NOVEL RESULTS OF AN ORTHOGONAL $(\alpha - F)$-CONVEX CONTRACTION MAPPINGS

GUNASEKARAN NALLASELLI[1], ARUL JOSEPH GNANAPRAKASAM[2], GUNASEELAN MANI[3], OZGUR EGE[4,*]

Abstract. The main goal of this article is to introduce the idea of $(\alpha_\perp - F)$-convex contraction in the context of orthogonal metric spaces and to provide some novel fixed point results in that recently described spaces. Additionally, we offer a case study to illustrate the originality of the outcomes. As an application of our key finding, we investigate the solution of a nonlinear Volterra integral equation.

1. Introduction

In 1971, Ćirić [1] investigated a class of generalized contractions, which includes the Banach's contractions and the mappings which satisfy

$$d(Tx, Ty) \leq a(d(x, Tx) + d(y, Ty)),\ 0 < a < \frac{1}{2}.$$

In 1974, Ćirić [2] introduced the quasi-contraction

$$d(Tx, Ty) \leq q \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

for some $q < 1$. Also, he proved some fixed point results in these quasi-contraction of the above all possible values (self-map on a metric space). In 1981, Istrătescu [3] introduced a "convexity condition" by proving the generalization of the Banach contraction principle. In 2011, Alghamdi et al. [4] obtained the generalization of the Banach contraction principle to the class of convex contractions on non-normal cone metric spaces. Ghorbanian et al. [5] proved some ordered fixed point results for convex contractions and special mappings which satisfy some contraction conditions and are not necessarily continuous. Khan et al. [6] recently addressed the concepts of the (α, p)-convex contraction and asymptotically T2-regular sequence and showed that the (α, p)-convex contraction reduces to two-sided convex contraction. Additionally, they demonstrated through instances the independence between the concepts of asymptotically T-regular and T2-regular sequences. We refer readers to the researchers in [[7] [10]] for additional details in this manner.

In 2012, Wardowski [11] introduced a new type of contraction called F-contraction and prove a new fixed point theorem concerning F-contraction. Samet et al. [12] introduced a new concept of (α, ψ)-contractive type mappings and established fixed point theorems for such mappings in complete metric spaces. Further, more details (see [13]- [21]). Very recently, Gordji et al. [22] introduced the orthogonal set (in short, O-set) and its properties. Many researchers proved fixed point results used in the O-sets in various metric spaces, see

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*: Corresponding author.

([23]- [31]). Touail and Moutawakil [31] introduced generalized orthogonal sets and $\perp_{\Psi F}$-contractions. They proved some fixed point theorems and gave an application to a differential equation. Mehmood et al. [32] proved some fixed point results for self-maps in the setting of two metrics satisfying F-Lipschitzian conditions of rational-type where F is considered as a semi-Wardowski function with constant $\tau \in \mathcal{R}$ instead of $\tau > 0$. Later on, Ramezani [33] introduced the concepts of generalized convex contractions on orthogonal metric spaces and established some fixed point results.

In this article, we introduce the notion of $(\alpha_\perp - F)$-convex contraction in the background of orthogonal metric space (OMS) inspired by the work of Mahendra Singh, Khan, and Kang [15]. We also provide a case study to illustrate the originality of the outcomes. We investigate the numerical illustration of a nonlinear Volterra integral equation to satisfy all conditions of the fixed point theorem.

Throughout this paper, we use the notations $\mathbb{R}$ represents $(-\infty, +\infty)$, $\mathbb{R}_+$ is $(0, +\infty)$ and $\mathbb{R}_+^0$ represents $[0, +\infty)$ respectively.

Gordji et al. [22] introduced the following new notion of O-set in 2017.

Definition 1.1. *[22] Let Λ be a nonempty set and $\perp \subseteq \Lambda \times \Lambda$ be a binary relation. If $\perp$ satisfies the following conditions:*

$$\exists\ \mathtt{m}_0 \in \Lambda : (\forall \mathtt{m} \in \Lambda, \mathtt{m} \perp \mathtt{m}_0) \quad or \quad (\forall \mathtt{m} \in \Lambda, \mathtt{m}_0 \perp \mathtt{m}),$$

then it is called an orthogonal set (briefly O-set) and it is denoted by $(\Lambda, \perp)$.

Example 1.1. *Let $(\Lambda, \mathcal{G})$ be a metric space and $\Gamma : \Lambda \to \Lambda$ be a Picard operator, that is, there exists $\mathtt{m}^* \in \Lambda$ such that $\lim_{\beta \to \infty} \Gamma^\beta(\ell) = \mathtt{m}^*$ for all $\ell \in \Lambda$. We define $\mathtt{m} \perp \ell$ if*

$$\lim_{\beta \to \infty} (\mathtt{m}, \Gamma^\beta(\ell)) = 0.$$

Then $(\Lambda, \perp)$ is an O-set.

Definition 1.2. *[22] Let $(\Lambda, \perp)$ be an O-set. A sequence $\{\mathtt{m}_\beta\}$ is called an orthogonal sequence (briefly, O-sequence) if*

$$(\forall\ \beta \in \mathbb{N}, \mathtt{m}_\beta \perp \mathtt{m}_{\beta+1}) \quad or \quad (\forall\ \beta \in \mathbb{N}, \mathtt{m}_{\beta+1} \perp \mathtt{m}_\beta).$$

Definition 1.3. *[22] The triplet $(\Lambda, \perp, \mathcal{G})$ is called an orthogonal metric space if $(\Lambda, \perp)$ is an O-set and $(\Lambda, \mathcal{G})$ is a metric space.*

Definition 1.4. *[22] Let $(\Lambda, \perp, \mathcal{G})$ be an orthogonal metric space. Then, a mapping $\Gamma : \Lambda \to \Lambda$ is said to be orthogonal continuous (or $\perp$-continuous) in $\mathtt{m} \in \Lambda$ if for each O-sequence $\{\mathtt{m}_\beta\}$ in Λ with $\mathtt{m}_\beta \to \mathtt{m}$ as $\beta \to \infty$, we have $\Gamma(\mathtt{m}_\beta) \to \Gamma(\mathtt{m})$ as $\beta \to \infty$. Also, Γ is said to be $\perp$-continuous on Λ if Γ is $\perp$-continuous in each $\mathtt{m} \in \Lambda$.*

Definition 1.5. *[22] Let $(\Lambda, \perp, \mathcal{G})$ be an orthogonal metric space. Then, Λ is said to be an orthogonal complete (briefly, O-complete) if every O-Cauchy sequence is convergent.*

Definition 1.6. *[22] Let $(\Lambda, \perp)$ be an O-set. A mapping $\Gamma : \Lambda \to \Lambda$ is said to be $\perp$-preserving if $\Gamma\mathtt{m} \perp \Gamma\ell$ whenever $\mathtt{m} \perp \ell$. Also $\Gamma : \Lambda \to \Lambda$ is said to be weakly $\perp$-preserving if $\Gamma(\mathtt{m}) \perp \Gamma(\ell)$ or $\Gamma(\ell) \perp \Gamma(\mathtt{m})$ whenever $\mathtt{m} \perp \ell$.*

Wardowski [11] introduced the following new notion of F-contraction in 2012.

Definition 1.7. *[11] Let $F \in \Im$ be the set of all mapping, $F : \mathbb{R}_+ \to \mathbb{R}$ satisfying the stipulations:*

(F_1) F *is strictly non decreasing, i.e.,* $\forall\ \delta, \epsilon \in \mathbb{R}_+$ *such that* $\delta < \epsilon, F(\delta) < F(\epsilon)$;

(F_2) *For each sequence* $\{\delta_\beta\} \in \mathbb{N}$, $\lim_{\beta\to\infty} \delta_\beta = 0 \Leftrightarrow \lim_{\beta\to\infty} F(\delta_\beta) = -\infty$;

(F_3) $\exists\ \mathbf{k} \in (0, 1)$ *such that* $\lim_{\delta\to 0^+} \delta^{\mathbf{k}} F(\delta) = 0$.

Definition 1.8. *[11] We say that a self-map* Γ *on* Λ *is an orthogonal* F*-contraction on* $(\Lambda, \mathcal{G})$ *if* $\exists\ F \in \Im$ *and* $\mu > 0$ *such that*

$$\mathcal{G}(\Gamma\mathtt{m}, \Gamma\ell) > 0 \implies \mu + F(\mathcal{G}(\Gamma\mathtt{m}, \Gamma\ell)) \leq F(\mathcal{G}(\mathtt{m}, \ell)), \tag{1}$$

$\forall\ \mathtt{m}, \ell \in \Lambda$ *with* $\mathtt{m}\perp\ell$.

Example 1.2. *[11] Suppose the functions* $F : \mathbb{R}_+ \to \mathbb{R}$ *are in* $\Im$.

(*i*) $F(\delta) = ln\delta$;

(*ii*) $F(\delta) = ln\delta + \delta$;

(*iii*) $F(\delta) = \frac{-1}{\sqrt{\delta}}$;

(*iv*) $F(\delta) = ln(\delta^2 + \delta)$.

Definition 1.9. *[30] A self-map* $\Gamma : \Lambda \to \Lambda$ *defined on a non-void O-set* Λ *and a mapping* $\alpha : \Lambda \times \Lambda \to [0, \infty)$. *Then,* Γ *is said to be an orthogonal* α*-admissible(shortly,* $\alpha_\perp$*-admissible) if* $\mathtt{m}, \ell \in \Lambda$ *with* $\mathtt{m}\perp\ell$, $\alpha(\mathtt{m}, \ell) \geq 1 \Rightarrow \alpha(\Gamma\mathtt{m}, \Gamma\ell) \geq 1$.

Definition 1.10. *[30] Let* $\Gamma : \Lambda \to \Lambda$ *be a self-map and a mapping* $\alpha : \Lambda \times \Lambda \to (-\infty, +\infty)$. *Then,* Γ *is called an orthogonal triangular* α*-admissible(shortly,* $\triangle_{\alpha_\perp}$*-admissible) if*

(Γ_1) $\alpha(\mathtt{m}, \ell) \geq 1 \Rightarrow \alpha(\Gamma\mathtt{m}, \Gamma\ell) \geq 1$, $\forall\ \mathtt{m}, \ell \in \Lambda$ *with* $\mathtt{m}\perp\ell$;

(Γ_2) $\alpha(\mathtt{m}, \mathtt{o}) \geq 1$ *and* $\alpha(\mathtt{o}, \ell) \geq 1$ *imply* $\alpha(\mathtt{m}, \ell) \geq 1$, $\forall\ \mathtt{m}, \ell, \mathtt{o} \in \Lambda$ *with* $\mathtt{m}\perp\mathtt{o}$ *and* $\mathtt{o}\perp\ell$ *imply* $\mathtt{m}\perp\ell$.

Example 1.3. *Let* $\Lambda = [0, \infty)$ *and define* $\Gamma : \Lambda \to \Lambda$ *by* $\Gamma\mathtt{m} = ln(1 + \mathtt{m})\ \forall\ \mathtt{m} \in \Lambda$. *Define* $\alpha : \Lambda \times \Lambda \to [0, \infty)$ *by*

$$\alpha(\mathtt{m}, \ell) = \begin{cases} 1 + \mathtt{m}, & \text{if } \mathtt{m} \geq \ell, \\ 0, & \text{else.} \end{cases}$$

Then, Γ *is* $\alpha_\perp$*-admissible as* $\alpha(\mathtt{m}, \ell) \geq 1 \Rightarrow \alpha(\Gamma\mathtt{m}, \Gamma\ell) \geq 1$ *for* $\mathtt{m} \geq \ell$ *and* $\alpha(\mathtt{m}, \ell) = \alpha(\ell, \mathtt{m})$ $\forall\ \mathtt{m} = \ell$.

Definition 1.11. *Let* $\Lambda \neq \emptyset$ *and let* Γ *be an* $\alpha_\perp$*-admissible mapping on* Λ. *Then* Λ *has the hypothesis(H) if for each* $\mathtt{m}, \ell \in Fix(\Gamma)$ *with* $\mathtt{m}\perp\ell$, $\exists\ \mathtt{o} \in \Lambda$ *such that* $\alpha(\mathtt{m}, \mathtt{o}) \geq 1$ *and* $\alpha(\mathtt{o}, \ell) \geq 1$ *with* $\mathtt{m}\perp\mathtt{o}$ *and* $\mathtt{o}\perp\ell \implies \alpha(\mathtt{m}, \ell) \geq 1$ *with* $\mathtt{m}\perp\ell$.

Definition 1.12. *Let* Γ *be a self-map on an orthogonal metric space* $(\Lambda, \mathcal{G})$. *Then, we say that* Γ *is an orthogonal orbitally continuous on* Λ *if* $\lim_{\mathfrak{k}\to\infty} \Gamma^{\beta_\mathfrak{k}}\mathtt{m} = \mathtt{o}$ *implies that* $\lim_{\mathfrak{k}\to\infty} \Gamma^{\beta_\mathfrak{k}}\mathtt{m} = \Gamma\mathtt{o}$.

A self-map $\Gamma : \Lambda \to \Lambda$ *on a non-void O-set* Λ. *Define* $Fix(\Gamma) = \{\mathtt{m} : \Gamma\mathtt{m} = \mathtt{m}\ \text{for all}\ \mathtt{m} \in \Lambda\}$.

In the next section, we define an orthogonal $\alpha - F$-convex contraction and prove a fixed results of the above mentioned contraction in metric space with an orthogonal concepts.

2. Orthogonal $(\alpha - F)$-convex Contraction

This section will discuss the beauty of orthogonal $(\alpha - F)$-convex contractions. Assume that Γ represents a mapping on $(\Lambda, \perp, \mathcal{G})$. We denote

$$\mathcal{M}^v(\mathtt{m}, \ell) = \max\{\mathcal{G}^v(\mathtt{m}, \ell), \mathcal{G}^v(\Gamma\mathtt{m}, \Gamma\ell), \mathcal{G}^v(\mathtt{m}, \Gamma\mathtt{m}), \mathcal{G}^v(\Gamma\mathtt{m}, \Gamma^2\mathtt{m}), \mathcal{G}^v(\ell, \Gamma\ell), \mathcal{G}^v(\Gamma\ell, \Gamma^2\ell)\}. \tag{2}$$

Definition 2.1. *We say that a self-map Γ on Λ is an orthogonal $(\alpha - F)$-convex contraction(shortly, $(\alpha_\perp - F)$-convex contraction) if $\exists$ two mappings $\alpha : \Lambda \times \Lambda \to \mathbb{R}^0_+$ and $F \in \Im$ such that*

$$\mathcal{G}^\upsilon(\Gamma^2\mathtt{m}, \Gamma^2\ell) > 0 \implies \mu + F(\alpha(\mathtt{m}, \ell)\mathcal{G}^\upsilon(\Gamma^2\mathtt{m}, \Gamma^2\ell)) \leq F(\mathcal{M}^\upsilon(\mathtt{m}, \ell)), \tag{3}$$

$\forall$ $\mathtt{m}, \ell \in \Lambda$ with $\mathtt{m}\perp\ell$, where $\upsilon \in [1, \infty)$ and $\mu > 0$.

Example 2.1. *Let $\Lambda = [0, 1]$ with $\mathcal{G}(\mathtt{m}, \ell) = |\mathtt{m} - \ell|$. Define a mapping $\Gamma : \Lambda \to \Lambda$ by $\Gamma\mathtt{m} = \frac{\mathtt{m}^2}{2} + \frac{1}{4}$ $\forall$ $\mathtt{m} \in \Lambda$ with $\alpha(\mathtt{m}, \ell) = 1$ for all $\mathtt{m}, \ell \in \Lambda$ with $\mathtt{m}\perp\ell$. Then, Γ is $\alpha_\perp$-admissible. Now, we get Γ is non-expansive, since we obtain*

$$|\Gamma\mathtt{m} - \Gamma\ell| = \frac{1}{2}|\mathtt{m}^2 - \ell^2| \leq |\mathtt{m} - \ell| \ \forall\ \mathtt{m}, \ell \in \Lambda \text{ with } \mathtt{m}\perp\ell.$$

Setting $F \in \Im$ such that $F(\mathfrak{x}) = ln\mathfrak{x}, \mathfrak{x} > 0$. Then, $\forall$ $\mathtt{m}, \ell \in \Lambda$ with $\mathtt{m}\perp\ell$ and $\mathtt{m} \neq \ell$, we obtain

$$\begin{aligned}
\alpha(\mathtt{m}, \ell|\Gamma^2\mathtt{m}, \Gamma^2\ell| &= |\Gamma^2\mathtt{m}, \Gamma^2\ell| \\
&= \frac{1}{8}(|(\mathtt{m}^4 + \mathtt{m}^2) - (\ell^4 + \ell^2)|) \\
&\leq \frac{1}{8}(|\mathtt{m}^4 - \ell^4| + |\mathtt{m}^2 - \ell^2|) \\
&\leq \frac{1}{2}|\Gamma\mathtt{m} - \Gamma\ell| + \frac{1}{4}|\mathtt{m} - \ell| \\
&\leq \frac{3}{4}\max\{|\Gamma\mathtt{m} - \Gamma\ell|, |\mathtt{m} - \ell|\} \\
&\leq e^{-\mu}\flat^1(\mathtt{m}, \ell),
\end{aligned}$$

where $-\mu = In(\frac{3}{4})$. Applying logarithm on both sides, we have

$$\mu + F(\alpha(\mathtt{m}, \ell)\mathcal{G}(\Gamma^2\mathtt{m}, \Gamma^2\ell)) \leq F(\flat^1(\mathtt{m}, \ell)).$$

We conclude that Γ is an $(\alpha_\perp - F)$-convex contraction with $\upsilon = 1$.

3. Fixed point results of an $(\alpha_\perp - F)$-convex contraction

First, we prove the following lemma using an $(\alpha_\perp - F)$-convex contraction.

Lemma 3.1. *Let $(\Lambda, \perp, \mathcal{G})$ be an OMS and $\Gamma : \Lambda \to \Lambda$ be an $(\alpha_\perp - F)$-convex contraction the following affirmations hold:*

- *(i)* *Γ is $\alpha_\perp$-admissible;*
- *(ii)* *$\exists$ $\mathtt{m}_0 \in \Lambda$ such that $\alpha(\mathtt{m}_0, \Gamma\mathtt{m}_0) \geq 1$;*
- *(iii)* *$\perp$-preserving.*

Define an O-sequence $\{\mathtt{m}_\beta\}$ in Λ by $\mathtt{m}_{\beta+1} = \Gamma\mathtt{m}_\beta = \Gamma^{\beta+1}\mathtt{m}_0$ for all $\beta \geq 0$, then $\{\mathcal{G}^\upsilon(\mathtt{m}_\beta, \mathtt{m}_{\beta+1})\}$ is strictly non-increasing sequence in Λ.

Proof. By the definition of orthogonality, there exists $\mathtt{m}_0 \in \Lambda$ be such that

$$(\forall \ell \in \Lambda, \mathtt{m}_0\perp\ell) \text{ or } (\forall \ell \in \Lambda, \ell\perp\mathtt{m}_0).$$

It follows that $\mathtt{m}_0\perp\Gamma(\mathtt{m}_0)$ or $\Gamma(\mathtt{m}_0)\perp\mathtt{m}_0$. Let

$$\mathtt{m}_1 := \Gamma(\mathtt{m}_0); \mathtt{m}_2 = \Gamma(\mathtt{m}_1) = \Gamma^2(\mathtt{m}_0); \ldots; \mathtt{m}_{\beta+1} = \Gamma(\mathtt{m}_\beta) = \Gamma^{\beta+1}(\mathtt{m}_0)$$

for all $\beta \in \mathbb{N} \cup \{0\}$.

If $\mathtt{m}_\beta = \mathtt{m}_{\beta+1}$ for any $\beta \in \mathbb{N} \cup \{0\}$, then, it is clear that Λ_β is a fixed point of Γ. Assume that $\mathtt{m}_\beta \neq \mathtt{m}_{\beta+1}$ for all $\beta \in \mathbb{N} \cup \{0\}$. Thus, we have $\mathcal{G}(\mathtt{m}_\beta, \mathtt{m}_{\beta+1}) > 0$ for all $\beta \in \mathbb{N} \cup \{0\}$. Since Γ is $\perp$-preserving, we have

$$\mathtt{m}_\beta \perp \mathtt{m}_{\beta+1} \quad \text{or} \quad \mathtt{m}_{\beta+1} \perp \mathtt{m}_\beta \tag{4}$$

for all $\beta \in \mathbb{N} \cup \{0\}$. This implies that $\{\mathtt{m}_\beta\}$ is an O-sequence. Postulating that $\mathtt{m}_\beta \neq \mathtt{m}_{\beta+1}$ $\forall\ \beta \geq 0$. Then, $\mathcal{G}(\mathtt{m}_\beta, \mathtt{m}_{\beta+1}) > 0\ \forall\ \beta \geq 0$. Letting $\mathfrak{v} = \max\{\mathcal{G}^v(\mathtt{m}_0, \mathtt{m}_1), \mathcal{G}^v(\mathtt{m}_1, \mathtt{m}_2)\}$. From (2), taking $\mathtt{m} = \mathtt{m}_0$ and $\ell = \mathtt{m}_1$, we obtain

$$\begin{aligned} \mathcal{M}^v(\mathtt{m}_0, \mathtt{m}_1) &= \max\{\mathcal{G}^v(\mathtt{m}_0, \mathtt{m}_1), \mathcal{G}^v(\Gamma\mathtt{m}_0, \Gamma\mathtt{m}_1), \mathcal{G}^v(\mathtt{m}_0, \Gamma\mathtt{m}_0), \mathcal{G}^v(\Gamma\mathtt{m}_0, \Gamma^2\mathtt{m}_0), \mathcal{G}^v(\mathtt{m}_1, \Gamma\mathtt{m}_1), \mathcal{G}^v(\Gamma\mathtt{m}_1, \Gamma^2\mathtt{m}_1)\} \\ &= \max\{\mathcal{G}^v(\mathtt{m}_0, \mathtt{m}_1), \mathcal{G}^v(\mathtt{m}_1, \mathtt{m}_2), \mathcal{G}^v(\mathtt{m}_0, \mathtt{m}_1), \mathcal{G}^v(\mathtt{m}_1, \mathtt{m}_2), \mathcal{G}^v(\mathtt{m}_1, \mathtt{m}_2), \mathcal{G}^v(\mathtt{m}_2, \mathtt{m}_3)\} \\ &= \max\{\mathcal{G}^v(\mathtt{m}_0, \mathtt{m}_1), \mathcal{G}^v(\mathtt{m}_1, \mathtt{m}_2), \mathcal{G}^v(\mathtt{m}_2, \mathtt{m}_3)\}. \end{aligned} \tag{5}$$

By (F_1) and $\alpha(\mathtt{m}_0, \mathtt{m}_1) \geq 1$, by (3) and (5), we obtain

$$\begin{aligned} F(\mathcal{G}^v(\mathtt{m}_2, \mathtt{m}_3)) &= F(\mathcal{G}^v(\Gamma^2\mathtt{m}_0, \Gamma^2\mathtt{m}_1)) \\ &\leq F(\alpha(\mathtt{m}_0, \mathtt{m}_1)\mathcal{G}^v(\Gamma^2\mathtt{m}_0, \Gamma^2\mathtt{m}_1)) \\ &\leq F(\mathcal{M}^v(\mathtt{m}_0, \mathtt{m}_1)) - \mu \\ &= F(\max\{\mathcal{G}^v(\mathtt{m}_0, \mathtt{m}_1), \mathcal{G}^v(\mathtt{m}_1, \mathtt{m}_2), \mathcal{G}^v(\mathtt{m}_2, \mathtt{m}_3)\}) - \mu \\ &\leq F(\max\{\mathfrak{v}, \mathcal{G}^v(\mathtt{m}_2, \mathtt{m}_3)\}) - \mu. \end{aligned} \tag{6}$$

If $\max\{\mathfrak{v}, \mathcal{G}^v(\mathtt{m}_2, \mathtt{m}_3)\} = \mathcal{G}^v(\mathtt{m}_2, \mathtt{m}_3)$, then (6) gives

$$F(\mathcal{G}^v(\mathtt{m}_2, \mathtt{m}_3)) \leq F(\mathcal{G}^v(\mathtt{m}_2, \mathtt{m}_3)) - \mu < F(\mathcal{G}^v(\mathtt{m}_2, \mathtt{m}_3)).$$

This is a contradiction. It follows that

$$F(\mathcal{G}^v(\mathtt{m}_2, \mathtt{m}_3)) \leq F(\mathfrak{v}) - \mu < F(\mathfrak{v}).$$

Since $\mu > 0$ and by (F_1), we have

$$\mathcal{G}^v(\mathtt{m}_2, \mathtt{m}_3) < \mathfrak{v} = \max\{\mathcal{G}^v(\mathtt{m}_0, \mathtt{m}_1), \mathcal{G}^v(\mathtt{m}_1, \mathtt{m}_2)\}.$$

Again, by (2) taking with $\mathtt{m} = \mathtt{m}_1$ and $\ell = \mathtt{m}_2$, we get

$$\begin{aligned} \mathcal{M}^v(\mathtt{m}_1, \mathtt{m}_2) &= \max\{\mathcal{G}^v(\mathtt{m}_1, \mathtt{m}_2), \mathcal{G}^v(\Gamma\mathtt{m}_1, \Gamma\mathtt{m}_2), \mathcal{G}^v(\mathtt{m}_1, \Gamma\mathtt{m}_1), \mathcal{G}^v(\Gamma\mathtt{m}_1, \Gamma^2\mathtt{m}_1), \mathcal{G}^v(\mathtt{m}_2, \Gamma\mathtt{m}_2), \mathcal{G}^v(\Gamma\mathtt{m}_2, \Gamma^2\mathtt{m}_2)\} \\ &= \max\{\mathcal{G}^v(\mathtt{m}_1, \mathtt{m}_2), \mathcal{G}^v(\mathtt{m}_2, \mathtt{m}_3), \mathcal{G}^v(\mathtt{m}_1, \mathtt{m}_2), \mathcal{G}^v(\mathtt{m}_2, \mathtt{m}_3), \mathcal{G}^v(\mathtt{m}_2, \mathtt{m}_3), \mathcal{G}^v(\mathtt{m}_3, \mathtt{m}_4)\} \\ &= \max\{\mathcal{G}^v(\mathtt{m}_1, \mathtt{m}_2), \mathcal{G}^v(\mathtt{m}_2, \mathtt{m}_3), \mathcal{G}^v(\mathtt{m}_3, \mathtt{m}_4)\}. \end{aligned} \tag{7}$$

By (3) and (7), we obtain

$$\begin{aligned} F(\mathcal{G}^v(\mathtt{m}_3, \mathtt{m}_4)) &= F(\mathcal{G}^v(\Gamma^2\mathtt{m}_1, \Gamma^2\mathtt{m}_2)) \\ &\leq F(\alpha(\mathtt{m}_1, \mathtt{m}_2)\mathcal{G}^v(\Gamma^2\mathtt{m}_1, \Gamma^2\mathtt{m}_2)) \\ &\leq F(\mathcal{M}^v(\mathtt{m}_1, \mathtt{m}_2)) - \mu \\ &= F(\max\{\mathcal{G}^v(\mathtt{m}_1, \mathtt{m}_2), \mathcal{G}^v(\mathtt{m}_2, \mathtt{m}_3), \mathcal{G}^v(\mathtt{m}_3, \mathtt{m}_4)\}) - \mu. \end{aligned}$$

If $\max\{\mathcal{G}^v(\mathtt{m}_1, \mathtt{m}_2), \mathcal{G}^v(\mathtt{m}_2, \mathtt{m}_3), \mathcal{G}^v(\mathtt{m}_3, \mathtt{m}_4)\} = \mathcal{G}^v(\mathtt{m}_3, \mathtt{m}_4)$, then we obtain

$$F(\mathcal{G}^v(\mathtt{m}_3, \mathtt{m}_4)) \leq F(\mathcal{G}^v(\mathtt{m}_3, \mathtt{m}_4)) - \mu < F(\mathcal{G}^v(\mathtt{m}_3, \mathtt{m}_4)).$$

Which is a contradiction. We obtain

$$\max\{\mathcal{G}^v(\mathtt{m}_1, \mathtt{m}_2), \mathcal{G}^v(\mathtt{m}_2, \mathtt{m}_3)\} > \mathcal{G}^v(\mathtt{m}_3, \mathtt{m}_4).$$

Therefore,

$$\mathfrak{v} > \mathcal{G}^{\upsilon}(\mathtt{m}_2, \mathtt{m}_3) > \mathcal{G}^{\upsilon}(\mathtt{m}_3, \mathtt{m}_4).$$

Continuing in this way, inductively prove that the non-increasing O-sequence $\{\mathcal{G}^{\upsilon}(\mathtt{m}_\beta, \mathtt{m}_{\beta+1})\}$ is strictly in Λ. □

Theorem 3.2. *Let $(\Lambda, \perp, \mathcal{G})$ be an O-complete metric space and $\Gamma : \Lambda \to \Lambda$ be an $(\alpha_\perp - F)$-convex contraction the following affirmations hold:*

(i) Γ is $\alpha_\perp$-admissible;
(ii) $\exists\ \mathtt{m}_0 \in \Lambda$ such that $\alpha(\mathtt{m}_0, \Gamma\mathtt{m}_0) \geq 1$;
(iii) Γ is $\perp$-continuous or, $\perp$-orbitally continuous on Λ;
(iv) $\perp$-preserving.

Then Γ has a fixed point in Λ. Moreover, for any $\mathtt{m}_0 \in \Lambda$ if $\mathtt{m}_{\beta+1} = \Gamma^{\beta+1}\mathtt{m}_0 \neq \Gamma\mathtt{m}_\beta$ for all $\beta \in \mathbb{N} \cup \{0\}$, then $\lim_{\beta\to\infty} \Gamma^\beta \mathtt{m}_0 = \mathfrak{o}$.

Proof. By the definition of orthogonality, there exists $\mathtt{m}_0 \in \Lambda$ be such that

$$(\forall \ell \in \Lambda, \mathtt{m}_0 \perp \ell) \text{ or } (\forall \ell \in \Lambda, \ell \perp \mathtt{m}_0).$$

It follows that $\mathtt{m}_0 \perp \Gamma(\mathtt{m}_0)$ or $\Gamma(\mathtt{m}_0) \perp \mathtt{m}_0$. Let

$$\mathtt{m}_1 := \Gamma(\mathtt{m}_0); \mathtt{m}_2 = \Gamma(\mathtt{m}_1) = \Gamma^2(\mathtt{m}_0); \ldots ; \mathtt{m}_{\beta+1} = \Gamma(\mathtt{m}_\beta) = \Gamma^{\beta+1}(\mathtt{m}_0)$$

for all $\beta \in \mathbb{N} \cup \{0\}$.

If $\mathtt{m}_\beta = \mathtt{m}_{\beta+1}$ for any $\beta \in \mathbb{N} \cup \{0\}$, then it is clear that Λ_β is a fixed point of Γ. Assume that $\mathtt{m}_\beta \neq \mathtt{m}_{\beta+1}$ for all $\beta \in \mathbb{N} \cup \{0\}$. Thus, we have $\mathcal{G}(\mathtt{m}_\beta, \mathtt{m}_{\beta+1}) > 0$ for all $\beta \in \mathbb{N} \cup \{0\}$. Since Γ is $\perp$-preserving, we have

$$\mathtt{m}_\beta \perp \mathtt{m}_{\beta+1} \quad \text{or} \quad \mathtt{m}_{\beta+1} \perp \mathtt{m}_\beta \tag{8}$$

for all $\beta \in \mathbb{N} \cup \{0\}$. This implies that $\{\mathtt{m}_\beta\}$ is an O-sequence.

Now, we postulate that $\mathtt{m}_\beta \neq \mathtt{m}_{\beta+1}\ \forall\ \beta \geq 0$. Then, $\mathcal{G}(\mathtt{m}_\beta, \mathtt{m}_{\beta+1}) > 0\ \forall\ \beta \geq 0$. By (i), we have $\alpha(\mathtt{m}_0, \Gamma\mathtt{m}_0) \geq 1 \Rightarrow \alpha(\mathtt{m}_1, \mathtt{m}_2) = \alpha(\Gamma\mathtt{m}_0, \Gamma^2\mathtt{m}_0) \geq 1$. Therefore, inductively shows that $\alpha(\mathtt{m}_\beta, \mathtt{m}_{\beta+1}) = \alpha(\Gamma^\beta\mathtt{m}_0, \Gamma^{\beta+1}\mathtt{m}_0) \geq 1\ \forall\ \beta \geq 0$. Letting $\mathfrak{v} = \max\{\mathcal{G}^{\upsilon}(\mathtt{m}_0, \mathtt{m}_1), \mathcal{G}^{\upsilon}(\mathtt{m}_1, \mathtt{m}_2)\}$.
Now form (2), taking $\mathtt{m} = \mathtt{m}_{\beta-2}$ and $\ell = \mathtt{m}_{\beta-1}$ with $\beta \geq 2$, we have

$$\begin{aligned}
\mathcal{M}^{\upsilon}(\mathtt{m}_{\beta-2}, \mathtt{m}_{\beta-1}) &= \max\{\mathcal{G}^{\upsilon}(\mathtt{m}_{\beta-2}, \mathtt{m}_{\beta-1}), \mathcal{G}^{\upsilon}(\Gamma\mathtt{m}_{\beta-2}, \Gamma\mathtt{m}_{\beta-1}), \mathcal{G}^{\upsilon}(\mathtt{m}_{\beta-2}, \Gamma\mathtt{m}_{\beta-2}), \mathcal{G}^{\upsilon}(\Gamma\mathtt{m}_{\beta-2}, \Gamma^2\mathtt{m}_{\beta-2}), \\
&\quad \mathcal{G}^{\upsilon}(\mathtt{m}_{\beta-1}, \Gamma\mathtt{m}_{\beta-1}), \mathcal{G}^{\upsilon}(\Gamma\mathtt{m}_{\beta-1}, \Gamma^2\mathtt{m}_{\beta-1})\} \\
&= \max\{\mathcal{G}^{\upsilon}(\mathtt{m}_{\beta-2}, \mathtt{m}_{\beta-1}), \mathcal{G}^{\upsilon}(\mathtt{m}_{\beta-1}, \mathtt{m}_\beta), \mathcal{G}^{\upsilon}(\mathtt{m}_{\beta-2}, \mathtt{m}_{\beta-1}), \mathcal{G}^{\upsilon}(\mathtt{m}_{\beta-1}, \mathtt{m}_\beta), \\
&\quad \mathcal{G}^{\upsilon}(\mathtt{m}_{\beta-1}, \mathtt{m}_\beta), \mathcal{G}^{\upsilon}(\mathtt{m}_\beta, \mathtt{m}_{\beta+1})\} \\
&= \max\{\mathcal{G}^{\upsilon}(\mathtt{m}_{\beta-2}, \mathtt{m}_{\beta-1}), \mathcal{G}^{\upsilon}(\mathtt{m}_{\beta-1}, \mathtt{m}_\beta), \mathcal{G}^{\upsilon}(\mathtt{m}_\beta, \mathtt{m}_{\beta+1})\}.
\end{aligned}$$

Since Γ is an $(\alpha_\perp - F)$-convex contraction mapping, we have

$$\begin{aligned}
F(\mathcal{G}^{\upsilon}(\mathtt{m}_\beta, \mathtt{m}_{\beta+1})) &= F(\mathcal{G}^{\upsilon}(\Gamma^2\mathtt{m}_{\beta-2}, \Gamma^2\mathtt{m}_{\beta-1})) \\
&\leq F(\alpha(\mathtt{m}_{\beta-2}, \mathtt{m}_{\beta-1})\mathcal{G}^{\upsilon}(\Gamma^2\mathtt{m}_{\beta-2}, \Gamma^2\mathtt{m}_{\beta-1})) \\
&\leq F(\mathcal{M}^{\upsilon}(\mathtt{m}_{\beta-2}, \mathtt{m}_{\beta-1})) - \mu \\
&\leq F(\max\{\mathcal{G}^{\upsilon}(\mathtt{m}_{\beta-2}, \mathtt{m}_{\beta-1}), \mathcal{G}^{\upsilon}(\mathtt{m}_{\beta-1}, \mathtt{m}_\beta), \mathcal{G}^{\upsilon}(\mathtt{m}_\beta, \mathtt{m}_{\beta+1})\}) - \mu.
\end{aligned}$$

If $\max\{\mathcal{G}^{\upsilon}(\mathtt{m}_{\beta-2}, \mathtt{m}_{\beta-1}), \mathcal{G}^{\upsilon}(\mathtt{m}_{\beta-1}, \mathtt{m}_\beta), \mathcal{G}^{\upsilon}(\mathtt{m}_\beta, \mathtt{m}_{\beta+1})\} = \mathcal{G}^{\upsilon}(\mathtt{m}_\beta, \mathtt{m}_{\beta+1})$, then we obtain

$$F(\mathcal{G}^{\upsilon}(\mathtt{m}_{\beta}, \mathtt{m}_{\beta+1})) \leq F(\mathcal{G}^{\upsilon}(\mathtt{m}_{\beta}, \mathtt{m}_{\beta+1})) - \mu < F(\mathcal{G}^{\upsilon}(\mathtt{m}_{\beta}, \mathtt{m}_{\beta+1})).$$

This is a contradiction. Therefore

$$F(\mathcal{G}^{\upsilon}(\mathtt{m}_{\beta}, \mathtt{m}_{\beta+1})) \leq F(\max\{\mathcal{G}^{\upsilon}(\mathtt{m}_{\beta-2}, \mathtt{m}_{\beta-1}), \mathcal{G}^{\upsilon}(\mathtt{m}_{\beta-1}, \mathtt{m}_{\beta})\}) - \mu.$$

Since $\{\mathcal{G}^{\upsilon}(\mathtt{m}_{\beta}, \mathtt{m}_{\beta+1})\}$ is strictly non-increasing. Therefore, we obtain

$$F(\mathcal{G}^{\upsilon}(\mathtt{m}_{\beta}, \mathtt{m}_{\beta+1})) \leq F(\mathcal{G}^{\upsilon}(\mathtt{m}_{\beta-2}, \mathtt{m}_{\beta-1})) - \mu \leq ... \leq F(\mathfrak{v}) - J\mu, \tag{9}$$

whenever $\beta = 2J$ or $\beta = 2J + 1$ for $J \geq 1$.

From (7), we obtain

$$\lim_{\beta\to\infty} F(\mathcal{G}^{\upsilon}(\mathtt{m}_{\beta}, \mathtt{m}_{\beta+1})) = -\infty. \tag{10}$$

Therefore, by (F2) and by equation (10), we have

$$\lim_{\beta\to\infty} \mathcal{G}(\mathtt{m}_{\beta}, \mathtt{m}_{\beta+1}) = 0. \tag{11}$$

By (F3), $\exists\ 0 < \mathbf{k} < 1$ such that

$$\lim_{\beta\to\infty} [\mathcal{G}^{\upsilon}(\mathtt{m}_{\beta}, \mathtt{m}_{\beta+1})]^{\mathbf{k}} F(\mathcal{G}^{\upsilon}(\mathtt{m}_{\beta}, \mathtt{m}_{\beta+1})) = 0. \tag{12}$$

Also, by equation (9), we get

$$[\mathcal{G}^{\upsilon}(\mathtt{m}_{\beta}, \mathtt{m}_{\beta+1})]^{\mathbf{k}}[F(\mathcal{G}^{\upsilon}(\mathtt{m}_{\beta}, \mathtt{m}_{\beta+1})) - F(\mathfrak{v})] \leq -[\mathcal{G}^{\upsilon}(\mathtt{m}_{\beta}, \mathtt{m}_{\beta+1})]^{\mathbf{k}}\ J\mu \leq 0, \tag{13}$$

where $\beta = 2J$ or $\beta = 2J + 1$ for $J \geq 1$. Setting $\beta \to \infty$ in (13) along with (11) and (12), we have

$$\lim_{\beta\to\infty}\ J[\mathcal{G}(\mathtt{m}_{\beta}, \mathtt{m}_{\beta+1})]^{\mathbf{k}} = 0. \tag{14}$$

Now, we arise two cases.

Case-(i): If β is even and $\beta \geq 2$, then by equation (14), we have

$$\lim_{\beta\to\infty} \beta[\mathcal{G}(\mathtt{m}_{\beta}, \mathtt{m}_{\beta+1})]^{\mathbf{k}} = 0. \tag{15}$$

Case-(ii): If β is odd and $\beta \geq 3$, then by equation (14), we have

$$\lim_{\beta\to\infty} (\beta - 1)[\mathcal{G}(\mathtt{m}_{\beta}, \mathtt{m}_{\beta+1})]^{\mathbf{k}} = 0. \tag{16}$$

Using (11), (16) gives

$$\lim_{\beta\to\infty} \beta[\mathcal{G}(\mathtt{m}_{\beta}, \mathtt{m}_{\beta+1})]^{\mathbf{k}} = 0. \tag{17}$$

We conclude the above cases that, $\exists\ \beta_1 \in \mathbb{N}$ such that

$$\beta[\mathcal{G}(\mathtt{m}_{\beta}, \mathtt{m}_{\beta+1})]^{\mathbf{k}} \leq 1\ \forall\ \beta \geq \beta_1.$$

Therefore, we obtain

$$\mathcal{G}(\mathtt{m}_{\beta}, \mathtt{m}_{\beta+1}) \leq \frac{1}{\beta^{\frac{1}{\mathbf{k}}}},\ \forall\ \beta \geq \beta_1.$$

Now, to prove the O-sequence $\{\mathtt{m}_{\beta}\}$ is a Cauchy. $\forall\ \upsilon > \mathfrak{q} \geq \beta_1$, we have

$$\mathcal{G}(\mathtt{m}_{\upsilon}, \mathtt{m}_{\mathfrak{q}}) \leq \mathcal{G}(\mathtt{m}_{\upsilon}, \mathtt{m}_{\upsilon-1}) + \mathcal{G}(\mathtt{m}_{\upsilon-1}, \mathtt{m}_{\upsilon-2}) + ... + \mathcal{G}(\mathtt{m}_{\mathfrak{q}+1}, \mathtt{m}_{\mathfrak{q}}) < \sum_{\mathfrak{k}=\mathfrak{q}}^{\infty} \mathcal{G}(\mathtt{m}_{\mathfrak{k}}, \mathtt{m}_{\mathfrak{k}+1}) \leq \sum_{\mathfrak{k}=\mathfrak{q}}^{\infty} \frac{1}{\mathfrak{k}^{\frac{1}{\mathbf{k}}}}.$$

Taking $\mathfrak{q} \to \infty$, we get $\lim_{\upsilon,\mathfrak{q}\to\infty} \mathcal{G}(\mathtt{m}_\upsilon, \mathtt{m}_\mathfrak{q}) = 0$, since $\sum_{\mathfrak{k}=\mathfrak{q}}^{\infty} \frac{1}{\mathfrak{k}^{\frac{1}{k}}}$ is convergent. This proves that the O-sequence $\{\mathtt{m}_\beta\}$ is a Cauchy in Λ. By O-completeness property, $\exists\ \mathfrak{o} \in \Lambda$ such that $\lim_{\beta\to\infty} \mathtt{m}_\beta = \mathfrak{o}$. Next, to prove $\mathfrak{o}$ is a fixed point of Γ. By (iii), we obtain

$$\mathcal{G}(\mathfrak{o}, \Gamma\mathfrak{o}) = \lim_{\beta\to\infty} \mathcal{G}(\mathtt{m}_\beta, \Gamma\mathtt{m}_\beta) = \lim_{\beta\to\infty} \mathcal{G}(\mathtt{m}_\beta, \mathtt{m}_{\beta+1}) = 0.$$

This implies that $\mathfrak{o}$ is a fixed point of Γ.

Also, by (iii), we get

$$\mathtt{m}_{\beta+1} = \Gamma\mathtt{m}_\beta = \Gamma(\Gamma^\beta \mathtt{m}_0) \to \Gamma\mathfrak{o} \text{ as } \beta \to \infty.$$

By O-completeness, we obtain $\Gamma\mathfrak{o} = \mathfrak{o}$. Therefore, $Fix(\Gamma) \neq 0$.

To prove the uniqueness property of fixed point, let $\ell^* \in \Lambda$ be a fixed point of Γ. Then we have $\Gamma^\beta(\mathfrak{o}^*) = \mathfrak{o}^*$ and $\Gamma^\beta(\ell^*) = \ell^*$ for all $\beta \in \mathbb{N}$. By the choice of $\mathfrak{o}_0$ in the first part of proof, we have

$$[\mathfrak{o}_0 \perp \mathfrak{o}^* \text{ and } \mathfrak{o}_0 \perp \ell^*] \ \textit{or}\ [\mathfrak{o}^* \perp \mathfrak{o}_0 \text{ and } \ell^* \perp \mathfrak{o}_0].$$

Since Γ is $\perp$-preserving, we have

$$[\Gamma^\beta \mathfrak{o}_0 \perp \Gamma^\beta \mathfrak{o}^* \text{ and } \Gamma^\beta \mathfrak{o}_0 \perp \Gamma^\beta \ell^*] \ \textit{or}\ [\Gamma^\beta \mathfrak{o}^* \perp \Gamma^\beta \mathfrak{o}_0 \text{ and } \Gamma^\beta \ell^* \perp \Gamma^\beta \mathfrak{o}_0].$$

for all $\beta \in \mathbb{N}$. Therefore, by the triangle inequality, we have

$$\begin{aligned}
\mathcal{G}(\mathfrak{o}^*, \ell^*) &= \mathcal{G}(\Gamma^\beta \mathfrak{o}^*, \Gamma^\beta \ell^*) \\
&\leq \mathcal{G}(\Gamma^\beta \mathfrak{o}^*, \Gamma^\beta \mathfrak{o}_0) + \mathcal{G}(\Gamma^\beta \mathfrak{o}_0, \Gamma^\beta \ell^*) \\
&\leq \mathcal{G}(\mathfrak{o}^*, \mathfrak{o}_0) + \mathcal{G}(\mathtt{m}\mathfrak{o}_0, \ell^*) \\
&\leq \mathcal{G}(\mathfrak{o}^*, \ell^*).
\end{aligned}$$

This is a contradiction. Thus it follows that $\mathfrak{o}^* = \ell^*$. Finally, let $\mathfrak{o} \in \Lambda$ be arbitrary. Similarly, we have

$$[\mathfrak{o}_0 \perp \mathfrak{o}^* \text{ and } \mathfrak{o}_0 \perp \mathfrak{o}] \ \textit{or}\ [\mathfrak{o}^* \perp \mathfrak{o}_0 \text{ and } \mathfrak{o} \perp \mathfrak{o}_0].$$

Since Γ is $\perp$-preserving, we have

$$[\Gamma^\beta \mathfrak{o}_0 \perp \Gamma^\beta \mathfrak{o}^* \text{ and } \Gamma^\beta \mathfrak{o}_0 \perp \Gamma^\beta \mathfrak{o}] \ \textit{or}\ [\Gamma^\beta \mathfrak{o}^* \perp \Gamma^\beta \mathfrak{o}_0 \text{ and } \Gamma^\beta \mathfrak{o} \perp \Gamma^\beta \mathfrak{o}_0].$$

for all $\beta \in \mathbb{N}$. Hence, for all $\beta \in \mathbb{N}$, we get

$$\begin{aligned}
\mathcal{G}(\mathfrak{o}^*, \Gamma^\beta \mathfrak{o}) &= \mathcal{G}(\Gamma^\beta \mathfrak{o}^*, \Gamma^\beta \mathfrak{o}) \\
&\leq \mathcal{G}(\Gamma^\beta \mathfrak{o}^*, \Gamma^\beta \mathfrak{o}_0) + \mathcal{G}(\Gamma^\beta \mathfrak{o}_0, \Gamma^\beta \mathfrak{o}) \\
&\leq \mathcal{G}(\mathfrak{o}^*, \mathfrak{o}_0) + \mathcal{G}(\mathtt{m}\mathfrak{o}_0, \mathfrak{o}) \\
&\leq \mathcal{G}(\mathfrak{o}^*, \mathfrak{o}).
\end{aligned}$$

Hence the proof is completed. □

Corollary 3.3. *Let* $(\Lambda, \perp, \mathcal{G})$ *be an O-complete metric space and a mapping* $\alpha : \Lambda \times \Lambda \to [0, \infty)$. *Postulating that* $\Gamma : \Lambda \to \Lambda$ *be a self-map the following affirmations hold* *(i)* $\forall$ $\mathtt{m}, \ell \in \Lambda$ *with* $\mathtt{m} \perp \ell$,

$$\begin{aligned} &\mathcal{G}^{\upsilon}(\Gamma^2\mathtt{m}, \Gamma^2\ell) > 0 \\ &\implies \alpha(\mathtt{m}, \ell)\mathcal{G}(\Gamma^2\mathtt{m}, \Gamma^2\ell) \leq \Bbbk \max\{\mathcal{G}(\mathtt{m}, \ell), \mathcal{G}(\Gamma\mathtt{m}, \Gamma\ell), \mathcal{G}(\mathtt{m}, \Gamma\mathtt{m}), \mathcal{G}(\Gamma\mathtt{m}, \Gamma^2\mathtt{m}), \\ &\mathcal{G}(\ell, \Gamma\ell), \mathcal{G}(\Gamma\ell, \Gamma^2\ell)\} \end{aligned} \tag{18}$$

where $\Bbbk \in [0, 1)$*;*

(ii) Γ *is* $\alpha_\perp$*-admissible;*

(iii) $\exists$ $\mathtt{m}_0 \in \Lambda$ *such that* $\alpha(\mathtt{m}_0, \Gamma\mathtt{m}_0) \geq 1$*;*

(iv) Γ *is* $\perp$*-continuous or,* $\perp$*-orbitally continuous on* Λ*;*

(v) $\perp$*-preserving.*

Then, Γ *has a fixed point in* Λ*. Moreover, for any* $\mathtt{m}_0 \in \Lambda$ *if* $\mathtt{m}_{\beta+1} = \Gamma^{\beta+1}\mathtt{m}_0 \neq \Gamma^\beta\mathtt{m}_0$ $\forall$ $\beta \in \mathbb{N} \cup \{0\}$*, then* $\lim_{\beta\to\infty} \Gamma^\beta\mathtt{m}_0 = \mathfrak{o}$*.*

Proof. Setting $F(\mathfrak{x}) = In(\mathfrak{x}), \mathfrak{x} > 0$. Obviously, $F \in \Im$. Applying logarithm on both sides of (18), we get

$$\begin{aligned} &- In\Bbbk + In\alpha(\mathtt{m}, \ell)\mathcal{G}(\Gamma^2\mathtt{m}, \Gamma^2\ell) \\ &\leq In(\max\{\mathcal{G}(\mathtt{m}, \ell), \mathcal{G}(\Gamma\mathtt{m}, \Gamma\ell), \mathcal{G}(\mathtt{m}, \Gamma\mathtt{m}), \mathcal{G}(\Gamma\mathtt{m}, \Gamma^2\mathtt{m}), \mathcal{G}(\ell, \Gamma\ell), \mathcal{G}(\Gamma\ell, \Gamma^2\ell)\}), \end{aligned}$$

which implies that

$$\mu + F(\alpha(\mathtt{m}, \ell)\mathcal{G}(\Gamma^2\mathtt{m}, \Gamma^2\ell)) \leq F(\mathcal{M}^1(\mathtt{m}, \ell))$$

$\forall$ $\mathtt{m}, \ell \in \Lambda$ with $\mathtt{m} \perp \ell$ and $\mathtt{m} \neq \ell$ where $\mu = -In\Bbbk$. It follows that Γ is an $(\alpha_\perp - F)$-convex contraction with $\upsilon = 1$. Thus, all the affirmations of Theorem (3.2) are hold and hence, Γ has a unique fixed point in Λ. □

Corollary 3.4. *Let* $(\Lambda, \perp, \mathcal{G})$ *be an O-complete metric space and a mapping* $\alpha : \Lambda \times \Lambda \to [0, \infty)$*. Postulating that* $\Gamma : \Lambda \to \Lambda$ *be a self-map the following affirmations hold:*

(i) $\forall$ $\mathtt{m}, \ell \in \Lambda$ *with* $\mathtt{m} \perp \ell$,

$$\begin{aligned} \mathcal{G}^{\upsilon}(\Gamma^2\mathtt{m}, \Gamma^2\ell) > 0 \implies &\alpha(\mathtt{m}, \ell)\mathcal{G}(\Gamma^2\mathtt{m}, \Gamma^2\ell) \\ &\leq \alpha_1\mathcal{G}(\mathtt{m}, \ell) + \alpha_2\mathcal{G}(\Gamma\mathtt{m}, \Gamma\ell) + \alpha_3\mathcal{G}(\mathtt{m}, \Gamma\mathtt{m}) \\ &+ \alpha_4\mathcal{G}(\Gamma\mathtt{m}, \Gamma^2\mathtt{m}) + \alpha_5\mathcal{G}(\ell, \Gamma\ell) + \alpha_6\mathcal{G}(\Gamma\ell, \Gamma^2\ell), \end{aligned}$$

where $0 \leq \alpha_\mathfrak{k} < 1, \mathfrak{k} = 1, 2, ..., 6$ *such that* $\sum_{\mathfrak{k}=1}^{6} \alpha_\mathfrak{k} < 1$*;*

(ii) Γ *is* $\alpha_\perp$*-admissible;*

(iii) $\exists$ $\mathtt{m}_0 \in \Lambda$ *such that* $\alpha(\mathtt{m}_0, \Gamma\mathtt{m}_0) \geq 1$*;*

(iv) Γ *is* $\perp$*-continuous or,* $\perp$*-orbitally continuous on* Λ*;*

(v) $\perp$*-preserving.*

Then, Γ *has a fixed point in* Λ*. Moreover, for any* $\mathtt{m}_0 \in \Lambda$ *if* $\mathtt{m}_{\beta+1} = \Gamma^{\beta+1}\mathtt{m}_0 \neq \Gamma^\beta\mathtt{m}_0$ $\forall$ $\beta \in \mathbb{N} \cup \{0\}$*, then* $\lim_{\beta\to\infty} \Gamma^\beta\mathtt{m}_0 = \mathfrak{o}$*.*

Proof. Setting $F(\mathfrak{x}) = In(\mathfrak{x}), \mathfrak{x} > 0$. Obviously, $F \in \Im$. $\forall\ \mathtt{m}, \ell \in \Lambda$ with $\mathtt{m} \perp \ell$ and $\mathtt{m} \neq \ell$, we obtain

$$\begin{aligned}\alpha(\mathtt{m},\ell)\mathcal{G}(\Gamma^2\mathtt{m},\Gamma^2\ell) &= \mathcal{G}(\Gamma^2\mathtt{m},\Gamma^2\ell)\\ &\leq \alpha_1\mathcal{G}(\mathtt{m},\ell)+\alpha_2\mathcal{G}(\Gamma\mathtt{m},\Gamma\ell)+\alpha_3\mathcal{G}(\mathtt{m},\Gamma\mathtt{m})\\ &\quad+\alpha_4\mathcal{G}(\Gamma\mathtt{m},\Gamma^2\mathtt{m})+\alpha_5\mathcal{G}(\ell,\Gamma\ell)+\alpha_6\mathcal{G}(\Gamma\ell,\Gamma^2\ell)\\ &\leq \Bbbk\max\Big\{\mathcal{G}(\mathtt{m},\ell),\mathcal{G}(\Gamma\mathtt{m},\Gamma\ell),\mathcal{G}(\mathtt{m},\Gamma\mathtt{m}),\mathcal{G}(\Gamma\mathtt{m},\Gamma^2\mathtt{m}),\\ &\mathcal{G}(\ell,\Gamma\ell),\mathcal{G}(\Gamma\ell,\Gamma^2\ell)\Big\},\end{aligned}$$

where $\Bbbk = \sum_{\mathfrak{k}=1}^{6}\alpha_{\mathfrak{k}} < 1$. By Corollary (3.3), Γ has a unique fixed point in Λ. □

Corollary 3.5. *A $\perp$-continuous self-map Γ on an O-complete metric space $(\Lambda, \perp, \mathcal{G})$. If $\exists\ \Bbbk \in [0,1)$ satisfying the following inequality*

$$\mathcal{G}^{v}(\Gamma^2\mathtt{m},\Gamma^2\ell) > 0 \implies$$

$$\mathcal{G}(\Gamma^2\mathtt{m},\Gamma^2\ell) \leq \Bbbk\max\Big\{\mathcal{G}(\mathtt{m},\ell),\mathcal{G}(\Gamma\mathtt{m},\Gamma\ell),\mathcal{G}(\mathtt{m},\Gamma\mathtt{m}),\mathcal{G}(\Gamma\mathtt{m},\Gamma^2\mathtt{m}),\mathcal{G}(\ell,\Gamma\ell),\mathcal{G}(\Gamma\ell,\Gamma^2\ell)\Big\}$$

$\forall\ \mathtt{m}, \ell \in \Lambda$ with $\mathtt{m}\perp\ell$, then Γ has a unique fixed point in Λ.

4. Application

In this section, we prove the existence of fixed point for $(\alpha_\perp - F)$-convex contraction to nonlinear integral equation of Volterra type

$$\mathfrak{x}(\tau) = \int_0^{\tau} \mathcal{J}(\tau, \flat, \mathfrak{x}(\flat))\mathcal{G}\flat + \gamma(\tau), \tau \in [0, \mathcal{P}]. \tag{19}$$

Consider the following assumptions:

($\flat_1$) Here $\mathcal{J} : [0,\mathcal{P}] \times [0,\mathcal{P}] \times \mathcal{R} \to \mathcal{R}, \gamma : [0,\mathcal{P}] \to \mathcal{R}$ are continuous functions and $\mathbb{I} = [0,\mathcal{P}], \mathcal{P} > 0$.

($\flat_2$) $\exists$ a strictly increasing O-sequence $\{\mathtt{m}_\beta\}_{\mathfrak{n}\in\mathcal{N}\cup(0)}$ satisfying for any $\mathfrak{n} \in \mathcal{N}$ such that

$$|\mathcal{J}(\tau,\flat,\mathtt{m}) - \mathcal{J}(\tau,\flat,\ell)| \leq \mathfrak{e}^{-\mu-\tau}|\mathtt{m}-\ell| \tag{20}$$

for all $\tau, \flat \in \mathbb{I}, \mu \in (0,1)$ and $\mathtt{m}, \ell \in \mathcal{R}$.

Let the set of all continuous functions $\mathcal{J} = \mathcal{C}(\mathbb{I},\mathcal{R})$ defined on $[0,\mathcal{P}]$ endowed with the O-complete metric space. Define $\mathcal{G} : \mathcal{J} \times \mathcal{J} \to \mathcal{R}$ by

$$\mathcal{M}(\mathtt{m},\ell) = \max_{\tau\in[0,\mathcal{P}]}\{|\mathtt{m}(\tau) - \ell(\tau)|\mathfrak{e}^{-\tau}\} \tag{21}$$

for all $\mathtt{m}, \ell \in \mathcal{J} = \mathcal{C}(\mathbb{I},\mathcal{R})$ with $((\mathcal{C}(\mathbb{I},\mathcal{R})),\mathcal{G})$ is an O-complete metric space.

Theorem 4.1. *If ($\flat_1$) and ($\flat_2$) are fulfilled, then the non-linear integral equation of Volterra type Equation (19) has a unique solution in $(\mathcal{C}(\mathbb{I},\mathcal{R}))$.*

Proof. For any $\mathtt{m} \in \mathcal{J}$ is a solution of (19) iff $\mathtt{m} \in \mathcal{J}$ is a solution of the integral equation

$$\mathtt{m}(\tau) = \int_0^{\tau} (\mathcal{J}(\tau,\flat,\mathtt{m}(\flat))\mathcal{G}\flat + \gamma(\tau). \tag{22}$$

Then, (19) is equivalent to prove $\Gamma(\mathtt{m}) = \mathtt{m}$ for $\mathtt{m} \in \mathcal{J}$. Define a relation $\perp$ on $\mathcal{J}$ by

$$\mathtt{m} \perp \ell \Leftrightarrow \mathtt{m}(\tau)\ell(\tau) \geq 0, \tag{23}$$

for all $\tau \in [0, \mathcal{P}]$. Since $\mathcal{J}$ is an orthogonal for all $\mathtt{m} \in \mathcal{J}$, $\exists\ \ell(\tau) = 0$, $\forall\ \tau \in [0, \mathcal{P}]$ such that $\mathtt{m}(\tau)\ell(\tau) = 0$: We examine

$$\mathcal{M}(\mathtt{m}, \ell) = \max_{\tau \in [0,\mathcal{P}]} \{|\mathtt{m}(\tau) - \ell(\tau)|\mathfrak{e}^{-\tau}\},$$

for all $\mathtt{m}, \ell \in \mathcal{J}$. So, the triplet $(\mathcal{J}, \perp, \mathcal{G})$ is an O-complete metric space. This implies that Γ is $\perp$-continuous. Now, to prove Γ is $\perp$-preserving. Let $\mathtt{m}(\tau) \perp \ell(\tau)$, $\forall\ \tau \in [0, \mathcal{P}]$. Now, we have

$$\Gamma\mathtt{m}(\tau) = \int_0^\tau \mathcal{J}(\tau, \flat, \mathtt{m}(\flat))\mathcal{G}\flat + \gamma(\tau) > 0. \tag{24}$$

which yields that $\Gamma\mathtt{m}(\tau) \perp \Gamma\ell(\tau)$, i.e., Γ is $\perp$-preserving.

Define a mapping $\alpha : \mathcal{J} \times \mathcal{J} \to \mathcal{R}^+$ by $\alpha(\mathtt{m}, \ell) = 1\ \forall\ \mathtt{m}, \ell \in \mathcal{J}$. Therefore, Γ is $\alpha_\perp$-admissible. Letting $F \in \Im$ such that $F(\mathfrak{x}) = In(\mathfrak{x}), \mathfrak{x} > 0$. In this point, from Definition 2.1, we get,

$$\begin{aligned}
\mu + F(\alpha(\mathtt{m}, \ell)\mathcal{G}^\upsilon(\Gamma^2\mathtt{m}, \Gamma^2\ell)) \leq F(\mathcal{M}^\upsilon(\mathtt{m}, \ell)) &\implies \mu + F(|\mathcal{P}^2\mathtt{m} - \mathcal{P}^2\ell|\mathfrak{e}^{-\tau}) \leq F(|\mathtt{m} - \ell|\mathfrak{e}^{-\tau}) \\
&\implies \mu + In|\mathcal{P}^2\mathtt{m} - \mathcal{P}^2\ell|\mathfrak{e}^{-\tau} \leq In|\mathtt{m} - \ell|\mathfrak{e}^{-\tau} \\
&\implies \mu \leq In\frac{|\mathtt{m} - \ell|\mathfrak{e}^{-\tau}}{|\mathcal{P}^2\mathtt{m} - \mathcal{P}^2\ell|\mathfrak{e}^{-\tau}} \\
&\implies \mathfrak{e}^\mu < \frac{|\mathtt{m} - \ell|\mathfrak{e}^{-\tau}}{|\mathcal{P}^2\mathtt{m} - \mathcal{P}^2\ell|\mathfrak{e}^{-\tau}} \\
&\implies |\mathcal{P}^2\mathtt{m} - \mathcal{P}^2\ell| < \frac{|\mathtt{m} - \ell|}{\mathfrak{e}^\mu}.
\end{aligned}$$

Thus,

$$\mathfrak{e}^{-\tau}|\mathcal{P}^2\mathtt{m} - \mathcal{P}^2\ell| < \frac{|\mathtt{m} - \ell|\mathfrak{e}^{-\tau}}{\mathfrak{e}^\mu}, \tag{25}$$

for all $\mathtt{m}, \ell \in \mathcal{C}(\mathbb{I})$. We conclude that for any $\flat \in \mathbb{I}$, we get

$$\begin{aligned}
|\mathtt{m}(\flat) - \ell(\flat)| &\leq \mathfrak{e}^\flat \max \mathfrak{e}^{-\flat}|\mathtt{m}(\flat) - \ell(\flat)| \\
&< \mathfrak{e}^\flat \mathcal{M}(\mathtt{m}, \ell) \\
&\leq \mathfrak{e}^\mathcal{P} \mathcal{M}(\mathtt{m}, \ell).
\end{aligned} \tag{26}$$

Therefore due to ($\flat_2$), we obtain,

$$\begin{aligned}
|\Gamma\mathtt{m}(\tau) - \Gamma\ell(\tau)| &\leq |\int_0^\tau \mathcal{J}(\tau,\flat,\mathtt{m}(\flat))\mathcal{G}\flat - \int_0^\tau \mathcal{J}(\tau,\flat,\ell(\flat))\mathcal{G}\flat| \\
&\leq \int_0^\tau |\mathcal{J}(\tau,\flat,\mathtt{m}(\flat)) - \mathcal{J}(\tau,\flat,\ell(\flat))|\mathcal{G}\flat \\
&\leq \mathfrak{e}^{-\mu-\tau}\int_0^\tau \max_{\flat\in[0,\mathcal{P}]} |\mathtt{m}(\flat) - \ell(\flat)|\mathfrak{e}^{-\flat}\mathfrak{e}^{\flat}\mathcal{G}\flat \\
&\leq \mathfrak{e}^{-\mu-\tau}\mathcal{M}(\mathtt{m},\ell)\int_0^\tau \mathfrak{e}^{\flat}\mathcal{G}\flat \\
&\leq \mathfrak{e}^{-\mu-\tau}\mathcal{M}(\mathtt{m},\ell)[\mathfrak{e}^{\tau} - 1] \\
&\leq \mathfrak{e}^{-\mu}(1 - \mathfrak{e}^{-\tau})\mathcal{M}(\mathtt{m},\ell) \\
&< \mathfrak{e}^{-\mu}\mathcal{M}(\mathtt{m},\ell); \tau \in \mathbb{I}.
\end{aligned} \tag{27}$$

By O-sequence $\{\mathtt{m}_\beta\}$, we have,

$$\mathfrak{e}^{-\tau}|\Gamma\mathtt{m}(\tau) - \Gamma\ell(\tau)| \leq \frac{\mathcal{M}(\mathtt{m},\ell)\mathfrak{e}^{-\tau}}{\mathfrak{e}^{\mu}}, \tau \in \mathbb{I}. \tag{28}$$

Letting supremum in (28), we have,

$$\mu + F(\alpha(\mathtt{m},\ell)\mathcal{G}(\Gamma\mathtt{m},\Gamma\ell)) \leq F(\mathcal{M}(\mathtt{m},\ell))$$

Therefore, Γ has a unique solution by Theorem (3.2). □

5. Obtaining a numerical solution to integral equations

Example 5.1. *Let $\mathcal{J} = \{\mathfrak{f}(\tau)/\mathfrak{f}(\tau)$ be a continuous function defined on $[0,1]\}$, i.e., $\mathcal{J} = \mathcal{C}[0,1]$.*
Define $\mathcal{G} : \mathcal{J}\times\mathcal{J} \to \mathcal{R}$ by $\mathcal{G}(\mathfrak{a},\mathfrak{b}) = \sup_{\tau\in[0,1]}\{|\mathtt{m}(\tau) - \ell(\tau)|\}$ for all $\mathtt{m},\ell \in \mathcal{J}$.
Clearly, $(\mathcal{J},\mathcal{G})$ is an O-complete metric space. Define $\mathcal{O} : \mathcal{J} \to \mathcal{J}$ by:

$$\mathcal{O}\mathtt{m}(\tau) = \gamma(\tau) + \int_0^1 \mathcal{J}(\tau,\flat,\mathtt{m}(\flat))\mathcal{G}\flat; \mathtt{m}(\tau) \in \mathcal{J}. \tag{29}$$

Define a mapping $F : (0,\infty) \to \mathcal{R}$ defined by $F(\mathfrak{x}) = In(\mathfrak{x}), \mathfrak{x} > 0$ and a mapping $\alpha : \mathcal{J}\times\mathcal{J} \to \mathcal{R}^+$ by $\alpha(\mathtt{m},\ell) = 1\ \forall\ \mathtt{m},\ell \in \mathcal{J}$. Taking $\gamma(\tau) = \frac{5}{8}\tau$ and $\mathcal{J}(\tau,\flat,\mathtt{m}(\flat)) = \frac{\tau}{4}(1+\mathtt{m}(\flat))$. Then Eq. (29) reduces to $\mathcal{O}\mathtt{m}(\tau) = \frac{5}{8}\tau + \int_0^1 \frac{\tau}{4}(1+\mathtt{m}(\flat))\mathcal{G}\flat$, where $\frac{5}{8}\tau, \frac{\tau}{4}(1+\mathtt{m}(\flat))$ are continuous functions and $\mathcal{O}\mathtt{m} \in \mathcal{C}[0,1]$. Let us assume that $|\frac{\tau}{4}| \leq \mathfrak{e}^{-\mu-\tau}$. To prove that $\mathcal{O}$ is a ($\alpha_\perp$-F)-contraction, we need to prove $\mathcal{G}(\mathcal{O}\mathtt{m},\mathcal{O}\ell) \leq \mathcal{M}(\mathtt{m},\ell)\mathfrak{e}^{-\mu}$.

Since

$$\begin{aligned}
\mu + F(\alpha(\mathtt{m},\ell)\mathcal{G}^{\upsilon}(\Gamma^2\mathtt{m},\Gamma^2\ell)) \leq F(\mathcal{M}^{\upsilon}(\mathtt{m},\ell)) &\implies \mu + In(\mathcal{G}(\mathcal{O}\mathtt{m},\mathcal{O}\ell)) \leq In(\mathcal{M}(\mathtt{m},\ell)) \\
&\implies In\frac{|\mathcal{O}\mathtt{m} - \mathcal{O}\ell|}{|\mathtt{m} - \ell|} \leq -\mu \\
&\implies |\mathcal{O}\mathtt{m} - \mathcal{O}\ell| \leq |\mathtt{m} - \ell|\mathfrak{e}^{-\mu} \\
&\implies \mathcal{G}(\mathcal{O}\mathtt{m},\mathcal{O}\ell) \leq \mathcal{M}(\mathtt{m},\ell)\mathfrak{e}^{-\mu}.
\end{aligned}$$

Consider

$$
\begin{aligned}
|\mathcal{O}\mathtt{m}(\tau) - \mathcal{O}\ell(\tau)| &= |\int_0^1 \mathcal{J}(\tau, \flat, \mathtt{m}(\flat))\mathcal{G}\flat - \int_0^1 \mathcal{J}(\tau, \flat, \ell(\flat))\mathcal{G}\flat| \\
&\leq \int_0^1 |\mathcal{J}(\tau, \flat, \mathtt{m}(\flat)) - \mathcal{J}(\tau, \flat, \ell(\flat))|\mathcal{G}\flat \\
&\leq \int_0^1 |\frac{\tau}{4}(1 + \mathtt{m}(\flat)) - \frac{\tau}{4}(1 + \ell(\flat))|\mathcal{G}\flat \\
&\leq \int_0^1 |\frac{\tau}{4}(\mathtt{m}(\flat) - \ell(\flat))|\mathcal{G}\flat \\
&\leq \int_0^1 |\frac{\tau}{4}||\mathtt{m}(\flat) - \ell(\flat)|\mathcal{G}\flat, \qquad (30)
\end{aligned}
$$

then,

$$
\begin{aligned}
\sup_{\mathfrak{t}\in[0,1]} |\mathcal{O}\mathtt{m}(\tau) - \mathcal{O}\ell(\tau)| &\leq \int_0^1 |\frac{\tau}{4}||\mathtt{m}(\flat) - \ell(\flat)|\mathcal{G}\flat \\
&\leq \mathfrak{e}^{-\mu-\tau} \int_0^1 |\mathtt{m}(\flat) - \ell(\flat)|\mathcal{G}\flat \\
&\leq \mathfrak{e}^{-\mu}|\mathtt{m}(\flat) - \ell(\flat)|\mathfrak{e}^{-\tau} \int_0^1 \mathcal{G}\flat \\
&\leq \mathfrak{e}^{-\mu}|\mathtt{m}(\flat) - \ell(\flat)|\mathfrak{e}^{-\tau}. \qquad (31)
\end{aligned}
$$

Therefore, $\mathcal{G}(\mathcal{O}\mathtt{m}, \mathcal{O}\ell) \leq \mathfrak{e}^{-\mu}\mathcal{M}(\mathtt{m}, \ell)$.

Therefore, all the hypothesis of Theorem (3.2) *are satisfied and* $\mathcal{O}$ *has a unique fixed point and the nonlinear integral equation of Volterra type equation* (29) *has a unique solution.*

Verify that $\mathtt{m}(\tau) = \tau$ *is the exact solution of the Eq.* (29). *Utilizing the iteration process, we get*

$$
\mathtt{m}_{\beta+1}(\tau) = \mathcal{O}\mathtt{m}_\beta(\tau) = \frac{5}{8}\tau + \frac{\tau}{4}\int_0^1 (1 + \mathtt{m}_\beta(\flat))\mathcal{G}\flat.
$$

Let $\mathtt{m}_0(\tau)=0$ *be the initial condition. Letting* $\beta=0,1,2,\ldots$ *in Eq.*(28) *successively, we obtain,*

$$\begin{aligned}
\mathtt{m}_1(\tau)&=0.875\tau,\\
\mathtt{m}_2(\tau)&=0.984375\tau,\\
\mathtt{m}_3(\tau)&=0.998046875\tau,\\
\mathtt{m}_4(\tau)&=0.999755859375\tau,\\
\mathtt{m}_5(\tau)&=0.999969482421875\tau,\\
\mathtt{m}_6(\tau)&=0.999996185302734375\tau,\\
\mathtt{m}_7(\tau)&=0.999999523162841796875\tau,\\
\mathtt{m}_8(\tau)&=0.999999940395355224609375\tau,\\
\mathtt{m}_9(\tau)&=0.999999992549419403076171875\tau,\\
\mathtt{m}_{10}(\tau)&=0.999999999068677425384521484375\tau,\\
\mathtt{m}_{11}(\tau)&=0.999999999883584678173065185546880\tau,\\
\mathtt{m}_{12}(\tau)&=\tau.
\end{aligned}$$

Therefore $\mathtt{m}(\tau)=\tau$ *is the exact solution.*
By Theorem (3.2)*, we proved that the integral equation of Volterra type Eq.*(29) *has a unique solution. Then, the Eq.* (29) *has unique solution.*

6. Conclusion

In this paper, we improved the results of Mahendra Singh et al. [15] by conferring examples of an $(\alpha_\perp-\digamma)$-convex contraction of the self-map in the framework of $\alpha_\perp$-admissible. The notions of an $(\alpha_\perp-\digamma)$-convex contraction extend other well known metrical fixed point theorems within the literature.

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[1] Department of Mathematics, College of Engineering and Technology, Faculty of Engineering and Technology, SRM Institute of Science and Technology, SRM Nagar, Kattankulathur - 603203, Kanchipuram, Chennai, Tamil Nadu, India.

E-mail address: gn4255@srmist.edu.in

[2] Department of Mathematics, College of Engineering and Technology, Faculty of Engineering and Technology, SRM Institute of Science and Technology, SRM Nagar, Kattankulathur - 603203, Kanchipuram, Chennai, Tamil Nadu, India.
E-mail address: aruljoseph.alex@gmail.com

[3] Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Chennai 602105, Tamil Nadu, India.
E-mail address: mathsguna@yahoo.com

[4] Department of Mathematics, Faculty of Science, Ege University, Bornova, Izmir, 35100, Turkey.
E-mail address: ozgur.ege@ege.edu.tr