DIVISIBILITY OF CERTAIN SUMS INVOLVING CENTRAL q-BINOMIAL COEFFICIENTS

YIFAN CHEN AND XIAOXIA WANG*

ABSTRACT. In this paper, we shall give a generalization of Ni and Pan's q-congruence, originally conjectured by Guo, on certain sums involving central q-binomial coefficients.

1. INTRODUCTION

In [12], Ramanujan obtained 17 curious convergent series concerning $1/\pi$. Actually, there are a number of similar representations for $1/\pi$ which are not listed in [12] as well. For example, the identity

$$\sum_{k=0}^{\infty} \frac{4k+1}{(-64)^k} \binom{2k}{k}^3 = \frac{2}{\pi} \tag{1.1}$$

was first proved by Bauer [1] in 1859. Nowadays, Ramanujan-type series for $1/\pi$ has been developed greatly, partly because they play an important role in fast algorithms for computing decimal digits of π. Recently, Guillera [8] gave a new method to prove Ramanujan-type series.

In the last few years, the truncated Ramanujan-type series attracted many researchers' attentions. Van Hamme [13] conjectured 13 congruences on truncated Ramanujan-type series, such as

$$\sum_{k=0}^{(p-1)/2} \frac{4k+1}{(-64)^k} \binom{2k}{k}^3 \equiv p(-1)^{(p-1)/2} \pmod{p^3}, \tag{1.2}$$

where p is an odd prime. Now all of the 13 conjectures by Van Hamme have been confirmed by different methods. Some history of the development of Van Hamme's conjectures can be found in [5, 8, 15, 19], and we refer the reader to [9, 11, 14, 16–18, 20] for some other interesting q-congruences.

Recently, Ni and Pan [10] proved the following extension of (1.2): for any integers n and r with $n \geq 2$ and $r \geq 1$,

$$\sum_{k=0}^{n-1} (4k+1) \binom{2k}{k}^r (-4)^{r(n-k-1)} \equiv 0 \pmod{2^{r-2} n \binom{2n}{n}}. \tag{1.3}$$

2010 *Mathematics Subject Classification.* Primary 11B65; Secondary 05A10, 05A30.
Key words and phrases. congruences; q-congruences; q-binomial coefficients; cyclotomic polynomials.
This work is supported by Natural Science Foundation of Shanghai (22ZR1424100).

6 Apr 2023 19:18:53 PDT
221114-Wang-3 Version 2 - Submitted to Rocky Mountain J. Math.

They also gave a q-analogue of (1.3): for any integers n and r with $n \geq 2$ and $r \geq 2$, modulo $(1+q^{n-1})^{2r-2}[n]\begin{bmatrix}2n-1\\n-1\end{bmatrix}$,

$$\sum_{k=0}^{n-1}(-1)^k q^{k^2+(r-2)k}[4k+1]\begin{bmatrix}2k\\k\end{bmatrix}^{2r-1}(-q^{k+1};q)_{n-k-1}^{4r-2} \equiv 0, \tag{1.4}$$

$$\frac{1}{1+q^{n-1}}\sum_{k=0}^{n-1} q^{(r-2)k}[4k+1]\begin{bmatrix}2k\\k\end{bmatrix}^{2r}(-q^{k+1};q)_{n-k-1}^{4r} \equiv 0, \tag{1.5}$$

which were originally conjectured by Guo [4, Conjecture 5.4]. Here, the q-binomial coefficient is defined as

$$\begin{bmatrix}n\\k\end{bmatrix} = \begin{bmatrix}n\\k\end{bmatrix}_q = \begin{cases}\dfrac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, & 0 \leq k \leq n;\\ 0, & \text{otherwise,}\end{cases}$$

with the *q-shifted factorial* $(a;q)_0 = 1$, $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n \in \mathbb{Z}^+$ and $[n] = [n]_q = 1+q+\cdots+q^{n-1}$ is the *q-integer*.

Recently, Guo and Wang [7] proved the following generalizations of (1.4) and (1.5): for any integers n, r and s with $n \geq 2$, $r \geq 1$ and $s \geq 0$, modulo $(1+q^{2n-2})^{2r-2}[n]_{q^2}\begin{bmatrix}2n-1\\n-1\end{bmatrix}_{q^2}$,

$$\sum_{k=0}^{n-1}(-1)^k q^{2k^2+(2r-4s-4)k}[4k+1]^{2s}[4k+1]_{q^2}\begin{bmatrix}2k\\k\end{bmatrix}_{q^2}^{2r-1}(-q^{2k+2};q^2)_{n-k-1}^{4r-2} \equiv 0,$$

$$\frac{1}{1+q^{2n-2}}\sum_{k=0}^{n-1} q^{(2r-4s-4)k}[4k+1]^{2s}[4k+1]_{q^2}\begin{bmatrix}2k\\k\end{bmatrix}_{q^2}^{2r}(-q^{2k+2};q^2)_{n-k-1}^{4r} \equiv 0.$$

Motivated by the work just mentioned, we shall establish another different generalizations of (1.4) and (1.5) as follows.

Theorem 1.1. *For any integers n, r, m and l with $n \geq 1$, $r \geq 2$, $0 \leq m \leq 2r-3$ and $0 \leq l \leq n-1$, modulo $(1+q^{n-1})^{2r-3-m}(1+q^{n-l-1})^{1+m}[n]\begin{bmatrix}2n-1\\n-1\end{bmatrix}$,*

$$\sum_{k=l}^{n-1}(-1)^k q^{(k-l)^2+(r-2)(k-l)-2kml}[4k+1]\frac{(q;q^2)_k^{2r-2-m}(q;q^2)_{k+l}^{1+m}(-q;q)_{n-1}^{4r-2}}{(q^2;q^2)_k^{2r-2-m}(q^2;q^2)_{k-l}^{1+m}(q;q^2)_l^m} \equiv 0, \tag{1.6}$$

$$\frac{1}{1+q^{n-1}}\sum_{k=l}^{n-1} q^{l(l-2k-1)+(r-2)(k-l)-2kml}[4k+1]\frac{(q;q^2)_k^{2r-1-m}(q;q^2)_{k+l}^{1+m}(-q;q)_{n-1}^{4r}}{(q^2;q^2)_k^{2r-1-m}(q^2;q^2)_{k-l}^{1+m}(q;q^2)_l^m} \equiv 0. \tag{1.7}$$

Notice that

$$\frac{(q;q^2)_k}{(q^2;q^2)_k} = \begin{bmatrix}2k\\k\end{bmatrix}\frac{1}{(-q;q)_k^2},$$

therefore, (1.6) and (1.7) can be expressed in terms of q-binomial coefficients: modulo $(1+q^{n-1})^{2r-3-m}(1+q^{n-l-1})^{1+m}[n]{2n-1 \brack n-1}$,

$$\frac{1}{(q;q^2)_l^m}\sum_{k=l}^{n-1}(-1)^k q^{(k-l)^2+(r-2)(k-l)-2kml}[4k+1]{2k \brack k}^{2r-2-m}{2k-2l \brack k-l}^{1+m} \times (q^{2k-2l+1};q^2)_{2l}^{1+m}(-q^{k-l+1};q)_{n-k+l-1}^{2+2m}(-q^{k+1};q)_{n-k-1}^{4r-4-2m}\equiv 0,$$

$$\frac{1}{(1+q^{n-1})(q;q^2)_l^m}\sum_{k=l}^{n-1}q^{l(l-2k-1)+(r-2)(k-l)-2kml}[4k+1]{2k \brack k}^{2r-1-m}{2k-2l \brack k-l}^{1+m} \times (q^{2k-2l+1};q^2)_{2l}^{1+m}(-q^{k-l+1};q)_{n-k+l-1}^{2+2m}(-q^{k+1};q)_{n-k-1}^{4r-2-2m}\equiv 0.$$

Obviously, (1.4) and (1.5) can be deduced from Theorem 1.1 just by taking $l=0$.

The rest of the paper is arranged as follows. In Section 2, we shall present some useful preliminaries. The main proof of Theorem 1.1 will be shown in Section 3.

2. Preliminaries

To show the main results, we list some lemmas firstly. The following Lemma 2.1 is a special case of [10, Lemma 3.2].

Lemma 2.1. *Let s, t be non-negative integers and d an odd integer with $0\le t\le d-1$. Then*

$$\frac{(q;q^2)_{sd+t}}{(q^2;q^2)_{sd+t}}\equiv\frac{1}{4^s}\binom{2s}{s}\frac{(q;q^2)_t}{(q^2;q^2)_t}\qquad (\text{mod } \Phi_d(q)).$$

Here $\Phi_n(q)$ stands for the n-th *cyclotomic polynomial* in q:

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(n,k)=1}}(q-\zeta^k),$$

with ζ an n-th primitive root of unity.

We now give some useful notations used earlier by Guo and Wang [7]. For any positive integer n, let

$$S(n)=\{d\ge 3: d \text{ is odd and } \left\lfloor\frac{n-\frac{d+1}{2}}{d}\right\rfloor=\left\lfloor\frac{n}{d}\right\rfloor\},$$

where $\lfloor x\rfloor$ denotes the greatest integer which does not exceed x. Notice that if $d>2n-1$, the number $(d+1)/2$ is greater than n, as a result, $d\notin S(n)$, therefore, $S(n)$ is a finite set actually. Let

$$A_n(q)=\prod_{d\in S(n)}\Phi_d(q),$$
$$C_n(q)=\prod_{\substack{d\mid n,\ d>1\\ d \text{ is odd}}}\Phi_d(q).$$

It is easy to see that if $d \mid n$, then $d \notin S(n)$, which means the polynomials $A_n(q)$ and $C_n(q)$ are relatively prime due to the property of cyclotomic polynomial.

We now give the key lemma, which is a generalization of [7, Lemma 2.2].

Lemma 2.2. *Let $v_0(q), v_1(q), \ldots$ be a sequence of rational functions in q. For any positive odd integer d, positive integer n and l with $0 \le l \le n-1$, if $v_0(q), v_1(q), \ldots$ satisfies the following conditions:*

(i) *$v_k(q)$ is $\Phi_d(q)$-integral for each $k \ge 0$, i.e. the denominator of $v_k(q)$ is relatively prime to $\Phi_d(q)$;*

(ii) *For any non-negative integers s and t with $0 \le t \le d-1$,*

$$v_{sd+t}(q) \equiv u_s(q) v_t(q) \pmod{\Phi_d(q)},$$

where $u_s(q)$ is a $\Phi_d(q)$-integral rational function only dependent on s;

(iii)

$$\sum_{k=l}^{(d-1)/2-l} \frac{(q;q^2)_{k+l}}{(q^2;q^2)_{k-l}} v_k(q) \equiv 0 \pmod{\Phi_d(q)}.$$

Then, for any positive integer n,

$$\sum_{k=l}^{n-1} \frac{(q;q^2)_{k+l}}{(q^2;q^2)_{k-l}} v_k(q) \equiv 0 \pmod{A_n(q)C_n(q)}. \tag{2.1}$$

Proof. For $d \in S(n)$, we can write $n = ud + v$ with $(d+1)/2 \le v \le d-1$. Therefore, for any $n \le k \le ud + d + l - 1$, $(q;q^2)_{k+l}/(q^2;q^2)_{k-l}$ is divisible by $\Phi_d(q)$. As a result, we obtain

$$\begin{aligned}
&\sum_{k=l}^{n-1} \frac{(q;q^2)_{k+l}}{(q^2;q^2)_{k-l}} v_k(q) \\
&\equiv \sum_{s=0}^{u} \sum_{t=l}^{d+l-1} \frac{(q;q^2)_{sd+t+l}}{(q^2;q^2)_{sd+t-l}} v_{sd+t}(q) \\
&\equiv \sum_{s=0}^{u} \frac{1}{4^s}\binom{2s}{s} u_s(q) \sum_{t=l}^{d+l-1} \frac{(q;q^2)_{t+l}}{(q^2;q^2)_{t-l}} v_t(q) \equiv 0 \pmod{\Phi_d(q)},
\end{aligned}$$

where the second relation is due to Lemma 2.1, and in the last congruence, we have used the condition (iii) and the fact that $(q;q^2)_{t+l} v_t(q)/(q^2;q^2)_{t-l}$ is congruent to 0 modulo $\Phi_d(q)$ for $(d+1)/2 - l \le t \le d + l - 1$. This proves that (2.1) is true modulo $A_n(q)$.

On the other hand, for $d \mid n$, we assume that $u = n/d$. Firstly, we need to prove that

$$\sum_{k=l}^{n-1} \frac{(q;q^2)_{k+l}}{(q^2;q^2)_{k-l}} v_k(q) \equiv \sum_{s=0}^{u-1} \sum_{t=l}^{d+l-1} \frac{(q;q^2)_{sd+t+l}}{(q^2;q^2)_{sd+t-l}} v_{sd+t}(q) \pmod{\Phi_d(q)},$$

6 Apr 2023 19:18:53 PDT
221114-Wang-3 Version 2 - Submitted to Rocky Mountain J. Math.

which is trivial for $l=0$. Now, we should to verify that for $1\le l\le n-1$ and $n\le k\le n+l-1$,

$$\frac{(q;q^2)_{k+l}}{(q^2;q^2)_{k-l}}\equiv 0 \pmod{\Phi_d(q)}.$$

It can be easily seen that the numerator of $(q;q^2)_{k+l}/(q^2;q^2)_{k-l}$ must have the factor $(q;q^2)_{n+1}$, and the denominator doesn't have the factor $(q^2;q^2)_n$. As a result, the reduced form of $(q;q^2)_{k+l}/(q^2;q^2)_{k-l}$ is congruent to 0 modulo $\Phi_d(q)$.

Therefore, we obtain

$$\begin{aligned}\sum_{k=l}^{n-1}\frac{(q;q^2)_{k+l}}{(q^2;q^2)_{k-l}}v_k(q) &\equiv \sum_{s=0}^{u-1}\sum_{t=l}^{d+l-1}\frac{(q;q^2)_{sd+t+l}}{(q^2;q^2)_{sd+t-l}}v_{sd+t}(q)\\ &\equiv \sum_{s=0}^{u}\frac{1}{4^s}\binom{2s}{s}u_s(q)\sum_{t=l}^{d+l-1}\frac{(q;q^2)_{t+l}}{(q^2;q^2)_{t-l}}v_t(q)\equiv 0 \pmod{\Phi_d(q)}.\end{aligned}$$

This proves that (2.1) is also true modulo $C_n(q)$. Now we complete the proof of this Lemma for the polynomials $A_n(q)$ and $C_n(q)$ are relatively prime. □

We also need the following result, which is just a simple generalization of Guo and Schlosser [6, Lemma 3.1].

Lemma 2.3. *Let d be a positive odd integer. Then, for any nonnegative integers l and k with $l\le k\le (d-1)/2-l$, we have*

$$\frac{(q;q^2)_{(d-1)/2-k+l}}{(q^2;q^2)_{(d-1)/2-k-l}}\equiv(-1)^{(d-1)/2}\frac{(q;q^2)_{k+l}}{(q^2;q^2)_{k-l}}q^{(d-1)^2/4+k-4kl-l} \pmod{\Phi_d(q)}. \tag{2.2}$$

Proof. Notice that

$$\begin{aligned}\frac{(q;q^2)_{(d-1)/2-k+l}}{(q^2;q^2)_{(d-1)/2-k-l}} &= \frac{(q;q^2)_{(d-1)/2}}{(q^2;q^2)_{(d-1)/2}}\frac{(1-q^{d+1-2k-2l})\dots(1-q^{d-1})}{(1-q^{d-2k+2l})\dots(1-q^{d-2})}\\ &\equiv \frac{(q;q^2)_{(d-1)/2}}{(q^2;q^2)_{(d-1)/2}}\frac{(1-q^{2k+2l-1})\dots(1-q^{1})}{(1-q^{2k-2l})\dots(1-q^{2})}q^{-4kl+k-l} \pmod{\Phi_d(q)}.\end{aligned}$$

Then (2.2) holds due to the fact that

$$\frac{(q;q^2)_{(d-1)/2}}{(q^2;q^2)_{(d-1)/2}}\equiv(-1)^{(d-1)/2}q^{(d-1)^2/4} \pmod{\Phi_d(q)}.$$

□

3. Proof of Theorem 1.1

In order to make the proof of Theorem 1.1 clear, we present the following congruences firstly.

6 Apr 2023 19:18:53 PDT
221114-Wang-3 Version 2 - Submitted to Rocky Mountain J. Math.

Theorem 3.1. *For any integers n, r, m and l with $n \geq 1$, $r \geq 2$, $0 \leq m \leq 2r-3$ and $0 \leq l \leq n-1$, modulo $A_n(q)C_n(q)$,*

$$\frac{1}{(q;q^2)_l^m}\sum_{k=l}^{n-1}(-1)^k q^{(k-l)^2+(r-2)(k-l)-2kml}[4k+1]\frac{(q;q^2)_k^{2r-2-m}(q;q^2)_{k+l}^{1+m}}{(q^2;q^2)_k^{2r-2-m}(q^2;q^2)_{k-l}^{1+m}} \equiv 0, \tag{3.1}$$

$$\frac{1}{(q;q^2)_l^m}\sum_{k=l}^{n-1} q^{l(l-2k-1)+(r-2)(k-l)-2kml}[4k+1]\frac{(q;q^2)_k^{2r-1-m}(q;q^2)_{k+l}^{1+m}}{(q^2;q^2)_k^{2r-1-m}(q^2;q^2)_{k-l}^{1+m}} \equiv 0. \tag{3.2}$$

Proof. Here we only prove (3.1), for the reason that (3.2) can be verified by following the same steps used in the proof of (3.1).

For any integer $k \geq 0$, let

$$v_k(q) = (-1)^k q^{(k-l)^2+(r-2)(k-l)-2kml}[4k+1]\frac{(q;q^2)_k^{2r-2-m}(q;q^2)_{k+l}^{m}}{(q^2;q^2)_k^{2r-2-m}(q^2;q^2)_{k-l}^{m}(q;q^2)_l^m}.$$

We shall show that the sequence $v_0(q), v_1(q), \dots$ satisfies the requirements (i), (ii) and (iii) of Lemma 2.2.

Note that

$$(q;q^2)_{k+l} = (q;q^2)_{k-l}(q^{2k-2l+1};q^2)_{2l}$$

and

$$\frac{(q;q^2)_{k-l}}{(q^2;q^2)_{k-l}} = \begin{bmatrix} 2k-2l \\ k-l \end{bmatrix}\frac{1}{(-q;q)_{k-l}^2},$$

it is obviously that for $l \leq k \leq n-1$, the denominator of $(q;q^2)_{k+l}/((q^2;q^2)_{k-l}(q;q^2)_l)$ with the reduced form is always relatively prime to $\Phi_d(q)$. Therefore, for any odd d, the rational function $v_k(q)$ is $\Phi_d(q)$-integral, since the relation

$$\frac{(q;q^2)_k}{(q^2;q^2)_k} = \begin{bmatrix} 2k \\ k \end{bmatrix}\frac{1}{(-q;q)_k^2},$$

and $(-q;q)_k$ is relatively prime to $\Phi_d(q)$.

By applying Lemma 2.1, we can get that for non-negative integers s and t with $0 \leq t \leq d-1$,

$$\begin{aligned} v_{sd+t}(q) &= (-1)^{sd+t} q^{(sd+t-l)^2+(r-2)(sd+t-l)-2(sd+t)ml}\frac{[4(sd+t)+1](q;q^2)_{sd+t}^{2r-2-m}(q;q^2)_{sd+t+l}^{m}}{(q^2;q^2)_{sd+t}^{2r-2-m}(q^2;q^2)_{sd+t-l}^{m}(q;q^2)_l^m} \\ &\equiv (-1)^s \frac{1}{4^{(2r-2)s}}\binom{2s}{s}^{2r-2} v_t(q) \pmod{\Phi_d(q)}. \end{aligned}$$

Apparently, $(-1)^s\binom{2s}{s}^{2r-2}/4^{(2r-2)s}$ is a $\Phi_d(q)$-integral rational function only dependent on s.

We now start to verify the requirement (iii) of Lemma 2.2, i.e., modulo $\Phi_d(q)$,

$$\sum_{k=l}^{(d-1)/2-l} (-1)^k q^{(k-l)^2+(r-2)(k-l)-2kml}[4k+1]\frac{(q;q^2)_k^{2r-2-m}(q;q^2)_{k+l}^{1+m}}{(q^2;q^2)_k^{2r-2-m}(q^2;q^2)_{k-l}^{1+m}(q;q^2)_l^m} \equiv 0. \tag{3.3}$$

In fact, by Lemma 2.3, it can be easily shown that, for $l \le k \le (d-1)/2-l$, the k-th and $((d-1)/2-k)$-th terms on the left-hand side of (3.3) cancel each other modulo $\Phi_d(q)$, because

$$\frac{(q;q^2)_{k+l}}{(q^2;q^2)_{k-l}}v_k(q) \equiv -\frac{(q;q^2)_{(d-1)/2-k+l}}{(q^2;q^2)_{(d-1)/2-k-l}}v_{(d-1)/2-k}(q) \pmod{\Phi_d(q)}.$$

This means (3.3) is true. So far, we have finished verifying all of the requirements of Lemma 2.2. Hence, we access to the conclusion that (3.1) is true modulo $A_n(q)C_n(q)$. □

Now we begin to prove Theorem 1.1.

Proof of Theorem 1.1. For $l \le k \le n-2$, the q-factorial $(-q^{k+1};q)_{n-k-1}$ contains the factor $1+q^{n-1}$. For $k=n-1$, we have

$$\begin{bmatrix}2k\\k\end{bmatrix} = \begin{bmatrix}2n-2\\n-1\end{bmatrix} = (1+q^{n-1})\begin{bmatrix}2n-3\\n-2\end{bmatrix},$$

$$\begin{bmatrix}2k-2l\\k-l\end{bmatrix} = \begin{bmatrix}2n-2-2l\\n-1-l\end{bmatrix} = (1+q^{n-1-l})\begin{bmatrix}2n-3-2l\\n-2-l\end{bmatrix}.$$

Therefore, the left-hand sides of (1.6) and (1.7) are both divisible by $(1+q^{n-1})^{2r-2-m}(1+q^{n-l-1})^{1+m}$.

In what follows, we shall prove that the left-hand sides of (1.6) and (1.7) are both divisible by $[n]\left[{2n-1 \atop n-1}\right]$. From Chen and Hou's [2, Lemma 1] result

$$\begin{bmatrix}n\\k\end{bmatrix}_q = \prod_{d\in D_{n,k}} \Phi_d(q), \quad \text{with} \quad D_{n,k} := \left\{ d \ge 2 : \left\lfloor \frac{k}{d} \right\rfloor + \left\lfloor \frac{n-k}{d} \right\rfloor < \left\lfloor \frac{n}{d} \right\rfloor \right\}$$

and the well known relation

$$[n] = \prod_{d>1,\ d\mid n} \Phi_d(q),$$

obviously, we have

$$[n]\begin{bmatrix}2n-1\\n-1\end{bmatrix} = A_n(q)C_n(q) \prod_{\substack{d\mid n,\ d>1,\\ d \text{ is even}}} \Phi_d(q) \cdot \prod_{\substack{d\in D_{2n-1,n-1}\\ d \text{ is even}}} \Phi_d(q), \tag{3.4}$$

for $1 < d \in D_{2n-1,n-1}$ is odd if and only if $d \in S(n)$.

By Theorem 3.1, the left-hand sides of (1.6) and (1.7) are congruent to 0 modulo $A_n(q)C_n(q)$. It remains to show that (1.6) and (1.7) also hold modulo

$$\prod_{\substack{d\mid n,\ d>1\\ d \text{ is even}}} \Phi_d(q) \cdot \prod_{\substack{d\in D_{2n-1,n-1}\\ d \text{ is even}}} \Phi_d(q).$$

For this proof, we refer the reader to Guo and Wang's similar proof of [7, Theorem 1.1].

By taking $q^2 \to q$ in the equation at the top of page 8 of [7], we obtain

$$[n]\begin{bmatrix}2n-1\\ n-1\end{bmatrix} = (1+q^{n-1})[2n-1]\begin{bmatrix}2n-3\\ n-2\end{bmatrix}.$$

Notice that $(1+q^{n-1})$ is relatively prime to $[2n-1]\left[\begin{smallmatrix}2n-3\\ n-2\end{smallmatrix}\right]$, the least common multiple of $(1+q^{n-1})^{2r-2-m}(1+q^{n-l-1})^{1+m}$ and $[n]\left[\begin{smallmatrix}2n-1\\ n-1\end{smallmatrix}\right]$ is $(1+q^{n-1})^{2r-3-m}(1+q^{n-l-1})^{1+m}[n]\left[\begin{smallmatrix}2n-1\\ n-1\end{smallmatrix}\right]$. Now we obtain that (1.6) and (1.7) hold modulo $(1+q^{n-1})^{2r-3-m}(1+q^{n-l-1})^{1+m}[n]\left[\begin{smallmatrix}2n-1\\ n-1\end{smallmatrix}\right]$. So we finish proving this theorem. □

References

[1] Bauer, G.: Von den coefficienten der reihen von kugelfunctionen einer variablen. J. Reine Angew. Math. **56** (1859), 101-121.

[2] Chen, W.Y.C., Hou, Q.-H.: Factors of the Gaussian coefficients. Discrete Math. **306** (2006), 1446-1449.

[3] Guillera, J.: A method for proving Ramanujan's series for $1/\pi$. Ramanujan J. **52** (2020), 421-431.

[4] Guo, V.J.W.: A q-analogue of a Ramanujan-type supercongruence involving central binomial coefficients. J. Math. Anal. Appl. **458** (2018), 590-600.

[5] Guo, V.J.W.: A further q-analogue of Van Hamme's (H.2) supercongruence for primes $p \equiv 3$ (mod 4). Int. J. Number Theory **17** (2021), 1201-1206.

[6] Guo, V.J.W., Schlosser, M.J.: Some q-supercongruences from transformation formulas for basic hypergeometric series. Constr. Approx. **53** (2021), 155-200.

[7] Guo, V.J.W., Wang, S.-D.: Factors of certain sums involving central q-binomial coefficients. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **116** (2022), Art. 46.

[8] Liu, Y., Wang, X.: Further q-analogues of the (G.2) supercongruence of Van Hamme. Ramanujan J. **59** (2022), 791-802.

[9] Liu, Y., Wang, X.: Some q-supercongruences from a quadratic transformation by Rahman. Results Math. **77** (2022), Art. 44.

[10] Ni, H.-X., Pan, H.: Divisibility of some binomial sums. Acta Arith. **194** (2020), 367-381.

[11] Ni, H.-X., Pan, H.: Some symmetric q-congruence modulo the square of a cyclotomic polynomial. J. Math. Anal. Appl. **481** (2020), Art. 123372.

[12] Ramanujan, S.: Modular equations and approximation to π. Quart. J. Math. **45** (1914), 350-372.

[13] Van Hamme, L.: Some conjectures concerning partial sums of generalized hypergeometric series, in: p-Adic Functional Analysis (Nijmegen, 1996), Lecture Notes in Pure and Appl. Math. 192, Dekker, New York. (1997), 223-236.

[14] Wang, X., Xu, C.: q-Supercongruences on Triple and Quadruple Sums. Results Math. **78** (2023), Art. 27.

[15] Wei, C.: A further q-analogue of Van Hamme's (H.2) supercongruence for any prime $p \equiv 1$ (mod 4). Results Math. **76** (2021), Art. 92.

[16] Wei, C.: Some q-supercongruences modulo the fourth power of a cyclotomic polynomial. J. Combin. Theory Ser. A **182** (2021), Art. 105469.

[17] Wei, C.: q-Supercongruences from Gasper and Rahman' summation formula. Adv. in Appl. Math. **139** (2022), Art. 102376.

[18] Xu, C., Wang, X.: Proofs of Guo and Schlosser's two conjectures. Period. Math. Hungar. **85** (2) (2022), 474-480.

[19] Zudilin, W.: Ramanujan-type supercongruences. J. Number Theory **129** (2009), 1848-1857.

[20] Zudilin, W.: Congruences for q-binomial coefficients. Ann. Combin. **23** (2019), 1123-1135.

6 Apr 2023 19:18:53 PDT
221114-Wang-3 Version 2 - Submitted to Rocky Mountain J. Math.

DEPARTMENT OF MATHEMATICS, SHANGHAI UNIVERSITY, SHANGHAI 200444, P. R. CHINA
E-mail address: * Corresponding author. xiaoxiawang@shu.edu.cn (X. Wang), chenyf576@shu.edu.cn (Y. Chen).