IDEAL FACTORIZATION IN STRONGLY DISCRETE INDEPENDENT RINGS OF KRULL TYPE, II

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Abstract. A ZPUI domain D is an integral domain with property (#): every nonzero proper ideal I of D can be written as $I = JP_1 \cdots P_n$, where J is an invertible ideal of D and $\{P_1, \dots, P_n\}$ is a nonempty collection of pairwise comaximal prime ideals of D. In this paper, among other things, we study two types of natural generalizations of ZPUI domains: (i) the J in the property (#) is principal and (ii) the property (#) holds for all nonzero principal ideals of D. For example, we show that (1) D satisfies (i) if and only if D is a ZPUI domain whose invertible ideals are principal and (2) D satisfies (ii) if and only if D is an h-local domain in which each maximal ideal is invertible. We also study the w-operation analogs of these two properties.

1. Introduction

This is a continuation of our work [16] on ideal factorization in strongly discrete independent rings of Krull type. For an easy understanding of the introduction, we first review the notion of d-, v-, t-, and w-operation. Let D be an integral domain with quotient field K. A D-submodule A of K is called a *fractional ideal* of D if $dA \subseteq D$ for some $0 \neq d \in D$. An (integral) ideal I of D is a fractional ideal of D with $I \subseteq D$. Let $F(D)$ (resp., $f(D)$) be the set of nonzero fractional (resp., nonzero finitely generated fractional) ideals of D. For $A \in F(D)$, let $A^{-1} = \{x \in K \mid xA \subseteq D\}$; then $A^{-1} \in F(D)$. Hence, if we set

- $A^d = A$,
- $A^v = (A^{-1})^{-1}$,
- $A^t = \bigcup\{I^v \mid I \subseteq A \text{ and } I \in f(D)\}$, and
- $A^w = \{x \in K \mid xJ \subseteq A \text{ for some } J \in f(D) \text{ with } J^v = D\}$,

then the d-, v-, t-, and w-operation are well defined. It is easy to see that $I \subseteq I^w \subseteq I^t \subseteq I^v$ for all $I \in F(D)$. Let $* = d, v, t$, or w. An $I \in F(D)$ is called a *$*$-ideal* if $I^* = I$. A $*$-ideal is a *maximal $*$-ideal* if it is maximal among proper integral $*$-ideals. An $A \in F(D)$ is said to be *$*$-invertible* if $(II^{-1})^* = D$. Two ideals I, J of D are said to be *pairwise $*$-comaximal* if $(I + J)^* = D$. An ideal I of D is *$*$-unidirectional* if I is contained in a unique maximal $*$-ideal of D. We usually omit the d-operation, e.g., a d-ideal is just called an ideal.

In [34], Olberding introduced the notion of ZPUI domains which is a generalization of the ring of integers. We recall that D is a *ZPUI domain* if and only if every nonzero proper ideal I of D can be written as $I = JP_1 \cdots P_n$, where J is

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an invertible ideal of D and $\{P_1, \ldots, P_n\}$ is a nonempty collection of pairwise comaximal prime ideals of D by [34, Theorem 2.3] and [33, Theorem 5.2]. In [16], we used the w-operation to define a w-ZPUI domain, which is a generalization of ZPUI domains, that D is a *w-ZPUI domain* if every nonzero proper w-ideal I of D can be written as $I = (JP_1 \cdots P_n)^w$ for some w-invertible ideal J of D and a nonempty collection $\{P_1, \ldots, P_n\}$ of pairwise w-comaximal prime w-ideals of D. Then D is a ZPUI domain if and only if D is a w-ZPUI domain whose maximal ideals are t-ideals [16, Corollary 3.8]. In this paper, we continue to study the (w-)ZPUI domains and three new classes of generalizations.

A Dedekind domain is an integral domain in which each (nonzero) ideal is a finite product of prime ideals. A π-domain is an integral domain each of whose (nonzero) principal ideals is a finite product of prime ideals. Clearly, a Dedekind domain is a π-domain. However, a π-domain need not be a Dedekind domain (e.g., the polynomial ring over a Dedekind domain that is not a field). It is known that D is a Krull domain if and only if each nonzero principal ideal of D is a finite w-product of prime ideals (cf. [31, Theorem 3.9]). Note that a Dedekind domain (resp., Krull domain) is a ZPUI domain (resp., w-ZPUI domain). A nonzero principal ideal is invertible. Hence, we have the following natural questions.

Question 1.1. *Let D be an integral domain and $* = d$ or w.*

1. *What if every nonzero proper $*$-ideal I of D can be written as $I = (aP_1 \cdots P_n)^*$, where $0 \neq a \in D$ and $\{P_1, \ldots, P_n\}$ is a nonempty collection of pairwise $*$-comaximal prime $*$-ideals of D?*
2. *What if every nonzero proper principal ideal I of D can be written as $I = (JP_1 \cdots P_n)^*$, where J is a $*$-invertible ideal of D and $\{P_1, \ldots, P_n\}$ is a nonempty collection of pairwise $*$-comaximal prime $*$-ideals of D?*
3. *What if every nonzero proper principal ideal I of D can be written as $I = (aP_1 \cdots P_n)^*$, where $0 \neq a \in D$ and $\{P_1, \ldots, P_n\}$ is a nonempty collection of pairwise $*$-comaximal prime $*$-ideals of D?*

Note that a d-ZPUI domain is just the ZPUI domain. It is clear that "D satisfies the property of Question 1.1(1)" $\Rightarrow$ "D is a $*$-ZPUI domain" $\Rightarrow$ "D satisfies the property of Question 1.1(2)". In this paper, we are going to answer Question 1.1.

In section 2, we answer Question 1.1(1). Among other things, we show that D is a Bézout ZPUI domain (resp., GCD w-ZPUI domain) if and only if D satisfies the property stated in Question 1.1(1) for $* = d$ (resp., $* = w$). Section 3 is devoted to the study of integral domains satisfying the condition of Question 1.1(2). For example, we show that D is a weakly Matlis domain whose maximal t-ideals are t-invertible if and only if D satisfies the condition of Question 1.1(2) for $* = w$. We also study integral domains satisfying the statements of Question 1.1(3) with additional conditions.

Finally, in section 4, we raise questions of how integral domains possessing a property similar to π-domains behave and provide an answer to one of them ourselves. We end this paper with a diagram that shows the relationships between several kinds of integral domains with some ideal factorization properties which are studied in this paper.

For easy reference of the reader, we end the introduction with a couple of already well-known results on the d-, t-, and w-operation. Let $* = d, t,$ or w and $*$-Max(D)

be the set of maximal $*$-ideals of D. Then $*$-Max$(D) \neq \emptyset$ if and only if D is not a field; each maximal $*$-ideal of D is a prime ideal; each proper $*$-ideal of D is contained in a maximal $*$-ideal; each prime ideal of D minimal over a $*$-ideal is a $*$-ideal, whence each height-one prime ideal is a $*$-ideal; $D = \bigcap_{P \in *\text{-Max}(D)} D_P$; $I^w = \bigcap_{P \in t\text{-Max}(D)} ID_P$ for all $I \in F(D)$; and t-Max$(D) = w$-Max(D) (see, for example, [4]). An integral domain D is said to be of *finite $*$-character* if each nonzero nonunit of D is contained in only finitely many maximal $*$-ideals. We say that D is *$*$-independent* if no two elements of $*$-Max(D) contain a common nonzero prime $*$-ideal. Hence, D is an *h-local domain* (resp., a *weakly Matlis domain*) if D is of finite d-character and d-independent (resp., of finite t-character and t-independent).

We say that D is a *Prüfer v-multiplication domain* (PvMD) if each nonzero finitely generated ideal of D is t-invertible. A PvMD is said to be a *ring of Krull type* (resp., an *independent ring of Krull type*) if it is of finite t-character (resp., weakly Matlis). It is known that D is a PvMD if and only if D_P is a valuation domain for all maximal t-ideals P of D, if and only if $D[X]$, the polynomial ring over D, is a PvMD [30]; and a Prüfer domain is a PvMD whose maximal ideals are t-ideals. We mean by t-dim$(D) = 0$ (resp., t-dim$(D) = 1$) that D is a field (resp., D is not a field and each prime t-ideal of D is a maximal t-ideal). The class of integral domains D with t-dim$(D) = 1$ includes Krull domains and integral domains of (Krull) dimension one.

Let Inv$^t(D)$ (resp., Inv(D), Prin(D)) be the set of t-invertible fractional t-ideals (resp., invertible fractional ideals, nonzero principal fractional ideals) of an integral domain D. Then Inv$^t(D)$ forms a group under the t-product $I * J = (IJ)^t$; Inv(D) and Prin(D) become subgroups of Inv$^t(D)$; and Prin$(D) \subseteq$ Inv(D). The t-class group $Cl_t(D)$ (resp., Picard group Pic(D)) is the factor group Inv$^t(D)$/Prin(D) (resp., Inv(D)/Prin(D)). It is clear that Pic(D) is a subgroup of $Cl_t(D)$ and Pic$(D) = Cl_t(D)$ when each maximal ideal of D is a t-ideal (e.g., D is a Prüfer domain or one-dimensional). It is well known and easy to see that D is a Bézout domain (resp., GCD domain) if and only if D is a Prüfer domain (resp., PvMD) with $Cl_t(D) = \{0\}$.

2. (w-)ZPUI domains with trivial class group

Let D be an integral domain and $* = d$ or w on D. In this section, we are going to give an answer to Question 1.1(1). That is, we study some ideal factorization properties of D with the property that every nonzero proper $*$-ideal of D can be written as a $*$-product of a nonzero principal ideal and a nonempty collection of pairwise $*$-comaximal prime $*$-ideals.

Recall that D is a ZPUI domain (resp., w-ZPUI domain) if and only if D is a strongly discrete h-local Prüfer domain (resp., a strongly discrete independent ring of Krull type) [36, Theorem 1.1] (resp., [16, Theorem 2.5]) and that both an h-local Prüfer domain and an independent ring of Krull type are weakly Matlis domains.

Lemma 2.1. *Let D be a $*$-ZPUI domain for $* = d$ or w. Then the following hold.*

(1) *If P is a prime $*$-ideal of D, then there exists a two-generated ideal I of D such that $P = \sqrt{I}$ and $ID_P = PD_P$.*

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(2) *A prime $*$-ideal of D is $*$-invertible if and only if it is a maximal $*$-ideal of D.*
(3) *Let I be a proper $*$-invertible $*$ideal of D. Then I is contained in only finitely many maximal $*$-ideals $M_1, \dots, M_n$ of D and $I = (JM_1 \cdots M_n)^*$ for some ideal J of D that is a product of $*$-invertible $*$-unidirectional ideals of D.*

Proof. Note that $d = t$ on a Prüfer domain and $w = t$ on a PvMD. Note also that D is a strongly discrete independent ring of Krull type [16, Theorem 2.5]. Hence, $* = t$ on a $*$-ZPUI domain.

(1) Let P be a prime t-ideal of D. Then $PD_P = aD_P$ for some $a \in P$ [16, Proposition 2.2]. Now let $M_1, \dots, M_n$ be the set of maximal t-ideals of D containing a. Note that exactly one of M_i, say M_1, contains P. Then by the prime avoidance lemma, we can choose $b \in P \setminus (M_2 \cup \cdots M_n)$ and let $I = (a, b)$. Then $\sqrt{I} = P$ and $ID_P = PD_P$.

(2) Each maximal t-ideal of D is t-invertible by [16, Theorem 2.5] and [19, Lemma 3.1 and Theorem 3.5]. For the converse, let P be a prime t-ideal of D that is t-invertible. Thus, P is a maximal t-ideal [28, Proposition 1.3].

(3) Since D is weakly Matlis, I is contained in only finitely many maximal t-ideals of D, say, $M_1, \dots, M_n$. Note that each M_i is t-invertible by (2) and $(M_1 \cdots M_n)^t = M_1 \cap \cdots \cap M_n$ [6, Lemma 2.5], so $I = (JM_1 \cdots M_n)^t$ for some ideal J of D. Now, let $I_n = JD_{M_i} \cap D$ for $i = 1, \dots, n$. Then I_n is a t-unidirectional t-ideal of D [6, Lemma 2.3 and Lemma 2.5],

$$J^t = \bigcap_{M \in t\text{-Max}(D)} JD_M = I_1 \cap \cdots \cap I_n = (I_1 \cdots I_n)^t.$$

Moreover, since I is t-invertible, J is t-invertible, and thus $I_1, \dots, I_n$ are all t-invertible. □

A nonzero nonunit x of an integral domain D is said to be *homogeneous* if xD is t-unidirectional. As in [15], we call D a *homogenous factorization domain* (*HoFD*) if each nonzero nonunit of D is a finite product of pairwise t-comaximal homogeneous elements of D. Then D is an HoFD if and only if D is a weakly Matlis domain with $Cl_t(D) = \{0\}$ [5, Theorem 3.4].

Theorem 2.2. *Let D be an integral domain. Then the following are equivalent.*
(1) *Every nonzero proper ideal of D can be written as a product of a principal ideal and a nonempty collection of distinct prime ideals.*
(2) *D is a Bézout ZPUI domain.*
(3) *D is a ZPUI domain and every invertible ideal of D is principal.*
(4) *Every nonzero proper ideal of D can be written as a product of a principal ideal and a nonempty collection of pairwise comaximal prime ideals.*

In this case, D is an HoFD. Moreover, if $a \in D$ is a nonzero nonunit, then there is an element $b \in D$ such that $aD \subsetneq bD$ and $\frac{a}{b}$ is a finite product of distinct prime elements each of which generates a maximal ideal.

Proof. (1)⇒(2) Suppose that (1) holds. Then D is a ZPUI domain by definition. Next, choose a maximal ideal M of D. Then $M^2 = aP_1 \cdots P_n$ for some $a \in D$

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and distinct prime ideals $P_1 \cdots P_n$. Note that M is a maximal ideal, so $n = 1$ and $P_1 = M$, and since M is invertible by Lemma 2.1.(2), $M = aD$. Thus, every maximal ideal of D is principal.

Now let I be a proper invertible ideal of D. Then $I = aP_1 \cdots P_n$ for some $a \in D$ and distinct prime ideals $P_1, \ldots, P_n$ of D. Since I is invertible, so are $P_1, \ldots, P_n$. Therefore $P_1, \ldots, P_n$ are maximal ideals of D by Lemma 2.1.(2), and by the argument preceding this paragraph, they are principal ideals. So I, being a product of finitely many principal ideals of D, is a principal ideal of D. Hence, every invertible ideal of D is principal. Since D is a Prüfer domain, it means that every finitely generated ideal of D is principal, so D is a Bézout domain.

(2)$\Rightarrow$(3) This follows from the fact that every invertible ideal is finitely generated.

(3) $\Rightarrow$ (4) $\Rightarrow$(1) Clear.

In this case, D is an h-local Bézout domain, and hence D is a weakly Matlis domain with $Cl_t(D) = \{0\}$. Thus, D is an HoFD. Moreover, $aD = JQ_1 \cdots Q_n$ for some ideal J of D and distinct maximal ideals $Q_1, \ldots, Q_n$ of D. Note that J and Q_i are invertible, so by (3), $J = bD$ and $Q_i = p_iD$ for some $b, p_i \in D$. Thus, $aD \subsetneq bD$, $p_1, \ldots, p_n$ are distinct, each p_iD is a maximal ideal, and $\frac{a}{b} = up_1 \cdots p_n$ for some unit $u \in D$. □

It is worth noting that (i) every maximal ideal of a Bézout ZPUI domain is principal and (ii) the condition "distinct" of Theorem 2.2(1) is necessary (for example, if D is a Dedekind domain that is not a PID, then every nonzero proper ideal of D can be written as a principal ideal and a nonempty collection of prime ideals but D is not a Bézout domain).

Let $D[X]$ be the polynomial ring over an integral domain D. For $f \in D[X]$, let $c(f)$ denote the ideal of D generated by the coefficients of f. Then $S = \{f \in D[X] \mid c(f) = D\}$ is a saturated multiplicative subset of $D[X]$ and $D(X) := D[X]_S$, called the *Nagata ring* of D, is an overring of $D[X]$.

Corollary 2.3. *An integral domain D is a ZPUI domain if and only if every nonzero proper ideal of $D(X)$, the Nagata ring of D, can be written as a product of a principal ideal and a nonempty collection of distinct prime ideals.*

Proof. It is known that D is a ZPUI domain if and only if $D(X)$ is a ZPUI domain, a ZPUI domain is a Prüfer domain, and D is a Prüfer domain if and only if $D(X)$ is a Bézout domain [1, Theorem 8]. Thus, the result is from Theorem 2.2. □

Let $D[X]$ be the polynomial ring over an integral domain D and $N_v = \{f \in D[X] \mid f \neq 0 \text{ and } c(f)_v = D\}$. Then N_v is a saturated multiplicative subset of $D[X]$, and hence $D[X]_{N_v}$, called the *t-Nagata ring* of D, is an overring of $D[X]$, $D(X) \subseteq D[X]_{N_v}$, and $Cl_t(D[X]_{N_v}) = \{0\}$ [30, Propositions 2.1, 2.2, and Theorem 2.14].

Corollary 2.4. *An integral domain D is a w-ZPUI domain if and only if every nonzero proper ideal of $D[X]_{N_v}$ can be written as a product of a principal ideal and a nonempty collection of distinct prime ideals.*

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Proof. It is known that D is a w-ZPUI domain if and only if $D[X]_{N_v}$ is a ZPUI domain [16, Theorem 3.5] and $Cl_t(D[X]_{N_v}) = \{0\}$. Thus, the result follows from Theorem 2.2. □

The next result is a w-ZPUI domain analog of Theorem 2.2.

Corollary 2.5. *Let D be an integral domain. Then the following are equivalent.*

1. *Every proper w-ideal of D can be written as a w-product of a principal ideal and a nonempty collection of pairwise w-comaximal prime ideals.*
2. *D is a w-ZPUI domain with $Cl_t(D) = \{0\}$.*
3. *D is a w-ZPUI and a GCD domain.*
4. *Every proper w-ideal of D can be written as a w-product of a principal ideal and a nonempty collection of distinct prime w-ideals.*
5. *$D[X]$ is a w-ZPUI domain with $Cl_t(D) = \{0\}$.*

In this case, D is an HoFD. Moreover, if $a \in D$ is a nonzero nonunit, then there is an element $b \in D$ such that $aD \subsetneq bD$ and $\frac{a}{b}$ is a finite product of distinct prime elements.

Proof. (1) $\Rightarrow$ (4) Clear.

(4) $\Rightarrow$ (2) D is a w-ZPUI domain [16, Theorem 3.5]. Moreover, every w-invertible w-ideal of D is principal by the argument of the proof of (1) $\Rightarrow$ (2) in Theorem 2.2. Thus, $Cl_t(D) = \{0\}$.

(2) $\Rightarrow$ (1) Let I be a proper w-ideal of D. Then $I = (JP_1 \cdots P_n)^w$ for some w-invertible ideal J of D and $P_1, \dots, P_n$ are pairwise w-comaximal prime w-ideals of D. Note that $J^w = aD$ for some $a \in D$ by (2). Thus, $I = (aP_1 \cdots P_n)^w$.

(2) $\Leftrightarrow$ (3) This follows from the fact that a w-ZPUI domain is a PvMD [16, Theorem 3.5] and a GCD domain is just a PvMD with trivial class group.

(2) $\Leftrightarrow$ (5) This follows from [16, Theorem 3.5] that D is a w-ZPUI domain if and only if $D[X]$ is a w-ZPUI domain.

In this case, D is a weakly Matlis domain with $Cl_t(D) = \{0\}$. Thus, D is an HoFD. Moreover, $aD = (JQ_1 \cdots Q_n)^w$ for some ideal J of D and distinct maximal w-ideals $Q_1, \dots, Q_n$ of D. Note that J and Q_i are w-invertible, so by (3), $J^w = bD$ and $Q_i = p_iD$ for some $b, p_i \in D$. Thus, $a \in bD$, $p_1, \dots, p_n$ are distinct, and

$$aD = (bDp_1D \cdots p_nD)^w = ((bp_1 \cdots p_n)D)^w = (bp_1 \cdots p_n)D.$$

Thus, $\frac{a}{b} = up_1 \cdots p_n$ for some unit $u \in D$. □

Let S be a multiplicative subset of an integral domain D. We say that S is *splitting* if, for each $0 \neq d \in D$, there is an $s \in S$ such that $d = sa$ for some $a \in D$ with $(a, s')^t = D$ for all $s' \in S$. A multiplicative subset of a Noetherian domain generated by a set of (nonzero) prime elements is an easy example of splitting sets. Splitting sets were introduced by Anderson et al. [3].

Proposition 2.6. *Let $A \subseteq B$ be an extension of integral domains, X be an indeterminate over B, and $D = A + XB[X]$, i.e., $D = \{f \in B[X] \mid f(0) \in A\}$.*

1. *The following statements are equivalent.*
    (a) *D is a ZPUI domain.*

(b) *A is a strongly discrete valuation domain and B is the quotient field of A.*
(c) *D is a Bézout ZPUI domain.*

(2) *D is a GCD w-ZPUI domain if and only if A is a GCD w-ZPUI domain, $B = A_S$ for a splitting set S of A, and* $|\{P \in t\text{-}Max(D) \mid P \cap S \neq \emptyset\}| \leq 1$.

Proof. (1) (a) ⇔ (b) [16, Theorem 4.1]. (b) ⇒ (c) A valuation domain is a Bézout domain, so D is a Bézout domain [18, Corollary 4.13]. (c) ⇒ (a) Clear.

(2) This follows from the following two observations: (i) D is a GCD domain if and only if A is a GCD domain and $B = A_S$ for some splitting set S of A [8, Theorem 2.10] and (ii) if S is a splitting set of A, then $A + XA_S[X]$ is a w-ZPUI domain if and only if A is a w-ZPUI domain and $|\{P \in t\text{-Max}(D) \mid P \cap S \neq \emptyset\}| \leq 1$ [16, Theorem 4.2]. □

Recall that D is a Dedekind domain (resp., Krull domain) if and only if D is a ZPUI domain (resp., w-ZPUI domain) with $\dim(D) \leq 1$ (resp., t-$\dim(D) \leq 1$) and that a Krull domain (resp., Dedekind domain) is a UFD (resp., PID) if and only if its divisor class group is trivial. Hence, a UFD (resp., PID) is a GCD w-ZPUI domain (resp., Bézout ZPUI domain).

Example 2.7. (1) There exists a Bézout ZPUI domain of whose Krull dimension and number of maximal ideals are both arbitrarily large [35, Example 3.5].

(2) A ZPUI domain (resp., w-ZPUI domain) is a Prüfer domain (resp. PvMD), so it is a Bézout domain (resp. GCD domain) if and only if its class group is trivial. Thus, by [16, Corollary 3.9], there exists a Bézout ZPUI domain (resp., GCD w-ZPUI domain) that is not a PID (resp., UFD).

As shown in [36, Theorem 1.1], an integral domain D is a ZPUI domain if and only if every nonzero proper ideal I of D can be written as $I = JP_1 \cdots P_n$, where J is an invertible ideal of D and $\{P_1, \dots, P_n\}$ is a nonempty collection of prime ideals of D: the condition that "$P_1, \dots, P_n$ are pairwise comaximal" can be dropped. The situation is a little different for domains described in Question 1.1. For instance, a Dedekind domain D has the property (#) that every nonzero proper ideal of D can be written as a product of a principal ideal and a nonempty collection of prime ideals. However, D is not a Bézout domain in general, so the property (#) is a weaker condition than the condition of Theorem 2.2. We next state some properties of integral domains with property (#) and their w-operation analog.

Proposition 2.8. *Let D be an integral domain in which every nonzero proper w-ideal I can be written as a w-product of a principal ideal and a nonempty collection of prime w-ideals. Then*

(1) *D is a w-ZPUI domain.*
(2) *If P is a prime w-ideal of D that is not a maximal w-ideal, then there exists an element $a \in D$ such that P is the unique minimal prime ideal of aD and* $aD_P = PD_P$.
(3) *Let M be a maximal w-ideal of D that properly contains a nonzero prime w-ideal. Then $(M^l)^w$ is a principal ideal for some integer $l \geq 1$.*

Proof. (1) This follows from [16, Theorem 3.5].

(2) and (3) Now choose a maximal w-ideal M of D that properly contains a nonzero prime w-ideal P of D. Then by Lemma 2.1.(1), there exists a finitely generated ideal I of D such that $P = \sqrt{I}$ and $IR_P = PR_P$. By assumption, $I^w = (aM_1 \cdots M_n)^w$ for some $a \in D$ and (invertible) prime w-ideals $M_1, \ldots, M_n$. By Lemma 2.1.(2), $M_1, \ldots, M_n$ are maximal w-ideals of D containing I. In fact, since D is an independent ring of Krull type and $P = \sqrt{I^w} = \sqrt{aD} \cap M_1 \cap \cdots \cap M_n$, we have $P = \sqrt{aD}$ and $M = M_i$ for each i. In particular, $I^w = (aM^n)^w$ and $PD_P = ID_P = aD_P$. Now, by a similar reasoning, $aD = (bM^l)^w$ for some $b \in D$ and integer $l \geq 1$. Thus, $(M^l)^w$ is a principal ideal. □

Corollary 2.9. *Let D be an integral domain in which every nonzero proper ideal can be written as a product of a principal ideal and a nonempty collection of prime ideals. Then the following hold.*

1. *D is a ZPUI domain.*
2. *Let P be a nonmaximal prime ideal of D. Then there exists an element $a \in D$ such that $P = \sqrt{aD}$ and $aD_P = PD_P$.*
3. *Let M be a maximal ideal of D whose height is greater than 1. Then M^l is a principal ideal for some integer $l \geq 1$.*

Proof. Since $w = d$ on a ZPUI domain, the conclusion follows from Proposition 2.8. □

As we noted in the remark before Proposition 2.8, a Dedekind domain satisfies the statements of Proposition 2.8 and Corollary 2.9, while the ideal class group of a Dedekind domain need not be trivial. It is clear that if D is a ZPUI domain (resp., w-ZPUI domain) with $Cl_t(D) = \{0\}$, then every nonzero proper ideal (resp., w-ideal) of D can be written as a product (resp., w-product) of a principal ideal and a nonempty collection of prime ideals (resp., prime w-ideals).

3. Weakly $*$-ZPUI domains

An integral domain D is a π-domain if each nonzero principal ideal of D can be written as a finite product of prime ideals of D. Hence, a π-domain is a weak version of a Dedekind domain. In this section, we study a weak version of ZPUI domains and w-ZPUI domains. Motivated by Lemma 2.1, we define weakly ZPUI domains (resp., weakly w-ZPUI domains) as follows, so that a ZPUI domain (resp., w-ZPUI domain) is a weakly ZPUI domain (resp., weakly w-ZPUI domain).

Definition 3.1. *Let D be an integral domain and $*$ be one of d-, w- and t-operation on D. We say that D is a weakly $*$-ZPUI domain if every nonzero proper principal ideal aD of D can be written as $aD = (JP_1 \cdots P_n)^*$, where $\{P_1, \ldots, P_n\}$ is the set of maximal $*$-ideals of D containing a and J is either D or a product of $*$-invertible $*$-unidirectional ideals.*

It is clear that a weakly ZPUI domain is both a weakly t-ZPUI domain and a weakly w-ZPUI domain. Hence, by Lemma 3.2 below, weakly ZPUI domain $\Rightarrow$ weakly t-ZPUI domain $\Leftrightarrow$ weakly w-ZPUI domain.

Lemma 3.2. *An integral domain D is a weakly t-ZPUI domain if and only if it is a weakly w-ZPUI domain.*

Proof. Assume that D is a weakly t-ZPUI domain, and let $a \in D$ be a nonzero nonunit. Then $aD = (JP_1 \cdots P_n)^t$, where $\{P_1, \ldots, P_n\}$ is the set of maximal t-ideals of D containing a and J is either D or a product of t-invertible t-unidirectional ideals. Recall that t-Max$(D) = w$-Max(D), so a t-invertible t-ideal is w-invertible. Hence, $JP_1 \cdots P_n$ is w-invertible, and thus $aD = (JP_1 \cdots P_n)^w$. Thus, D is a weakly w-ZPUI domain. The converse can be proved by the same argument. □

Theorem 3.3. *The following are equivalent for an integral domain D.*

(1) *D is a weakly t-ZPUI domain.*
(2) *D is a weakly Matlis domain and each maximal w-ideal of D is w-invertible.*
(3) *D is a weakly Matlis domain and each maximal t-ideal of D is t-invertible.*
(4) *D is a weakly w-ZPUI domain.*

Proof. (1) ⇔ (4) Lemma 3.2.

(2) ⇔ (3) This follows from [4, Theorem 2.16] by which t-Max$(D) = w$-Max(D).

(2)⇒(4) Suppose that (2) holds. Choose a nonzero nonunit a of D, and let $P_1, \ldots, P_n$ be the set of maximal t-ideals of D containing a. For each $i \in \{1, \ldots, n\}$, $P_iD_{P_i}$ is a principal ideal since P_i is w-invertible ideal of D, so $aD_{P_i} = a_iP_iD_{P_i}$ for some $a_i \in D$. Set $J_i = a_iD_{P_i} \cap D$, $J = J_1 \cdots J_n$ and $I = JP_1 \cdots P_n$. Then J_i is either D or a w-invertible t-unidirectional ideal for every $i \in \{1, \ldots, n\}$ [6, Theorem 3.3]. Hence $(aD)D_P = ID_P$ for every maximal t-ideal P of D, so $aD = I^w$.

(4)⇒(2) Assume that (4) is true. It is clear that every nonzero proper principal ideal of D is a finite w-product of w-unidirectional w-ideals of D. Hence, D is a weakly Matlis domain [6, Theorem 2.1]. Next, let $P_0 \in t$-Max(D), and choose a nonzero $a \in P_0$. Then $aD = (JP_0P_1 \cdots P_n)^w$, where $P_0, P_1, \ldots, P_n$ are the maximal t-ideals of D containing a, and J is either D or a product of w-invertible t-unidirectional ideals. Since aD is w-invertible, so is P_0. Hence every maximal w-ideal of D is w-invertible. □

Let D be an integral domain, X be an indeterminate over D, and $D[X]$ be the polynomial ring over D. Then a nonzero prime ideal Q of $D[X]$ is called an *upper to zero* in $D[X]$ if $Q \cap D = (0)$, and D is a *UMT-domain* if every upper to zero in $D[X]$ is a maximal t-ideal of $D[X]$. It is known that D is a PvMD if and only if D is an integrally closed UMT-domain [28, Proposition 3.2].

Corollary 3.4. *Let D be a UMT-domain and $D[X]$ be the polynomial ring over D. Then D is a weakly w-ZPUI domain if and only if $D[X]$ is a weakly w-ZPUI domain.*

Proof. By [22, Proposition 2.2], D is weakly Matlis if and only if $D[X]$ is weakly Matlis. Moreover, if Q is an upper to zero in $D[X]$, then Q is t-invertible [28, Theorem 1.4]. On the other hand, if Q is a prime ideal of $D[X]$ that is not upper to zero, then Q is a maximal t-ideal of $D[X]$ if and only if $Q = (Q \cap D)[X]$ and $Q \cap D$ is a maximal t-ideal of D ([28, Proposition 1.1] and [30, Proposition 2.2]). Note also that $(Q \cap D)[X]$ is t-invertible in $D[X]$ if and only if $Q \cap D$ is t-invertible in D [30, Proposition 2.2]. Thus, the result follows from Theorem 3.3. □

It is easy to see that if $D[X]$ is a weakly Matlis domain, then D is weakly Matlis [22, Proposition 2.2] and if D is a weakly Matlis domain that is also a UMT-domain,

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then $D[X]$ is a weakly Matlis domain [22, Proposition 2.2]. However, in general, D being weakly Matlis does not imply that $D[X]$ is weakly Matlis [22, Example 2.5].

Corollary 3.5. *Let $D[X]$ be the polynomial ring over an integral domain D. Then $D[X]$ is a weakly w-ZPUI domain if and only if D is a weakly w-ZPUI domain and $D[X]$ is a weakly Matlis domain.*

Proof. ($\Rightarrow$) By Theorem 3.3, $D[X]$ is weakly Matlis, so D is weakly Matlis. Next, let P be a maximal t-ideal of D. Then $P[X]$ is a maximal t-ideal of $D[X]$ [28, Proposition 1.1], and hence $P[X]$ is t-invertible by Theorem 3.3, so P is t-invertible. Thus, D is a weakly w-ZPUI domain by Theorem 3.3.

($\Leftarrow$) Let Q be a maximal t-ideal of $D[X]$. If $Q \cap D \neq (0)$, then $Q \cap D$ is a maximal t-ideal of D and $Q = (Q \cap D)[X]$. Hence, $Q \cap D$ is t-invertible by Theorem 3.3, and thus Q is t-invertible. Next, assume that $Q \cap D = (0)$. Then Q is t-invertible ([28, Proposition 1.1], [30, Proposition 2.2]). Thus, $D[X]$ is a weakly w-ZPUI domain by Theorem 3.3. □

The next example shows that (i) the localization of a weakly w-ZPUI domain need not be a weakly w-ZPUI domain and (ii) $D[X]$ a weakly w-ZPUI domain does not implies that D is a UMT-domain.

Example 3.6. (cf. [7, Example 2b]) Let $\mathbb{Q}$ be the field of rational numbers, Y, Z be indeterminates over $\mathbb{Q}$, $\mathbb{Q}[[Y,Z]]$ be the power series ring over $\mathbb{Q}$, p be a prime number, and $D = \mathbb{Z}_{p\mathbb{Z}} + (Y,Z)\mathbb{Q}[[Y,Z]]$. Then D is a quasilocal ring whose maximal ideal is principal. Let K be the quotient field of D, T be an indeterminate over K, and $R = D + TK[[T]]$. Then R is a quasilocal ring whose maximal ideal is principal. Hence, R is a weakly Matlis domain whose maximal t-ideals are t-invertible. Thus, by Theorem 3.3, R is a weakly w-ZPUI domain.

(1) However, if $N = \{p^n \mid n \geq 0\}$, then $R_N = \mathbb{Q}[[Y,Z]] + TK[[T]]$ is not weakly Matlis, and thus R_N is not a weakly w-ZPUI domain.

(2) Note that R is integrally closed by [11, Theorem 2.1(b)], and R is quasilocal whose maximal ideal is principal, but R is not a valuation domain. Moreover, $R[X]$, the polynomial ring over R, is a weakly Matlis domain by [22, Corollary 2.3]. Thus, $R[X]$ is a weakly w-ZPUI domain by Theorem 3.3.

The following corollaries are immediate consequences of Theorem 3.3.

Corollary 3.7. *The following statements are equivalent for an integral domain D.*

(1) *D is a weakly ZPUI domain.*
(2) *D is an h-local domain in which each maximal ideal is invertible.*
(3) *D is a weakly w-ZPUI domain in which each maximal ideal is a t-ideal.*

Proof. (1) $\Rightarrow$ (3) Clearly, D is a weakly w-ZPUI domain. Moreover, if M is a maximal ideal of D, then M is invertible by the same argument as in the proof of (4) $\Rightarrow$ (2) of Theorem 3.3. Thus, M is a t-ideal.

(3) $\Rightarrow$ (2) D is weakly Matlis by Theorem 3.3, and since each maximal ideal of D is a t-ideal, D is h-local. Moreover, each maximal ideal is invertible by Theorem 3.3 again.

(2) $\Rightarrow$ (1) An invertible ideal is a t-ideal, so $d = w$ on D and an h-local domain is weakly Matlis. Thus, by the proof of (2) $\Rightarrow$ (4) in Theorem 3.3, D is a weakly Matlis ZPUI domain. □

It is known that D is a Krull domain if and only if every nonzero principal ideal of D can be written as a finite w-product of prime ideals ([31, Theorem 3.9] and the fact that $t = w$ on a Krull domain). Hence, a Krull domain D is a field or a weakly w-ZPUI domain with t-dim$(D) = 1$.

Corollary 3.8. *An integral domain D is a Krull domain if and only if D is a weakly w-ZPUI domain and t-dim$(D) \leq 1$.*

Proof. ($\Rightarrow$) Clear. ($\Leftarrow$) If t-dim$(D) = 0$, then D is a field, and hence D is a Krull domain. Next, assume that t-dim$(D) = 1$. Then every prime t-ideal of D is t-invertible by Theorem 3.3. Thus, D is a Krull domain [31, Theorem 3.6]. □

Corollary 3.9. *Let D be an integral domain with dim$(D) = 1$. Then the following statements are equivalent.*

(1) *D is a Dedekind domain.*
(2) *D is a weakly ZPUI domain.*
(3) *D is a ZPUI domain.*
(4) *D is a weakly w-ZPUI domain.*
(5) *D is a w-ZPUI domain.*

Proof. (1) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (4) Clear.

(4) $\Rightarrow$ (1) By Corollary 3.8, D is a Krull domain. Thus, dim$(D) = 1$ implies that D is a Dedekind domain.

(3) $\Rightarrow$ (5) $\Rightarrow$ (4) Clear. □

A ZPUI domain is a Prüfer domain, and hence integrally closed. The next example shows that (i) a weakly ZPUI domain need not be integrally closed and (ii) a weakly ZPUI domain that is also a valuation domain need not be a ZPUI domain.

Example 3.10. (1) Let $D = \mathbb{Z}_{2\mathbb{Z}} + X\mathbb{Q}(\sqrt{2})[[X]]$. Then D is quasilocal with principal maximal ideal $M = 2D$. Clearly, Spec(D) is linearly ordered. Thus, D is a weakly ZPUI domain by Corollary 3.7. Moreover, D is not integrally closed, and hence D is not a ZPUI domain.

(2) Let V be a valuation domain with principal maximal ideal and idempotent nonzero nonmaximal prime ideal. Then V is a weakly ZPUI domain by Corollary 3.7 but not a ZPUI domain because V is not strongly discrete. Moreover, V is an h-local Prüfer domain.

Hence, to discuss the weakly ZPUI domains along with ZPUI domains, we need to consider a strongly discrete Prüfer domain.

Corollary 3.11. *Let D be a strongly discrete Prüfer domain (resp., strongly discrete PvMD). Then D is a ZPUI domain (resp., w-ZPUI domain) if and only if D is a weakly ZPUI domain (resp., weakly w-ZPUI domain).*

Proof. (1) A ZPUI domain case: If D is a weakly ZPUI domain, then, by Corollary 3.7, D is a strongly discrete h-local Prüfer domain. Thus, D is a ZPUI domain [34, Theorem 2.3]. The converse is clear by definition.

(2) A w-ZPUI domain case: Let D be a weakly w-ZPUI domain. Then D is weakly Matlis by Theorem 3.3 and D is a strongly discrete PvMD by assumption. Thus, D is a w-ZPUI domain [16, Corollary 3.10]. The converse is clear. □

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Recall from [27, Theorem 3.1] that D is a PvMD on which $t = v$ if and only if D is an independent ring of Krull type whose maximal t-ideals are t-invertible.

Corollary 3.12. *Let D be a PvMD, $D[X]$ be the polynomial ring over D, and $N_v = \{f \in D[X] \mid f \neq 0$ and $c(f)_v = D\}$. Then the following statements are equivalent.*

(1) *D is a weakly w-ZPUI domain.*
(2) *$t = v$ on D.*
(3) *D is an independent ring of Krull type whose maximal t-ideals are t-invertible.*
(4) *Each nonzero ideal of $D[X]_{N_v}$ is divisorial.*
(5) *$D[X]$ is a weakly w-ZPUI domain.*
(6) *$D[X]_{N_v}$ is an h-local Prüfer domain whose maximal ideals are principal.*
(7) *$D[X]_{N_v}$ is a weakly ZPUI domain.*

Proof. (1) ⇔ (2) ⇔ (3) These follow from Theorem 3.3 and [27, Theorem 3.1].

(2) ⇔ (4) [29, Theorem 2.10].

(2) ⇔ (5) Note that $D[X]$ is a PvMD since D is a PvMD, so $t = v$ on D if and only if $t = v$ on $D[X]$ by [29, Corollary 3.6]. Thus, $t = v$ on D if and only if $D[X]$ is a weakly w-ZPUI domain by the equivalence of (1) and (2).

(3) ⇔ (6) This follows from [16, Lemma 2.1 and Lemma 2.2].

(6) ⇔ (7) Recall that $D[X]_{N_v}$ is a Prüfer domain and each invertible ideal of $D[X]_{N_v}$ is principal [30, Theorems 2.14 and 3.7]. Thus, the result follows from Corollary 3.7. □

Let D be an integral domain. Then D is a PvMD if and only if D is integrally closed and $w = t$ [30, Theorem 3.5]. Hence, by Corollary 3.12, D is an independent ring of Krull type whose maximal t-ideals are t-invertible if and only if D is an integrally closed and $w = v$ on D (see [19, Theorem 3.3]).

Corollary 3.13. *The following are equivalent for a Prüfer domain D.*

(1) *D is a weakly ZPUI domain.*
(2) *Every nonzro ideal of D is divisorial.*
(3) *D is an h-local Prüfer domain whose maximal ideals are invertible.*
(4) *Each nonzero ideal of $D(X)$ is divisorial.*

Proof. A Prüfer domain is a PvMD whose maximal ideals are t-ideals, so $d = w = t$ on a Prüfer domain. Thus the result follows from Corollary 3.12. □

Let D be an almost Dedekind domain. Then D_M is a weakly ZPUI domain for all maximal ideals M of D, while D is a weakly ZPUI domain if and only if D is of finite character. The next result shows that this is true of a weakly $*$-ZPUI domain for $* = d$ or w.

Proposition 3.14. *Let D be an integral domain and $* = d$ or w. Then D is a weakly $*$-ZPUI domain if and only if D is of $*$-finite character and D_M is a weakly ZPUI domain for each maximal $*$-ideal M of D.*

Proof. Let D be a weakly $*$-ZPUI domain. Then D is of $*$-finite character by Theorem 3.3 and Corollary 3.7. Furthermore, if M is a maximal $*$-ideal of D, then D_M is a weakly ZPUI domain by the definition of a weakly $*$-ZPUI domain.

Conversely, assume that D is of $*$-finite character and D_M is a weakly ZPUI domain for each maximal $*$-ideal M of D. Let a be a nonzero nonunit of D, and let $M_1, \dots, M_n$ be the set of maximal $*$-ideals of D containing a. Then $aD_{M_i} = I_i M_i D_{M_i}$ for some ideal I_i of D for each $i \in \{1, \dots, n\}$. Let $I = \prod_{i=1}^{n} (I_i D_{M_i} \cap D) M_i$. Then the maximal $*$-ideals of D containing I are $M_1, \dots, M_n$. Moreover, $I_i D_{M_i} \cap D$ is either equal to D or a $*$-unidirectional ideal of D [6, Lemma 2.3]. Since $aD_{M_i} = ID_{M_i}$ for each $i \in \{1, \dots, n\}$, we conclude that $aD = I^*$. Hence, D is a weakly $*$-ZPUI domain. □

Motivated by Theorem 2.2 and Corollary 2.5, we next give some characterizations of the weakly ZPUI domain (resp., weakly w-ZPUI domain) which is also a Bézout domain (resp., GCD domain).

Theorem 3.15. *The following statements are equivalent for a GCD domain D.*

(1) *D is a weakly w-ZPUI domain.*
(2) *D is a weakly w-ZPUI domain that is also a HoFD.*
(3) *D is a weakly Matlis domain whose maximal ideals are principal.*
(4) *D is an HoFD and for each nonzero nonunit $a \in D$, there is an element $b \in D$ such that $aD \subsetneq bD$ and $\frac{a}{b}D$ is a finite product of distinct principal maximal w-ideals of D.*

Proof. We first note that $Cl_t(D) = \{0\}$ because D is a GCD domain.

(1) ⇒ (2) Note that the product of two homogenous elements contained in the same maximal t-ideal M is a homogeneous element contained in M. Now choose a nonzero nonunit a of D. Then $aD = (IP_1 \cdots P_n)^w$ for some ideal I that is either D or a w-product of w-unidirectional w-invertible w-ideals and maximal w-ideals $P_1, \dots, P_n$ of D containing a. Since D is a GCD domain, $I, P_1, \dots, P_n$ are all principal, and hence a is a product of w-unidirectional elements of D. Thus, D is an HoFD.

(2) ⇒ (3) ⇒ (1) These follow from Theorem 3.3 because a nonzero principal ideal is t-invertible and $Cl_t(D) = \{0\}$.

(2) ⇒ (4) $Cl_t(D) = \{0\}$, so the result follows directly from the definition of a weakly w-ZPUI domain.

(4) ⇒ (3) Let D be a weakly Matlis domain whose maximal ideals are principal. Then D is a weakly w-ZPUI domain by Theorem 3.3. Thus D is a weakly ZPUI domain by Corollary 3.7.

(5) ⇒ (1) Let a be a nonzero nonunit of D. Then $a = a_1 \cdots a_n$ for some homogeneous elements $a_1, \dots, a_n$ of D such that M_i is the unique maximal t-ideal of D containing a_i for each $i \in \{1, \dots, n\}$. Hence, D is weakly Matlis. Now, let P be a maximal t-ideal of D. Then P contains a homogeneous element, say, a, and there is an element $b \in D$ such that $aD \subsetneq bD$ and $\frac{a}{b}D$ is a finite product of principal maximal t-ideals. Note that a is homogeneous, so $\frac{a}{b}D = P$. □

The next result is a weakly ZPUI domain analog of Theorem 3.15.

Corollary 3.16. *The following statements are equivalent for a Bézout domain D.*

(1) *D is a weakly ZPUI domain.*
(2) *D is a weakly ZPUI domain that is also a HoFD.*
(3) *D is an h-local domain whose maximal ideals are principal.*
(4) *D is a weakly Matlis domain whose maximal ideals are principal.*
(5) *D is an HoFD and for each nonzero nonunit $a \in D$, there is an element $b \in D$ such that $aD \subsetneq bD$ and $\frac{a}{b}D$ is a finite product of distinct maximal principal ideals of D.*

Proof. A Bézout domain is a GCD domain whose maximal ideals are t-ideals, so $d = w$. Moreover, an h-local domain is a weakly Matlis domain whose maximal ideals are t-idels. Thus, the result is an immediate consequence of Theorem 3.15. □

Example 3.17. The assumption that D is an HoFD cannot be dropped in Theorem 3.15.(5). Let $D = \mathbb{Z} + X\mathbb{Q}[X]$. Note that D is a Bézout domain whose maximal ideals are principal, so for each nonzero nonunit $a \in D$, there is an element $b \in D$ such that $a \in bD$ and $\frac{a}{b}D$ is a finite product of distinct maximal principal ideals of D. But since D is not an h-local domain, D cannot be a weakly ZPUI domain by Corollary 3.7.

A nonzero nonunit $a \in D$ is said to be a *pseudo-irreducible* element if it is impossible to factor a as $a = bc$ with b and c comaximal nonunits. Following [32], we say that D is a *comaximal factorization domain* (*CFD*) if every nonzero nonunit of D can be written as a product of pairwise comaximal pseudo-irreducible elements. A CFD D is called a *unique comaximal factorization domain* (*UCFD*) if the products of pairwise comaximal pseudo-irreducible elements are unique (up to order and units).

Proposition 3.18. *A weakly ZPUI domain D with $Cl_t(D) = \{0\}$ is a UCFD. Hence, a Bézout ZPUI domain is a UCFD.*

Proof. Let $a, b \in D$ be nonzero nonunits such that aD and bD are both unidirectional. It is clear that a and b are pseudo-irreducible. Moreover, if $(a, b) \subsetneq D$, then ab is pseudo-irreducible. Note that every nonzero proper principal ideal of D can be written as a finite product of pairwise comaximal unidirectional principal ideals, so every nonzero nonunit of D can be written uniquely as a finite product of pseudo-irreducible elements of D. Thus, D is a UCFD. Moreover, a Bézout ZPUI domain is a weakly ZPUI domain with trivial t-class group, and hence it is a UCFD. □

In [12], Brewer and Heinzer studied integral domains for which (i) each nonzero ideal ((ii) each nonzero principal ideal) can be written as a product $Q_1 \cdots Q_n$, where the Q_i are pairwise comaximal and each Q_i has prime radical. Clearly, (i) implies (ii), but (ii) does not imply (i) [12, Example 7]. Moreover, if D satisfies (ii), then D is *treed*, i.e., $\text{Spec}(D_M)$ is linearly ordered under inclusion for all maximal ideals M of D [12, Theorem 1]. It is easy to see that if D is a ZPUI domain (resp., weakly ZPUI domain that is treed), then each unidirectional ideal of D has a prime radical, and thus D satisfies (i) (resp., (ii)).

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4. Weak and w-weak π-domains

Let D be a π-domain that is not a Dedekind domain. Then (i) every nonzero proper principal ideal of D can be written as a product of an invertible ideal and a nonempty collection of prime ideals and (ii) D is a weakly w-ZPUI domain by Theorem 3.3, but D is not a weakly ZPUI domain by Corollary 3.7. We next list four ideal factorization properties which are generalizations of π-domains, Krull domains, and (w-)ZPUI domains.

Question 4.1. Let D be an integral domain. What could be said of integral domains with the following ideal factorization properties?

(1) For each nonzero nonunit $a \in D$, there is an element $b \in D$ such that $aD \subsetneq bD$ and $\frac{a}{b}D$ is a (i) finite product (resp., (ii) w-product) of prime ideals of D. (e.g., π-domains (resp., Krull domains))
(2) For each nonzero nonunit $a \in D$, there is an element $b \in D$ such that $aD \subsetneq bD$ and $\frac{a}{b}D$ is a (i) finite product (resp., (ii) w-product) of pairwise comaximal prime ideals (resp., w-comaximal prime w-ideals) of D. (e.g., PIDs (resp., UFDs))
(3) For each nonzero proper ideal (principal ideal) I of D, there is an invertible ideal J of D such that $I \subsetneq J$ and IJ^{-1} is a finite product of prime ideals (resp., pairwise comaximal prime ideals) of D. (e.g., ZPUI domains, Dedekind domains (resp., ZPUI Bézout domains))
(4) For each nonzero proper ideal (principal ideal) I of D, there is a w-invertible ideal J of D such that $I \subsetneq J^w$ and $(IJ^{-1})^w$ is a finite w-product of prime ideals (resp., pairwise w-comaximal prime ideals) of D. (e.g., w-ZPUI domains, Krull domains (resp., w-ZPUI GCD domains)

In this section, we briefly study integral domains satisfying the property described in Question 4.1.(1).

Definition 4.2. *Let D be an integral domain and $* = d$ or w. We will say that D is a $*$-weak π-domain if, for each nonzero nonunit $a \in D$, there is an element $b \in D$ such that $a \in bD$ and $\frac{a}{b}D$ is a finite $*$-product of prime $*$-ideals of D.*

Recall that a nonzero nonunit a of an integral domain D is *irreducible* if for each $b, c \in D$ such that $a = bc$, one of b and c is a unit of D. An integral domain in which each nonzero nonunit is a finite product of irreducible elements is said to be *atomic*. An integral domain D is said to be *Furstenberg* if each nonzero nonunit of D is divided by an irreducible element of D [17]. It is clear that every atomic domain is Furstenberg, but the converse does not hold in general. For instance, the domain D in Example 3.17 is Furstenberg, but it is not atomic. Our next result is that every $*$-weak π-domain is Furstenberg. First, consider the following simple lemma.

Lemma 4.3. *Let $P_1, \dots, P_n$ be nonzero prime $*$-ideals of an integral domain D and $I_1 \cdots, I_m$ be nonzero proper $*$-invertible $*$-ideals of D such that $(P_1 \cdots P_n)^* =$*

$(I_1 \cdots I_m)^*$. *Then there exists a partition* $\{A_i\}_{i=1}^m$ *of* $\{1, \dots, n\}$ *such that* $I_i = (\prod_{j \in A_i} P_j)^*$ *for each* $i \in \{1, \dots, m\}$.

Proof. We use induction on n. Notice that $P_1, \dots, P_n$ are $*$-invertible ideals, since $I_1 \cdots I_m$ is a $*$-invertible ideal. When $n = 1$, it follows that without loss of generality $I_1 \subseteq P_1$, so $I_1 = (P_1 I)^*$ for some ideal I of D. If $m > 1$, then $D = (I I_2 \cdots I_m)^*$, a contradiction. Hence $m = 1$ and $I_1 = P_1$, and the statement holds for $n = 1$. To initiate the induction process, suppose that there exists $k \in \mathbb{N}$ such that the statement holds when $n = k$. If $n = k + 1$, then without loss of generality $I_m \subseteq P_{k+1}$, so $I_m = (P_{k+1} J)^*$ for some $*$-invertible ideal J of D. Hence $(P_1 \cdots P_k)^* = (I_1 \cdots I_{m-1} J)^*$, and there exists a partition $\{A_i\}_{i=1}^m$ of $\{1, \dots, k\}$ such that $I_i = (\prod_{j \in A_i} P_j)^*$ for each $i \in \{1, \dots, m-1\}$ and $J = (\prod_{j \in A_m} P_j)^*$. Adjoining $k+1$ to A_m, we have the desired partition of $\{1, \dots, n\}$. □

It is clear that a d-weak π-domain is a w-weak π-domain but not vice versa. For convenience, we call a d-weak π-domain a weak π-domain.

Proposition 4.4. *An integral domain D is a weak π-domain (resp., w-weak π-domain) if and only if it is Furstenberg and each principal ideal generated by an irreducible element is a product of prime ideals (resp., w-product of maximal w-ideals) of D.*

Proof. We only need to show the necessity of the statement. In fact, it suffices to show that D is Furstenberg, for every w-invertible prime w-ideal is a maximal w-ideal ([28, Proposition 1.3], [14, Proposition 2.5]). Let D be a weak π-domain and a be a nonzero nonunit of D. Then there exists an element b of D, distinct prime ideals $P_1, \dots, P_n$ of D, and $r_1, \dots, r_n \in \mathbb{N}$ such that $\frac{a}{b} D = (P_1^{r_1} \cdots P_n^{r_n})^*$. If $\frac{a}{b}$ is an irreducible element of D, then a is divided by an irreducible element, so we have nothing to prove. Suppose that $\frac{a}{b}$ is not irreducible. Then $\frac{a}{b} = c_1 d_1$ for some nonzero nonunits c_1, d_1 of D. Hence by Lemma 4.3, there exist integers $a_{11}, \dots, a_{1n}$ such that $0 \leq a_{1j} \leq a_j$ for $j = 1, \dots, n$, $c_1 D = (\prod_{j=1}^n P_j^{a_{1j}})^*$, $d_1 D = (\prod_{j=1}^n P_j^{a_j - a_{1j}})^*$, and $\sum_{j=1}^n a_{1j} < \sum_{j=1}^n a_j$. If c_1 is irreducible, then we are done. Otherwise, $c_1 = c_2 d_2$ for some nonzero nonnuits c_2, d_2 of D. Again by Lemma 4.3, there exist integers $a_{21}, \dots, a_{2n}$ such that $0 \leq a_{2j} \leq a_{1j}$ for $j = 1, \dots, n$, $c_1 D = (\prod_{j=1}^n P_j^{a_{2j}})^*$, $d_1 D = (\prod_{j=1}^n P_j^{a_{1j} - a_{2j}})^*$, and $\sum_{j=1}^n a_{2j} < \sum_{j=1}^n a_{1j}$. Iterating, we conclude that $\frac{a}{b}$ is divided by an irreducible element. □

Corollary 4.5. *An integral domain is a π-domain (resp., Krull domain) if and only if it is an atomic weak π-domain (resp., w-weak π-domain).*

Proof. This follows because D is an atomic $*$-weak π-domain if and only if each principal ideal of D is a finite $*$-product of prime $*$-ideals by Proposition 4.4. □

Corollary 4.6. *Let D be an integral domain. If $D[X]$ is a w-weak π-domain, then so is D.*

Proof. $D[X]$ is Furstenberg by Proposition 4.4, and so is D [25, Proposition 4.7]. Now let d be an irreducible element of D. Then it is also an irreducible element of $D[X]$, and $dD[X] = (Q_1^{a_1} \cdots Q_n^{a_n})^w$ for some $a_i \in \mathbb{N}$ and distinct maximal w-ideals $Q_1, \dots, Q_n$ of $D[X]$ by Proposition 4.4. Note that $(\{Q_i^{a_i}\})^w$ are Q_i-primary w-ideals of R [37, Theorem 3.1], so we have $dD[X] = (Q_1^{a_1})^w \cap \cdots \cap (Q_n^{a_n})^w$ [6, Lemma 2.5]. Therefore, none of Q_i is upper to zero in $D[X]$, and hence $dD[X] = \prod_{i=1}^{n} ((Q_i \cap D)^{a_i})^w[X]$ [28, Proposition 1.1]. Hence $dD = \prod_{i=1}^{n} ((Q_i \cap D)^{a_i})^w$. Note that $Q_i \cap D$ is a prime t-ideal [14, Lemma 2.8], that is also t-invertible for each $i \in \{1, \dots, n\}$. Therefore D must be a w-weak π-domain by Proposition 4.4 and [28, Proposition 1.3]. □

An integral domain D is *divided* if for each prime ideal P of D and an element a of D, either $a \in P$ or $P \subseteq aD$. If D_M is divided for each maximal ideal (resp., each maximal t-ideal) M of D, we say that D is *locally divided* (resp., t-locally divided) (see, for example, [10] and [13]). Note that one-dimensional domains and Prüfer domains are locally divided, PvMDs are t-locally divided, and D is locally divided if and only if D is t-locally divided whose maximal ideals are t-ideals.

Proposition 4.7. *Let D be an integral domain. Then we have the following.*

(1) *If D is locally divided, then each invertible prime ideal of D is maximal.*
(2) *If $Cl_t(D) = \{0\}$, then D is a w-weak π-domain if and only if every nonzero nonunit of D is contained in a principal prime ideal of D.*

Proof. (1) If P is an invertible prime ideal of a locally divided domain D, then let M be a maximal ideal of D that contains P. It follows that PD_M is an invertible prime ideal of a divided domain R_M. Now let $a \in D_M \setminus PD_M$. Then $PD_M \subsetneq aD_M$, so $PD_M = aID_M$ for some ideal I of D contained in M. Since $a \notin PD_M$, we must have $ID_M \subseteq PD_M$. Hence $PD_M = aPD_M$, and since P is invertible, a is a unit of R_M. Hence $PD_M = MD_M$, and $P = PD_M \cap D = MD_M \cap D = M$. Thus, P is a maximal ideal.

(2) Suppose that $Cl_t(D) = \{0\}$. If D is a w-weak π-domain, then for each nonzero nonunit $r \in D$, rD is a w-product of a principal ideal and a nonempty collection of finitely many prime w-ideals. Each of such prime w-ideals must be t-invertible since rD is invertible. Since $Cl_t(D) = \{0\}$, every t-invertible ideal is principal. Hence r is contained in a principal prime ideal of D. Conversely, assume that every nonzero nonunit of D is contained in a principal prime ideal of D. If $a \in D$ is a nonzero nonunit, then $a \in pD$ for some principal prime pD, so $a = pd$ for some $d \in D$. Clearly, $aD \subsetneq dD$ and $\frac{a}{d} = p$. Thus, D is a w-weak π-domain. □

The following is an immediate corollary of Proposition 4.7.

Corollary 4.8. *Let D be an integral domain with $Cl_t(D) = \{0\}$.*

(1) *Let D be a t-locally divided domain (e.g., a GCD domain). Then D is a w-weak π-domain if and only if every maximal w-ideal of D is principal.*

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(2) *Let D be a locally divided domain (e.g., a Bézout domain). Then the following are equivalent.*
   (a) *D is a weak π-domain.*
   (b) *D is a w-weak π-domain.*
   (c) *Every maximal ideal of D is a principal ideal.*

It is not difficult to see that if D is a GCD w-weak π-domain, then $D[X]$ is a w-weak π-domain by Corollary 4.8. In the remainder of this section, we focus on how a new $*$-weak π-domain can be obtained from the old ones via $D+M$ construction and localization.

Proposition 4.9. *Let T be a quasilocal domain such that $T = K + M$ for some field K and a maximal ideal M of T. Let D be a proper subdomain of K, and let $R = D + M$.*

(1) *If D is not a field, then R is a weak π-domain if and only if D is a weak π-domain.*
(2) *If T is an atomic domain and D is a field, then R is not a weak π-domain.*

Proof. (1) Note first that every irreducible element of R is of the form $a+m$, where a is an irreducible element of D and m is an element of M. Indeed, let $a \in D$ and $m \in M$ so $a+m$ is an irreducible element of $D+M$. If $a = 0$, then for any nonzero nonunit b of D, we have $a + m = b\left(\frac{m}{b}\right)$ and neither b nor $\frac{m}{b}$ is a unit of R (note that $\frac{m}{b} \in Km \subseteq M \subseteq R$), a contradiction. Hence a is nonzero. If $a = bc$ for some $b, c \in D$, then $a+m = b\left(c+\frac{m}{b}\right)$, so either b or c is a unit of D [9, Lemma 4.17(2)]. Hence a is an irreducible element of D. Note also that if d is an irreducible element of D, then it is an irreducible element of R [9, Lemma 4.17(2)].

Suppose that R is a weak π-domain. Then D is Furstenberg. Indeed, if d is a nonzero nonunit of D, then d is divided by an irreducible element of R, say $a + m$ for some $a \in D, m \in M$. As mentioned in the first paragraph of this proof, a is an irreducible element of D that divides d. Hence D is Furstenberg. On the other hand, let c be an irreducible element of D. Then c is also an irreducible element of R, so $cR = Q_1 \cdots Q_n$ for some prime ideals $Q_1, \dots, Q_n$ of R by Proposition 4.4. Since $cR = cD + M$, we must have $M \subseteq Q_i$ for all $i \in \{1, \dots, n\}$. Hence for each i, there exists a prime ideal P_i of D such that $Q_i = P_i + M$ [20, Lemma 1.1.4]. It follows that $cD + M = P_1 \cdots P_n + M$, so $cD = P_1 \cdots P_n$. Hence D is a weak π-domain by Proposition 4.4.

Conversely, suppose that D is a weak π-domain. Then for each nonzero nonunit element $r \in R$, $r = a + m$ for some $a \in D, m \in M$. If $a = 0$, then r is divided by any irreducible element of D, which is also an irreducible element of R. If $a \neq 0$, then a is divided by an irreducible element b of D, and $r = b\left(\frac{a}{b} + \frac{m}{b}\right)$. Hence R is Furstenberg. On the other hand, choose an irreducible element of R. Then such an element is of the form $a + m$ for some irreducible element a of D. Then $(a+m)R = aR$ since $1+\frac{m}{a}$ is a unit of R [9, Lemma 4.17(2)]. Then by Proposition

4.4, $cD = P_1 \cdots P_n$ for some prime ideals $P_1, \ldots, P_n$ of D. Hence $cR = Q_1 \cdots Q_n$ where $Q_i = P_i + M$ is a prime ideal of R for each $i \in \{1, \ldots, n\}$. Thus R is a weak π-domain by Proposition 4.4.

(2) Since T is atomic and D is a field, R must be atomic [2, Proposition 1.2(a)]. On the other hand, R and T have the same complete integral closure since they share a nonzero proper ideal M. Hence R is not completely integrally closed, so R cannot be a Krull domain. Thus R is not a weak π-domain by Corollary 4.5. □

Note that localization of a weak π-domain may not be a weak π-domain. Indeed, let V be as in Example 3.10.(2). Then since V is a divided domain, Corollary 4.8 tells us that V is a weak π-domain, while the localization of V with respect to its idempotent nonzero prime ideal cannot be a weak π-domain. We next show that if S is a splitting set of a weak π-domain D, then so is D_S. We first list some properties of splitting sets.

Lemma 4.10. *Let D be an integral domain and S be a multiplicative subset of D. Then we have the following.*

(1) *Let S be a splitting set and x be an element of D, so $x = as$ for some $a \in D$ and $s \in S$ such that $aD \cap tD = atD$ for each $t \in S$. Then x is an irreducible element of D_S if and only if a is an irreducible element of D.*

(2) *Let S be a multiplicative set generated by prime elements of D, i.e., there exists a set T that consists of prime elements of D such that each element of S is of the form $up_1 \cdots p_n$ where u is a unit of D and $p_1, \ldots, p_n \in T$. Then S is a splitting set if and only if $\bigcap_{n\in\mathbb{N}} p^n D = (0)$ for each prime element $p \in S$, and $\bigcap_{n\in\mathbb{N}} p_n D = (0)$ for each sequence $\{p_n\}$ of nonassociative prime elements of S.*

Proof. These are part of [3, Lemma 1.2, Corollary 1.4, Proposition 1.6]. □

Using the notion of splitting sets, we can answer the question concerning localization of weak π-domains.

Proposition 4.11. *Let S be a splitting set of an integral domain D generated by prime elements of D. If D is a weak π-domain, then so is D_S.*

Proof. Let $\frac{x}{s}$ be an irreducible element of D_S for some $x \in D, s \in S$. Then $x = as$ for some $a \in D$ and $s \in S$ such that $aR \cap s'D = as'D$ for each $s' \in S$, and a is an irreducible element of D by Lemma 4.10.(1). By Proposition 4.4, $aD = P_1^{a_1} \cdots P_n^{a_n}$ for some prime ideals $P_1, \ldots, P_n$ of D and $a_1, \ldots, a_n \in \mathbb{N}$. Then the ideal of D_S generated by $\frac{x}{s}$ is a product of prime ideals of D_S. On the other hand, D is Furstenberg by Proposition 4.4. Applying the proof of [25, Proposition 6.4], it follows that D_S is Furstenberg. Hence by Proposition 4.4, D_S is a weak π-domain. □

We next give an example which shows that Proposition 4.11 fails in general if S is merely a multiplicative subset S of R generated by primes.

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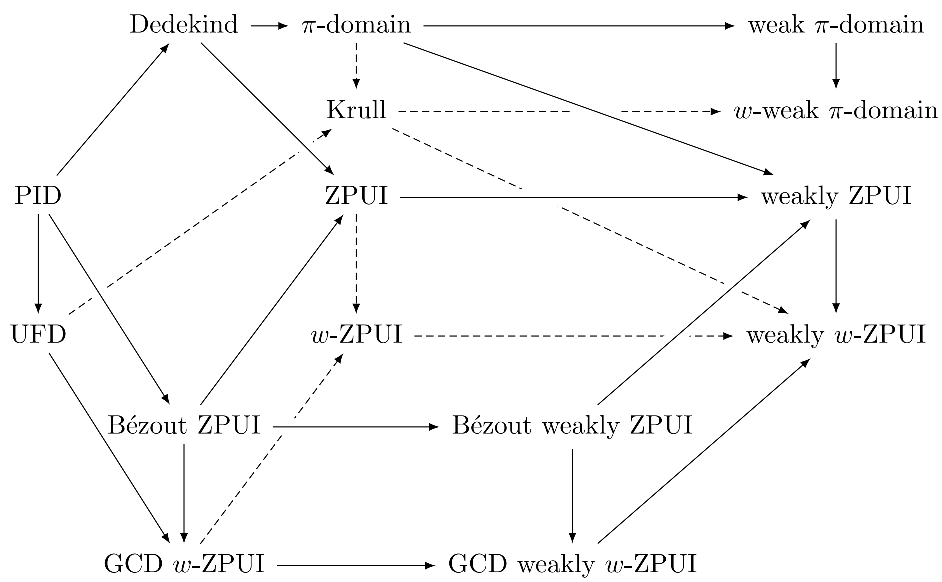
Example 4.12. Let $R = \mathbb{Z}_{2\mathbb{Z}} + X\mathbb{C}[[X]]$. Then R is a weak π-domain by Proposition 4.9. On the other hand, let S be the multiplicative subset of R generated by 2. Then $R_S = \mathbb{Q} + X\mathbb{C}[[X]]$ is not a weak π-domain by Proposition 4.9. Note that since $X \in \bigcap_{n\in\mathbb{N}} 2^n R$, S is not a splitting set of R by Lemma 4.10.(2) as mentioned in [25, Example 6.5].

We end this paper with a diagram showing the implications between various classes of domains dealt with in this work.



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