RECURRENCE OF WEIGHTED COMPOSITION OPERATOR

NOUREDDINE KARIM AND MOHAMED AMOUCH

Abstract. A linear bounded operator T on a Hilbert space X is called hypercyclic if there exists a vector $x \in X$ whose orbit under T; $\{T^n x;\ n \in \mathbb{N}\}$ is dense in X. The operator T is called recurrent (super-recurrent, respectively) if, for every non-empty open subset $U \subset X$, there is an integer n such that $T^n U \cap U \neq \emptyset$ (there is an integer n and a scalar λ such that $\lambda T^n U \cap U \neq \emptyset$, respectively). In this paper, we prove that if ϕ is an invertible increasing map and $C_{u,\phi}$ a bounded weighted composition operator on $\ell^2(\mathbb{Z})$, then $C_{u,\phi}$ is hypercyclic if and only if $C_{u,\phi}$ is recurrent. Furthermore, under the same conditions, we characterize the super-recurrent composition operator $C_{u,\phi}$ acting on $\ell^2(\mathbb{Z})$ in terms of its weight u and symbol ϕ.

1. Introduction

A linear bounded operator T acting on a topological vector space X is called hypercyclic if there exists a vector $x \in X$ such that its orbit under T,

$$\mathcal{O}(x,T) := \{T^n x;\ n \geq 0\}$$

is dense in X. Such a vector is called a hypercyclic vector for the operator T. Note that this notion makes sense only if the space X is separable. It is well known after Birkhoff's theorem (see [4]) that an operator T on a metrizable and separable Baire space X is hypercyclic if and only if it is topologically transitive, that is, if for any pair of non-empty open subsets U, V of X there exists some $n \in \mathbb{N}$ such that

$$T^n U \cap V \neq \emptyset,$$

see [10]. The operator T is called supercyclic if for every pair U, V of non-empty open subsets of X, there would be an integer n and a non-zero scalar λ such that

$$\lambda T^n(U) \cap V \neq \emptyset.$$

The study of the phenomenon of hypercyclicity has its origins in the papers by Birkhoff [5] and MacLane [12]. It is associated with the invariant subset problem. An operator T lacks nontrivial invariant closed subsets if and only if all non-zero vectors are hypercyclic for T. It is known that such operators exist on Banach spaces [14, 9]. However, their existence remains an open problem in Hilbert spaces. This fact has attracted the interest of many mathematicians to study the hypercyclicity and dynamics of linear operators in the recent two decades. Especially the hypercyclicity of some classes of operators, such as the weighted shifts and composition operators. The first example of a hypercyclic operator on a Banach space is due to S.Rolewicz in 1969. Indeed, for every scalar λ with $|\lambda| > 1$, he showed that λB is hypercyclic, where B is the unweighted backward shift operator on $\ell^2(\mathbb{N})$. For more information about the main concepts of linear dynamics, see [1, 13, 15, 6, 11].

Another notion in topological dynamics with topological transitivity is that of recurrence. This notion was introduced by Poincaré and Birkhoff and studied by Costakis, Manoussos, and Parissis in [8]. A bounded linear operator T on a topological vector space X is recurrent, if for any non-empty open subset U of X there exists some $n \in \mathbb{N}$ such that

$$T^n U \cap U \neq \emptyset,$$

2020 *Mathematics Subject Classification.* 47A16, 37B02, 37B20, 30K20.

Key words and phrases. Hypercyclic operator, Recurrent operator, Weighted composition operator, Weighted shift operator.

27 Sep 2023 13:45:33 PDT
221122-karim Version 2 - Submitted to Rocky Mountain J. Math.

see [8]. A vector $x \in X$ is said to be recurrent if there exists a sequence $(n_k)_{k\in\mathbb{N}}$ such that

$$T^{n_k}x \to x, \text{ as } k \to \infty.$$

Recall that an operator T is recurrent if and only if the set of all recurrent vectors of T, noted by $Rec(T)$, is dense in X, see [8].
Recently, in [2] it was introduced the notion of super-recurrence indicating that the operator T is super-recurrent if, for every open subset U of X, there is an integer n and a non-zero scalar λ such that

$$\lambda T^n(U) \cap U \neq \emptyset.$$

In [15] and [16], Salas characterized the hypercyclicity and the supercyclicity of the weighted shift on the spaces $\ell^2(\mathbb{Z})$, respectively. In their paper [7], Costakis and Parissis showed that a weighted backward shift on $\ell^2(\mathbb{Z})$ is recurrent if and only if it is hypercyclic. In [3], the authors gave a characterization of the hypercyclicity of weighted composition operators on $\ell^2(\mathbb{Z})$ in terms of their weight functions and symbols. In this work, we prove that if ϕ is an invertible increasing map and $C_{u,\phi}$ a bounded weighted composition operator on $\ell^2(\mathbb{Z})$, then $C_{u,\phi}$ is hypercyclic if and only if $C_{u,\phi}$ is recurrent. Furthermore, we characterize the super-recurrence of $C_{u,\phi}$ acting on $\ell^2(\mathbb{Z})$.

This paper is organized as follows. In the second section, we present some preliminary results which will be used in the sequel. In the third section, we characterize the recurrence of composition operators $C_{u,\phi}$ acting on $\ell^2(\mathbb{Z})$ whose symbols ϕ are invertible, increasing maps on $\mathbb{Z}$. As a consequence, we show that $C_{u,\phi}$ is recurrent if and only if it is hypercyclic. In the fourth section, we study the super-recurrence of $C_{u,\phi}$ on $\ell^2(\mathbb{Z})$ and give a necessary and sufficient condition in terms of the weight function and symbol of $C_{u,\phi}$ to be super-recurrent on $\ell^2(\mathbb{Z})$.

2. Preliminary results

Let $\ell^2(\mathbb{N})$ and $\ell^2(\mathbb{Z})$ be the Hilbert spaces:

$$\ell^2(\mathbb{N}) = \{x = (x_i)_{i\in\mathbb{N}} \subset \mathbb{C};\ \|x\|_2 = (\sum_{i\in\mathbb{N}} |x_i|^2)^{1/2} < \infty\},$$

$$\ell^2(\mathbb{Z}) = \{x = (x_i)_{i\in\mathbb{Z}} \subset \mathbb{C};\ \|x\|_2 = (\sum_{i\in\mathbb{Z}} |x_i|^2)^{1/2} < \infty\}.$$

Let $(e_n)_{n\in\mathbb{N}}$ (resp. $(e_n)_{n\in\mathbb{Z}}$) be the canonical basis of $\ell^2(\mathbb{N})$ (resp. $\ell^2(\mathbb{Z})$).

Definition 2.1. The unilateral forward shift operator is the operator defined on $\ell^2(\mathbb{N})$ by $Se_n = e_{n+1}$ for all $n \in \mathbb{N}$, and the unilateral backward shift operator is the operator defined on $\ell^2(\mathbb{N})$ by $Be_n = e_{n-1}$ for $n \geq 1$ and $Be_0 = 0$.

Now, let $\omega = (w_n)_{n\in\mathbb{Z}}$ be a sequence of positive numbers.

Definition 2.2. The bilateral weighted forward shift operator with weights $\omega = (w_n)_{n\in\mathbb{Z}}$ is defined by $S_\omega e_n = w_n e_{n+1}$ for all $n \in \mathbb{N}$, and the bilateral weighted backward shift operator with weights $\omega = (w_n)_{n\in\mathbb{Z}}$ is defined by $B_\omega e_n = w_n e_{n-1}$ for $n \geq 1$ and $B_\omega e_0 = 0$.

In [15], Salas proved the following results about hypercyclicity of the bilateral weighted forward shift operator and the bilateral weighted forward shift operator:

Theorem 2.3. [15] *Let S_ω be a bilateral weighted forward shift with positive weight sequence $\omega = (w_n)_{n\in\mathbb{Z}}$. Then S_ω is hypercyclic if and only if given $\varepsilon > 0$, and $q \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that for all $j \in [\![-q, q]\!]$,*

$$\prod_{s=0}^{n-1} w_{s+j} < \varepsilon \text{ and } \prod_{s=1}^{n} w_{j-s} > 1/\varepsilon.$$

Corollary 2.4. [15, Theorem 2.1] *Let B_ω be a bilateral weighted backward shift with positive weight sequence $\omega = (w_n)_{n\in\mathbb{Z}}$. Then B_ω is hypercyclic if and only if given $\varepsilon > 0$ and $q \in \mathbb{N}$, there exists n arbitrarily large such that for all $|j| \leq q$*

$$\prod_{s=0}^{n-1} w_{s+j} > 1/\varepsilon \text{ and } \prod_{s=1}^{n} w_{j-s} < \varepsilon.$$

In [16], Salas proved the following results about supercyclicity of the bilateral weighted forward shift operator:

Theorem 2.5. [16, Theorem 3.1] *Let B_ω be a bilateral weighted backward shift with positive weight sequence $\omega = (w_n)_{n\in\mathbb{Z}}$. Then B_ω is supercyclic if and only if every $q \in \mathbb{N}$, we have that*

$$\liminf_{n\to\infty} \max\Big(\frac{\prod_{j+1-n\leq k\leq j} w_k}{\prod_{h+1\leq k\leq h+n} w_k};\ |j|,|h| \leq q\Big) = 0.$$

Definition 2.6. [17] Let $(X, \mathfrak{L}, m)$ be a measure space. Then a measurable mapping ϕ from X into X is said to be nonsingular if

$$m(\phi^{-1}(S)) = 0 \text{ whenever } m(S) = 0.$$

Definition 2.7. [17] Let u be a function on $\mathbb{Z}$ and $\phi : \mathbb{Z} \to \mathbb{Z}$ a nonsingular transformation. Then a weighted composition operator $C_{u,\phi}$ on $\ell^2(\mathbb{Z})$ is defined by:

$$\begin{array}{cccc} C_{u,\phi} & : & \ell^2(\mathbb{Z}) & \to & \ell^2(\mathbb{Z}) \\ & & f & \mapsto & u.f\circ\phi \end{array}.$$

If $(e_n)_{n\in\mathbb{Z}}$ is the canonical basis of $\ell^2(\mathbb{Z})$, and $f = \sum_{i\in\mathbb{Z}} f(i)e_i \in \ell^2(\mathbb{Z})$, then $C_{u,\phi}(f) = \sum_{i\in\mathbb{Z}} u(i).f(\phi(i))e_i$.

The non-singularity of ϕ is a necessary condition for it to induce a composition operator on $\ell^2(\mathbb{Z})$. The transformation ϕ is usually called the symbol, and u is called the weight of the weighted composition operator. The weighted composition operator $C_{u,\phi}$ is bounded if and only if $\sup\{H(n);\ n \in \mathbb{Z}\} < \infty$, where $H(n) = \sum_{j\in B(n)} |u(j)|^2$, $B(n) = \{j \in supp(u);\ \phi(j) = n\}$ and $supp(u) = \{j \in \mathbb{Z};\ u(j) \neq 0\}$. For more informations, the reader is referred to [17]. Throughout this paper and without loss of generality, we may and will assume that the weight function u is nonnegative. Further, we use the notations $\phi^n = \phi\circ\phi\circ...\circ\phi$, n times and $\phi^{-n} = \phi^{-1}\circ\phi^{-1}\circ...\circ\phi^{-1}$, n times.

Theorem 2.8. [3] *Let u be a function such that $u(k) \neq 0$ for all $k \in \mathbb{Z}$ and ϕ be an invertible increasing map and let $C_{u,\phi}$ a bounded weighted composition operator on $\ell^2(\mathbb{Z})$. Then $C_{u,\phi}$ is hypercyclic if and only if for each $\varepsilon > 0$ and $q \in \mathbb{N}$, there exists $n \in \mathbb{N}$ arbitrarily large such that for all $j \in [\![-q, q]\!]$:*

$$\prod_{i=1}^{n} |u\circ\phi^{-i}(j)| < \varepsilon \text{ and } \prod_{i=0}^{n-1} |u\circ\phi^{i}(j)| > 1/\varepsilon.$$

3. Main result

Proposition 3.1. *Let $C_{u,\phi}$ be a recurrent weighted composition operator on $\ell^2(\mathbb{Z})$. Then for all $k \in \mathbb{Z}$, $u(k) \neq 0$.*

Proof. Suppose that there exists $k_0 \in \mathbb{Z}$ such that $u(k_0) = 0$. Let f be a recurrent vector for $C_{u,\phi}$, then there is a sequence of integers $(n_k)_{k\in\mathbb{N}}$ such that $C_{u,\phi}^{n_k} f \to f$. Furthermore, for every $n \in \mathbb{N}$ we have

$$\begin{aligned} C_{u,\phi}^{n} f(k_0) &= \left(\prod_{i=0}^{n-1} u(\phi^i(k_0))\right) f(\phi^n(k_0)) \\ &= u(k_0)u(\phi(k_0))...u(\phi^{n-1}(k_0))f(\phi^n(k_0)) \\ &= 0, \end{aligned}$$

then $f(k_0) = 0$. Thus $Rec(C_{u,\phi}) \subseteq \{f \in \ell^2(\mathbb{Z});\, f(k_0) = 0\}$ which contradicts that $Rec(C_{u,\phi})$ is dense in $\ell^2(\mathbb{Z})$. □

Remark 3.2. If $\phi = id_{\mathbb{Z}}$, then $C_{u,\phi}$ cannot be hypercyclic [3]. However, we can notice that $C_{u,\phi}$ is recurrent if and only if there is some $(n_l) \subset \mathbb{N}$ such that $u^{n_l}(k) \to 1$ for every $k \in \mathbb{Z}$.

Example 3.3. Let $u(k) = \alpha$ such that $|\alpha| = 1$, for every $k \in \mathbb{Z}$ and $\phi = id_{\mathbb{Z}}$. Then the composition operator $C_{u,\phi}$ on $\ell^2(\mathbb{Z})$ is recurrent but not hypercyclic.

Theorem 3.4. *Let u be a function such that $u(k) \neq 0$ for all $k \in \mathbb{Z}$ and ϕ be an invertible increasing map such that $\phi \neq id_{\mathbb{Z}}$ and let $C_{u,\phi}$ be a bounded weighted composition operator on $\ell^2(\mathbb{Z})$. Then $C_{u,\phi}$ is hypercyclic if and only if $C_{u,\phi}$ is recurrent.*

For the proof of the Theorem 3.4, we need the following lemma:

Lemma 3.5. *If $C_{u,\phi}$ is a bilateral weighted composition operator, then for every $j \in \mathbb{Z}$ we have:*

$$C^n_{u,\phi} e_j = (\prod_{i=0}^{n-1} u \circ \phi^i).e_{\phi^{-n}(j)}.$$

Proof of Lemma 3.5. Let $k \in \mathbb{Z}$, $C_{u,\phi}(e_j)k = u(k).e_j(\phi(k))$, with

$$e_j(\phi(k)) = \begin{cases} 1 & if\ j = \phi(k); \\ 0 & otherwise. \end{cases} = \begin{cases} 1 & if\ \phi^{-1}(j) = k; \\ 0 & otherwise. \end{cases} = e_{\phi^{-1}(j)}(k),$$

then $C_{u,\phi}(e_j)k = u(k).e_{\phi^{-1}(j)}(k) = (u.e_{\phi^{-1}(j)})(k)$, so $C_{u,\phi}(e_j) = u.e_{\phi^{-1}(j)}$. Then by induction we find that $C^n_{u,\phi}(e_j) = (\prod_{i=0}^{n-1} u \circ \phi^i).e_{\phi^{-n}(j)}$. □

Proof of Theorem 3.4. We only have to prove that if $C_{u,\phi}$ is recurrent, then $C_{u,\phi}$ is hypercyclic since the converse implication is always true. So assume that ϕ is increasing and $C_{u,\phi}$ recurrent. Let q be a positive integer and consider $\varepsilon > 0$. Choose $\delta > 0$ such that $\delta/(1-\delta) < \varepsilon$ and $\delta < 1$. Consider the open ball $B(\sum_{|j|\leq q} e_j, \delta)$. Since $C_{u,\phi}$ is recurrent there exists a positive integer n sufficiently large, such that $\phi^n(j) > 2q$ and $\phi^{-n}(j) < -2q$, with

$$B(\sum_{|j|\leq q} e_j, \delta) \cap C^{-n}_{u,\phi}(B(\sum_{|j|\leq q} e_j, \delta)) \neq \emptyset.$$

Hence there exists $f \in \ell^2(\mathbb{Z})$, such that $f \in B(\sum_{|j|\leq q} e_j, \delta)$ and $C^n_{u,\phi} f \in B(\sum_{|j|\leq q} e_j, \delta)$, so we have

$$\|f - \sum_{|j|\leq q} e_j\| < \delta, \tag{3.1}$$

and

$$\|C^n_{u,\phi} f - \sum_{|j|\leq q} e_j\| < \delta. \tag{3.2}$$

From (3.1) we have that:

$$\begin{cases} |\langle f, e_j\rangle| > 1-\delta, & \text{if } j \in [\![-q, q]\!]; \\ |\langle f, e_j\rangle| < \delta, & \text{if } j \notin [\![-q, q]\!]. \end{cases}$$

Note that we have

$$\begin{aligned} C^n_{u,\phi}(\langle f, e_j\rangle e_j) &= \langle f, e_j\rangle.C^n_{u,\phi}(e_j) \\ &= \langle f, e_j\rangle.(\prod_{i=0}^{n-1} u \circ \phi^i).e_{\phi^{-n}(j)},\ by\ lemma\ 3.5 \\ &= \langle f, e_j\rangle.(\prod_{i=0}^{n-1} u \circ \phi^i)(\phi^{-n}(j)) = \langle f, e_j\rangle.(\prod_{i=1}^{n} u \circ \phi^{-i})(j), \end{aligned}$$

so

$$|\langle f, e_j\rangle|.\prod_{i=1}^{n} |u \circ \phi^{-i}(j)| = \|C^n_{u,\phi}(\langle f, e_j\rangle e_j)\|.$$

Now from (3.2), $\|C^n_{u,\phi}(\langle f, e_j\rangle e_j)\| < \delta$, for $j \in [\![-q, q]\!]$. Thus if $j \in [\![-q, q]\!]$, we have that

$$\|C^n_{u,\phi}(\langle f, e_j\rangle e_j)\| = |\langle f, e_j\rangle| \prod_{i=1}^{n} |u \circ \phi^{-i}(j)| < \varepsilon,$$

since it is assumed that $\phi^{-n}(j) < -2q$ for each $j \in [\![-q, q]\!]$. Hence,

$$\prod_{i=1}^{n} |u \circ \phi^{-i}(j)| < \frac{\delta}{|\langle f, e_j\rangle|} < \frac{\delta}{1-\delta} < \varepsilon$$

for each $j \in [\![-q, q]\!]$. Moreover, for all $j \in [\![-q, q]\!]$, always after (3.2)

$$\begin{aligned} \delta &> \|C^n_{u,\phi} f - e_j\| \\ &= \|(\prod_{i=0}^{n-1} u \circ \phi^i) f \circ \phi^n - e_j\| \\ &\geq |(\prod_{i=0}^{n-1} u \circ \phi^i)(j) f \circ \phi^n(j) - 1| \\ &\geq 1 - |(\prod_{i=0}^{n-1} u \circ \phi^i)(j)||f \circ \phi^n(j)|. \end{aligned}$$

Since $\phi^n(j) > 2q$ for each $j \in [\![-q, q]\!]$, thus giving $|f \circ \phi^n(j)| < \delta$ and hence

$$\prod_{i=0}^{n-1} |u \circ \phi^i(j)| > \frac{1-\delta}{|f \circ \phi^n(j)|} > \frac{1-\delta}{\delta} > \frac{1}{\varepsilon}.$$

Finally, we get

$$\prod_{i=1}^{n} |u \circ \phi^{-i}(j)| < \varepsilon \text{ and } \prod_{i=0}^{n-1} |u \circ \phi^i(j)| > \frac{1}{\varepsilon},$$

for each $j \in [\![-q, q]\!]$, so from Theorem 2.8, $C_{u,\phi}$ is hypercyclic. □

Example 3.6. Let $\omega = (w_n)_{n\in\mathbb{Z}}$ be sequence of positive numbers, and let B_ω be the bilateral weighted backward shift given by $B_\omega e_n = w_n e_{n-1}$. Then B_ω is hypercyclic if and only if B_ω is recurrent. Indeed, let $\phi(n) = n + 1$ and $u(n) = w_{n+1}$, then

$$\begin{aligned} B_\omega e_n(j) &= w_n e_{n-1}(j) = u(n-1) e_{n-1}(j) \\ &= \begin{cases} u(n-1), & if\ j = n-1; \\ 0, & otherwise. \end{cases} = u(j) e_n(j+1) = u(j).(e_n \circ \phi)(j) \\ &= C_{u,\phi} e_n(j), \end{aligned}$$

so $B_\omega = C_{u,\phi}$ is a weighted composition on $\ell^2(\mathbb{Z})$, then B_ω is hypercyclic if and only if B_ω is recurrent.

4. Super-recurrence of weighted composition operator on $\ell^2(\mathbb{Z})$

Similarly to Proposition 3.1 we have that:

Proposition 4.1. *Let $C_{u,\phi}$ be a super-recurrent weighted composition operator on $\ell^2(\mathbb{Z})$. Then for all $k \in \mathbb{Z}$, $u(k) \neq 0$.*

Proof. The idea of the proof is similar to that of Proposition 3.1. □

Theorem 4.2. *Let u be a function such that $u(k) \neq 0$ for all $k \in \mathbb{Z}$ and ϕ be an invertible increasing map and let $C_{u,\phi}$ be a bounded weighted composition operator on $\ell^2(\mathbb{Z})$. Then $C_{u,\phi}$ is super-recurrent if and only if for each $\varepsilon > 0$ and $q \in \mathbb{N}$, there exists $n \in \mathbb{N}$ arbitrarily large such that for all $j, h \in [\![-q, q]\!]$:*

$$\big(\prod_{i=1}^{n} |u \circ \phi^{-i}(j)|\big) \times \big(\prod_{i=0}^{n-1} |u \circ \phi^i(h)|\big)^{-1} < \varepsilon.$$

Proof. First if $C_{u,\phi}$ is super-recurrent, let $1/2 > \varepsilon > 0$ and $q \in \mathbb{N}$. Consider the open ball $B(\sum_{|j|\leq q} e_j, \delta)$. Since $C_{u,\phi}$ is super-recurrent there exists a positive integer n sufficiently large and a non-zero scalar λ, such that $\phi^n(j) > 2q$ and $\phi^{-n}(j) < -2q$, with

$$B(\sum_{|j|\leq q} e_j, \delta) \cap \lambda C^n_{u,\phi}(B(\sum_{|j|\leq q} e_j, \delta)) \neq \emptyset.$$

Then, there exists a vector f such that

$$\|f - \sum_{|j|\leq q} e_j\| \leq \varepsilon. \tag{4.1}$$

and

$$\|\lambda C^n_{u,\phi} f - \sum_{|j|\leq q} e_j\| \leq \varepsilon. \tag{4.2}$$

Thus from 4.1 we have $|\langle f, e_j\rangle| > 1/2$ for $j \in [\![-q, q]\!]$ and

$$1/2|\lambda| \prod_{i=1}^{n} |u \circ \phi^{-i}(j)| \leq |\lambda| \prod_{i=1}^{n} |u \circ \phi^{-i}(j)||\langle f, e_j\rangle| < \varepsilon.$$

On the other hand, from 4.2 we have that for $h \in [\![-q, q]\!]$, we have $|f \circ \phi^n(h)| < \varepsilon$ and

$$1/2 < 1 - |\lambda \prod_{i=0}^{n-1} u \circ \phi^i(h).f \circ \phi^n(h) - 1|$$

$$\leq |\lambda \prod_{i=0}^{n-1} u \circ \phi^i(h).f \circ \phi^n(h)| \leq |\lambda| \prod_{i=0}^{n-1} |u \circ \phi^i(h)|.\varepsilon.$$

Consequently, for all $j, h \in [\![-q, q]\!]$,

$$\big(\prod_{i=1}^{n} |u \circ \phi^{-i}(j)|\big) \times \big(\prod_{i=0}^{n-1} |u \circ \phi^i(h)|\big)^{-1} \leq 4\varepsilon^2.$$

If ϕ is a decreasing transformation. Then in this case, we can find an integer n large enough such that $\phi^n(j) < -2q$ and $\phi^{-n}(j) > 2q$ for each $j \in [\![-q, q]\!]$ and the proof proceed similarly as first case.
To see the converse, we will prove the following lemma:

Lemma 4.3. *Let $C_{u,\phi}$ be a composition operator on $\ell^2(\mathbb{Z})$. Assume that if $\varepsilon > 0$ and vectors g, h are in span$\{e_j;\ |j| \leq q\}$, then there exist an integer n and a vector u in span$\{e_j;\ |\phi^n(j)| \leq q\}$ such that*

1. $\|u\| \times \|C^n_{u,\phi}(h)\| < \varepsilon$,
2. $\|C^n_{u,\phi}(u) - g\| < \varepsilon$.

Then T is supercyclic.

Proof. First note that we can find a sequence of non-zero scalars λ_n such that $\|\lambda_n C^n_{u,\phi}(h)\| < \varepsilon$ and $\|\lambda_n^{-1} u\| < \varepsilon$. Indeed, assume that $\alpha = \|u\|$ and $\beta_n = \|C^n_{u,\phi}(f)\|$ are not both 0. If $\alpha\beta_n \neq 0$, put $\lambda_n := \beta_n^{1/2}\alpha^{-1/2}$. Otherwise, take $\lambda_n = 2^n\beta_n$ if $\alpha = 0$ and $\lambda_n = 2^{-k}\alpha^{-1}$ if $\beta_n = 0$.
Now let $D := \{g_k = \sum_{|j|\leq k} \langle g_k, e_j\rangle e_j;\ k \in \mathbb{N}\}$ which is dense in $\ell^2(\mathbb{Z})$. We will construct f to be equal to $\sum_{k=1}^{\infty} \lambda_k^{-1} f_k$, such that

$$\lim_{k\to\infty} \|\lambda_k T^{n_k}(f_k) - g_k\|$$

where n_k is a rapidly increasing sequence to be specified.

Let $n_1 = 0$ and $f_1 = \lambda_1^{-1} g_1$. Assume that for $1 \leq j \leq k$, the number n_j and the vector f_j in span$\{e_i;\ |\phi^{n_j}(i)| \leq j\}$ have been chosen. Let $M = \|C_{u,\phi}\|$, we distingue to cases

1. if $M > 1$, put $\varepsilon = M^{-n_k}.2^{-k-1}$;
2. if $M \leq 1$, we take $N \in \mathbb{R}^+$ such that $M + N > 1$ and we put $\varepsilon = (M + N)^{-n_k}.2^{-k-1}$.

27 Sep 2023 13:45:33 PDT
221122-karim Version 2 - Submitted to Rocky Mountain J. Math.

We now choose n and u by applying the hypothesis to ε, and the vectors $g = g_{k+1}$, $h = \lambda_1^{-1} f_1 + ... + \lambda_k^{-1} f_k$.

Let the n and u so obtained be denoted by n_{k+1} and f_{k+1} respectively. We also ask that $n_k + \sum_{i=1}^{k+1} i < n_{k+1}$ to insure that the supports of the f_i's are pairwise disjoint. Then

$$\|\lambda_{k+1}^{-1} f_{k+1}\| < \varepsilon,$$

$$\|C_{u,\phi}^{n_{k+1}}(f_{k+1}) - g_{k+1}\| < \varepsilon,$$

and

$$\|\lambda_{k+1} C_{u,\phi}^{n_{k+1}}\big(\sum_{j=1}^{k} \lambda_j^{-1} f_j\big)\| < \varepsilon.$$

It follows that

$$\|\lambda_k C_{u,\phi}^{n_k}\big(\sum_{j=1}^{\infty} \lambda_j^{-1} f_j\big) - g_k\| \leq \|\lambda_k C_{u,\phi}^{n_k}\big(\sum_{j=1}^{k-1} \lambda_j^{-1} f_j\big)\| + \|\lambda_k C_{u,\phi}^{n_k}(\lambda_k^{-1} f_k) - g_k\| + \sum_{k+1}^{\infty} \|\lambda_k T^{n_k}(\lambda_j^{-1} f_j)\|$$

$$\leq 2^{-k+2}$$

□

Now we show the remaining implication in Theorem 4.2, first we have that if $f = \sum_{|j|\leq q} \langle f, e_j \rangle e_j$, then

$$\|C_{u,\phi}^{n} f\| \leq \max\big(\prod_{i=1}^{n} |u \circ \phi^{-i}(j)|;\ |j| \leq q\big).\|f\|. \tag{4.3}$$

But such a vector f is also in the domain of the operator $C_{u,\phi}^{-n}$, and it satisfies

$$\|C_{u,\phi}^{-n} f\| \leq \max\big((\prod_{i=0}^{n-1} |u \circ \phi^{i}(j)|)^{-1};\ |j| \leq q\big).\|f\|. \tag{4.4}$$

Let $\varepsilon > 0$ and $q \in \mathbb{N}$. Assume that there is n satisfies

$$\big(\prod_{i=1}^{n} |u \circ \phi^{-i}(j)|\big) \times \big(\prod_{i=0}^{n-1} |u \circ \phi^{i}(h)|\big)^{-1} < \varepsilon,$$

for all $|j| \leq q$. If vectors f, g are in $span\{e_j;\ |j| \leq q\}$, the inequalities 4.3 and 4.4 imply that

$$\|C_{u,\phi}^{-n} g\|.\|C_{u,\phi}^{n} h\| \leq \varepsilon.\|g\|.\|h\|.$$

By setting $C_{u,\phi}^{-n} g = u$, we see that the conditions in the hypothesis of Lemma 4.3 are satisfied. Thus $C_{u,\phi}$ is supercyclic and hence super-recurrent; this completes the proof of Theorem 4.2. □

Remark 4.4. In the proof of Theorem 4.2, we have proved that the operator $C_{u,\phi}$ is super-recurrent if and only if it is supercyclic.

Example 4.5. Let $\omega = (w_n)_{n\in\mathbb{Z}}$ be sequence of positive numbers, and let B_ω be the bilateral weighted backward shift given by $B_\omega e_n = w_n e_{n-1}$. Then the following assertions are equivalent:

(1) B_ω is super-recurrent;
(2) B_ω is supercyclic;

$$\liminf_{n\to\infty} \max\big(\frac{\prod_{j+1-n\leq k\leq j} w_k}{\prod_{h+1\leq k\leq h+n} w_k};\ |j|, |h| \leq q\big) = 0$$

for all $q \in \mathbb{N}$.

Indeed, if $\phi(n) = n + 1$ and $u(n) = w_{n+1}$, then $B_\omega = C_{u,\phi}$ is a weighted composition on $\ell^2(\mathbb{Z})$, then we can use Theorem 4.2.

As we can see, the Theorem 4.2 and Theorem [16, Theorem 3.1] are compatible.

References

[1] Abakumov E., Gordon J.: Common hypercyclic vectors for multiples of backward shift, J. Funct. Anal., 200(2), (2003), 494-504.
[2] Amouch, M., Benchiheb, O. On super-recurrent operators. arXiv:2102.12170v1[math.FA]24 Feb 2021.
[3] Azimi M.R.: Hypercyclic Weighted Composition Operators on $\ell^2(\mathbb{Z})$, Cankaya University Journal of Science and Engineering, Volume 14, No. 2 (2017) 125-133.
[4] Birkhoff G. D.: Surface transformations and their dynamical applications. ActaMath., 43 (1922) 1-119.
[5] Birkhoff G.D.: Demonstration d'un theoreme elementaire sur les fonctions entieres, C. R. Acad. Sci. Paris 189 (1929) 473-475.
[6] Bayart F. and Matheron E.: Dynamics of linear operators, Cambridge Tracts in Mathematics, vol. 179, Cambridge University Press, Cambridge, 2009.
[7] Costakis G. and Parissis I.: Szemeredi's Theorem, Frequent Hypercyclicity and multiple Recurrence, MATH. SCAND. Vol. 110 (2012), 251-272.
[8] Costakis G., Manoussos A., and Parissis I.: Recurrent Linear Operators. Compl. Anal. Oper. (2014) 1601-1643.
[9] Enflo P.: On the invariant subspace problem for Banach spaces, Acta Math. 158 (1987), no. 3-4, 213-313.
[10] Grosse-Erdmann K.-G.: Universal families and hypercyclic operators. Bull. Amer. Math. Soc. (N.S.) 36 (1999), 345-381.
[11] Grosse-Erdmann K.-G., Peris Manguillot A.: Linear chaos, Universitext. Springer, London, 2011.
[12] MacLane G.R.: Sequences of derivatives and normal families, J. Anal. Math. 2 (1952) 72-87.
[13] Madore B. F., Martínez-Avendaño R. A.: Subspace hypercyclicity, J. Math. Anal. Appl., 373, (2011), 502-511.
[14] Read C.J.: The invariant subspace problem for a class of Banach spaces. II: Hypercyclic operators, Israel J. Math. 63 (1988) 1-40.
[15] Salas H.N.: Hypercyclic weighted shifts, Trans. Amer. Math. Soc. 347 (1995), 993-1004.
[16] Salas, H. N. (1999). Supercyclicity and weighted shifts. Studia Math, 135(1), 55-74.
[17] Singh R. K., Manhas J. S.: Composition operators on function spaces, North-Holland Mathematics Studies, North Holland Publishing Co., Amsterdam, (1993).

Noureddine Karim and Mohamed Amouch, University Chouaib Doukkali. Department of Mathematics, Faculty of Science Eljadida, Morocco
Email address: noureddinekarim1894@gmail.com
Email address: amouch.m@ucd.ac.ma