proof: Let $z^*\in \mathbf{S}^{1,\mathscr{H}}(\Omega)$ be a weak solution of the problem (7), i.e., $z^*\in\mathscr{P}$ and there exists $\zeta^*\in\mathbf{N}^{p'}_{\mathscr{M}}(z^*)$ such that

$$\begin{aligned}&\int_\Omega\left(|\nabla z^*|^{p-2}\nabla z^*+\mu(x)|\nabla z^*|^{q-2}\nabla z^*\right)\cdot\nabla(v-z^*)\,dx+\int_{\Gamma_b}g(x,v)\,d\Gamma\\&-\int_{\Gamma_b}g(x,z^*)\,d\Gamma+\int_\Omega\left(|z^*|^{p-2}z^*+\mu(x)|z^*|^{q-2}z^*\right)(v-z^*)\,dx\\&\geq\int_\Omega\zeta^*(x)\,(v-z^*)\,dx+\int_\Omega h(x,z^*,\nabla z^*)\,(v-z^*)\,dx\end{aligned}$$

for all $v\in\mathscr{P}$.

For any $z\in\mathscr{P}$ fixed, we insert $v=z$ into the above inequality to obtain

$$\begin{aligned}&\int_\Omega\left(|\nabla z^*|^{p-2}\nabla z^*+\mu(x)|\nabla z^*|^{q-2}\nabla z^*\right)\cdot\nabla(z-z^*)\,dx+\int_{\Gamma_b}g(x,z)\,d\Gamma\\&-\int_{\Gamma_b}g(x,z^*)\,d\Gamma+\int_\Omega\left(|z^*|^{p-2}z^*+\mu(x)|z^*|^{q-2}z^*\right)(z-z^*)\,dx\\&-\int_\Omega\zeta^*(x)\,(z-z^*)\,dx-\int_\Omega h(x,z^*,\nabla z^*)\,(z-z^*)\,dx\geq 0.\end{aligned}\tag{18}$$

Recall the function $\mathscr{Q}_{\mu,\omega}:\mathscr{P}\times L^{p'}(\Omega)\to\mathbb{R}$ is defined by

$$\begin{aligned}\mathscr{Q}_{\mu,\omega}(z,\zeta)=\sup_{v\in\mathscr{P}}\Bigg(&\int_\Omega\left(|\nabla z|^{p-2}\nabla z+\mu(x)|\nabla z|^{q-2}\nabla z\right)\cdot\nabla(z-v)\,dx+\int_\Omega\zeta(x)\,(v-z)\,dx\\&+\int_\Omega\left(|z|^{p-2}z+\mu(x)|z|^{q-2}z\right)(z-v)\,dx-\int_{\Gamma_b}g(x,v)\,d\Gamma+\int_{\Gamma_b}g(x,z)\,d\Gamma\\&+\int_\Omega h(x,z,\nabla z)\,(v-z)\,dx-\frac{\omega}{p}\|z-v\|^p_{p,\Omega}\Bigg),\end{aligned}$$

it follows from Remark 3 and the definition of $\Upsilon_{\mu,\omega}$ that there exists $\zeta_z\in\mathbf{N}^{p'}_{\mathscr{M}}(z)$ such that

$$\begin{aligned}&\Upsilon_{\mu,\omega}(z)\\=&\mathscr{Q}_{\mu,\omega}(z,\zeta_z)\\\geq&\int_\Omega\left(|\nabla z|^{p-2}\nabla z+\mu(x)|\nabla z|^{q-2}\nabla z\right)\cdot\nabla(z-z^*)\,dx+\int_\Omega\zeta_z(x)\,(z^*-z)\,dx\\&+\int_\Omega\left(|z|^{p-2}z+\mu(x)|z|^{q-2}z\right)(z^*-z)\,dx-\int_{\Gamma_b}g(x,z^*)\,d\Gamma+\int_{\Gamma_b}g(x,z)\,d\Gamma\\&+\int_\Omega h(x,z,\nabla z)\,(z^*-z)\,dx-\frac{\omega}{p}\|z-z^*\|^p_{p,\Omega}\end{aligned}\tag{19}$$

By the hypothesis $\mathbf{A}(\mathscr{M}^*)$, we have

$$\begin{aligned}
&\int_\Omega \zeta_z(x)\,(z^*-z)\,dx + \int_\Omega \zeta^*(x)\,(z-z^*)\,dx \\
&= \int_\Omega (\zeta_z(x)-\zeta^*(x))\,(z^*-z)\,dx \\
&\geq -\int_\Omega c_{\mathscr{M}}|z^*-z|^p\,dx \\
&= -c_{\mathscr{M}}\|z-z^*\|^p_{p,\Omega}.
\end{aligned} \tag{20}$$

Moreover, we have

$$\begin{aligned}
&\int_\Omega \left(|\nabla z|^{p-2}\nabla z + \mu(x)|\nabla z|^{q-2}\nabla z\right)\cdot\nabla(z-z^*)\,dx \\
&\quad -\int_\Omega \left(|\nabla z^*|^{p-2}\nabla z^* + \mu(x)|\nabla z^*|^{q-2}\nabla z^*\right)\cdot\nabla(z-z^*)\,dx \\
&= \int_\Omega \left(|\nabla z|^{p-2}\nabla z - |\nabla z^*|^{p-2}\nabla z^*\right)\cdot\nabla(z-z^*)\,dx \\
&\quad +\int_\Omega \mu(x)\left(|\nabla z|^{q-2}\nabla z - |\nabla z^*|^{q-2}\nabla z^*\right)\cdot\nabla(z-z^*)\,dx \\
&\geq \int_\Omega a(p)|\nabla(z-z^*)|^p\,dx + \int_\Omega \mu(x)a(q)|\nabla(z-z^*)|^q\,dx \\
&= a(p)\|\nabla(z-z^*)\|^p_{p,\Omega} + a(q)\|\nabla(z-z^*)\|^q_{q,\mu,\Omega},
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
&\int_\Omega \left(|z|^{p-2}z+\mu(x)|z|^{q-2}z\right)(z-z^*)\,dx - \int_\Omega \left(|z^*|^{p-2}z^*+\mu(x)|z^*|^{q-2}z^*\right)(z-z^*)\,dx \\
&= \int_\Omega \left(|z|^{p-2}-|z^*|^{p-2}z^*\right)(z-z^*)\,dx + \int_\Omega \mu\left(|z|^{q-2}z-|z^*|^{q-2}z^*\right)(z-z^*)\,dx \\
&\geq a(p)\|z-z^*\|^p_{p,\Omega} + a(q)\|z-z^*\|^q_{q,\mu,\Omega}.
\end{aligned} \tag{22}$$

By the condition $\mathbf{A}(h)$(iii), one has

$$\begin{aligned}
&\int_\Omega h(x,z,\nabla z)\,(z^*-z)\,dx + \int_\Omega h(x,z^*,\nabla z^*)\,(z-z^*)\,dx \\
&= \int_\Omega (h(x,z,\nabla z)-h(x,z^*,\nabla z))\,(z^*-z)\,dx \\
&\quad +\int_\Omega (h(x,z^*,\nabla z)-h(x,z^*,\nabla z^*))\,(z^*-z)\,dx \\
&\geq -\int_\Omega e_h|z-z^*|^p\,dx - \int_\Omega f_h|\nabla(z-z^*)|^{p-1}|z-z^*|\,dx
\end{aligned}$$

Applying Hölder's inequality gives

$$\begin{aligned}
&\int_\Omega h(x,z,\nabla z)\,(z^*-z)\,dx + \int_\Omega h(x,z^*,\nabla z^*)\,(z-z^*)\,dx \\
&\geq -e_h\|z-z^*\|^p_{p,\Omega} - f_h\|\nabla(z-z^*)\|^{p-1}_{p,\Omega}\|z-z^*\|_{p,\Omega}.
\end{aligned} \tag{23}$$

From (18), (20)–(23), we have

$$\begin{aligned}
&\int_\Omega \left(|\nabla z|^{p-2}\nabla z+\mu(x)|\nabla z|^{q-2}\nabla z\right)\cdot\nabla(z-z^*)\,dx+\int_\Omega \zeta_z(x)\,(z^*-z)\,dx\\
&-\int_{\Gamma_b} g(x,z^*)\,d\Gamma+\int_{\Gamma_b} g(x,z)\,d\Gamma+\int_\Omega h(x,z,\nabla z)\,(z^*-z)\,dx\\
&\geq a(p)\left(\|\nabla(z-z^*)\|_{p,\Omega}^p+\|z-z^*\|_{p,\Omega}^p\right)+a(q)\left(\|\nabla(z-z^*)\|_{q,\mu,\Omega}^q+\|z-z^*\|_{p,\Omega}^p\right)\\
&-(e_h+c_{\mathscr{M}})\|z-z^*\|_{p,\Omega}^p-f_h\|\nabla(z-z^*)\|_{p,\Omega}^{p-1}\|z-z^*\|_{p,\Omega}. \quad (24)
\end{aligned}$$

Combining (2), (19) and (24), one has

$$\begin{aligned}
&\Upsilon_{\mu,\omega}(z)\\
&\geq a(p)\|\nabla(z-z^*)\|_{p,\Omega}^p+\left(a(p)-e_h-c_{\mathscr{M}}\right)\|z-z^*\|_{p,\Omega}^p-\frac{\omega}{p}\|z-z^*\|_{p,\Omega}^p\\
&\quad -f_h\|\nabla(z-z^*)\|_{p,\Omega}^{p-1}\|z-z^*\|_{p,\Omega}+a(q)\left(\|\nabla(z-z^*)\|_{q,\mu,\Omega}^q+\|z-z^*\|_{q,\mu,\Omega}^q\right)\\
&\geq a(p)\|\nabla(z-z^*)\|_{p,\Omega}^p+\left(a(p)-e_h-c_{\mathscr{M}}-\frac{\omega}{p}\right)\|z-z^*\|_{p,\Omega}^p\\
&\quad -f_h\lambda_p^{\frac{1}{p}}\|\nabla(z-z^*)\|_{p,\Omega}^p+a(q)\left(\|\nabla(z-z^*)\|_{q,\mu,\Omega}^q+\|z-z^*\|_{q,\mu,\Omega}^q\right)\\
&\geq L_0\left(\|\nabla(z-z^*)\|_{p,\Omega}^p+\|z-z^*\|_{p,\Omega}^p+\|\nabla(z-z^*)\|_{q,\mu,\Omega}^q+\|z-z^*\|_{q,\mu,\Omega}^q\right)\\
&\geq L_0\min\left\{\|z-z^*\|_{1,\mathscr{H}}^p,\|z-z^*\|_{1,\mathscr{H}}^q\right\}, \quad (25)
\end{aligned}$$

where $L_0>0$ is defined by

$$L_0:=\min\left\{a(p)-f_h\lambda_p^{\frac{1}{p}},a(p)-e_h-c_{\mathscr{M}}-\frac{\omega}{p},a(q)\right\}.$$

Set

$$\mathscr{E}(z)=\frac{\Upsilon_{\mu,\omega}(z)}{L_0}.$$

Then, the inequality (25) implies that

$$\|z-z^*\|_{1,\mathscr{H}}\leq\max\left\{\mathscr{E}^{\frac{1}{p}}(z),\mathscr{E}^{\frac{1}{q}}(z)\right\}.$$

Therefore, the inequality (17) is valid. ■

The following results derive upper-bound error estimates for the problem (7) under the norm $\|\cdot\|_{p,\Omega}$.

Theorem 5.2. *Let $z^*\in\mathbf{S}^{1,\mathscr{H}}(\Omega)$ be a weak solution of the problem (7). Assume that all assumptions of Lemma 3.3 and the hypothesis $\mathbf{A}(\mathscr{M}^*)$ hold. Assume further that $\omega>0$ satisfies*

$$a(p)\lambda_p^{-1}-f_h\lambda_p^{\frac{1-p}{p}}+a(p)-e_h-c_{\mathscr{M}}-\frac{\omega}{p}>0.$$

Then, for each $z \in \mathscr{P}$, we have

$$\|z-z^*\|_{p,\Omega} \leq \left[\frac{\Upsilon_{\mu,\omega}(z)}{a(p)\lambda_p^{-1} - f_h\lambda_p^{\frac{1-p}{p}} + a(p) - e_h - c_{\mathscr{M}} - \frac{\omega}{p}}\right]^{\frac{1}{p}}. \tag{26}$$

Proof: Let $z^* \in \mathbf{S}^{1,\mathscr{H}}(\Omega)$ be a weak solution of the problem (7). Using a similar method as in the first part of the demonstration of Theorem 5.1 leads to the expressions (18)–(20) and (23). Since

$$\int_\Omega \mu(x)\left(|\nabla z|^{q-2}\nabla z - |\nabla z^*|^{q-2}\nabla z^*\right)\cdot\nabla(z-z^*)\,dx \geq 0,$$

taking

$$\int_\Omega \left(|\nabla z|^{p-2}\nabla z - |\nabla z^*|^{p-2}\nabla z^*\right)\cdot\nabla(z-z^*)\,dx \geq a(p)\|\nabla(z-z^*)\|_{p,\Omega}^p,$$

into account (21) gives

$$\begin{aligned}&\int_\Omega \left(|\nabla z|^{p-2}\nabla z + \mu(x)|\nabla z|^{q-2}\nabla z\right)\cdot\nabla(z-z^*)\,dx\\ &-\int_\Omega \left(|\nabla z^*|^{p-2}\nabla z^* + \mu(x)|\nabla z^*|^{q-2}\nabla z^*\right)\cdot\nabla(z-z^*)\,dx \geq a(p)\|\nabla(z-z^*)\|_{p,\Omega}^p.\end{aligned} \tag{27}$$

From (2), (18), (20), (23) and (27), we obtain

$$\begin{aligned}&\int_\Omega \left(|\nabla z|^{p-2}\nabla z + \mu(x)|\nabla z|^{q-2}\nabla z\right)\cdot\nabla(z-z^*)\,dx + \int_\Omega \zeta_z(x)\,(z^*-z)\,dx\\
&\qquad \int_\Omega \left(|z|^{p-2}z + \mu(x)|z|^{q-2}z\right)(z-z^*)\,dx - \int_{\Gamma_b} g(x,z^*)\,d\Gamma + \int_{\Gamma_b} g(x,z)\,d\Gamma\\
&\qquad + \int_\Omega h(x,z,\nabla z)\,(z^*-z)\,dx\\
&\geq a(p)\|\nabla(u-z^*)\|_{p,\Omega}^p + (a(p)-e_h-c_{\mathscr{M}})\|z-z^*\|_{p,\Omega}^p - f_h\lambda_p^{\frac{1}{p}}\|\nabla(z-z^*)\|_{p,\Omega}^p\\
&= \left(a(p) - f_h\lambda_p^{\frac{1}{p}}\right)\|\nabla(z-z^*)\|_{p,\Omega}^p + (a(p)-e_h-c_{\mathscr{M}})\|z-z^*\|_{p,\Omega}^p\\
&\geq \left(a(p) - f_h\lambda_p^{\frac{1}{p}}\right)\lambda_p^{-1}\|z-z^*\|_{p,\Omega}^p + (a(p)-e_h-c_{\mathscr{M}})\|z-z^*\|_{p,\Omega}^p\\
&= \left(a(p)\lambda_p^{-1} - f_h\lambda_p^{\frac{1-p}{p}} + a(p) - e_h - c_{\mathscr{M}}\right)\|z-z^*\|_{p,\Omega}^p.\end{aligned} \tag{28}$$

Combining (19) and (28), one has

$$\Upsilon_{\mu,\omega}(z) \geq \left(a(p)\lambda_p^{-1} - f_h\lambda_p^{\frac{1-p}{p}} + a(p) - e_h - c_{\mathscr{M}} - \frac{\omega}{p}\right)\|z-z^*\|_{p,\Omega}^p. \tag{29}$$

Thus, the desired inequality (26) holds. ■

Theorem 5.3. *Let $z^* \in \mathbf{S}^{1,\mathscr{H}}(\Omega)$ be a weak solution of the problem (7). Assume that the hypotheses of Theorem 5.2 hold. Then, for each $z \in \mathscr{P}$ and all $\omega, \delta > 0$, we have*

$$\|z - z^*\|_{p,\Omega} \leq \left[\frac{2^{p-1}\Theta^{\delta}_{\Upsilon_{\mu,\omega}}(z)}{\min\left\{ a(p)\lambda_p^{-1} - f_h\lambda_p^{\frac{1-p}{p}} + a(p) - e_h - c_{\mathscr{M}} - \frac{\omega}{p}, \delta\beta^{*-1} \right\}} \right]^{\frac{1}{p}}, \tag{30}$$

where $\beta^* > 0$ is the constant such that $\|z\|^p_{p,\Omega} \leq \beta^*\|z\|^p_{1,\mathscr{H}}$ for all $z \in \mathbf{S}^{1,\mathscr{H}}(\Omega)$ due to the continuity of the embedding of $\mathbf{S}^{1,\mathscr{H}}(\Omega)$ into $L^p(\Omega)$.

Proof: Let $z^* \in \mathbf{S}^{1,\mathscr{H}}(\Omega)$ be a weak solution of the problem (7). By the definition of the function $\Theta^{\delta}_{\Upsilon_{\mu,\omega}}$ and the inequality (29), for any $z \in \mathscr{P}$ we get

$$\begin{aligned}
&\Theta^{\delta}_{\Upsilon_{\mu,\omega}}(z)\\
&= \inf_{w\in\mathscr{P}} \left\{ \Upsilon_{\mu,\omega}(w) + \delta\|z-w\|^p_{1,\mathscr{H}} \right\}\\
&\geq \inf_{w\in\mathscr{P}} \left\{ \left(a(p)\lambda_p^{-1} - f_h\lambda_p^{\frac{1-p}{p}} + a(p) - e_h - c_{\mathscr{M}} - \frac{\omega}{p} \right) \|w-z^*\|^p_{p,\Omega} + \delta\|z-w\|^p_{1,\mathscr{H}} \right\}\\
&\geq \inf_{w\in\mathscr{P}} \left\{ \left(a(p)\lambda_p^{-1} - f_h\lambda_p^{\frac{1-p}{p}} + a(p) - e_h - c_{\mathscr{M}} - \frac{\omega}{p} \right) \|w-z^*\|^p_{p,\Omega} + \delta\beta^{*-1}\|z-w\|^p_{p,\Omega} \right\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\Theta^{\delta}_{\Upsilon_{\mu,\omega}}(z) \geq \min&\left\{ a(p)\lambda_p^{-1} - f_h\lambda_p^{\frac{1-p}{p}} + a(p) - e_h - c_{\mathscr{M}} - \frac{\omega}{p}, \delta\beta^{*-1} \right\}\\
&\times \inf_{w\in\mathscr{P}} \left\{ \|w-z^*\|^p_{p,\Omega} + \|z-w\|^p_{p,\Omega} \right\}.
\end{aligned} \tag{31}$$

By applying the following inequality

$$\|w-z^*\|^p_{p,\Omega} + \|z-w\|^p_{p,\Omega} \geq \frac{1}{2^{p-1}}\|z-z^*\|^p_{p,\Omega},$$

it follows from (31) that

$$\Theta^{\delta}_{\Upsilon_{\mu,\omega}}(z) \geq \frac{1}{2^{p-1}} \min\left\{ a(p)\lambda_p^{-1} - f_h\lambda_p^{\frac{1-p}{p}} + a(p) - e_h - c_{\mathscr{M}} - \frac{\omega}{p}, \delta\beta^{*-1} \right\} \|z-z^*\|^p_{p,\Omega}.$$

This implies that the inequality (30) holds. ■

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