On the energy equality for axisymmetric weak solutions to the 3D Navier-Stokes equations

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Abstract

In this paper, we are focus on the energy equality for axisymmetric weak solutions of the 3D Navier-Stokes equations. The classical Shinbrot condition says that if the weak solution u of the Navier-Stokes equations belongs $L^q(0,T;L^p(\mathbb{R}^3))$ with $\frac{1}{q}+\frac{1}{p}=\frac{1}{2}$ and $p\geq 4$, then u must satisfy the energy equality. A novel point is that, for the axisymmetric Navier-Stokes equations, the Shinbrot condition can be relaxed as follows: if $\tilde{u}=u^r e_r+u^z e_z\in L^q(0,T;L^p(\mathbb{R}^3))$ with $\frac{1}{q}+\frac{1}{p}=\frac{1}{2}$ and $p\geq 4$, then u must satisfy the energy equality. Furthermore, some other interesting results will be obtained.

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1 Introduction

We are concerned with the energy equality for weak solutions of the Navier-Stokes equations:

$$\begin{cases}\partial_t u-\Delta u+u\cdot\nabla u+\nabla p=0\,, & \text{in } \mathbb{R}^3\times(0,T)\,,\\ \nabla\cdot u=0\,, & \text{in } \mathbb{R}^3\times(0,T)\,,\\ u(x,0)=u_0(x)\,, & \text{in } \mathbb{R}^3\,,\end{cases}\tag{1.1}$$

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where u stands for the velocity field of the flow and p represents the pressure of the fluid, respectively.

Concerning the Navier-Stokes equations (1.1), it is well known that, for any finite energy initial data there exists at least one weak solution satisfying the energy inequality. Weak solutions obeying the energy inequality are called Leray–Hopf solutions, see J. Leray and E. Hopf [13, 9]. However, the regularity problem of weak solutions is an outstanding open problem in mathematical fluid mechanics. This problem is so difficult that one investigates the solution with some special structure. A interesting case of global well-posedness to (1.1) is for data which is axisymmetric and without swirl (i.e., the case when u^θ in (1.7)). In this case, M.R. Ukhovskii and V.I. Yudovich [20], and independently O.A. Ladyzhenskaya [10] proved the existence of solutions, uniqueness and regularity. If the swirl is not zero, in general, the global well-posedness of (1.1) are still open. Refer to [4, 5, 6, 7, 11, 12, 22] for this subject.

We know that LerayHopf weak solution enjoys the energy inequality:

$$\frac{1}{2}\int_{\mathbb{R}^3} |u(s,t)|^2 dx + \int_0^t \int_{\mathbb{R}^3} |\nabla u(x,\tau)|^2 dx d\tau \leq \frac{1}{2}\int_{\mathbb{R}^3} |u_0(x)|^2 dx\,. \tag{1.2}$$

A important question of whether such solutions satisfy the energy equality is open, and only conditional criteria are available. In [18] M. Shinbrot shows that if a weak solution u to the Navier-Stokes equations (1.1) satisfies

$$u \in L^q(0,T;L^p(\mathbb{R}^3))\,, \tag{1.3}$$

where

$$\frac{2}{q} + \frac{2}{p} = 1, \quad p \geq 4\,,$$

then it satisfies the energy equality. This result is a generalization of previous results due to G. Prodi [17] and J.L. Lions [16], where these authors proved the above result for $p = q = 4$.

In [14, 15, 19], the authors established energy equality under assumptions on the size and/or structure of the singularity set in addition to the integrability of the solution, and proved that any solution to the 3-dimensional NavierStokes Equations which is Type-I in time must satisfy the energy equality at the first blowup time.

Recently, H. Beirao da Veiga and the author of this paper [2] generalized the above criterion to the case of $p < 4$:

$$u \in L^q(0,T;L^p(\mathbb{R}^3))\,, \tag{1.4}$$

where

$$\frac{1}{q} + \frac{3}{p} = 1, \quad 3 < p < 4\,.$$

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Another line is to establish some criteria via the gradient of the velocity. In [3], L.C. Berselli and E. Chiodaroli established the following criterion:

$$\nabla u \in L^q(0,T;L^p(\mathbb{R}^3)) \quad \text{with} \quad \begin{cases} \frac{1}{q}+\frac{3}{p}=2\,, & \frac{3}{2}<p<\frac{9}{5}\,,\\ \frac{1}{q}+\frac{6}{5p}=1\,, & \frac{9}{5}\le p\le 3\,,\\ \frac{1}{q}+\frac{2}{p+2}=1\,, & p>3\,. \end{cases} \tag{1.5}$$

Later on, H. Beirao da Veiga and the author of this paper [1] improved the above results for $p>3$ to

$$\nabla u \in L^q(0,T;L^p(\mathbb{R}^3)) \quad \text{with} \quad \frac{1}{q}+\frac{6}{5p}=1\,.$$

Recently, Y. Wang, X. Mei and Y. Huang[21] established an energy conservation criterion via a combination of the velocity and the gradient of velocity. In the following, we will extend their results to the axisymmetric Navier-Stokes equations, as a corollary, we obtain some interesting results.

In the present paper, we consider the energy equality for axisymmetric weak solutions of the Navier-Stokes equations. Recall the cylindrical coordinates given by

$$\begin{cases} x_1=r\cos\theta\,,\\ x_2=r\sin\theta\,,\\ x_3=z\,. \end{cases} \tag{1.6}$$

By an axisymmetric solutions of Navier-Stokes equations, we mean a solution of (1.1) with the form:

$$u(t,x)=u^r(t,r,z)e_r+u^\theta(t,r,z)e_\theta+u^z(t,r,z)e_z\,, \tag{1.7}$$

where

$$e_r=(\cos\theta\,,\sin\theta\,,0)\,,\quad e_r=(-\sin\theta\,,\cos\theta\,,0)\,,\quad e_z=(0\,,0\,,1)\,.$$

For the axisymmetric solutions, we can rewrite (1.1) as follows.

$$\begin{cases} \frac{\tilde{D}}{Dt}u^r-\left(\partial_r^2+\partial_z^2+\frac{\partial_r}{r}-\frac{1}{r^2}\right)u^r-\frac{(u^\theta)^2}{r}+\partial_r p=0\,, & \text{in } \mathbb{R}^3\times(0,T)\,,\\ \frac{\tilde{D}}{Dt}u^\theta-\left(\partial_r^2+\partial_z^2+\frac{\partial_r}{r}-\frac{1}{r^2}\right)u^\theta+\frac{u^\theta u^r}{r}=0\,, & \text{in } \mathbb{R}^3\times(0,T)\,,\\ \frac{\tilde{D}}{Dt}u^z-\left(\partial_r^2+\partial_z^2+\frac{\partial_r}{r}\right)u+\partial_z p=0\,, & \text{in } \mathbb{R}^3\times(0,T)\,,\\ \partial_r u^r+\frac{1}{r}u^r+\partial_z u^z=0\,, & \text{in } \mathbb{R}^3\times(0,T)\,,\\ (u^r,u^\theta,u^z)|_{t=0}=(u_0^r,u_0^\theta,u_0^z)\,, & \text{in } \mathbb{R}^3\,, \end{cases} \tag{1.8}$$

where

$$\frac{\tilde{D}}{Dt}=\partial_t+u^r\partial_r+u^z\partial_z\,.$$

Compared with the classical Navier-Stokes equations (1.8), it is natural to conjecture some better criteria for the axisymmetric Navier-Stokes equations. A very interesting finding that one only needs to impose the condition on the components $\tilde{u} = u^r e_r + u^z e_z$ ($ru_0^\theta \in L^\infty(\mathbb{R}^3)$ is needed), see below. As far as we know, this is a first result on the energy conservation law for the axisymmetric Navier-Stokes equations.

To state our results, we first recall the definition of the weak solution.

Definition 1.1. Let $u_0 \in L^2(\mathbb{R}^3)$ with $\nabla \cdot u = 0$. The vector field u is called a Leray-Hopf weak solution of (1.1) in $(0, T)$ if u satisfies

(1) $u \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3))$;

(2) (u, p) solves (1.1) in the sense of distributions.

(3) u satisfies the energy inequality for $t \in [0, T)$,

$$\frac{1}{2}\int_{\mathbb{R}^3} |u(s,t)|^2 dx + \int_0^t \int_{\mathbb{R}^3} |\nabla u(x,\tau)|^2 dx d\tau \leq \frac{1}{2}\int_{\mathbb{R}^3} |u_0(x)|^2 dx\,. \tag{1.9}$$

We shall establish the following theorem.

Theorem 1.2. *Let u be a axisymmetric weak solution to the 3D Navier-Stokes equations* (1.8)*, and $ru_0^\theta \in L^\infty(\mathbb{R}^3)$. Then the energy equality holds if one of the following conditions is satisfied for $k, l \in (1, \infty)$:*

(1) $\tilde{u} \in L^{\frac{2k}{k-1}}(0, T; L^{\frac{2l}{l-1}}(\mathbb{R}^3))\,,\ \omega^\theta \in L^k(0, T; L^l(\mathbb{R}^3))\,,\ \omega^z \in L^{\frac{4k}{k+2}}(0, T; L^{\frac{4l}{l+2}}(\mathbb{R}^3))$*;*

(2) $\tilde{u} \in L^{\frac{2k}{k-1}}(0, T; L^{\frac{2l}{l-1}}(\mathbb{R}^3)) \cap L^{\frac{4k}{k+2}}(0, T; L^{\frac{4l}{l+2}}(\mathbb{R}^3))\,,\ \omega^\theta\,, \omega^z \in L^k(0, T; L^l(\mathbb{R}^3))$.

As a direct consequence of the above theorem, we have the following results.

Corollary 1.3. *Let $\delta > 0$ be given and $ru_0^\theta \in L^\infty(\mathbb{R}^3)$, then the energy equality is valid if one of the following conditions is satisfied:*

(1) $\tilde{u} 1_{r\leq\delta} \in L^q(0, T; L^p(\mathbb{R}^3))$ *with* $\frac{1}{q} + \frac{1}{p} = \frac{1}{2}$ *and* $p \geq 4$*;*

(2) $\tilde{u} 1_{r\leq\delta} \in L^q(0, T; L^p(\mathbb{R}^3))$ *with* $\frac{1}{q} + \frac{3}{p} = 1$ *and* $3 < p \leq 4$*;*

(3) $\omega^\theta \in L^q(0, T; L^p(\mathbb{R}^3))$, $\omega^z \in L^{\frac{4q}{q+2}}(0, T; L^{\frac{4p}{p+2}}(\mathbb{R}^3))$ *with* $\frac{1}{q} + \frac{6}{5p} = 1$ *and* $p \geq \frac{9}{5}$*;*

(4) $\omega^\theta \in L^q(0, T; L^p(\mathbb{R}^3))$, $\omega^z \in L^{\frac{4q}{q+2}}(0, T; L^{\frac{4p}{p+2}}(\mathbb{R}^3))$ *with* $\frac{1}{q} + \frac{3}{p} = 2$ *and* $\frac{3}{2} < p \leq \frac{9}{5}$*;*

(5) $\omega^\theta\,, \omega^z \in L^q(0, T; L^p(\mathbb{R}^3))$ *with* $\frac{1}{q} + \frac{6}{5p} = 1$ *and* $2 \leq p \leq 4$.

2 Some important observations

This section will give the explanations why the conditional criteria can only be imposed on the components u^r, u^z or ω^θ, ω^z. This is due to the following observations:

- u^θ enjoys a better proposition (see Lemma 3.3):

$$u^\theta \in L^4(0,T;L^4(\mathbb{R}^3))\,, \tag{2.1}$$

  which implies u^θ is a good component due to the Shinbrot condition.

- The term $u^r\partial_r u^\theta u^\theta$ belongs to $L^1(0,T;L^1(\mathbb{R}^3))$, which means the ω^r is a good component. Indeed, we have

$$\begin{aligned}
&\left|\int_0^t\int_{\mathbb{R}^3} u^r\partial_r u^\theta u^\theta dxds\right| \\
&=\left|2\pi\int_0^t\int_{-\infty}^{\infty}\int_0^{\infty} u^r\partial_r u^\theta u^\theta r dr dz ds\right| \\
&\le \|\frac{u^r}{r}\|_{L^2(0,T;L^2(\mathbb{R}^3))}\|\partial_r u^\theta\|_{L^2(0,T;L^2(\mathbb{R}^3))}\|ru^\theta\|_{L^\infty(\mathbb{R}\times(0,T))} \\
&\le \|\nabla u\|_{L^2(0,T;L^2(\mathbb{R}^3))}\|\nabla u^\theta\|_{L^2(0,T;L^2(\mathbb{R}^3))}\|ru^\theta\|_{L^\infty(\mathbb{R}\times(0,T))} < \infty\,.
\end{aligned} \tag{2.2}$$

- $\nabla\tilde{u}$ can be controlled by ω^θ (see Lemma 3.4):

$$\|\nabla\tilde{u}\|_{L^p(\mathbb{R}^3)} \le C\|\omega^\theta\|_{L^p(\mathbb{R}^3)}\,. \tag{2.3}$$

  Hence, we only need impose the condition on vorticity. We remark that (2.3) can be obtained by the following equations:

$$\begin{cases} \operatorname{div}\tilde{u} = 0\,, \\ \operatorname{curl}\tilde{u} = \omega^\theta e_\theta\,, \end{cases} \tag{2.4}$$

  that is

$$-\Delta\tilde{u} = \operatorname{curl}(\omega^\theta e_\theta)\,. \tag{2.5}$$

3 Some useful lemmas

Lemma 3.1 ([8], Lemma 2.2)**.** *Let u be a weak solution to (1.1) in $\mathbb{R}^3\times(0,T)$. Then u can be redefined on a set of zero Lebesgue measure in such a way that $u(t)\in L^2(\mathbb{R}^2)$ for all $t\in[0,T)$ and satisfies the identity*

$$\int_s^t\int_{\mathbb{R}^3}(u\cdot\phi_\tau - \nabla u\cdot\nabla\phi - u\cdot\nabla u\cdot\phi)\,dxd\tau = \int_{\mathbb{R}^3} u(t)\cdot\phi(t)dx - \int_{\mathbb{R}^3} u(s)\cdot\phi(s)dx \tag{3.1}$$

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for all $s \in [0,t]$, $t < T$ *and all* $\phi \in C_0^\infty(\mathbb{R}^3 \times [0,T))$ *with* $\nabla \cdot \phi = 0$.

Lemma 3.2 ([7], Lemma 2.1)**.** *Let* u *be an axisymmetric vector field. Then the following equalities hold:*

$$|\nabla \tilde{u}|^2 = \left|\frac{u^r}{r}\right|^2 + |\tilde{\nabla} u^r|^2 + |\tilde{\nabla} u^z|^2\,, \tag{3.2}$$

$$|\nabla (u^\theta e_\theta)|^2 = \left|\frac{u^\theta}{r}\right|^2 + |\tilde{\nabla} u^\theta|^2\,. \tag{3.3}$$

Lemma 3.3. *Suppose that* u *is a axisymmetric weak solution of the Navier-Stokes equations, if* $ru_0^\theta \in L^\infty(\mathbb{R}^3)$*, then* $ru^\theta \in L^\infty(\mathbb{R}^3 \times (0,T))$*. Moreover,* $u^\theta \in L^4(0,T;L^4(\mathbb{R}^3))$*.*

Proof. $ru^\theta \in L^\infty(\mathbb{R}^3 \times (0,T))$ follows from [4, Proposition 1]. From this estimate and Lemma 3.2, since $u \in L^2(0,T;H^1(\mathbb{R}^3))$, we derive that

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}^3} \left(u^\theta\right)^4 dxdt &= 2\pi \int_0^T \int_{-\infty}^\infty \int_0^\infty \left(u^\theta\right)^4 r\, drdzdt \\
&\le \|ru^\theta\|^2_{L^\infty(\mathbb{R}^3\times(0,T))} \int_0^T \int_{-\infty}^\infty \int_0^\infty \left(\frac{u^\theta}{r}\right)^2 r\, drdzdt \\
&\le \|ru^\theta\|^2_{L^\infty(\mathbb{R}^3\times(0,T))} \|\nabla u\|^2_{L^2(0,T;L^2(\mathbb{R}^3))} < \infty\,.
\end{aligned} \tag{3.4}$$

□

Lemma 3.4 ([7], Lemma 2.3)**.** *Let* $1 < p < \infty$*. Then we have*

$$\|\nabla \tilde{u}\|_{L^p(\mathbb{R}^3)} \le C\|\omega^\theta\|_{L^p(\mathbb{R}^3)}\,. \tag{3.5}$$

Lemma 3.5 ([22], Lemma 2.1)**.** *Suppose that* u *is a axisymmetric weak solution of the Navier-Stokes equations. Let* $\delta > 0$*, then*

$$\|u 1_{r\ge\delta}\|^4_{L^4(0,T;L^4(\mathbb{R}^3))} \le \frac{C}{\delta}\|u_0\|^4_{L^2(\mathbb{R}^3)}\,. \tag{3.6}$$

4 Proof of Theorem 1.2

Proof. It follows from [1, Lemma 5.1] that there exists a sequence $\{u_m\}$ such that

$$\lim_{m\infty} \|\tilde{u}_m - \tilde{u}\|_{L^{\frac{2k}{k-1}}(0,T;L^{\frac{2l}{l-1}}(\mathbb{R}^3))} \to 0\,, \quad \lim_{m\infty} \|\nabla\tilde{u}_m - \nabla\tilde{u}\|_{L^k(0,T;L^l(\mathbb{R}^3))} \to 0 \tag{4.1}$$

$$\lim_{m\infty} \|u_m^\theta - u^\theta\|_{L^4(0,T;L^4(\mathbb{R}^3))} \to 0\,, \quad \lim_{m\infty} \|\nabla u_m - \nabla u\|_{L^2(0,T;L^2(\mathbb{R}^3))} \to 0 \tag{4.2}$$

Taking $\phi = u_m^\epsilon = \int_0^t j_\epsilon(s-\tau) u_m \, d\tau$ in (3.1), where j_ϵ is an even, non-negative, infinitely differentiable function with support in $(-\epsilon, \epsilon)$, and $\int_{-\infty}^{+\infty} j_\epsilon(s) ds = 1$. We have

$$\int_s^t \int_{\mathbb{R}^3} \left(u \cdot \partial_s u_m^\epsilon - \nabla u \cdot \nabla u_m^\epsilon - u \cdot \nabla u \cdot u_m^\epsilon \right) dx d\tau = \int_{\mathbb{R}^3} u(t) \cdot u_m^\epsilon(t) dx - \int_{\mathbb{R}^3} u_0 \cdot u_m^\epsilon(0) dx$$

Following [3, 8], we have

$$\begin{aligned}
&\lim_{\epsilon \to 0} \lim_{m \to \infty} \int_{\mathbb{R}^3} u(t) \cdot u_m^\epsilon(t) dx = \frac{1}{2} \|u(t)\|_{L^2(\mathbb{R}^3)}^2 , \\
&\lim_{\epsilon \to 0} \lim_{m \to \infty} \int_{\mathbb{R}^3} u_0 \cdot u_m^\epsilon(0) dx = \frac{1}{2} \|u_0\|_{L^2(\mathbb{R}^3)}^2 , \\
&\lim_{\epsilon \to 0} \lim_{m \to \infty} \int_0^t \int_{\mathbb{R}^3} u \cdot \partial_s u_m^\epsilon dx ds = 0 , \\
&\lim_{\epsilon \to 0} \lim_{m \to \infty} \int_0^t \int_{\mathbb{R}^3} \nabla u(s) \cdot \nabla u_m^\epsilon(s) dx ds = \frac{1}{2} \|\nabla u\|_{L^2(0,T;L^2(\mathbb{R}^3))}^2
\end{aligned} \tag{4.3}$$

For the nonlinear term $\int_0^t \int_{\mathbb{R}^3} u \cdot \nabla u \cdot u_m^\epsilon dx ds$, we can rewrite it as follows:

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^3} u \cdot \nabla u \cdot u_m^\epsilon dx ds \\
=& \int_0^t \int_{\mathbb{R}^3} u \cdot \nabla u \cdot (u_m^\epsilon - u^\epsilon) dx ds + \int_0^t \int_{\mathbb{R}^3} u \cdot \nabla u \cdot (u^\epsilon - u) dx ds \\
&+ \int_0^t \int_{\mathbb{R}^3} u \cdot \nabla u \cdot u dx ds + \int_0^t \int_{\mathbb{R}^3} u \cdot \nabla u_m \cdot u_m dx ds \\
=& \int_0^t \int_{\mathbb{R}^3} u \cdot \nabla u \cdot (u_m^\epsilon - u^\epsilon) dx ds + \int_0^t \int_{\mathbb{R}^3} u \cdot \nabla u \cdot (u^\epsilon - u) dx ds \\
&+ \int_0^t \int_{\mathbb{R}^3} u \cdot \nabla (u - u_m) \cdot u dx ds + \int_0^t \int_{\mathbb{R}^3} u \cdot \nabla u_m \cdot (u - u_m) dx ds .
\end{aligned} \tag{4.4}$$

where we have used the relation

$$\int_0^t \int_{\mathbb{R}^3} u \cdot \nabla u_m \cdot u_m dx ds = 0 ,$$

which is due to the integration by parts and divergence-free condition. To estimate the term $\int_0^t \int_{\mathbb{R}^3} u \cdot \nabla u \cdot (u_m^\epsilon - u^\epsilon) dx ds$, we use the following equation:

$$\begin{aligned}
u \cdot \nabla u \cdot (u_m^\epsilon - u^\epsilon) =& \tilde{u} \cdot \tilde{\nabla} u \cdot (u_m^\epsilon - u^\epsilon) \\
=& \tilde{u} \cdot \tilde{\nabla} \tilde{u} \cdot (\tilde{u}_m^\epsilon - \tilde{u}^\epsilon) + (\tilde{u} \cdot \tilde{\nabla} u^\theta)(\tilde{u}_m^\epsilon - \tilde{u}^\epsilon)^\theta ,
\end{aligned} \tag{4.5}$$

where we used the fact $u \cdot \nabla = \tilde{u} \cdot \tilde{\nabla}$ since u is independent of θ. Thus one can rewrite it as follows:

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}^3} u \cdot \nabla u \cdot (u_m^\epsilon - u^\epsilon) dxds \\
&= \int_0^t \int_{\mathbb{R}^3} \tilde{u} \cdot \tilde{\nabla} \tilde{u} \cdot (\tilde{u}_m^\epsilon - \tilde{u}^\epsilon) dxds + \int_0^t \int_{\mathbb{R}^3} (\tilde{u} \cdot \tilde{\nabla} u^\theta)(\tilde{u}_m^\epsilon - \tilde{u}^\epsilon)^\theta dxds\,.
\end{aligned}
\tag{4.6}
$$

Using the integration by parts and divergence-free condition, one derives that

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}^3} u \cdot \nabla u \cdot (u_m^\epsilon - u^\epsilon) dxds \\
&= \int_0^t \int_{\mathbb{R}^3} \tilde{u} \cdot \tilde{\nabla} \tilde{u} \cdot (\tilde{u}_m^\epsilon - \tilde{u}^\epsilon) dxds - \int_0^t \int_{\mathbb{R}^3} \tilde{u} \cdot \tilde{\nabla} (\tilde{u}_m^\epsilon - \tilde{u}^\epsilon)^\theta u^\theta dxds \\
&= \int_0^t \int_{\mathbb{R}^3} \tilde{u} \cdot \tilde{\nabla} \tilde{u} \cdot (\tilde{u}_m^\epsilon - \tilde{u}^\epsilon) dxds - \int_0^t \int_{\mathbb{R}^3} u^r u^\theta \partial_r (u_m^\epsilon - u^\epsilon)^\theta dxds \\
&\quad - \int_0^t \int_{\mathbb{R}^3} u^z u^\theta \partial_z (u_m^\epsilon - u^\epsilon)^\theta dxds\,.
\end{aligned}
\tag{4.7}
$$

By using the Hölder inequality and Lemma 3.2, we have

$$
\begin{aligned}
&\left| \int_0^t \int_{\mathbb{R}^3} \tilde{u} \cdot \tilde{\nabla} \tilde{u} \cdot (\tilde{u}_m^\epsilon - \tilde{u}^\epsilon) dxds \right| \\
&\leq \|\tilde{u}\|_{L^{\frac{2k}{k-1}}(0,T;L^{\frac{2l}{l-1}}(\mathbb{R}^3))} \|\tilde{\nabla} \tilde{u}\|_{L^k(0,T;L^l(\mathbb{R}^3))} \|\tilde{u}_m^\epsilon - \tilde{u}^\epsilon\|_{L^{\frac{2k}{k-1}}(0,T;L^{\frac{2l}{l-1}}(\mathbb{R}^3))}\,,
\end{aligned}
\tag{4.8}
$$

and

$$
\begin{aligned}
&\left| \int_0^t \int_{\mathbb{R}^3} u^r u^\theta \partial_r (u_m^\epsilon - u^\epsilon)^\theta dxds \right| \\
&= \left| 2\pi \int_0^t \int_{-\infty}^\infty \int_0^\infty u^r u^\theta \partial_r (u_m^\epsilon - u^\epsilon)^\theta r dr dz ds \right| \\
&\leq \|\frac{u^r}{r}\|_{L^2(0,T;L^2(\mathbb{R}^3))} \|\partial_r (u_m^\epsilon - u^\epsilon)^\theta\|_{L^2(0,T;L^2(\mathbb{R}^3))} \|r u^\theta\|_{L^\infty(\mathbb{R}\times(0,T))} \\
&\leq \|\nabla u\|_{L^2(0,T;L^2(\mathbb{R}^3))} \|\nabla (u_m^\epsilon - u^\epsilon)\|_{L^2(0,T;L^2(\mathbb{R}^3))} \|r u^\theta\|_{L^\infty(\mathbb{R}\times(0,T))}\,,
\end{aligned}
\tag{4.9}
$$

and

$$
\begin{aligned}
&\left| \int_0^t \int_{\mathbb{R}^3} u^z u^\theta \partial_z (u_m^\epsilon - u^\epsilon)^\theta dxds \right| \\
&\leq \|u^z\|_{L^{\frac{2k}{k-1}}(0,T;L^{\frac{2l}{l-1}}(\mathbb{R}^3))} \|\partial_z (u_m^\epsilon - u^\epsilon)^\theta\|_{L^{\frac{4k}{k+2}}(0,T;L^{\frac{4l}{l+2}}(\mathbb{R}^3))} \|u^\theta\|_{L^4(0,T;L^4(\mathbb{R}^3))}\,,
\end{aligned}
\tag{4.10}
$$

or

$$
\begin{aligned}
&\left| \int_0^t \int_{\mathbb{R}^3} u^z \partial_z u^\theta (u_m^\epsilon - u^\epsilon)^\theta dxds \right| \\
&\leq \|u^z\|_{L^{\frac{4k}{k+2}}(0,T;L^{\frac{4l}{l+2}}(\mathbb{R}^3))} \|\partial_z (u_m^\epsilon - u^\epsilon)^\theta\|_{L^k(0,T;L^l(\mathbb{R}^3))} \|u^\theta\|_{L^4(0,T;L^4(\mathbb{R}^3))}\,.
\end{aligned}
\tag{4.11}
$$

According to the assumptions of Theorem 1.2, we can pass the limit to obtain

$$\lim_{\epsilon\to 0}\lim_{m\to\infty}\int_0^t\int_{\mathbb{R}^3} u\cdot\nabla u\cdot(u_m^\epsilon-u^\epsilon)dxds=0\,.$$

Similarly, we can obtain that

$$\lim_{\epsilon\to 0}\lim_{m\to\infty}\int_0^t\int_{\mathbb{R}^3} u\cdot\nabla u\cdot(u^\epsilon-u)dxds=0\,,$$

and

$$\lim_{\epsilon\to 0}\lim_{m\to\infty}\int_0^t\int_{\mathbb{R}^3} u\cdot\nabla(u-u_m)\cdot udxds=0\,,$$

and

$$\lim_{\epsilon\to 0}\lim_{m\to\infty}\int_0^t\int_{\mathbb{R}^3} u\cdot\nabla u_m\cdot(u-u_m)dxds=0\,.$$

Thus, we have

$$\lim_{\epsilon\to 0}\lim_{m\to\infty}\int_0^t\int_{\mathbb{R}^3} u\cdot\nabla u\cdot u_m^\epsilon dxds=0\,.$$

Therefore, we have

$$\frac{1}{2}\int_{\mathbb{R}^3}|u(s,t)|^2dx+\int_0^t\int_{\mathbb{R}^3}|\nabla u(x,\tau)|^2dxd\tau=\frac{1}{2}\int_{\mathbb{R}^3}|u_0(x)|^2dx\,. \tag{4.12}$$

□

5 Proof of Corollary 1.3

Proof. (1) is due to the Shinbrot condition, Lemmas 3.3 and 3.5.

(2) is due to [2, Theorem 1.1], Lemmas 3.3 and 3.5.

To prove (3), it follows from Theorem 1.2 that it is enough to prove $\tilde{u}\in L^{\frac{2q}{q-1}}(0,T;L^{\frac{2p}{p-1}}(\mathbb{R}^3))$. By means of the GagliardoNirenberg inequality, we obtain

$$\|\tilde{u}\|_{L^{\frac{2p}{p-1}}(\mathbb{R}^3)}\le C\|\tilde{u}\|_{L^2(\mathbb{R}^3)}^{\frac{5p-9}{5p-6}}\|\nabla\tilde{u}\|_{L^p(\mathbb{R}^3)}^{\frac{3}{5p-6}}\le C\|\tilde{u}\|_{L^2(\mathbb{R}^3)}^{\frac{5p-9}{5p-6}}\|\omega^\theta\|_{L^p(\mathbb{R}^3)}^{\frac{3}{5p-6}}\,.$$

Hence, we have

$$\|\tilde{u}\|_{L^{\frac{2q}{q-1}}(0,T;L^{\frac{2p}{p-1}}(\mathbb{R}^3))}\le C\|\tilde{u}\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}^{\frac{5p-9}{5p-6}}\|\omega^\theta\|_{L^q(0,T;L^p(\mathbb{R}^3))}^{\frac{3}{5p-6}}\,.$$

For (4), similarly, we have

$$\begin{aligned}\|\tilde{u}\|_{L^{\frac{2p}{p-1}}(\mathbb{R}^3)}\le&C\|\tilde{u}\|_{L^6(\mathbb{R}^3)}^{\frac{9-5p}{3(2-p)}}\|\nabla\tilde{u}\|_{L^p(\mathbb{R}^3)}^{\frac{2p-3}{3(2-p)}}\\ \le&C\|\nabla\tilde{u}\|_{L^2(\mathbb{R}^3)}^{\frac{9-5p}{3(2-p)}}\|\nabla\tilde{u}\|_{L^p(\mathbb{R}^3)}^{\frac{2p-3}{3(2-p)}}\le C\|\omega^\theta\|_{L^2(\mathbb{R}^3)}^{\frac{9-5p}{3(2-p)}}\|\omega^\theta\|_{L^p(\mathbb{R}^3)}^{\frac{2p-3}{3(2-p)}}\,,\end{aligned}$$

which implies

$$\|\tilde{u}\|_{L^{\frac{2q}{q-1}}(0,T;L^{\frac{2p}{p-1}}(\mathbb{R}^3))} \leq C\|\omega^\theta\|^{\frac{9-5p}{3(2-p)}}_{L^2(0,T;L^2(\mathbb{R}^3))}\|\omega^\theta\|^{\frac{2p-3}{3(2-p)}}_{L^q(0,T;L^p(\mathbb{R}^3))}.$$

Thus, (4) follows from Theorem 1.2.

To prove (5), it is enough to check if $\tilde{u} \in L^{\frac{4q}{q+2}}(0,T;L^{\frac{4p}{p+2}}(\mathbb{R}^3))$ since we have $\tilde{u} \in L^{\frac{2q}{q-1}}(0,T;L^{\frac{2p}{p-1}}(\mathbb{R}^3))$ following the proof of (3). When $2 \leq p \leq 4$, it is easy to obtain that

$$\|\tilde{u}\|_{L^{\frac{4p}{p+2}}(\mathbb{R}^3)} \leq C\|\tilde{u}\|^{2-\frac{p}{2}}_{L^2(\mathbb{R}^3)}\|\tilde{u}\|^{\frac{p}{2}-1}_{L^{\frac{2p}{p-1}}(\mathbb{R}^3)},$$

which derives that

$$\int_0^T \|\tilde{u}\|^{\frac{4q}{q+2}}_{L^{\frac{4p}{p+2}}(\mathbb{R}^3)} dt \leq C\|\tilde{u}\|^{\left(2-\frac{p}{2}\right)\frac{4q}{q+2}}_{L^\infty(0,T;L^2(\mathbb{R}^3))}\int_0^T \|\tilde{u}\|^{\left(\frac{p}{2}-1\right)\frac{4q}{q+2}}_{L^{\frac{2p}{p-1}}(\mathbb{R}^3)} dt.$$

From the assumptions in (5), one can easily check that

$$\left(\frac{p}{2}-1\right)\frac{4q}{q+2} \leq \frac{2q}{q-1}.$$

Therefore, $\tilde{u} \in L^{\frac{4q}{q+2}}(0,T;L^{\frac{4p}{p+2}}(\mathbb{R}^3))$, which implies (5). □

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