Positive periodic solutions to a generalized Lennard-Jones potential with indefinite weights *

Zhibo Cheng † Yuting Qian Juan Song

School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo 454000, China

Abstract—In this paper, we present sufficient conditions for the existence of positive periodic solutions to the generalized Lennard-Jones potential with indefinite weights

$$x'' = \frac{h(t)}{x^\rho} - \frac{g(t)}{x^\mu},$$

where h and g are sign-changing functions. The main tools are Krasnoselskiǐ's-Guo fixed point theorem and the positivity of the associated Green function.

Keywords—Krasnoselskiǐ's-Guo fixed point theorem; Lennard-Jones potential; Singular; Indefinite weight; Positive periodic solution.

MSC2020—34B16; 34C25.

1 Introduction

This paper deals with the problem of at least one positive periodic solution for the generalized Lennard-Jones potential with indefinite weights

$$x'' = \frac{h(t)}{x^\rho} - \frac{g(t)}{x^\mu}, \tag{1.1}$$

where ρ and μ are two positive constants, h and $g \in L^1(\mathbb{R}/T\mathbb{Z})$, and T is a positive constant. In eq. (1.1), the sign of the functions $h(t)$ and $g(t)$ are allowed to change. This means that the singularity associated with $\frac{h(t)}{x^\rho} - \frac{g(t)}{x^\mu}$ at $x = 0$ is an indefinite type.

*Research is supported by Technological innovation talents in universities and colleges in Henan Province (21HASTIT025), Natural Science Foundation of Henan Province (222300420449), Henan Province "Double first-class" discipline establishment engineering cultivation project (AQ20230718) and Innovative Research Team of Henan Polytechnic University (T2022-7).

†Corresponding author. E-mail: czb_1982@126.com.

The Lennard-Jones potential [15] is a very well-known empirical model in molecular dynamics to model the interaction potential between a pair of electrically neutral molecules or atoms. Its generalized expression is

$$V_{n,m}(x) = \frac{A}{|x|^n} - \frac{B}{|x|^m}.$$

On the right end of the equation, it presents two different terms. The first term describes the short-range repulsive force caused by overlapping electron orbitals (the so-called Pauli repulsion), while the second term models the long-range attractive force (van der Waals force). Thus, the generalized Lennard-Jones potential can be seen as differential equation (1.1) with attractive-repulsive singularities, see [1,16]. Furthermore in a different physical setting, positive periodic solutions of eq. (1.1) is equivalent to a similar breathing period controlled by the scattering length in a Bose-Einstein condensed state (the mathematical model is a nonlinear Schrödinger equation with cubic terms, see [18] for details). A third potential area of application is pulse propagation in nonlinear fibers, as described in [19, Section 5.4].

By this reason, this problem has been examined by many scholars before and several papers have given sufficient conditions for the existence of positive periodic solutions of eq. (1.1), see for instance [2, 5–8, 11–13, 17, 19] and the reference therein. First, Hakl and Torres paved the way for solving such formal problems in [11]. Chu et al. [7] in 2016 then further spread the results of [11] and proved in their Theorem 4.3 that eq. (1.1) had at least one twist periodic solution. The proof of [7] and [11] is based on the method of lower and upper functions and h and $g \in L^1(\mathbb{R}/T\mathbb{Z})$ are positive. Later in [13], Hakl and Zamora in 2017 investigated the existence of positive periodic solutions for a special form eq. (1.1) (i.e., Emden-Fowler equation)

$$x'' = \frac{h(t)}{x^{\rho}}, \tag{1.2}$$

where $h \in L^1(\mathbb{R}/T\mathbb{Z})$ and $\rho \geq 1$. After that, this attribute was extended by Godoy and Zamora in [8]. They obtained the existence of positive periodic solutions of eq. (1.2) if $0 < \rho < 1$. We also note that the proofs of [8] and [13] rely on a direct application of Leray-Schauder degree theory.

Originally motivated by the pioneer papers [7, 8, 11, 13], the objective of this paper is to consider eq. (1.1) in a unified way and then to derive new sufficient conditions when $h(t)$ and $g(t)$ are of indefinite sign. The main tools are Krasnoselskiĭ's-Guo fixed point theorem and the positivity of the associated Green function. Moreover, based on the sign of functions $h(t)$ and $g(t)$, we discuss eq. (1.1) in the following two cases:

(i) the case of sign-constant $h \in L^1(\mathbb{R}/T\mathbb{Z})$ and sign-changing $g \in L^1(\mathbb{R}/T\mathbb{Z})$ (in section 3);

(ii) the case of sign-changing $h \in L^1(\mathbb{R}/T\mathbb{Z})$ and sign-changing $g \in L^1(\mathbb{R}/T\mathbb{Z})$ (in section 4).

Finally, we finish the introduction with a statement. In this paper, since the functions h and g are Lebesgue integrable, i.e., they may be equal to infinity in some points or even undefined on some set of zero measure. Therefore, the solutions of eq. (1.1) are understood in a Carathéodory sense, and and all

1 Dec 2023 16:43:36 PST
221113-Cheng Version 3 - Submitted to Rocky Mountain J. Math.

the equalities and inequalities in this paper are understood almost everywhere.

2 Preliminaries

We can now start by introducing the Krasnoselskiĭ's-Guo fixed point theorem [10, P. 94], which will be used in the proofs of many of the theorems below.

Lemma 2.1. *Let X be a Banach space and $\mathcal{K}$ be a cone in X. Assume that Ω_1 and Ω_2 are open subsets of X with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$. Let*

$$\Phi : \mathcal{K} \cap (\overline{\Omega}_2 \backslash \Omega_1) \to \mathcal{K}$$

be a completely continuous operator such that one of the following conditions holds:

(i) $\|\Phi x\| \geq \|x\|$ for $x \in \mathcal{K} \cap \partial\Omega_1$ and $\|\Phi x\| \leq \|x\|$ for $x \in \mathcal{K} \cap \partial\Omega_2$;

(ii) $\|\Phi x\| \leq \|x\|$ for $x \in \mathcal{K} \cap \partial\Omega_1$ and $\|\Phi x\| \geq \|x\|$ for $x \in \mathcal{K} \cap \partial\Omega_2$.

Then Φ has a fixed point in $\mathcal{K} \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

The second important tool in the proof process is the concept of Green function. In what follows, we recall two lemmas about Green function and can be noted in [14, Lemmas 2.1-2.5]. Against this background, the usual structural mechanism of the Green functions is described in [3, 4].

Lemma 2.2. *If $N > 0$ is such that $N \neq \frac{2k\pi}{T}$ for any natural number k, then for any $f \in L^1(\mathbb{R}/T\mathbb{Z})$ the equation*

$$x'' + N^2 x = f(t)$$

admits a unique T-periodic solution, which can be written as follows

$$x(t) = \int_0^T G_1(t,s) f(s) ds,$$

where the Green's function $G_1(t,s)$ has the following form

$$G_1(t,s) = \begin{cases} \dfrac{\cos N(t-s-\frac{T}{2})}{2N \sin \frac{NT}{2}}, & 0 \leq s \leq t \leq T, \\ \dfrac{\cos N(t-s+\frac{T}{2})}{2N \sin \frac{NT}{2}}, & 0 \leq t < s \leq T. \end{cases}$$

Moreover, if $N < \frac{\pi}{T}$, then $G_1(t,s) > 0$ for any $(t,s) \in [0,T] \times [0,T]$ and $\int_0^T G_1(t,s) N^2 ds \equiv 1$.

Lemma 2.3. *If $N > 0$, then for any $f \in L^1(\mathbb{R}/T\mathbb{Z})$ the equation*

$$-x'' + N^2 x = f(t)$$

admits a unique T-periodic solution, which can be written as follows

$$x(t) = \int_0^T G_2(t,s) f(s) ds,$$

where the Green's function $G_2(t,s)$ *has the following form*

$$G_2(t,s)=\begin{cases}\dfrac{\exp(-N(s-t))+\exp(N(s-t-T))}{2N(1-\exp(-NT))}, & 0\le t<s\le T,\\ \dfrac{\exp(-N(s-t+T))+\exp(N(s-t))}{2N(1-\exp(-NT))}, & 0\le s\le t\le T.\end{cases}$$

Moreover, $G_2(t,s)>0$ *for any* $(t,s)\in[0,T]\times[0,T]$ *and* $\int_0^T G_2(t,s)N^2ds\equiv 1$.

Define

$$\begin{aligned}
&A_1:=\min_{0\le s,t\le T} G_1(t,s)=\frac{1}{2N}\cot\frac{NT}{2},\quad B_1:=\max_{0\le s,t\le T} G_1(t,s)=\frac{1}{2N\sin\frac{NT}{2}},\\
&A_2:=\min_{0\le s,t\le T} G_2(t,s)=\frac{\exp(-\frac{NT}{2})}{N(1-\exp(-NT))},\quad B_2:=\max_{0\le s,t\le T} G_2(t,s)=\frac{1+\exp(-NT)}{2N(1-\exp(-NT))},\\
&\sigma_1:=\frac{A_1}{B_1},\quad \sigma_2:=\frac{A_2}{B_2}.
\end{aligned}\tag{2.1}$$

It is evident that $0<A_1\le B_1$ and $0<\sigma_1\le 1$ from $N<\frac{\pi}{T}$, $0<A_2\le B_2$ and $0<\sigma_2\le 1$. Define

$$\mathcal{K}_i:=\{x\in C_T:\min_{t\in\mathbb{R}}x(t)\ge\sigma_i\|x\|\},\qquad i=1,\ 2,$$

where $C_T:=\{x\in C(\mathbb{R},\mathbb{R}):\ x(t+T)=x(t),\ \text{for all } t\in\mathbb{R}\}$ with norm $\|x\|:=\max_{t\in\mathbb{R}}|x(t)|$. It is easily verified that $\mathcal{K}_1$ and $\mathcal{K}_2$ are cones in C_T.

Finally, for a given periodic function $g(t)$, we denote

$$g^+(t):=\max\{0,g(t)\},\quad g^-(t):=-\min\{0,g(t)\},\quad \overline{g}:=\frac{1}{T}\int_0^T g(t)dt.$$

3 The case of sign-constant $h\in L^1(\mathbb{R}/T\mathbb{Z})$

In this section, we used the Krasnoselskiĭ's-Guo fixed point theorem to prove the existence of a positive periodic solution for eq. (1.1) in the case where $h\in L^1(\mathbb{R}/T\mathbb{Z})$ is a sign-constant function and $g\in L^1(\mathbb{R}/T\mathbb{Z})$ is a sign-changing function. According to the sign of the function h, we study the following two cases.

3.1 $h(t)\ge 0$ for a.e. $t\in[0,T]$ and $\overline{h}>0$

Case 1 $\rho>\mu$

Theorem 3.1. *Let* $h(t)\ge 0$ *for a.e.* $t\in[0,T]$, $\overline{h}>0$, *and* $g\in L^1(\mathbb{R}/T\mathbb{Z})$ *be a sign-changing function. Assume that there exists* $0<N<\frac{\pi}{T}$ *such that*

$$\overline{g^-}<\sigma_1^{1+\mu}\overline{g^+}\quad\text{and}\quad \overline{h}>\frac{1}{A_1T\sigma_1^{1+\rho}}\left(\frac{\|g^+\|_\infty}{N^2}\right)^{\frac{1+\rho}{1+\mu}},\tag{3.1}$$

where $\|g^+\|_\infty=ess\ \sup\{g^+(t):t\in[0,T]\}$.

If $\rho>\mu$, *then there exists at least one positive periodic solution to eq.* (1.1).

Proof. Writing eq. (1.1) as

$$x'' + N^2 x = \frac{h(t)}{x^\rho} - \frac{g(t)}{x^\mu} + N^2 x, \tag{3.2}$$

a T-periodic solution of eq. (3.2) is just a fixed point of the map $\Phi : C_T \to C_T$ defined by

$$(\Phi x)(t) := \int_0^T G_1(t,s)\left(\frac{h(s)}{x^\rho(s)} - \frac{g(s)}{x^\mu(s)} + N^2 x(s)\right) ds, \tag{3.3}$$

and we know that $G_1(t,s) > 0$ for all $(t,s) \in [0,T] \times [0,T]$ from Lemma 2.2.

Now we define two open sets

$$\Omega_1 := \{x \in C_T : \|x\| < r_1\} \quad \text{and} \quad \Omega_2 := \{x \in C_T : \|x\| < R_1\}.$$

Note that Φ is well-defined in the set $\mathcal{K}_1 \cap (\overline{\Omega}_2 \backslash \Omega_1)$, and it is a completely continuous operator by a standard application of Ascoli-Arzelà Theorem.

By (3.1), the positive constants r_1 and R_1 can be fixed such that

$$R_1 > r_1 = (A_1 T \overline{h})^{\frac{1}{1+\rho}} > \frac{1}{\sigma_1}\left(\frac{\|g^+\|_\infty}{N^2}\right)^{\frac{1}{1+\mu}}.$$

Step 1. We assert that $\Phi(\mathcal{K}_1 \cap (\overline{\Omega}_2 \backslash \Omega_1)) \subset \mathcal{K}_1$. In fact, for any $x \in \mathcal{K}_1 \cap (\overline{\Omega}_2 \backslash \Omega_1)$, we have

$$\sigma_1 r_1 \le x(t) \le R_1, \quad \text{for all} \quad t \in \mathbb{R}.$$

Since $r_1 > \frac{1}{\sigma_1}\left(\frac{\|g^+\|_\infty}{N^2}\right)^{\frac{1}{1+\mu}}$, we arrive at

$$\begin{aligned}
\frac{h(t)}{x^\rho(t)} - \frac{g(t)}{x^\mu(t)} + N^2 x(t) =& \frac{h(t)}{x^\rho(t)} - \frac{g^+(t)}{x^\mu(t)} + \frac{g^-(t)}{x^\mu(t)} + N^2 x(t) \\
>& -\frac{g^+(t)}{x^\mu(t)} + N^2 x(t) \\
>& -\frac{\|g^+\|_\infty}{(\sigma_1 r_1)^\mu} + N^2(\sigma_1 r_1) \\
>& 0,
\end{aligned} \tag{3.4}$$

for all $t \in \mathbb{R}$. It follows from (3.4) that

$$\begin{aligned}
\min_{t\in\mathbb{R}}(\Phi x)(t) =& \min_{t\in\mathbb{R}} \int_0^T G_1(t,s)\left(\frac{h(s)}{x^\rho(s)} - \frac{g(s)}{x^\mu(s)} + N^2 x(s)\right) ds \\
\ge& A_1 \int_0^T \left(\frac{h(s)}{x^\rho(s)} - \frac{g(s)}{x^\mu(s)} + N^2 x(s)\right) ds \\
=& \sigma_1 B_1 \int_0^T \left(\frac{h(s)}{x^\rho(s)} - \frac{g(s)}{x^\mu(s)} + N^2 x(s)\right) ds \\
\ge& \sigma_1 \max_{t\in\mathbb{R}} \int_0^T G_1(t,s)\left(\frac{h(s)}{x^\rho(s)} - \frac{g(s)}{x^\mu(s)} + N^2 x(s)\right) ds \\
\ge& \sigma_1 \|\Phi x\|.
\end{aligned}$$

Therefore, we get that $\Phi(\mathcal{K}_1 \cap (\overline{\Omega}_2 \backslash \Omega_1)) \subset \mathcal{K}_1$.

Step 2. We demonstrate that

$$\|\Phi x\| \le \|x\|, \quad \text{for} \;\; x \in \mathcal{K}_1 \cap \partial\Omega_2. \tag{3.5}$$

In fact, for any $x \in \mathcal{K}_1 \cap \partial\Omega_2$, it is evident that $\|x\| = R_1$ and

$$\sigma_1 R_1 \le x(t) \le R_1, \qquad \text{for all } t \in \mathbb{R}.$$

It follows from (2.1) and $\int_0^T G_1(t,s)N^2 ds \equiv 1$ that

$$\begin{aligned}(\Phi x)(t) &= \int_0^T G_1(t,s)\left(\frac{h(s)}{x^\rho(s)} - \frac{g(s)}{x^\mu(s)} + N^2 x(s)\right) ds \\ &= \int_0^T G_1(t,s)\left(\frac{h(s)}{x^\rho(s)} - \frac{g^+(s)}{x^\mu(s)} + \frac{g^-(s)}{x^\mu(s)} + N^2 x(s)\right) ds \\ &\le \frac{B_1 T\overline{h}}{(\sigma_1 R_1)^\rho} - \frac{A_1 T\overline{g^+}}{R_1^\mu} + \frac{B_1 T\overline{g^-}}{(\sigma_1 R_1)^\mu} + R_1 \\ &\le R_1,\end{aligned}$$

where $\frac{B_1 T\overline{h}}{(\sigma_1 R_1)^\rho} - \frac{A_1 T\overline{g^+}}{R_1^\mu} + \frac{B_1 T\overline{g^-}}{(\sigma_1 R_1)^\mu} \le 0$ holds, i.e.,

$$\frac{B_1\overline{h}}{\sigma_1^\rho} \le \left(A_1\overline{g^+} - \frac{B_1\overline{g^-}}{\sigma_1^\mu}\right) R_1^{\rho-\mu}$$

for sufficiently large R_1 and $A_1\overline{g^+} > \frac{B_1\overline{g^-}}{\sigma_1^\mu}$ from $\overline{g^-} < \sigma_1^{1+\mu}\overline{g^+}$. This implies that (3.5) holds.

Step 3. Let us demonstrate that

$$\|\Phi x\| \ge \|x\|, \quad \text{for} \;\; x \in \mathcal{K}_1 \cap \partial\Omega_1. \tag{3.6}$$

Since $x \in \mathcal{K}_1 \cap \partial\Omega_1$, we know that $\|x\| = r_1$ and

$$\sigma_1 r_1 \le x(t) \le r_1, \qquad \text{for all } t \in \mathbb{R}.$$

As can be seen from (2.1) and (3.4), we conclude that

$$\begin{aligned}(\Phi x)(t) &= \int_0^T G_1(t,s)\left(\frac{h(s)}{x^\rho(s)} - \frac{g^+(s)}{x^\mu(s)} + \frac{g^-(s)}{x^\mu(s)} + N^2 x(s)\right) ds \\ &> \int_0^T G_1(t,s)\frac{h(s)}{x^\rho(s)} ds \\ &\ge \frac{A_1 T\overline{h}}{r_1^\rho} = r_1\end{aligned}$$

since $r_1 = (A_1 T\overline{h})^{\frac{1}{1+\rho}}$ from definition of r_1. Hence, (3.6) holds.

It follows from Lemma 2.1 that Φ has a fixed point $x \in \mathcal{K}_1 \cap (\overline{\Omega}_2 \backslash \Omega_1)$. Obviously, the fixed point is a positive periodic solution of eq. (1.1) satisfying $x \in [\sigma_1 r_1, R_1]$. □

Remark 3.1. In the respect of (2.1), it is major to note that A_1 and σ_1 are functions of N and T. In fact, the sufficient condition (3.1) has an explicit (but rather cumbersome) expression as

$$\overline{g^-} < \cos^{1+\mu}\left(\frac{NT}{2}\right)\overline{g^+} \quad \text{and} \quad \overline{h} > \frac{2N\sin\left(\frac{NT}{2}\right)}{T\cos^{2+\rho}\left(\frac{NT}{2}\right)}\left(\frac{\|g^+\|_\infty}{N^2}\right)^{\frac{1+\rho}{1+\mu}}. \tag{3.7}$$

It is obvious that N plays an important role regarding (3.7). It is natural to wonder whether we can give an optimal N such that (3.7) holds. Taking $T=\pi$, $\rho=2$ and $\mu=1$, and let $F(N):=\frac{2N\sin\left(\frac{N\pi}{2}\right)}{T\cos^4\left(\frac{N\pi}{2}\right)}\left(\frac{1}{N^2}\right)^{\frac{3}{2}}$, $G(N):=\cos^2\left(\frac{N\pi}{2}\right)$. It is easy to verify that $F_{min}:=\min\{F(N):\ N\in(0,1)\}$ and $G_{max}:=\max\{G(N):\ N\in(0,1)\}$ are optimal value about (3.7). To facilitate the consideration of $F(N)$ and $G(N)$, we give the following figure

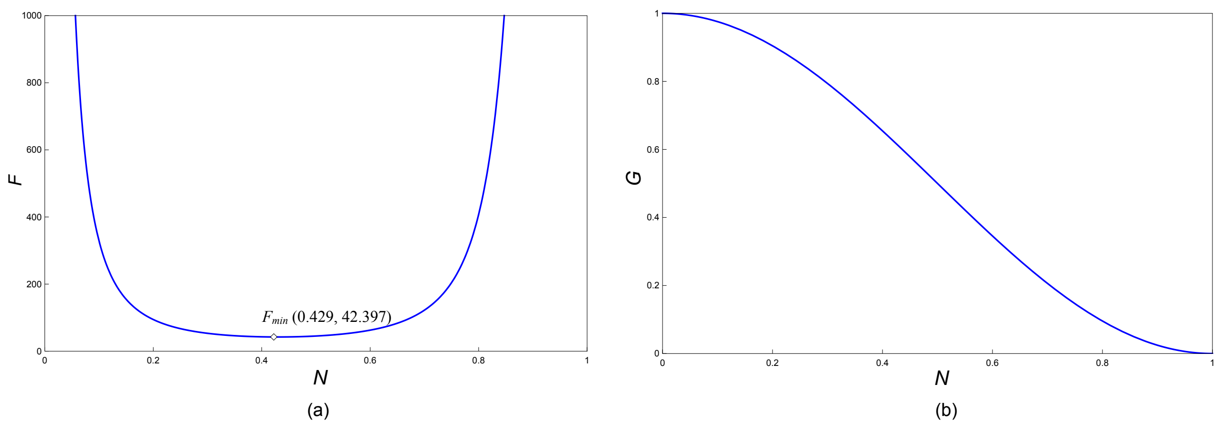

Figure 1: (a) The motion profile of $F(N)$, $F_{\min}=42.397$ and $N=0.429$. (b) $G(N)$ is monotonically decreasing about N.

By Figure 1, we can get that $N=0.429$ may be an optimal point to (3.7). Furthermore, it is easy to construct explicit examples. For instance, from the above analysis, taking $T=\pi,\ N=\frac{1}{2},\ \rho=2,\ \mu=1$ and the functions

$$h(t)=\cos 2t+120,$$

and

$$g(t)=\begin{cases}2\pi\cos 2t, & t\in[-\frac{\pi}{4},\frac{\pi}{4}],\\ \cos 2t, & t\in[\frac{\pi}{4},\frac{3\pi}{4}].\end{cases}$$

Then, we give

$$\overline{h}=\frac{1}{T}\int_0^\pi(\cos 2t+120)dt=120,\quad \overline{g^+}=\frac{1}{T}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}2\pi\cos 2t dt=2,\quad \overline{g^-}=-\frac{1}{T}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\cos 2t dt=\frac{1}{\pi},\quad \|g^+\|_\infty=2\pi.$$

Furthermore, we have

$$\overline{g^-}=\frac{1}{\pi}<\cos^{1+\mu}\left(\frac{NT}{2}\right)\overline{g^+}=1\ \text{ and }\ \overline{h}=120>\frac{2N\sin\frac{NT}{2}}{T\cos^{2+\rho}\left(\frac{NT}{2}\right)}\left(\frac{\|g^+\|_\infty}{N^2}\right)^{\frac{1+\rho}{1+\mu}}\approx 113.437.$$

The above computations show that (3.7) holds and the eq. (1.1) admits at least one positive π-periodic solution.

Case 2 $\rho<\mu$

Theorem 3.1 requires that $\rho>\mu$. In what follows, the existence of a positive periodic solution for eq. (1.1) is investigated under the condition of $\rho<\mu$.

Theorem 3.2. *Let $h(t) \geq 0$ for a.e. $t \in [0,T]$, $\overline{h} > 0$, and $g \in L^1(\mathbb{R}/T\mathbb{Z})$ be a sign-changing function. Assume that there exists $0 < N < \frac{\pi}{T}$ such that*

$$\overline{g^-} < \sigma_1^{1+\mu}\overline{g^+} \quad and \quad \frac{1}{\sigma_1}\left(\frac{\|g^+\|_\infty}{N^2}\right)^{\frac{1}{1+\mu}} < \left(\frac{\sigma_1^{1+\mu}\overline{g^+}-\overline{g^-}}{\overline{h}\sigma_1^{\mu-\rho}}\right)^{\frac{1}{\mu-\rho}} < (A_1T\overline{h})^{\frac{1}{1+\rho}}. \tag{3.8}$$

If $\rho < \mu$, then there exists at least one positive periodic solution to eq. (1.1).

Proof. Define

$$\Omega_3 := \{x \in C_T : \|x\| < r_2\} \quad \text{and} \quad \Omega_4 := \{x \in C_T : \|x\| < R_2\}.$$

By (3.8), the positive constants r_2 and R_2 can be fixed such that

$$R_2 = (A_1T\overline{h})^{\frac{1}{1+\rho}} > \left(\frac{\sigma_1^{1+\mu}\overline{g^+}-\overline{g^-}}{\overline{h}\sigma_1^{\mu-\rho}}\right)^{\frac{1}{\mu-\rho}} > r_2 > \frac{1}{\sigma_1}\left(\frac{\|g^+\|_\infty}{N^2}\right)^{\frac{1}{1+\mu}}.$$

By an analogous reasoning as in Step 1 of Theorem 3.1, we get that $\Phi(\mathcal{K}_1 \cap (\overline{\Omega}_4 \backslash \Omega_3) \subset \mathcal{K}_1$, here Φ is defined in (3.3).

Next, we demonstrate that

$$\|\Phi x\| \leq \|x\|, \quad \text{for} \quad x \in \mathcal{K}_1 \cap \partial\Omega_3. \tag{3.9}$$

In fact, for any $x \in \mathcal{K}_1 \cap \partial\Omega_3$, we see that $\|x\| = r_2$ and

$$\sigma_1 r_2 \leq x(t) \leq r_2, \qquad \text{for all } t \in \mathbb{R}.$$

As can be seen from (2.1) and $\int_0^T G_1(t,s)N^2 ds \equiv 1$ that

$$\begin{aligned}
(\Phi x)(t) =& \int_0^T G_1(t,s)\left(\frac{h(s)}{x^\rho(s)} - \frac{g(s)}{x^\mu(s)} + N^2 x(s)\right) ds \\
=& \int_0^T G_1(t,s)\left(\frac{h(s)}{x^\rho(s)} - \frac{g^+(s)}{x^\mu(s)} + \frac{g^-(s)}{x^\mu(s)} + N^2 x(s)\right) ds \\
\leq& \frac{B_1T\overline{h}}{(\sigma_1 r_2)^\rho} - \frac{A_1T\overline{g^+}}{r_2^\mu} + \frac{B_1T\overline{g^-}}{(\sigma_1 r_2)^\mu} + r_2 \\
\leq& r_2,
\end{aligned}$$

where $\frac{B_1T\overline{h}}{(\sigma_1 r_2)^\rho} - \frac{A_1T\overline{g^+}}{r_2^\mu} + \frac{B_1T\overline{g^-}}{(\sigma_1 r_2)^\mu} \leq 0$ holds, i.e.,

$$r_2^{\mu-\rho}\frac{B_1\overline{h}}{\sigma_1^\rho} \leq \left(A_1\overline{g^+} - \frac{B_1\overline{g^-}}{\sigma_1^\mu}\right)$$

because $\overline{g^-} < \sigma_1^{1+\mu}\overline{g^+}$ and $r_2 < \left(\frac{\sigma_1^{1+\mu}\overline{g^+}-\overline{g^-}}{\overline{h}\sigma_1^{\mu-\rho}}\right)^{\frac{1}{\mu-\rho}}$. This implies that (3.9) holds.

Finally, we demonstrate that

$$\|\Phi x\| \geq \|x\|, \quad \text{for} \quad x \in \mathcal{K}_1 \cap \partial\Omega_4. \tag{3.10}$$

Since $x \in \mathcal{K}_1 \cap \partial\Omega_4$, it is evident that $\|x\| = R_2$ and

$$\sigma_1 R_2 \leq x(t) \leq R_2, \qquad \text{for all } t \in \mathbb{R}.$$

According to (2.1) and (3.4), we arrive at

$$
\begin{aligned}
(\Phi x)(t) &= \int_0^T G_1(t,s)\left(\frac{h(s)}{x^\rho(s)} - \frac{g^+(s)}{x^\mu(s)} + \frac{g^-(s)}{x^\mu(s)} + N^2 x(s)\right) ds \\
&> \int_0^T G_1(t,s)\frac{h(s)}{x^\rho(s)} ds \\
&\geq \frac{A_1 T\overline{h}}{R_2^\rho} = R_2
\end{aligned}
$$

since $R_2 = (A_1 T\overline{h})^{\frac{1}{1+\rho}}$ from definition of R_2. Hence, (3.10) holds. The proof is completed. □

Remark 3.2. Sufficient condition (3.8) can be written explicitly as

$$
\overline{g^-} < \cos^{1+\mu}\left(\frac{NT}{2}\right)\overline{g^+} \quad \text{and} \quad \frac{1}{\cos\left(\frac{NT}{2}\right)}\left(\frac{\|g^+\|_\infty}{N^2}\right)^{\frac{1}{1+\mu}} < \left(\frac{\cos^{1+\mu}\left(\frac{NT}{2}\right)\overline{g^+} - \overline{g^-}}{\overline{h}\cos^{\mu-\rho}\left(\frac{NT}{2}\right)}\right)^{\frac{1}{\mu-\rho}} < \left(\frac{T\overline{h}}{2N}\cot\left(\frac{NT}{2}\right)\right)^{\frac{1}{1+\rho}}. \tag{3.11}
$$

From where it is easy to construct explicit examples. For instance, taking $T = \frac{\pi}{40},\ N = 10,\ \rho = 1,\ \mu = 2$ and the functions

$$
h(t) = \cos 80t + 20,
$$

and

$$
g(t) = \begin{cases} 20\pi \sin 80t, & t \in [0, \frac{\pi}{80}], \\ \pi \sin 80t, & t \in [\frac{\pi}{80}, \frac{\pi}{40}]. \end{cases}
$$

Then, we give

$$
\overline{h} = \frac{1}{T}\int_0^{\frac{\pi}{40}} (\cos 80t + 20)dt = 20, \quad \overline{g^+} = \frac{1}{T}\int_0^{\frac{\pi}{80}} 20\pi \sin 80t dt = 20, \quad \overline{g^-} = -\frac{1}{T}\int_{\frac{\pi}{80}}^{\frac{\pi}{40}} \pi \sin 80t dt = 1, \quad \|g^+\|_\infty = 20\pi.
$$

Furthermore, we have

$$
\overline{g^-} = 1 < \cos^{1+\mu}\left(\frac{NT}{2}\right)\overline{g^+} \approx 19.999,
$$

and

$$
\begin{aligned}
&\frac{1}{\cos\left(\frac{NT}{2}\right)}\left(\frac{\|g^+\|_\infty}{N^2}\right)^{\frac{1}{1+\mu}} = \frac{1}{\cos\left(\frac{\pi}{8}\right)}\left(\frac{\pi}{5}\right)^{\frac{1}{3}} \approx 0.857 \\
<& \left(\frac{\cos^{1+\mu}\left(\frac{NT}{2}\right)\overline{g^+} - \overline{g^-}}{\overline{h}\cos^{\mu-\rho}\left(\frac{NT}{2}\right)}\right)^{\frac{1}{\mu-\rho}} = \left(\frac{20\cos^3\left(\frac{\pi}{8}\right) - 1}{20\cos\left(\frac{\pi}{8}\right)}\right) \approx 0.950 \\
<& \left(\frac{T\overline{h}}{2N}\cot\left(\frac{NT}{2}\right)\right)^{\frac{1}{1+\rho}} = \left(\frac{\pi}{40}\cot\left(\frac{\pi}{8}\right)\right)^{\frac{1}{1+\rho}} \approx 3.385.
\end{aligned}
$$

The above computations show that (3.11) holds and the eq. (1.1) admits at least one positive $\frac{\pi}{40}$-periodic solution.

Remark 3.3. It should be emphasised here that the assumption $\rho \neq \mu$ is crucial for the main result of this paper. The conditions in Theorems 3.1 and 3.2 cannot be applied in the case $\rho = \mu$ or, equivalently, in the case $h \equiv 0$. Moreover, when $\rho = \mu$, eq. (1.1) simplifies to the classical generalized Emden-Fowler equation

$$x'' = \frac{f(t)}{x^\rho}, \tag{3.12}$$

where $f := h(t) - g(t) \in L^1(\mathbb{R}/T\mathbb{Z})$. The existence of periodic solutions to eq. (3.12) needs to be further investigated, and the published articles include [8, 9, 13, 20, 21].

3.2 $h(t) \leq 0$ for a.e. $t \in [0, T]$ and $\overline{h} < 0$

Case 1 $\rho > \mu$

Theorem 3.3. *Let $h(t) \leq 0$ for a.e. $t \in [0, T]$, $\overline{h} < 0$, and $g \in L^1(\mathbb{R}/T\mathbb{Z})$ be a sign-changing function. Assume that there exists $N > 0$ such that*

$$\overline{g^+} < \sigma_2^{1+\mu}\overline{g^-} \quad \text{and} \quad |\overline{h}| > \frac{1}{A_2 T \sigma_2^{1+\rho}} \left(\frac{\|g^-\|_\infty}{N^2}\right)^{\frac{1+\rho}{1+\mu}}. \tag{3.13}$$

If $\rho > \mu$, then there exists at least one positive T-periodic solution to eq. (1.1).

Proof. Writing eq. (1.1) as

$$-x'' + N^2 x = -\frac{h(t)}{x^\rho} + \frac{g(t)}{x^\mu} + N^2 x, \tag{3.14}$$

a T-periodic solution of eq. (3.14) is just a fixed point of the map Ψ defined by

$$(\Psi x)(t) := \int_0^T G_2(t,s)\left(-\frac{h(s)}{x^\rho(s)} + \frac{g(s)}{x^\mu(s)} + N^2 x(s)\right) ds, \tag{3.15}$$

and we know that $G_2(t, s) > 0$ for all $(t, s) \in [0, T] \times [0, T]$ by Lemma 2.3. In this respect, the steps of the proof are the same as in Theorem 3.1. □

Remark 3.4. Sufficient condition (3.13) can be written explicitly as

$$\overline{g^+} < \left(\frac{2\exp\left(-\frac{NT}{2}\right)}{1+\exp(-NT)}\right)^{1+\mu}\overline{g^-} \text{ and } |\overline{h}| > \frac{N(1-\exp(-NT))(1+\exp(-NT))^{1+\rho}}{2^{1+\rho}T\left(\exp\left(-\frac{NT}{2}\right)\right)^{2+\rho}}\left(\frac{\|g^-\|_\infty}{N^2}\right)^{\frac{1+\rho}{1+\mu}}. \tag{3.16}$$

In fact, taking $T = \pi$, $N = \frac{1}{2}$, $\rho = 2$, $\mu = 1$ and the functions

$$h(t) = \sin 2t - 1000,$$

and

$$g(t) = \begin{cases} \pi\cos 2t, & t \in [-\frac{\pi}{4}, \frac{\pi}{4}], \\ 10\pi\cos 2t, & t \in [\frac{\pi}{4}, \frac{3\pi}{4}]. \end{cases}$$

Then, we give

$$|\overline{h}| = \left|\frac{1}{T}\int_0^\pi (\sin 2t - 1000)dt\right| = 1000, \quad \overline{g^+} = \frac{1}{T}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \pi\cos 2t dt = 1, \quad \overline{g^-} = -\frac{1}{T}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} 10\pi\cos 2t dt = 10, \quad \|g^-\|_\infty = 10\pi.$$

1 Dec 2023 16:43:36 PST
221113-Cheng Version 3 - Submitted to Rocky Mountain J. Math.

Furthermore, we have

$$\overline{g^+} = 1 < \left(\frac{2\exp\left(-\frac{NT}{2}\right)}{1+\exp(-NT)} \right)^{1+\mu} \overline{g^-} = 5.699$$

and

$$|\overline{h}| = 1000 > \frac{N(1-\exp(-NT))(1+\exp(-NT))^{1+\rho}}{2^{1+\rho}T\left(\exp\left(-\frac{NT}{2}\right)\right)^{2+\rho}} \left(\frac{\|g^-\|_\infty}{N^2} \right)^{\frac{1+\rho}{1+\mu}} \approx 905.291.$$

The above computations show that (3.16) holds and the eq. (1.1) admits at least one positive π-periodic solution.

Case 2 $\rho < \mu$

Theorem 3.4. *Let $h(t) \leq 0$ for a.e. $t \in [0,T]$, $\overline{h} < 0$, and $g \in L^1(\mathbb{R}/T\mathbb{Z})$ be a sign-changing function. Assume that there exists $N > 0$ such that*

$$\overline{g^+} < \sigma_2^{1+\mu}\overline{g^-} \quad \text{and} \quad \frac{1}{\sigma_2}\left(\frac{\|g^-\|_\infty}{N^2} \right)^{\frac{1}{1+\mu}} < \left(\frac{\sigma_2^{1+\mu}\overline{g^-} - \overline{g^+}}{|\overline{h}|\sigma_2^{\mu-\rho}} \right)^{\frac{1}{\mu-\rho}} < (A_2 T|\overline{h}|)^{\frac{1}{1+\rho}}. \tag{3.17}$$

If $\rho < \mu$, then there exists at least one positive periodic solution to eq. (1.1).

Similar to the method of proof in Theorems 3.2, Theorem 3.4 can be obtained.

Remark 3.5. Sufficient condition (3.17) can be written explicitly as

$$\overline{g^+} < \left(\frac{2\exp\left(-\frac{NT}{2}\right)}{1+\exp(-NT)} \right)^{1+\mu} \overline{g^-}$$

and

$$\frac{1+\exp(-NT)}{2\exp\left(-\frac{NT}{2}\right)} \left(\frac{\|g^-\|_\infty}{N^2} \right)^{\frac{1}{1+\mu}} < \left(\frac{\left(\frac{2\exp\left(-\frac{NT}{2}\right)}{1+\exp(-NT)}\right)^{1+\mu}\overline{g^-} - \overline{g^+}}{|\overline{h}|\left(\frac{2\exp\left(-\frac{NT}{2}\right)}{1+\exp(-NT)}\right)^{\mu-\rho}} \right)^{\frac{1}{\mu-\rho}} < \left(\frac{T|\overline{h}|\exp\left(-\frac{NT}{2}\right)}{N(1-\exp(-NT))} \right)^{\frac{1}{1+\rho}}.$$

Remark 3.6. It is worth mentioning that the cases "g is positive and $\rho > \mu$" and "g is positive and $\rho < \mu$" are equivalent to the cases "h is negative and $\rho < \mu$ and "h is negative and $\rho > \mu$", respectively. Also the cases for "g is negative" are equivalent to the cases for "h is positive". Indeed, it is sufficient to swap g with h and ρ with μ.

Remark 3.7. Note that in [11], among others, the equation

$$x'' = \frac{h(t)}{x^\rho} - \frac{g(t)}{x^\mu}$$

is considered with $h(t) \geq 0$ and $g(t) \geq 0$ for a.e. $t \in [0,T]$, $\overline{h} > 0$, and $\overline{g} > 0$. Corollary 3.2 in [11] deals with the case $\rho < \mu$, while Corollary 3.4 deals with the case $\rho > \mu$. Therefore, Theorems 3.1 and 3.4 are more general than the corresponding Corollary 3.4 of [11]. Similarly, Theorems 3.2 and 3.3 extend and improve the corresponding Corollary 3.2 of [11].

4 The case of sign-changing $h \in L^1(\mathbb{R}/T\mathbb{Z})$

In this section, we used the Krasnoselskiĭ's-Guo fixed point theorem to prove the existence of a positive periodic solution for eq. (1.1) in the case where h and $g \in L^1(\mathbb{R}/T\mathbb{Z})$ are sign-changing.

4.1 The case $\rho > \mu$

Theorem 4.1. *Let h and $g \in L^1(\mathbb{R}/T\mathbb{Z})$ be sign-changing functions. Assume that there exist $0 < N < \frac{\pi}{T}$ and $\alpha \in (0,1)$ such that*

$$\overline{g^-} < \sigma_1^{1+\mu}\overline{g^+} \quad \textit{and} \quad \overline{h^+} > \frac{1}{A_1 T \sigma_1^{1+\rho}} \max\left\{\frac{\|h^-\|_\infty}{\alpha N^2}, \left(\frac{\|g^+\|_\infty}{(1-\alpha)N^2}\right)^{\frac{1+\rho}{1+\mu}}\right\}. \tag{4.1}$$

If $\rho > \mu$, then there exists at least one positive periodic solution to eq. (1.1).

Proof. Now we define two open sets

$$\Omega_5 := \{x \in C_T : \|x\| < r_3\} \quad \text{and} \quad \Omega_6 := \{x \in C_T : \|x\| < R_3\}.$$

By (4.1), the positive constants r_3 and R_3 can be fixed such that

$$R_3 > r_3 = (A_1 T\overline{h^+})^{\frac{1}{1+\rho}} > \frac{1}{\sigma_1}\max\left\{\left(\frac{\|h^-\|_\infty}{\alpha N^2}\right)^{\frac{1}{1+\rho}},\ \left(\frac{\|g^+\|_\infty}{(1-\alpha)N^2}\right)^{\frac{1}{1+\mu}}\right\}.$$

By an analogous reasoning as in Step 1 proof of Theorem 3.1, $\Phi(\mathcal{K}_1 \cap (\overline{\Omega}_6 \backslash \Omega_5) \subset \mathcal{K}_1$ is easily verified, where Φ is defined in (3.3).

Next, let us demonstrate that

$$\|\Phi x\| \geq \|x\|, \quad \text{for} \ \ x \in \mathcal{K}_1 \cap \partial\Omega_5. \tag{4.2}$$

Since $x \in \mathcal{K}_1 \cap \partial\Omega_5$, we see that $\|x\| = r_3$ and

$$\sigma_1 r_3 \leq x(t) \leq r_3, \qquad \text{for all } t \in \mathbb{R}.$$

Since $r_3 > \frac{1}{\sigma_1}\max\left\{\left(\frac{\|h^-\|_\infty}{\alpha N^2}\right)^{\frac{1}{1+\rho}},\ \left(\frac{\|g^+\|_\infty}{(1-\alpha)N^2}\right)^{\frac{1}{1+\mu}}\right\}$, we deduce

$$\begin{aligned}\frac{h(t)}{x^\rho(t)} - \frac{g(t)}{x^\mu(t)} + N^2 x(t) =& \frac{h^+(t)}{x^\rho(t)} - \frac{h^-(t)}{x^\rho(t)} - \frac{g^+(t)}{x^\mu(t)} + \frac{g^-(t)}{x^\mu(t)} + N^2 x(t)\\ >& -\frac{h^-(t)}{x^\rho(t)} - \frac{g^+(t)}{x^\mu(t)} + N^2 x(t)\\ >& -\frac{\|h^-\|_\infty}{(\sigma_1 r_3)^\rho} + \alpha N^2(\sigma_1 r_3) - \frac{\|g^+\|_\infty}{(\sigma_1 r_3)^\mu} + (1-\alpha)N^2(\sigma_1 r_3)\\ >&0,\end{aligned} \tag{4.3}$$

for all $t \in \mathbb{R}$. From (2.1) and (4.3), we get

$$\begin{aligned}(\Phi x)(t) &= \int_0^T G_1(t,s)\left(\frac{h^+(s)}{x^\rho(s)} - \frac{h^-(s)}{x^\rho(s)} - \frac{g^+(s)}{x^\mu(s)} + \frac{g^-(s)}{x^\mu(s)} + N^2 x(s)\right) ds \\ &> \int_0^T G_1(t,s)\frac{h^+(s)}{x^\rho(s)} ds \\ &\geq \frac{A_1 T\overline{h^+}}{r_3^\rho} = r_3\end{aligned}$$

since $r_3 = (A_1 T\overline{h^+})^{\frac{1}{1+\rho}}$ from definition of r_3. Thus, (4.2) holds.

Finally, we demonstrate that

$$\|\Phi x\| \leq \|x\|, \quad \text{for} \quad x \in \mathcal{K}_1 \cap \partial\Omega_6. \tag{4.4}$$

In fact, for any $x \in \mathcal{K}_1 \cap \partial\Omega_6$, it is evident that $\|x\| = R_3$ and

$$\sigma_1 R_3 \leq x(t) \leq R_3, \qquad \text{for all } t \in \mathbb{R}.$$

As can be seen from (2.1) and $\int_0^T G_1(t,s)N^2 ds \equiv 1$ that

$$\begin{aligned}(\Phi x)(t) &= \int_0^T G_1(t,s)\left(\frac{h(s)}{x^\rho(s)} - \frac{g(s)}{x^\mu(s)} + N^2 x(s)\right) ds \\ &= \int_0^T G_1(t,s)\left(\frac{h^+(s)}{x^\rho(s)} - \frac{h^-(s)}{x^\rho(s)} - \frac{g^+(s)}{x^\mu(s)} + \frac{g^-(s)}{x^\mu(s)} + N^2 x(s)\right) ds \\ &\leq \frac{B_1 T\overline{h^+}}{(\sigma_1 R_3)^\rho} - \frac{A_1 T\overline{h^-}}{R_3^\rho} - \frac{A_1 T\overline{g^+}}{R_3^\mu} + \frac{B_1 T\overline{g^-}}{(\sigma_1 R_3)^\mu} + R_3 \\ &\leq R_3,\end{aligned}$$

where $\frac{B_1 T\overline{h^+}}{(\sigma_1 R_3)^\rho} - \frac{A_1 T\overline{h^-}}{R_3^\rho} - \frac{A_1 T\overline{g^+}}{R_3^\mu} + \frac{B_1 T\overline{g^-}}{(\sigma_1 R_3)^\mu} \leq 0$ holds, i.e.,

$$\overline{h^+} - \sigma_1^{1+\rho}\overline{h^-} \leq (\sigma_1^{1+\mu}\overline{g^+} - \overline{g^-})(\sigma_1 R_3)^{\rho-\mu},$$

for sufficiently large R_3 and $\overline{g^-} < \sigma_1^{1+\mu}\overline{g^+}$. This implies that (4.4) holds. The proof is finished. □

From Theorem 4.1, we know that $0 < N < \frac{\pi}{T}$. In the following, we give a result similar to Theorem 4.1, in the absence of any restriction on $N > 0$.

Theorem 4.2. *Let h and $g \in L^1(\mathbb{R}/T\mathbb{Z})$ be sign-changing functions. Assume that there exist $N > 0$ and $\beta \in (0,1)$ such that*

$$\overline{g^+} < \sigma_2^{1+\mu}\overline{g^-} \quad \textit{and} \quad \overline{h^-} > \frac{1}{A_2 T\sigma_2^{1+\rho}} \max\left\{\frac{\|h^+\|_\infty}{\beta N^2}, \left(\frac{\|g^-\|_\infty}{(1-\beta)N^2}\right)^{\frac{1+\rho}{1+\mu}}\right\}.$$

If $\rho > \mu$, then there exists at least one positive periodic solution to eq. (1.1).

Writing eq. (1.1) in the form of (3.14). Define the map Ψ, where Ψ is introduced in (3.15). The steps of the proof are the same as in Theorem 4.1.

Remark 4.1. Notice that in the proof of Theorem 4.1, i.e., while passing through (4.3), the term N^2x is divided into $\alpha N^2 x$ and $(1-\alpha)N^2x$ with $\alpha \in (0,1)$ in order to be compared with h^- and g^+, respectively. It is natural to wonder whether we can give an optimal α such that (4.1) holds. Obviously, the optimal choice of α depends on h and g. In particular, the optimal α is such that

$$\left(\frac{\|h^-\|_\infty}{\alpha N^2}\right)^{\frac{1}{1+\rho}} = \left(\frac{\|g^+\|_\infty}{(1-\alpha)N^2}\right)^{\frac{1}{1+\mu}}.$$

we can obtain that such an α exists and it is unique, provided h^- and g^+ are not zero function. Similar to the above analysis, we can find the optimal value of β, which exists and is unique, provided that h^+ and g^- are nonzero functions.

Next, we try to find the specific optimal value α by an example. Taking $T=\pi,\ N=\frac{1}{2},\ \rho=2,\ \mu=1$ and the functions

$$h(t)=\begin{cases}140\pi\sin 2t, & t\in[0,\frac{\pi}{2}],\\ \sin 2t, & t\in[\frac{\pi}{2},\pi].\end{cases}$$

and

$$g(t)=\begin{cases}2\pi\cos 2t, & t\in[-\frac{\pi}{4},\frac{\pi}{4}],\\ \cos 2t, & t\in[\frac{\pi}{4},\frac{3\pi}{4}].\end{cases}$$

Then, we give

$$\overline{h^+}=140,\qquad \|h^-\|_\infty=1,\qquad \overline{g^+}=2,\qquad \overline{g^-}=\frac{1}{\pi},\qquad \|g^+\|_\infty=2\pi.$$

At this point, the optimal α is such that

$$\left(\frac{4}{\alpha}\right)^{\frac{1}{3}} = \left(\frac{8\pi}{(1-\alpha)}\right)^{\frac{1}{2}}.$$

Based on the software Matlab, we get the optimal value $\alpha\approx 0.0303$. Further, we have

$$\overline{g^-}=\frac{1}{\pi}<\cos^{1+\mu}\left(\frac{NT}{2}\right)\overline{g^+}=1 \ \text{ and }\ \overline{h^+}=140>\frac{1}{A_1T\sigma_1^{1+\rho}}\max\left\{\frac{\|h^-\|_\infty}{\alpha N^2},\left(\frac{\|g^+\|_\infty}{(1-\alpha)N^2}\right)^{\frac{1+\rho}{1+\mu}}\right\}\approx 118.854.$$

The above computations show that (4.1) holds and the eq. (1.1) admits at least one positive π-periodic solution.

4.2 The case $\rho<\mu$

Theorem 4.3. *Let h and $g\in L^1(\mathbb{R}/T\mathbb{Z})$ be sign-changing functions. Assume that there exist $0<N<\frac{\pi}{T}$ and $\alpha\in(0,1)$ such that*

$$\overline{g^-}<\sigma_1^{1+\mu}\overline{g^+}\ \text{ and }\ \frac{1}{\sigma_1}\max\left\{\left(\frac{\|h^-\|_\infty}{\alpha N^2}\right)^{\frac{1}{1+\rho}},\ \left(\frac{\|g^+\|_\infty}{(1-\alpha)N^2}\right)^{\frac{1}{1+\mu}}\right\}<\left(\frac{\sigma_1^{1+\mu}\overline{g^+}-\overline{g^-}}{(\overline{h^+}-\sigma_1^{1+\rho}\overline{h^-})\sigma_1^{\mu-\rho}}\right)^{\frac{1}{\mu-\rho}}<(A_1T\overline{h^+})^{\frac{1}{1+\rho}}. \tag{4.5}$$

If $\rho<\mu$, then there exists at least one positive periodic solution to eq. (1.1).

Proof. Define

$$\Omega_7 := \{x \in C_T : \|x\| < r_4\} \quad \text{and} \quad \Omega_8 := \{x \in C_T : \|x\| < R_4\}.$$

By (4.5), the positive constants r_4 and R_4 can be fixed such that

$$R_4 = (A_1 T\overline{h^+})^{\frac{1}{1+\rho}} > \left(\frac{\sigma_1^{1+\mu}\overline{g^+} - \overline{g^-}}{(\overline{h^+} - \sigma_1^{1+\rho}\overline{h^-})\sigma_1^{\mu-\rho}}\right)^{\frac{1}{\mu-\rho}} > r_4 > \frac{1}{\sigma_1}\max\left\{\left(\frac{\|h^-\|_\infty}{\alpha N^2}\right)^{\frac{1}{1+\rho}}, \left(\frac{\|g^+\|_\infty}{(1-\alpha)N^2}\right)^{\frac{1}{1+\mu}}\right\}.$$

By an analogous reasoning as in Steps of Theorem 4.1, we get that $\Phi(\mathcal{K}_1 \cap (\overline{\Omega}_8 \backslash \Omega_7) \subset \mathcal{K}_1$, where Φ is defined in (3.3).

Next, we demonstrate that

$$\|\Phi x\| \le \|x\|, \quad \text{for } x \in \mathcal{K}_1 \cap \partial\Omega_7. \tag{4.6}$$

Since $x \in \mathcal{K}_1 \cap \partial\Omega_7$, it is evident that $\|x\| = r_4$ and

$$\sigma_1 r_4 \le x(t) \le r_4, \quad \text{for all } t \in \mathbb{R}.$$

As can be seen from (4.3) and $\int_0^T G_1(t,s)N^2 ds \equiv 1$ that

$$\begin{aligned}
(\Phi x)(t) &= \int_0^T G_1(t,s)\left(\frac{h(s)}{x^\rho(s)} - \frac{g(s)}{x^\mu(s)} + N^2 x(s)\right) ds \\
&= \int_0^T G_1(t,s)\left(\frac{h^+(s)}{x^\rho(s)} - \frac{h^-(s)}{x^\rho(s)} - \frac{g^+(s)}{x^\mu(s)} + \frac{g^-(s)}{x^\mu(s)} + N^2 x(s)\right) ds \\
&\le \frac{B_1 T\overline{h^+}}{(\sigma_1 r_4)^\rho} - \frac{A_1 T\overline{h^-}}{(\sigma_1 r_4)^\rho} - \frac{A_1 T\overline{g^+}}{r_4^\mu} + \frac{B_1 T\overline{g^-}}{(\sigma_1 r_4)^\mu} + r_4 \\
&\le r_4,
\end{aligned}$$

where $\frac{B_1 T\overline{h^+}}{(\sigma_1 r_4)^\rho} - \frac{A_1 T\overline{h^-}}{(\sigma_1 r_4)^\rho} - \frac{A_1 T\overline{g^+}}{r_4^\mu} + \frac{B_1 T\overline{g^-}}{(\sigma_1 r_4)^\mu} \le 0$ holds, i.e.,

$$r_4^{\mu-\rho}\left(\frac{B_1\overline{h^+}}{\sigma_1^\rho} - A_1\overline{h^-}\right) \le \left(A_1\overline{g^+} - \frac{B_1\overline{g^-}}{\sigma_1^\mu}\right)$$

because $\overline{g^-} < \sigma_1^{1+\mu}\overline{g^+}$ and $r_4 < \left(\frac{\sigma_1^{1+\mu}\overline{g^+} - \overline{g^-}}{(\overline{h^+} - \sigma_1^{1+\rho}\overline{h^-})\sigma_1^{\mu-\rho}}\right)^{\frac{1}{\mu-\rho}}$. This implies that (4.6) holds.

Finally, we demonstrate that

$$\|\Phi x\| \ge \|x\|, \quad \text{for } x \in \mathcal{K}_1 \cap \partial\Omega_8. \tag{4.7}$$

Since $x \in \mathcal{K}_1 \cap \partial\Omega_8$, it is evident that $\|x\| = R_4$ and

$$\sigma_1 R_4 \le x(t) \le R_4, \quad \text{for all } t \in \mathbb{R}.$$

According to (2.1) and (4.3), we arrive at

$$\begin{aligned}
(\Phi x)(t) &= \int_0^T G_1(t,s)\left(\frac{h^+(s)}{x^\rho(s)} - \frac{h^-(s)}{x^\rho(s)} - \frac{g^+(s)}{x^\mu(s)} + \frac{g^-(s)}{x^\mu(s)} + N^2 x(s)\right) ds \\
&> \int_0^T G_1(t,s)\frac{h^+(s)}{x^\rho(s)} ds \\
&\ge \frac{A_1 T\overline{h^+}}{R_4^\rho} = R_4
\end{aligned}$$

since $R_4 = (A_1 T\overline{h^+})^{\frac{1}{1+\rho}}$ from definition of R_4. Hence, (4.7) holds. The proof is completed. □

By Theorems 4.2 and 4.3, we can derive the following conclusion.

Theorem 4.4. *Let h and $g \in L^1(\mathbb{R}/T\mathbb{Z})$ be sign-changing functions. Assume that there exist $N > 0$ and $\beta \in (0,1)$ such that*

$$\overline{g^+} < \sigma_2^{1+\mu}\overline{g^-} \quad \textit{and} \quad \frac{1}{\sigma_2}\max\left\{\left(\frac{\|h^+\|_\infty}{\beta N^2}\right)^{\frac{1}{1+\rho}}, \ \left(\frac{\|g^-\|_\infty}{(1-\beta)N^2}\right)^{\frac{1}{1+\mu}}\right\} < \left(\frac{\sigma_2^{1+\mu}\overline{g^-} - \overline{g^+}}{(\overline{h^-} - \sigma_2^{1+\rho}\overline{h^+})\sigma_2^{\mu-\rho}}\right)^{\frac{1}{\mu-\rho}} < (A_2T\overline{h^-})^{\frac{1}{1+\rho}}.$$

If $\rho < \mu$, then there exists at least one positive periodic solution to eq. (1.1).

Remark 4.2. It is worth to mention that the cases $h^- \equiv 0$ and $h^+ \equiv 0$ can be treated as a limit cases with $\alpha = 0$ and $\beta = 0$, respectively. In such a way one can naturally get the results of Section 3 from those established for the general cases in Section 4. That is, when $\alpha = 0$, we have that $h^- \equiv 0$, one can naturally derive the result of Theorems 3.1 and 3.2 from Theorems 4.1 and 4.3, respectively. Similarly, when $\beta = 0$, we have that $h^+ \equiv 0$, it is evident that Theorems 4.2 and 4.4 are generalized versions of Theorems 3.3 and 3.4, respectively.

Acknowledgments

The authors would like to thank the referee for invaluable comments and insightful suggestions.

CRediT authorship contribution statement

Zhibo Cheng : Conceptualization, Methodology, Writing-original draft, Revising. **Yuting Qian** : Software, Data curation, Revising-original draft. **Juan Song** : Writing-original draft.

References

[1] Bărbosu, M., Mioc, V., Pasca, D., Szenkovits, F.: The two-body problem with generalized Lennard-Jones potential. J. Math. Chem. **49**, 1961-1975 (2011)

[2] Bravo, J., Torres, P.: Periodic solutions of a singular equation with indefinite weight. Adv. Nonlinear Stud. **10**, 927-938 (2010)

[3] Cabada, A.: Green's functions in the theory of ordinary differential equations. SpringerBriefs in Mathematics. Springer, New York, 2014.

[4] Cabada, A., Cid, J., Máquez-Villamarín, B.: Computation of Greens functions for boundary value problems with Mathematica. Appl. Math. Comput. **219**, 1919-1936 (2012)

[5] Cheng, Z., Gu, L.: Positive periodic solution to a second-order differential equation with attractive-repulsive singularities. Rocky Mountain J. Math. **52**, 77-85 (2022)

[6] Cheng, Z., Ren, J.: Periodic solution for second order damped differential equations with attractive-repulsive singularities. Rocky Mountain J. Math. **48**, 753-768 (2018)

[7] Chu, J., Torres, P., Wang, F.: Twist periodic solutions for differential equations with a combined attractive-repulsive singularity. J. Math. Anal. Appl. **437**, 1070-1083 (2016)

[8] Godoy, J., Zamora, M.: A general result to the existence of a periodic solution to an indefinite equation with a weak singularity. J. Dynam. Differential Equations **31**, 451-468 (2018)

[9] Godoy, J., Zamora, M.: Periodic solutions for a second order differential equation with indefinite weak singularity. Proc. R. Soc. Edinb. Sect. A. Math. **149**, 1-18 (2019)

[10] Guo, D., Lakshmikantham, V.: Nonlinear Problems in Abstract Cones. Academic Press. Boston, 1988.

[11] Hakl, R., Torres, R.: On periodic solutions of second-order differential equations with attractive-repulsive singularities. J. Differential Equations **248**, 111-126 (2010)

[12] Hakl, R., Zamora, M.: Existence and multiplicity of periodic solutions to indefinite singular equations having a nonmonotone term with two singularities. Adv. Nonlinear Stud. **19**, 317-322 (2018)

[13] Hakl, R., Zamora, M.: Periodic solutions to second-order indefinite singular equations. J. Differential Equations **263**, 451-469 (2017)

[14] Han, W., Ren, J.: Some results on second-order neutral functional differential equations with infinite distributed delay. Nonlinear Anal. **70**, 1393-1406 (2009)

[15] Lennard-Jones, J.: On the determination of molecular fields. Proc. Roy. Soc. A. **106**, 463-477 (1924)

[16] Llibre, J., Long, Y.: Periodic solutions for the generalized anisotropic Lennard-Jones Hamiltonian. Qual. Theory Dyn. Syst. **14**, 291-311 (2015)

[17] Lu, S., Yu, X.: Periodic solutions for second order differential equations with indefinite singularities. Adv. Nonlinear Anal. **9**, 994-1007 (2020)

[18] Montesinos, G., Perez-García, V., Torres, P.: Stabilization of solitons of the multidimensional non-linear Schrödinger equation: Matter-wave breathers. Phys. D. **191**, 193-210 (2004)

[19] Torres, P.: Mathematical models with singularities-A zoo of singular creatures. Atlantis Press, 2015.

[20] Ureña, A.: A counterexample for singular equations with indefinite weight. Adv. Nonlinear Stud. **17**, 497-516 (2017)

[21] Ureña, A.: Periodic solutions of singular equations. Topol. Meth. Nonlinear Anal. **47**, 55-72 (2016)