On commuting d-tuples of m-expansive operators

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Abstract

Given a commuting d-tuple $\mathbb{A}$ in $B(\mathcal{H})^d$, if $\mathbb{A}$ is $2m$-expansive for some positive integer m, then $\mathbb{A}$ is $(2m-1)$-expansive; $\mathbb{A}$ is $2m$-expansive and n-expansive for some integer $n > 2m$ implies $\mathbb{A}$ is t-expansive for all $2m-1 \leq t \leq n$. Commuting products of commuting d-tuples of expansive operators are considered.

1. Introduction

Let $B(\mathcal{H})$ denote the algebra of operators, i.e. bounded linear transformations, on an infinite dimensional complex Hilbert space $\mathcal{H}$ (with inner product $\langle .,. \rangle$) into itself, and let $B(\mathcal{H})^d$ denote the product of d copies of $B(\mathcal{H})$ for some integer $d \geq 1$. For operators $A, B \in B(\mathcal{H})$, let L_A and $R_B \in B(B(\mathcal{H}))$ denote, respectively, the operators $L_A(X) = AX$ and $R_B(X) = XB$ of left multiplication by A and right multiplication by B. An operator $A \in B(\mathcal{H})$ is m-expansive for some positive integer m, A is m-expansive, if

$$\begin{aligned}
\triangle^m_{A^*,A}(I) &= (I - L_{A^*}R_A)^m(I) \\
&= \left(\sum_{j=0}^{m}(-1)^j \begin{pmatrix} m \\ j \end{pmatrix} L^j_{A^*}R^j_A\right)(I) \\
&= \sum_{j=0}^{m}(-1)^j \begin{pmatrix} m \\ j \end{pmatrix} A^{*j}A^j \\
&\leq 0
\end{aligned}$$

[8, 9, 4, 10]. Considered as a generalisation of m-isometric operators A

$$\triangle^m_{A^*,A}(I) = \sum_{j=0}^{m}(-1)^j \begin{pmatrix} m \\ j \end{pmatrix} A^{*j}A^j = 0$$

[1, 5], m-expansive operators share some (but by no means all) of the structural properties of m-isometric operators [4]. Following [6], see also [2, 10], a generalisation of m-expansive operators to commuting d-tuples $\mathbb{A} \in B(\mathcal{H})^d$, i.e. d-tuples $\mathbb{A} = (A_1, \cdots, A_d)$ such that $[A_i, A_j] = A_iA_j - A_jA_i = 0$ for all $1 \leq i, j \leq d$, is obtained as follows: a

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commuting d-tuple $\mathbb{A} = (A_1, \cdots, A_d)$ is m-expansive if

$$\begin{aligned} \triangle^m_{\mathbb{A}^*,\mathbb{A}}(I) &= (I - \mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})^m(I) \\ &= \left(\sum_{j=0}^m (-1)^j \begin{pmatrix} m \\ j \end{pmatrix} (\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})^j\right)(I) \\ &\leq 0, \end{aligned}$$

where

$$(\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})^j(X) = \left(\sum_{|\alpha|=j} \frac{j!}{\alpha!} \mathbb{L}^{\alpha}_{\mathbb{A}^*} \mathbb{R}^{\alpha}_{\mathbb{A}}\right)(X) = \left(\sum_{i=1}^d L_{A_i^*} R_{A_i}\right)^j (X),$$

for all integers $j \geq 0$ and operators $X \in B(\mathcal{X})$, and

$$\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d), \ \alpha_i \geq 0 \text{ for all } 1 \leq i \leq d, \ |\alpha| = \sum_{i=1}^d \alpha_i, \text{ and } \alpha! = \Pi_{i=1}^d \alpha_i!.$$

Commuting d-tuples $\mathbb{A}$ fail to satisfy many an m-isometric property satisfied by single linear operators [6]. Furthermore, even if a commuting m-tuple satisfies an m-isometric property, the property may fail the m-expansive test. For example, $\mathbb{A} \in m$-isometric implies $\mathbb{A} \in t$-isometric for all integers $t \geq m$. This fails for m-expansive $\mathbb{A}$:

$$\begin{aligned} \triangle^{m+1}_{\mathbb{A}^*,\mathbb{A}}(I) &= \triangle^m_{\mathbb{A}^*,\mathbb{A}}(I) - (\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}}) \triangle^m_{\mathbb{A}^*,\mathbb{A}}(I) \\ &= \triangle^m_{\mathbb{A}^*,\mathbb{A}}(I) - \sum_{i=1}^d \left(A_i^* (\triangle^m_{\mathbb{A}^8,\mathbb{A}}(I)) A_i\right), \end{aligned}$$

and the hypothesis $\triangle^m_{\mathbb{A}^*,\mathbb{A}}(I) \leq 0$ fails in general to guarantee $\triangle^{m+1}_{\mathbb{A}^*,\mathbb{A}}(I) \leq 0$, even for the case in which $d = 1$ and $A \in B(\mathcal{H})$. For example, if $\mathcal{H} = \ell^2(\mathbb{N}_0)$ with an orthonormal basis $\{e_n\}_{n=0}^\infty$ and A_α is the weighted shift $A_\alpha e_n = \alpha e_{n+1}$ for some real $\alpha > 1$, then $\triangle^m_{A_\alpha^*, A_\alpha}(I) = (1 - \alpha^2)^m$ and A_α is m-expansive for $m = 2n + 1$, but not m-expansive for $m = 2n$, for all positive integers n.

Recall that $\mathbb{A} \in B(\mathcal{H})^d$ is m-hyperexpansive if it is t-expansive for all $1 \leq t \leq m$ [7, 9]. It is well known that 2-expansive operators are 2-hyperexpansive [10]; again, if an operator $A \in B(\mathcal{H})$ is both 2-expansive and m-expansive for an integer $m > 2$, then A is m-hyperexpansive [4]. This paper proves that commuting d-tuples share this property. It is seen that, just as for single linear operators, $\mathbb{A}$ is $2m$-expansive implies $\mathbb{A}$ is $(2m - 1)$-expansive. Commuting products property A is m_1-isometric and B is m_2-isometric, where A and B commute, implies AB is $(m_1 + m_2 - 1)$-isometric [3, 5] does not extend to products of commuting expansive operators [6]: we prove a sufficient condition, in the spirit of results from [4], for the (suitably defined) product $\mathbb{AB} = \mathbb{BA}$, $\mathbb{A}$ is m_1-expansive and $\mathbb{B}$ is m_2-expansive, to be $(m_1 + m_2 - 1)$-expansive. The arguments we use to prove these results have their roots in the arguments used in papers of the ilk of [4, 5, 6], and depend upon a juducious use of the algebraic properties of the left/right multiplication operators.

2. Results

Throughout the following, the d-tuple $\mathbb{A} \in B(\mathcal{H})^d$ will be defined by $\mathbb{A} = (A_1, \cdots, A_d)$; the d-tuple $\mathbb{A}$ is said to be a commuting d-tuple if $[A_i, A_j] = A_iA_j - A_jA_i = 0$ for all $1 \le i,j \le d$. The d-tuples $\mathbb{A}, \mathbb{B} = (B_1, \cdots, B_d)$ are said to commute, $[\mathbb{A}, \mathbb{B}] = 0$, if $[A_i, B_j] = 0$ for all $1 \le i,j \le d$. Observe that if $X \in B(\mathcal{H})$ is a positive operator, $X \ge 0$, then, for all $x \in \mathcal{H}$,

$$\begin{aligned}\langle (\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})(X)x, x\rangle &= \left\langle (\sum_{i=1}^d L_{A_i^*}R_{A_i})(X)x, x\right\rangle \\ &= \sum_{i=1}^d \langle A_i^* X A_i x, x\rangle \\ &= \sum_{i=1}^d \langle X A_i x, A_i x\rangle \\ &\ge 0,\end{aligned}$$

i.e., if $X \in B(\mathcal{H})$ is a positive operator, then $(\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})(X)$ is a positive operator. In particular:

Lemma 2.1 *Given operators $B, C \in B(\mathcal{H})$ and an operator $\mathbb{A} \in B(\mathcal{H})^d$, if $B \le C$, then $(\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})(B) \le (\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})(C)$.*

We say in the following that an operator $\mathbb{A} \in B(\mathcal{H})^d$ is (m, X)-expansive for some operator $X \in B(\mathcal{H})$ if $\triangle^m_{\mathbb{A}^*,\mathbb{A}}(X) \le 0$. Let $\nabla_{\mathbb{A}^*,\mathbb{A}}$ be the operator

$$\nabla_{\mathbb{A}^*,\mathbb{A}}(X) = (\mathbb{L}_{\mathbb{A}^*}R_{\mathbb{A}} - I)(X) = -\triangle_{\mathbb{A}^*,\mathbb{A}}(X), \ X \in B(\mathcal{H}).$$

The following theorem says that if an $\mathbb{A} \in B(\mathcal{H})^d$ is both 2-expansive and m-expansive for an integer $m > 2$, then it is t-expansive for all $1 \le t \le m$.

Theorem 2.2 *If $\mathbb{A} \in B(\mathcal{H})^d$ is both $(2, X)$- expansive and (m, X)- expansive for some operator $X \in B(\mathcal{H})$ and an integer $m > 2$, then $\mathbb{A}$ is (m, X)-hyperexpansive.*

Proof. The proof proceeds in three steps, stated below as claims.

Claim I: $\triangle^2_{\mathbb{A}^*,\mathbb{A}}(X) \le 0$ implies $\triangle_{\mathbb{A}^*,\mathbb{A}}(X) \le 0$.

If $\mathbb{A}$ is $(2, X)$-expansive, then

$$\nabla^2_{\mathbb{A}^*,\mathbb{A}}(X) = \triangle^2_{\mathbb{A}^*,\mathbb{A}}(X) = \left(\sum_{j=0}^2 (-1)^j \begin{pmatrix} 2 \\ j \end{pmatrix} (\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{B}})^j\right)(X)$$

$$= \left(\sum_{j=0}^2 (-1)^j \begin{pmatrix} 2 \\ j \end{pmatrix} (\sum_{i=1}^d L_{A_i^*}R_{A_i})^j\right)(X) \le 0$$

$$\iff X - 2(\sum_{i=1}^d L_{A_i^*}R_{A_i})(X) + (\sum_{i=1}^d L_{A_i^*}R_{A_i})^2(X) \le 0$$

$$\iff (\sum_{i=1}^d L_{A_i^*}R_{A_i})^2(X) - 2\nabla_{\mathbb{A}^*,\mathbb{A}}(X) - X \le 0$$

11 Jul 2023 06:21:43 PDT
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$$\begin{aligned}
&\iff (\sum_{i=1}^{d} L_{A_i^*}R_{A_i})^2(X) \leq 2\nabla_{\mathbb{A}^*,\mathbb{A}}(X) + X \\
&\iff (\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})^2(X) \leq 2\nabla_{\mathbb{A}^*,\mathbb{A}}(X) + X \\
&\Longrightarrow (\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})^3(X) \leq 2(\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})\nabla_{\mathbb{A}^*,\mathbb{A}}(X) + (\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})(X) = 3\nabla_{\mathbb{A}^*,\mathbb{A}}(X) + X
\end{aligned}$$

(see Lemma 2.1). Repeating the argument, we have

$$(\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})^n(X) \leq n\nabla_{\mathbb{A}^*,\mathbb{A}}(X) + X,$$

equivalently,

$$\nabla_{\mathbb{A}^*,\mathbb{A}}(X) \geq \frac{1}{n}(\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})^n(X) - \frac{1}{n}X.$$

Letting $n \longrightarrow \infty$, this implies

$$\nabla_{\mathbb{A}^*,\mathbb{A}}(X) \geq 0, \text{ equivalently } \triangle_{\mathbb{A}^*,\mathbb{A}}(X) \leq 0.$$

(Thus, $\mathbb{A}$ is $(2, X)$-expansive if and only if it is $(2, X)$-hyperexpansive.)

Claim II: the sequence $\{(\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})^n\nabla_{\mathbb{A}^*,\mathbb{A}}(X)\}$ converges to an operator $Q \geq 0$.

The hypothesis $\mathbb{A}$ is $(2, X)$-expansive implies also that

$$\begin{aligned}
0 \geq \nabla^2_{\mathbb{A}^*,\mathbb{A}}(X) \quad &= \quad (\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}} - I)^2(X) = (\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})\left(\nabla_{\mathbb{A}^*,\mathbb{A}}(X)\right) - \nabla_{\mathbb{A}^*,\mathbb{A}}(X) \\
&\iff (\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})\left(\nabla_{\mathbb{A}^*,\mathbb{A}}(X)\right) \leq \nabla_{\mathbb{A}^*,\mathbb{A}}(X) \\
&\Longrightarrow (\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})^2\left(\nabla_{\mathbb{A}^*,\mathbb{A}}(X)\right) \leq (\mathbb{L}_{A^*} * \mathbb{R}_{\mathbb{A}})\left(\nabla_{\mathbb{A}^*,\mathbb{A}}(X)\right) \leq \nabla_{\mathbb{A}^*,\mathbb{A}}(X) \\
&\cdots \\
&\Longrightarrow (\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})^n\left(\nabla_{\mathbb{A}^*,\mathbb{A}}(X)\right) \leq (\mathbb{L}_{A^*} * \mathbb{R}_{\mathbb{A}})^{n-1}\left(\nabla_{\mathbb{A}^*,\mathbb{A}}(X)\right) \leq \cdots \leq \nabla_{\mathbb{A}^*,\mathbb{A}}(X)
\end{aligned}$$

for all positive integers n. Thus $\{(\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})^n\left(\nabla_{\mathbb{A}^*,\mathbb{A}}(X)\right)\}$ is a bounded below decreasing sequence of non-negative operators. (Recall from the proof of **Claim I** that $\nabla_{\mathbb{A}^*,\mathbb{A}}(X) \geq 0$.) Consequently, the sequence converges to a positive operator $Q \geq 0$.

Claim III: $\triangle^2_{\mathbb{A}^*,\mathbb{A}}(X) \leq 0$ and $\triangle^m_{\mathbb{A}^*,\mathbb{A}}(X) \leq 0$ for some integer $m > 2$ implies $\triangle^{m-1}_{\mathbb{A}^*,\mathbb{A}}(X) \leq 0$.

If $\triangle^m_{\mathbb{A}^*,\mathbb{A}}(X) \leq 0$ for some integer $m > 2$, then

$$\begin{aligned}
&\triangle^m_{\mathbb{A}^*,\mathbb{A}}(X) \leq 0 \iff \triangle^{m-1}_{\mathbb{A}^*,\mathbb{A}}(X) \leq (\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})\triangle^{m-1}_{\mathbb{A}^*,\mathbb{A}}(X) \\
\Longrightarrow\ &\triangle^{m-1}_{\mathbb{A}^*,\mathbb{A}}(X) \leq (\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})\triangle^{m-1}_{\mathbb{A}^*,\mathbb{A}}(X) \leq (\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})^2\triangle^{m-1}_{\mathbb{A}^*,\mathbb{A}}(X) \\
\cdots\ & \\
\Longrightarrow\ &\triangle^{m-1}_{\mathbb{A}^*,\mathbb{A}}(X) \leq (\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})\triangle^{m-1}_{\mathbb{A}^*,\mathbb{A}}(X) \leq \cdots \leq (\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})^n\triangle^{m-1}_{\mathbb{A}^*,\mathbb{A}}(X)
\end{aligned}$$

for all positive integers n. Since

$$\begin{aligned}
(\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})^n\triangle^{m-1}_{\mathbb{A}^*,\mathbb{A}}(X) &= \triangle^{m-2}_{\mathbb{A}^*,\mathbb{A}}\left((\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})^n\triangle_{\mathbb{A}^*,\mathbb{A}}(X)\right) \\
&= -\triangle^{m-2}_{\mathbb{A}^*,\mathbb{A}}\left((\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})^n\nabla_{\mathbb{A}^*,\mathbb{A}}(X)\right) \\
&= -\sum_{j=0}^{m-2}(-1)^j \binom{m-2}{j}\left((\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})^{n+j}\nabla_{\mathbb{A}^*,\mathbb{A}}(X)\right)
\end{aligned}$$

implies

$$\triangle^{m-1}_{\mathbb{A}^*,\mathbb{A}}(X) \leq -\sum_{j=0}^{m-2}(-1)^j \begin{pmatrix} m-2 \\ j \end{pmatrix} \left((\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})^{n+j}\nabla_{\mathbb{A}^*,\mathbb{A}}(X)\right),$$

we have

$$\begin{aligned}\triangle^{m-1}_{\mathbb{A}^*,\mathbb{A}}(X) &\leq \lim_{n\to\infty}\left(-\sum_{j=0}^{m-2}(-1)^j \begin{pmatrix} m-2 \\ j \end{pmatrix} \left((\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})^{n+j}\nabla_{\mathbb{A}^*,\mathbb{A}}(X)\right)\right)\\ &= -\sum_{j=0}^{m-2}(-1)^j \begin{pmatrix} m-2 \\ j \end{pmatrix} \lim_{n\to\infty}\left((\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})^{n+j}\nabla_{\mathbb{A}^*,\mathbb{A}}(X)\right)\\ &= \sum_{j=0}^{m-2}(-1)^{j+1} \begin{pmatrix} m-2 \\ j \end{pmatrix} Q = 0.\end{aligned}$$

Thus

$$\triangle^{m-1}_{\mathbb{A}^*,\mathbb{A}}(X) \leq 0,$$

Repeating the argument we eventually have that $\triangle^{t}_{\mathbb{A}^*,\mathbb{A}}(X) \leq 0$ for all $2 \leq t \leq m$. Hence $\mathbb{A}$ is (t, X)-expansive for all $1 \leq t \leq m$. □

It is known, see [6], that if an operator $A \in B(\mathcal{H})$ is m-expansive for an even positive integer m, then it is $(m-1)$-expansive. This extends to commuting operator tuples $\mathbb{A}$. (Observe that the argument of the proof of Theorem 2.2, **Claim III**, which says that $\mathbb{A}$ is (m, X)-expansive implies $\mathbb{A}$ is $(m-1, X))$-expansive for all positive integers m depends in an essential way upon our hypothesis that $\mathbb{A}$ is $(2, X)$-expansive.)

Theorem 2.3 *(i) If $\triangle^{m}_{\mathbb{A}^*,\mathbb{A}}(X) \leq 0$ for some operator $X \in B(\mathcal{H})$ and an even positive integer m, then $\triangle^{m-1}_{\mathbb{A}^*,\mathbb{A}}(X) \leq 0$.*

(ii) If $\triangle^{m}_{\mathbb{A}^*,\mathbb{A}}(X) \geq 0$ for some operator $X \in B(\mathcal{H})$ and an odd positive integer m, then $\triangle^{m-1}_{\mathbb{A}^*,\mathbb{A}}(X) \geq 0$.

Proof. The identity

$$(a-1)^m = a^m - \sum_{j=0}^{m} \begin{pmatrix} m \\ j \end{pmatrix} (a-1)^j$$

implies

$$\begin{aligned}\nabla^{m}_{\mathbb{A}^*,\mathbb{A}}(X) &= (\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}} - I)^m(X) = (\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})^m(X) - \left(\sum_{j=0}^{m} \begin{pmatrix} m \\ j \end{pmatrix} \nabla^{j}_{\mathbb{A}^*,\mathbb{A}}\right)(X)\\ &= (-1)^m\triangle^{m}_{\mathbb{A}^*,\mathbb{A}}(X).\end{aligned}$$

Let $\nabla^{m}_{\mathbb{A}^*,\mathbb{A}}(X) \leq 0$. Since

$$\nabla^{j}_{\mathbb{A}^*,\mathbb{A}}(Z) = (\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})(\nabla^{j-1}_{\mathbb{A}^*,\mathbb{A}}(Z)) - \nabla^{j-1}_{\mathbb{A}^*,\mathbb{A}}(Z),$$

for all $Z \in B(\mathcal{H})$ and integers $j \geq 1$,

$$(\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}}) \sum_{j=0}^{m-1} \begin{pmatrix} m \\ j \end{pmatrix} \nabla^{j}_{\mathbb{A}^*,\mathbb{A}}$$

$$= \sum_{j=0}^{m-1} \binom{m}{j} \nabla^{j+1}_{\mathbb{A}^*,\mathbb{A}} + \sum_{j=0}^{m-1} \binom{m}{j} \nabla^{j}_{\mathbb{A}^*,\mathbb{A}}$$
$$= \binom{m}{m-1} \nabla^{m}_{\mathbb{A}^*,\mathbb{A}} + \left(\sum_{j=0}^{m-2} \binom{m}{j} \nabla^{j+1}_{\mathbb{A}^*,\mathbb{A}} + \sum_{j=0}^{m-1} \binom{m}{j} \nabla^{j}_{\mathbb{A}^*,\mathbb{A}} \right)$$
$$= \binom{m}{m-1} \nabla^{m}_{\mathbb{A}^*,\mathbb{A}} + \sum_{j=0}^{m-1} \binom{m+1}{j} \nabla^{j}_{\mathbb{A}^*,\mathbb{A}},$$

and hence

$$\begin{aligned}(\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})^{m+1}(X) &\leq \binom{m}{m-1} \nabla^{m}_{\mathbb{A}^*,\mathbb{A}}(X) + \sum_{j=0}^{m-1} \binom{m+1}{j} \nabla^{j}_{\mathbb{A}^*,\mathbb{A}}(X) \\ &\leq \sum_{j=0}^{m-1} \binom{m+1}{j} \nabla^{j}_{\mathbb{A}^*,\mathbb{A}}(X) \\ &= \binom{m+1}{m-1} \nabla^{m-1}_{\mathbb{A}^*,\mathbb{A}}(X) + \sum_{j=0}^{m-2} \binom{m+1}{j} \nabla^{j}_{\mathbb{A}^*,\mathbb{A}}(X).\end{aligned}$$

An induction argument now proves that

$$(\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})^{n}(X) \leq \binom{n}{m-1} \nabla^{m-1}_{\mathbb{A}^*,\mathbb{A}}(X) + \sum_{j=0}^{m-2} \binom{n}{j} \nabla^{j}_{\mathbb{A}^*,\mathbb{A}}(X) \tag{1}$$

for all $n \geq m$.

(i). If m is even, then $\triangle^{m}_{\mathbb{A}^*,\mathbb{A}}(X) = \nabla^{m}_{\mathbb{A}^*,\mathbb{A}}(X)$ and inequality (1) implies

$$\frac{1}{\binom{n}{m-1}} \left[(\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})^{n}(X) - \sum_{j=0}^{m-2} \binom{n}{j} \nabla^{j}_{\mathbb{A}*,\mathbb{A}}(X) \right] \leq \nabla^{m-1}_{\mathbb{A}^*,\mathbb{A}}(X).$$

Letting $n \longrightarrow \infty$, and observing that $\lim_{n\to\infty} \frac{\binom{n}{j}}{\binom{n}{m-1}} = 0$ for all $0 \leq j \leq m-2$, we have

$$\nabla^{m-1}_{\mathbb{A}^*,\mathbb{A}}(X) \geq 0.$$

This implies $\triangle^{m-1}_{\mathbb{A}^*,\mathbb{A}}(X) \leq 0$.

(ii). If m is odd, then $\triangle^{m}_{\mathbb{A}^*,\mathbb{A}}(X) \geq 0$ is equivalent to $\nabla^{m}_{\mathbb{A}^*,\mathbb{A}}(X) \leq 0$, the argument above applies and we conclude that $\nabla^{m-1}_{\mathbb{A}^*,\mathbb{A}}(X) \geq 0$. Since $m-1$ is even, the proof is complete. □

Products of commuting d-tuples. The product $\mathbb{AB}$ of d-tuples $\mathbb{A} = (A_1, \cdots, A_d)$ and $\mathbb{B} = (B_1, \cdots, B_d)$ is the d^2-tuple

$$\mathbb{AB} = (A_1B_1, \cdots, A_1B_d, A_2B_1, \cdots, A_2B_d, \cdots, A_dB_1, \cdots, A_dB_d)$$

Given commuting operators $S, T \in B(\mathcal{H})$, $\triangle^m_{S^*,S}(I) = \triangle^n_{T^*,T}(I) = 0$ implies $\triangle^{m+n-1}_{S^*T^*,ST}(I) = 0$ [3, 5]. This does not extend to expansive operators $S, T \in B(\mathcal{H})$ (i.e., $[S,T] = 0$, $\triangle^m_{S^*,S}(I) \le 0$ and $\triangle^n_{T^*,T}(I) \le 0$ does not imply $\triangle^{m+n-1}_{S^*T^*,ST}(I) \le 0$ - see for example [4, Example 2.5(ii)]). Additional hypothses are required. Taking a cue from [4, Page 164], we say in the following that:

a sequence $\{X_j\}_{j=r_1}^{r_2}$ is a partial expansive sequence for $\mathbb{B} \in B(\mathcal{H})^d$ if $\triangle^{r_2-j}_{\mathbb{B}^*,\mathbb{B}}(X_j) \le 0$ for all $r_1 \le j \le r_2$.

We are, in the following, interested in sequences of type $X_j = X_j(X, \mathbb{A}^*, \mathbb{A}) = \triangle^j_{\mathbb{A}^*,\mathbb{A}}(X) \le 0$. Such partial expansive sequences occur naturally, especially for expansive operators $\mathbb{A}$ for which $\triangle^m_{\mathbb{A}^*,\mathbb{A}}(X) = 0$ (such operators have been called (m, X)-isometric in the literature); see [4, Page 164] for examples involving operators $A \in B(\mathcal{H})$, and, also, Remark 4.6(II) *infra.*

Theorem 2.4 *Given commuting d-tuples* $\mathbb{A}, \mathbb{B} \in B(\mathcal{H})^d$ *such that*

$$[\mathbb{A}, \mathbb{B}] = 0,\ \triangle^m_{\mathbb{A}^*,\mathbb{A}}(X) \le 0 \text{ and } \triangle^n_{\mathbb{B}^*,\mathbb{B}}(X) \le 0$$

for some operator $X \in B(\mathcal{H})$, *if the sequence* $\{\triangle^k_{\mathbb{A}^*,\mathbb{A}}(X)\}_{k=m}^{m+n-1}$ *is a partial expansive sequence for* $\mathbb{B}$ *and the sequence* $\{\triangle^k_{\mathbb{B}^*,\mathbb{B}}(X)\}_{k=0}^{m-1}$ *is a partial expansive sequence for* $\mathbb{A}$, *then* $\triangle^{m+n-1}_{\mathbb{A}^*\mathbb{B}^*,\mathbb{A}\mathbb{B}}(X) \le 0$.

Proof. By definition

$$\begin{aligned}
\triangle^t_{\mathbb{A}^*\mathbb{B}^*,\mathbb{A}\mathbb{B}}(X) &= (I - \mathbb{L}_{\mathbb{A}^*\mathbb{B}^*} * \mathbb{R}_{\mathbb{A}\mathbb{B}})^t(X) = (I - \mathbb{L}_{\mathbb{A}^*}\mathbb{L}_{\mathbb{B}^*} * \mathbb{R}_{\mathbb{A}}\mathbb{R}_{\mathbb{B}})^t(X) \\
&= [I - (\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})(\mathbb{L}_{\mathbb{B}^*} * \mathbb{R}_{\mathbb{B}})]^t(X), \text{ since } [\mathbb{A}, \mathbb{B}] = 0 \\
&= [(\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})(I - \mathbb{L}_{\mathbb{B}^*} * \mathbb{R}_{\mathbb{B}}) + (I - \mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})]^t(X) \\
&= \sum_{j=0}^{t} \binom{t}{j} (\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})^{t-j} \triangle^{t-j}_{\mathbb{B}^*,\mathbb{B}}\left(\triangle^j_{\mathbb{A}^*,\mathbb{A}}(X)\right) \\
&= \sum_{j=0}^{t} \binom{t}{j} (\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})^{t-j} \triangle^j_{\mathbb{A}^*,\mathbb{A}}\left(\triangle^{t-j}_{\mathbb{B}^*,\mathbb{B}}(X)\right).
\end{aligned}$$

By Lemma 2.1, if $Z \le 0$ for an operator $Z \in B(\mathcal{H})$, then

$$(\mathbb{L}_{A^*} * \mathbb{R}_{\mathbb{A}})^j(Z) = \left(\sum_{i=1}^{d} L_{A_i^*} R_{A_i}\right)^j (Z) \le 0$$

for all integers $j \ge 0$. Let $t = m+n-1$. The hypothesis $\{\triangle^j_{\mathbb{A}^*,\mathbb{A}}(X)\}_{j=m}^{m+n-1}$ is a partial expansive sequence for $\mathbb{B}$ then implies

$$\triangle^{m+n-1-j}_{\mathbb{B}^*,\mathbb{B}}\left(\triangle^j_{\mathbb{A}^*,\mathbb{A}}(X)\right) \le 0,\ m \le j \le m+n-1.$$

Hence

$$\begin{aligned}
\triangle^{m+n-1}_{\mathbb{A}^*\mathbb{B}^*,\mathbb{A}\mathbb{B}}(X) &= \sum_{j=0}^{m+n-1} \binom{m+n-1}{j} (\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})^{m+n-1-j} \triangle^{m+n-1-j}_{\mathbb{B}^*,\mathbb{B}}\left(\triangle^j_{\mathbb{A}^*,\mathbb{A}}(X)\right) \\
&\le \sum_{j=0}^{m-1} \binom{m+n-1}{j} (\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})^{m+n-1-j} \triangle^j_{\mathbb{A}^*,\mathbb{A}}\left(\triangle^{m+n-1-j}_{\mathbb{B}^*,\mathbb{B}}(X)\right).
\end{aligned}$$

11 Jul 2023 06:21:43 PDT
221120-Duggal Version 3 - Submitted to Rocky Mountain J. Math.

Considering now the hypothesis that the sequence $\{\triangle^j_{\mathbb{B}^*,\mathbb{B}}(X)\}_{j=0}^{m-1}$ is a partial expansive sequence for $\mathbb{A}$, we have

$$\triangle^j_{\mathbb{A}^*,\mathbb{A}}\left(\triangle^{m+n-1-j}_{\mathbb{B}^*,\mathbb{B}}(X)\right) \leq 0,\ 0 \leq j \leq m-1,$$

and hence

$$\triangle^{m+n-1}_{\mathbb{A}^*\mathbb{B}^*,\mathbb{A}\mathbb{B}}(X) \leq 0.$$

□

The hypotheses $\triangle^{m+n-1}_{\mathbb{A}^*,\mathbb{A}}(X) \leq 0$ and $\triangle^m_{\mathbb{A}^*,\mathbb{A}}(X) \leq 0$, as also the hypotheses $\triangle^{m+n-1}_{\mathbb{B}^*,\mathbb{B}}(X) \leq 0$ and $\triangle^n_{\mathbb{B}^*,\mathbb{B}}(X) \leq 0$, are an integral part of the argument of the proof of Theorem 2.4. We remark that the hypotheses $\mathbb{A}$ is both m and $m+n-1$ expansive does not in general imply $\mathbb{A}$ is r-expansive for all $m \leq r \leq m+n-1$. (Similarly, the hypothesis that $\mathbb{B}$ is both $m+n-1$ and n expansive does not imply $\mathbb{B}$ is r-expansive for all $n \leq r \leq m+n-1$.) Thus, if m is odd, $\mathbb{A} = (aI, \cdots, aI)$ for some positive real number a such that $da^2 > 1$, then

$$\triangle^m_{\mathbb{A}^*,\mathbb{A}}(I) = \sum_{j=0}^{m} \begin{pmatrix} m \\ j \end{pmatrix} d^j a^{2j} = (1-da^2)^m \leq 0.$$

However, $\mathbb{A}$ is not r-expansive for any positive even integer r. The situation for even m, as one might suspect, is very different.

Theorem 2.5 *If $\mathbb{A} \in B(\mathcal{H})^d$ is (r,X)-expansive for $r = m$ and $r = m+n-1$ for an operator $X \in B(\mathcal{H})$, even positive integer m and an integer $n > 1$, then $\mathbb{A}$ is (r,X)-expansive for all $m-1 \leq r \leq m+n-1$.*

Proof. A proof of the theorem may be obtained from an argument similar to that used to prove Theorem 2.2: in the following we prove the theorem using a slightly different argument (which makes clear that the essence of the argument of the proof of Theorem 2.2 lies in proving the hyperexpansivity of $(2,X)$-expansive operators).

Define $Y \in B(\mathcal{H})$ by $\triangle^{m-2}_{\mathbb{A}*,\mathbb{A}}(X) = Y$. Then $\triangle^2_{\mathbb{A}^*,\mathbb{A}}(Y) = \nabla^2_{\mathbb{A}^*,\mathbb{A}}(Y) \leq 0$, and an argument similar to that used to prove inequality (1) (of the proof of Theorem 2.3) shows that

$$(\mathbb{L}_{\mathbb{A}^*} * \mathbb{R}_{\mathbb{A}})^t(Y) - t\nabla_{\mathbb{A}^*,\mathbb{A}}(Y) - Y \leq 0$$

for all integers $t \geq 2$. Hence $\nabla_{\mathbb{A}^*,\mathbb{A}}(Y) \geq 0$ (equivalently, $\triangle_{\mathbb{A}^*,\mathbb{A}}(Y) = \triangle^{m-1}_{\mathbb{A}^*,\mathbb{A}}(X) \leq 0$). Now if n is even then set $\triangle^{m-1}(X) = Z$ and if n is odd then set $\triangle^m(X) = Z$. We have $\triangle^{n-1}_{\mathbb{A}^*,\mathbb{A}}(Z) \leq 0$ if n is even and $\triangle^{n-2}_{\mathbb{A}^*,\mathbb{A}}(Z) \leq 0$ if n is odd. In either case $\triangle^{m+n-2}_{\mathbb{A}^*,\mathbb{A}}(X) \leq 0$. Repeating the argument a finite number of times, the result follows □

Remark 2.6 (I) In closing. we start with a remark on commuting d-tuples $\mathbb{A}$ such that $\triangle^m_{\mathbb{A}^*,\mathbb{A}}(X) \geq 0$ for some odd positive integer m. (Operators $A \in B(\mathcal{H})$ such that $\triangle^m_{A^*,A}(I) \geq 0$ have been called m-contractive in the literature [9].) If we let $\nabla^{m-2}_{\mathbb{A}^*,\mathbb{A}}(X) = Y$, then $\triangle^m_{\mathbb{A}^*,\mathbb{A}}(X) \geq 0$ if and only if $\nabla^m_{A^*,A}(X) = \nabla^2_{\mathbb{A}^*,\mathbb{A}}(Y) \leq 0$. Arguing as in the proof above, this imples $\triangle_{\mathbb{A}^*,\mathbb{A}}(Y) = \triangle^{m-1}_{\mathbb{A}^*,\mathbb{A}}(X) \geq 0$. Assume now that $\triangle^n_{\mathbb{A}^*,\mathbb{A}}(X) \geq 0$ for an integer $n > m$. Set $\triangle^{m-1}_{\mathbb{A},\mathbb{A}}(X) = Z$ if n is odd and $\triangle^m_{\mathbb{A}^*,\mathbb{A}}(X) = Z$ if n is even. Then the preceding argument implies that $\triangle^{n-1}_{\mathbb{A}^*,\mathbb{A}}(X) \geq 0$. Repeating the argument, we have $\triangle^t_{\mathbb{A}^*,\mathbb{A}}(X) \geq 0$ for all $m-1 \leq t \leq n$.
(II) If $\mathbb{A}$ is both m-expansive and $(m+n-1)$-expansive for some even positive integer m and integer $n > 1$, then the conclusion of Theorem 2.5 implies (trivially) that

$\{X_j\}_{j=m-1}^{m+n-1} = \{\triangle^j_{\mathbb{A}^*,\mathbb{A}}(X)\}_{j=m-1}^{m+n-1}$ is a partial expansive sequence for $\mathbb{A}$. Again, if we let $\mathbb{I}$ denote the identity of $B(\mathcal{H})^d$, then $\triangle^t_{\frac{1}{d}\mathbb{I}^*,\frac{1}{d}\mathbb{I}}(X_j) = (1-\frac{1}{d})^t X_j \leq 0$ for all $m_1 \leq j \leq m+n-1$ and positive integers t; hence $\{X_j\}_{j=m-1}^{m+n-1}$ is a partial expansive sequence for $\frac{1}{d}\mathbb{I}$.

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