ROCKY MOUNTAIN JOURNAL OF MATHEMATICS
Vol. , No. , YEAR
https://doi.org/rmj.YEAR..PAGE

IMPLICIT FRACTIONAL DIFFERENTIAL EQUATIONS WITH ADVANCED ARGUMENTS AND THE CONVEX COMBINED CAPUTO DERIVATIVE

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ABSTRACT. The purpose of this article is to prove the existence and uniqueness results for a class of implicit fractional differential equations involving the combined Caputo fractional derivative with advanced arguments by using the fixed point theorems of Banach and nonlinear alternative of Leray-Schauder. We will also establish the Ulam stability and give some examples to show the applicability of our results.

1. Introduction

Fractional calculus has become a very important tool in modeling of many phenomena in applications and sciences such as physics, biology, finance, engineering, stability, controllability and rheology. It can better describe the memory properties of the physical process than the standard integer order calculus. For more details on the applications of fractional calculus, the reader is directed to the books of Baleanu *et al.* [5] and Graef *et al.* [8]. In [1, 2, 3], Abbas *et al.* studied several problems with advanced fractional differential and integral equations and presented various applications. In [6, 7, 9], the authors presented some results on the fractional differential equations with Riesz and Riesz-Caputo fractional derivatives. Salim *et al.* [12, 19, 20, 21, 24, 23] addressed the existence, stability, and uniqueness of solutions for diverse problems with fractional differential equations using various fractional derivatives and different types of conditions.

In this paper, we consider the convex combined Caputo fractional derivative ${}_0^CD_{\varkappa}^{\zeta_1,\zeta_2;\gamma}$ which is a convex combination of the left Caputo fractional derivative of order ζ_1 and the right Caputo fractional derivative of order ζ_2 on $[0,\varkappa]$. The main feature of the convex combined Caputo fractional operator is that it is a two sided operator, this property plays a decisive role in the fractional modeling. See [4], for more information.

Mathematician Ulam originally highlighted the stability problem in functional equations in a 1940 presentation at Wisconsin University. S. M. Ulam introduced the following challenge: ”Under what conditions does an additive mapping exist near an approximately additive mapping ?” [30]. The following year, in [10], Hyers provided an answer to Ulam's problem for additive functions defined on Banach spaces. In 1978, Rassias [26] demonstrated the existence of unique linear mappings near

2020 *Mathematics Subject Classification.* 26A33, 34A08.
Key words and phrases. Combined Caputo fractional derivative, implicit problem, existence, convex, fixed point, Ulam stability, advanced argument.

approximate additive mappings, generalizing Hyers' findings. Several research articles in the literature address the Ulam stabilities of various types of differential and integral equations, see [29]. Luo *et al.* [16, 15, 17] studied the Ulam stability of several differential fractional problems with some types of delay. In [22, 24, 13], the authors studied several problems with advanced fractional differential equations and presented various stability results and some applications.

The authors of [6] studied the existence of solution for the following boundary value problem:

$$\begin{cases} {}_0^{RC}D_{\varkappa}^{\nu}\varphi(\theta) = g(\theta,\varphi(\theta)), \quad \theta\in\Theta := [0,\varkappa], \\ \varphi(0)=\varphi_0, \quad \varphi(\varkappa)=\varphi_{\varkappa}, \end{cases}$$

where ${}_0^{RC}D_{\varkappa}^{\nu}$ is a Riesz-Caputo derivative of order $0<\nu\le 1$, $g:\Theta\times\mathbb{R}\to\mathbb{R}$ a continuous function and $\varphi_0,\varphi_{\varkappa}\in\mathbb{R}$. Their arguments are based on Leray-Schauder fixed point theorem, and Schauder's fixed point theorem.

In [14], Li and Wang discussed the following fractional problem:

$$\begin{gathered} {}_0^{RC}D_1^{\gamma}\varphi(\theta) = \Psi(\theta,\varphi(\theta)), \quad \theta\in[0,1], \quad 0<\gamma\le 1, \\ \varphi(0)=a, \quad \varphi(1)=b\varphi(\eta), \end{gathered}$$

where ${}_0^{RC}D_1^{\gamma}$ is the Riesz-Caputo derivative, $\Psi\in C([0,1]\times[0,+\infty),[0,+\infty)), 0<\eta<1, a>0, 0<b<2$. They found the positive solutions by applying the technique of monotone iterative.

Naas *et al.* [18] investigated the existence and uniqueness results of the following fractional differential equation with the Riesz-Caputo derivative:

$$\begin{cases} {}_0^{RC}D_T^{\vartheta}\varkappa(\theta) + \mathfrak{F}\left(\theta,\varkappa(\theta),{}_0^{RC}D_T^{\varsigma}\varkappa(\theta)\right) = 0, \theta\in\mathscr{J} := [0,T], \\ \varkappa(0)+\varkappa(T)=0, \quad \mu\varkappa'(0)+\sigma\varkappa'(T)=0, \end{cases}$$

where $1<\vartheta\le 2$ and $,0<\varsigma\le 1, {}_0^{RC}D_T^{\kappa}$ is the Riesz-Caputo fractional derivative of order $\kappa\in\{\vartheta,\varsigma\}$, $\mathfrak{F}:\mathscr{J}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, is a continuous function, and μ,σ are nonnegative constants with $\mu>\sigma$. The existence and uniqueness of solutions of the above cited problem are demonstrated with the Riesz-Caputo derivatives via Banach's, Schaefer's, and Krasnoselskii's fixed point theorems.

In this work, we investigate the existence, uniqueness and stability results for the following implicit fractional problem:

$$(1) \qquad {}_0^{C}D_{\varkappa}^{\zeta_1,\zeta_2;\gamma}\xi(\theta) = f(\theta,\xi^{\theta},\, {}_0^{C}D_{\varkappa}^{\zeta_1,\zeta_2;\gamma}\xi(\theta)), \quad if\ \theta\in\Theta := [0,\varkappa],$$

$$(2) \qquad \xi(0)=\xi_0,$$

$$(3) \qquad \xi(\theta)=\psi(\theta), \quad if\ \theta\in[\varkappa,\varkappa+\delta],$$

where ${}_0^{C}D_{\varkappa}^{\zeta_1,\zeta_2;\gamma}$ is the convex combined Caputo fractional derivative of order $\zeta_1,\zeta_2\in(0,1]$, $\gamma\in[0,1]$, $\delta>0$, $f:\Theta\times C([0,\delta],\mathbb{R})\times\mathbb{R}\to\mathbb{R}$ is a given function, $\psi\in C([\varkappa,\varkappa+\delta],\mathbb{R})$, and $\xi_0\in\mathbb{R}$. We denote by ξ^{θ} the elements of $C([0,\delta],\mathbb{R})$ defined by

$$\xi^{\theta} = \xi(\theta+s) : s\in[0,\delta].$$

This paper is organized as follows: Section 2 introduces some preliminaries, definitions and lemmas. In section 3, we give some existence and uniqueness results for the problem (1)-(3) that are based on Banach contraction principle and the nonlinear alternative of Leray-Schauder fixed point theorem. In section 4 we prove that the problem (1)-(3) is Ulam stable. Finally we present some examples to show the validity of our results.

2. Preliminaries

In this section, we recall some notations, definitions and previous results which are used throughout this paper.

We denote by $C(\Theta,\mathbb{R})$ the Banach space of all continuous functions from Θ into $\mathbb{R}$ with the norm

$$\|\xi\|_{\Theta} = \sup\{|\xi(\theta)| : \theta \in \Theta\}.$$

Let $C([\varkappa,\varkappa+\delta],\mathbb{R})$ be the Banach space with the norm

$$\|\xi\|_{[\varkappa,\varkappa+\delta]} = \sup\{|\xi(\theta)| : \theta \in [\varkappa,\varkappa+\delta]\},$$

and $C([0,\delta],\mathbb{R})$ be the Banach space with the norm

$$\|\xi\|_{[0,\delta]} = \sup\{|\xi(\theta)| : \theta \in [0,\delta]\}.$$

Let

$$\Upsilon = \{\xi : [0,\varkappa+\delta] \to \mathbb{R} : \xi|_{\Theta} \in C(\Theta,\mathbb{R}) \text{ and } \xi|_{[\varkappa,\varkappa+\delta]} \in C([\varkappa,\varkappa+\delta],\mathbb{R})\}.$$

We note that Υ is a Banach space with the norm

$$\|\xi\|_{\Upsilon} = \sup_{\theta\in[0,\varkappa+\delta]} |\xi(\theta)|.$$

Definition 2.1 ([11]). *Let $\zeta_1 > 0$. The left and right Riemann-Liouville fractional integrals of a function $\varphi \in C(\Theta,\mathbb{R})$ of order ζ_1 are given respectively by*

$$ {}_0I_{\theta}^{\zeta_1}\varphi(\theta) = \frac{1}{\Gamma(\zeta_1)}\int_0^{\theta}(\theta-\rho)^{\zeta_1-1}\varphi(\rho)d\rho,$$

and

$$ {}_{\theta}I_{\varkappa}^{\zeta_1}\varphi(\theta) = \frac{1}{\Gamma(\zeta_1)}\int_{\theta}^{\varkappa}(\rho-\theta)^{\zeta_1-1}\varphi(\rho)d\rho.$$

Definition 2.2 ([4, 25]). *Let $\zeta_1,\zeta_2 > 0$. The combined Riemann fractional integral of a function $\varphi \in C(\Theta,\mathbb{R})$ of order (ζ_1,ζ_2) is defined by*

$$ {}_0I_{\varkappa}^{\zeta_1,\zeta_2}\varphi(\theta) = {}_0I_{\theta}^{\zeta_1}\varphi(\theta) + {}_{\theta}I_{\varkappa}^{\zeta_2}\varphi(\theta),$$

where ${}_0I_{\theta}^{\zeta_1}$ and ${}_{\theta}I_{\varkappa}^{\zeta_2}$ are the left and right fractional integrals of Riemann-Liouville of order ζ_1 and ζ_2 respectively.

Definition 2.3 ([11]). *Let $\zeta_1 \in (n,n+1]$, $n \in \mathbb{N}$. The left and right Caputo fractional derivatives of a function $\varphi \in C^{n+1}(\Theta,\mathbb{R})$ of order ζ_1 are given respectively by*

$$ {}_0^CD_{\theta}^{\zeta_1}\varphi(\theta) = \frac{1}{\Gamma(n+1-\zeta_1)}\int_0^{\theta}(\theta-\rho)^{n-\zeta_1}\varphi^{(n+1)}(\rho)d\rho,$$

and

$$ {}_{\theta}^{C}D_{\varkappa}^{\zeta_1}\varphi(\theta) = \frac{(-1)^{n+1}}{\Gamma(n+1-\zeta_1)}\int_{\theta}^{\varkappa}(\rho-\theta)^{n-\zeta_1}\varphi^{(n+1)}(\rho)d\rho. $$

Definition 2.4 ([4, 25])**.** *Let $\zeta_1,\zeta_2 \in (n,n+1], \gamma \in [0,1]$. The combined Caputo fractional derivative of a function $\varphi \in C^{n+1}(\Theta,\mathbb{R})$ of order (ζ_1,ζ_2) is given by*

$$ {}_{0}^{C}D_{\varkappa}^{\zeta_1,\zeta_2;\gamma}\varphi(\theta) = \gamma\,{}_{0}^{C}D_{\theta}^{\zeta_1}\varphi(\theta) + (-1)^{n+1}(1-\gamma){}_{\theta}^{C}D_{\varkappa}^{\zeta_2}\varphi(\theta), $$

where ${}_{0}^{C}D_{\theta}^{\zeta_1}$ is the left Caputo derivative and ${}_{\theta}^{C}D_{\varkappa}^{\zeta_2}$ is the right one.

Lemma 2.5 ([11])**.** *If $\xi \in C^{n+1}(\Theta,\mathbb{R})$ and $\zeta_1,\zeta_2 \in (n,n+1]$, $\gamma \in [0,1]$, then we have*

$$ {}_{0}I_{\theta}^{\zeta_1}\,{}_{0}^{C}D_{\theta}^{\zeta_1}\xi(\theta) = \xi(\theta) - \sum_{k=0}^{n}\frac{\xi^{(k)}(0)}{k!}\theta^k, $$

and

$$ {}_{\theta}I_{\varkappa}^{\zeta_2}\,{}_{\theta}^{C}D_{\varkappa}^{\zeta_2}\xi(\theta) = (-1)^{n+1}\left[\xi(\theta) - \sum_{k=0}^{n}\frac{(-1)^k\xi^{(k)}(\varkappa)}{k!}(\varkappa-\theta)^k\right]. $$

Consequently, we may have

$$ {}_{0}I_{\varkappa}^{\zeta_1,\zeta_2}\,{}_{0}^{C}D_{\varkappa}^{\zeta_1,\zeta_2;\gamma}\xi(\theta) = \gamma\,{}_{0}I_{\theta}^{\zeta_1}\,{}_{0}^{C}D_{\theta}^{\zeta_1}\xi(\theta) + (-1)^{n+1}(1-\gamma)\,{}_{\theta}I_{\varkappa}^{\zeta_2}\,{}_{\theta}^{C}D_{\varkappa}^{\zeta_2}\xi(\theta)). $$

In particular, if $0 < \zeta_1,\zeta_2 \leq 1$, then we obtain

$$ {}_{0}I_{\varkappa}^{\zeta_1,\zeta_2}\,{}_{0}^{C}D_{\varkappa}^{\zeta_1,\zeta_2;\gamma}\xi(\theta) = \xi(\theta) - \gamma\xi(0) - (1-\gamma)\xi(\varkappa). $$

Remark 2.6. *If we take $\gamma = \frac{1}{2}$ and $\zeta_1 = \zeta_2$, then the combined Caputo fractional derivative coincides with the Riesz-Caputo derivative.*

2.1. *Some Fixed Point Theorems.*

Theorem 2.7 (Banach's fixed point theorem [28])**.** *Let E be a Banach space and $\mathscr{H} : E \longrightarrow E$ a contraction, i.e. there exists $k \in [0,1)$ such that*

$$ \|\mathscr{H}(\xi_1) - \mathscr{H}(\xi_2)\| \leq k\|\xi_1 - \xi_2\|, \qquad for\ all\ \xi_1,\xi_2 \in E. $$

Then $\mathscr{H}$ has a unique fixed point.

Theorem 2.8 (Nonlinear alternative of Leray-Schauder [28])**.** *Let E be a Banach space and C a nonempty convex subset of E. Let U be a nonempty open subset of C, with $0 \in U$ and $\mathscr{H} : \bar{U} \to C$ a continuous and compact operator.*
Then, either
(a) $\mathscr{H}$ has fixed points or
(b) there exist $\chi \in \partial U$ and $\varpi(0,1)$ with $\chi = \varpi\mathscr{H}(\chi)$.

3. Existence Results

Consider the following fractional differential problem:

$$ {}_{0}^{C}D_{\varkappa}^{\zeta_1,\zeta_2;\gamma}\xi(\theta) = \mu(\theta), \quad if\ \theta \in \Theta,\ 0 < \zeta_1, \zeta_2 \le 1, \gamma \in [0,1], \tag{4} $$

$$ \xi(0) = \xi_0, \tag{5} $$

$$ \xi(\theta) = \psi(\theta), \quad if\ \theta \in [\varkappa, \varkappa+\delta],\ \delta > 0, \tag{6} $$

where μ is a continuous function, and $\psi \in C([\varkappa, \varkappa+\delta], \mathbb{R})$.

Lemma 3.1. *Let $\zeta_1, \zeta_2 \in (0,1], \gamma \in [0,1]$, and $\mu : \Theta \to \mathbb{R}$ be continuous. Then, the problem (4)-(6) has a unique solution given by*

$$ \xi(\theta) = \begin{cases} \xi_0 - \dfrac{1}{\Gamma(\zeta_2)} \displaystyle\int_0^{\varkappa} s^{\zeta_2-1}\mu(s)ds + \dfrac{1}{\Gamma(\zeta_1)} \displaystyle\int_0^{\theta} (\theta-s)^{\zeta_1-1}\mu(s)ds \\ + \dfrac{1}{\Gamma(\zeta_2)} \displaystyle\int_{\theta}^{\varkappa} (s-\theta)^{\zeta_2-1}\mu(s)ds, \qquad if\ \theta \in \Theta, \\ \psi(\theta), \qquad if\ \theta \in [\varkappa, \varkappa+\delta]. \end{cases} \tag{7} $$

Proof. Suppose that ξ satisfies (4)-(6), then

$$ {}_{0}^{C}D_{\varkappa}^{\zeta_1,\zeta_2;\gamma}\xi(\theta) = \mu(\theta). $$

By Lemma 2.5, we have

$$ {}_{0}I_{\varkappa}^{\zeta_1,\zeta_2}\, {}_{0}^{C}D_{\varkappa}^{\zeta_1,\zeta_2;\gamma}\xi(\theta) = \xi(\theta) - \gamma\xi(0) - (1-\gamma)\xi(\varkappa), $$

this implies that

$$ \begin{aligned} \xi(\theta) &= \gamma\xi(0) + (1-\gamma)\xi(\varkappa) + {}_{0}I_{\varkappa}^{\zeta_1,\zeta_2}\mu(\theta) \\ &= \gamma\xi(0) + (1-\gamma)\xi(\varkappa) + \frac{1}{\Gamma(\zeta_1)} \int_0^{\theta} (\theta-s)^{\zeta_1-1}\mu(s)ds \\ &\quad + \frac{1}{\Gamma(\zeta_2)} \int_{\theta}^{\varkappa} (s-\theta)^{\zeta_2-1}\mu(s)ds. \end{aligned} $$

For $\theta = 0$, we have

$$ \xi(\varkappa)(1-\gamma) = \xi_0(1-\gamma) - \frac{1}{\Gamma(\zeta_2)} \int_0^{\varkappa} s^{\zeta_2-1}\mu(s)ds. $$

Then, the final solution is given by:

$$ \begin{aligned} \xi(\theta) = \xi_0 &- \frac{1}{\Gamma(\zeta_2)} \int_0^{\varkappa} s^{\zeta_2-1}\mu(s)ds + \frac{1}{\Gamma(\zeta_1)} \int_0^{\theta} (\theta-s)^{\zeta_1-1}\mu(s)ds \\ &+ \frac{1}{\Gamma(\zeta_2)} \int_{\theta}^{\varkappa} (s-\theta)^{\zeta_2-1}\mu(s)ds. \end{aligned} $$

Conversely, we can easily prove by lemma 2.5 that if ξ satisfies equation (7), then it satisfied the problem (4)-(6). □

Lemma 3.2. *Let $f:\Theta\times C([0,\delta],\mathbb{R})\times\mathbb{R}\to\mathbb{R}$ be a continuous function. Then the problem (1)-(3) is equivalent to the following integral equation:*

$$\xi(\theta)=\begin{cases}\xi_0-\dfrac{1}{\Gamma(\zeta_2)}\displaystyle\int_0^{\varkappa}s^{\zeta_2-1}f(s,\xi^s,\wp(s))ds+\dfrac{1}{\Gamma(\zeta_1)}\displaystyle\int_0^{\theta}(\theta-s)^{\zeta_1-1}f(s,\xi^s,\wp(s))ds\\ +\dfrac{1}{\Gamma(\zeta_2)}\displaystyle\int_\theta^{\varkappa}(s-\theta)^{\zeta_2-1}f(s,\xi^s,\wp(s))ds, \qquad if\ \theta\in\Theta,\\ \psi(\theta), \qquad if\ \theta\in[\varkappa,\varkappa+\delta],\end{cases}$$

where $\wp\in C(\Theta,\mathbb{R})$ satisfies the following functional equation

$$\wp(\theta)=f(\theta,\xi^\theta,\wp(\theta)).$$

Let us assume the following assumptions:

(*B1*) The function $f:\Theta\times C([0,\delta],\mathbb{R})\times\mathbb{R}\to\mathbb{R}$ is continuous.

(*B2*) There exist constants $\lambda_1>0$ and $0<\lambda_2<1$ such that

$$|f(\theta,\chi,\beta)-f(\theta,\bar\chi,\bar\beta)|\le\lambda_1\|\chi-\bar\chi\|_{[0,\delta]}+\lambda_2|\beta-\bar\beta|,$$

for any $\chi,\bar\chi\in C([0,\delta],\mathbb{R})$, $\beta,\bar\beta\in\mathbb{R}$ and $\theta\in\Theta$.

We are now in a position to prove the existence result of the problem (1)-(3) based on the Banach contraction principle.

Theorem 3.3. *Assume that the assumptions $(B1)$-$(B2)$ hold. If*

$$\frac{2\lambda_1\varkappa^{\zeta_2}}{(1-\lambda_2)\Gamma(\zeta_2+1)}+\frac{\lambda_1\varkappa^{\zeta_1}}{(1-\lambda_2)\Gamma(\zeta_1+1)}<1, \tag{8}$$

then the problem (1)-(3) has a unique solution on Θ.

Proof. Consider the operator $A:\Upsilon\longrightarrow\Upsilon$ defined by:

$$A\xi(\theta)=\begin{cases}\xi_0-\dfrac{1}{\Gamma(\zeta_2)}\displaystyle\int_0^{\varkappa}s^{\zeta_2-1}\wp(s)ds+\dfrac{1}{\Gamma(\zeta_1)}\displaystyle\int_0^{\theta}(\theta-s)^{\zeta_1-1}\wp(s)ds\\ +\dfrac{1}{\Gamma(\zeta_2)}\displaystyle\int_\theta^{\varkappa}(s-\theta)^{\zeta_2-1}\wp(s)ds, \qquad \theta\in\Theta,\\ \psi(\theta), \qquad \theta\in[\varkappa,\varkappa+\delta].\end{cases}$$

Clearly, the fixed points of the operator A are solutions of the problem (1)-(3).
Let $\xi,z\in\Upsilon$. If $\theta\in[\varkappa,\varkappa+\delta]$, then

$$|A\xi(\theta)-Az(\theta)|=0.$$

If $\theta\in\Theta$, we have

$$|A\xi(\theta)-Az(\theta)|\le\frac{1}{\Gamma(\zeta_2)}\int_0^{\varkappa}s^{\zeta_2-1}|\wp(s)-h(s)|ds$$

$$+\frac{1}{\Gamma(\zeta_1)}\int_0^\theta (\theta-s)^{\zeta_1-1}|\wp(s)-h(s)|ds$$
$$+\frac{1}{\Gamma(\zeta_2)}\int_\theta^\varkappa (s-\theta)^{\zeta_2-1}|\wp(s)-h(s)|ds,$$

where $\wp$ and h are two functions verifying the functional equations:

$$\wp(\theta)=f(\theta,\xi^\theta,\wp(\theta)),$$
$$h(\theta)=f(\theta,z^\theta,h(\theta)).$$

By $(B2)$, we have

$$\begin{aligned}|\wp(\theta)-h(\theta)| &= |f(\theta,\xi^\theta,\wp(\theta))-f(\theta,z^\theta,h(\theta))|\\ &\leq \lambda_1\|\xi^\theta-z^\theta\|_{[0,\delta]}+\lambda_2|\wp(\theta)-h(\theta)|.\end{aligned}$$

Thus,

$$|\wp(\theta)-h(\theta)|\leq\frac{\lambda_1}{1-\lambda_2}\|\xi^\theta-z^\theta\|_{[0,\delta]}.$$

Then, for each $\theta\in\Theta$, we have

$$\begin{aligned}|A\xi(\theta)-Az(\theta)| &\leq \frac{\lambda_1}{(1-\lambda_2)\Gamma(\zeta_2)}\int_0^\varkappa s^{\zeta_2-1}\|\xi^s-z^s\|_{[0,\delta]}ds\\ &+\frac{\lambda_1}{(1-\lambda_2)\Gamma(\zeta_1)}\int_0^\theta(\theta-s)^{\zeta_1-1}\|\xi^s-z^s\|_{[0,\delta]}ds\\ &+\frac{\lambda_1}{(1-\lambda_2)\Gamma(\zeta_2)}\int_\theta^\varkappa(s-\theta)^{\zeta_2-1}\|\xi^s-z^s\|_{[0,\delta]}ds\\ &\leq\left[\frac{\lambda_1\varkappa^{\zeta_2}}{(1-\lambda_2)\Gamma(\zeta_2+1)}+\frac{\lambda_1\varkappa^{\zeta_1}}{(1-\lambda_2)\Gamma(\zeta_1+1)}+\frac{\lambda_1\varkappa^{\zeta_2}}{(1-\lambda_2)\Gamma(\zeta_2+1)}\right]\|\xi-z\|_\Upsilon\\ &\leq\left[\frac{2\lambda_1\varkappa^{\zeta_2}}{(1-\lambda_2)\Gamma(\zeta_2+1)}+\frac{\lambda_1\varkappa^{\zeta_1}}{(1-\lambda_2)\Gamma(\zeta_1+1)}\right]\|\xi-z\|_\Upsilon.\end{aligned}$$

Thus,

$$\|A\xi-Az\|_\Upsilon\leq\left[\frac{2\lambda_1\varkappa^{\zeta_2}}{(1-\lambda_2)\Gamma(\zeta_2+1)}+\frac{\lambda_1\varkappa^{\zeta_1}}{(1-\lambda_2)\Gamma(\zeta_1+1)}\right]\|\xi-z\|_\Upsilon.$$

Consequently, by the Banach contraction principle, the operator A has a unique fixed point which is a solution of the fractional problem (1)-(3). □

Remark 3.4. *Let us put*

$$q_1(\theta)=|f(\theta,0,0)|,\ \lambda_1=q_2^*,\ \lambda_2=q_3^*.$$

Then, the condition $(B2)$ implies that

$$|f(\theta,\chi,\beta)|\leq q_1(\theta)+q_2^*\|\chi\|_{[0,\delta]}+q_3^*|\beta|,$$

for $\theta \in \Theta$, $\chi \in C([0,\delta],\mathbb{R})$, $\beta \in \mathbb{R}$ *and* $q_1 \in C(\Theta,\mathbb{R}_+)$, *with*

$$q_1^* = \sup_{\theta\in\Theta} q_1(\theta).$$

Our second existence result for the problem (1)-(3) is based on Leray-Schauder's fixed point theorem.

Theorem 3.5. *Assume that the assumptions* $(B1)$*-*$(B2)$ *hold. If*

$$\frac{2q_2^*\varkappa^{\zeta_2}}{(1-q_3^*)\Gamma(\zeta_2+1)} + \frac{q_2^*\varkappa^{\zeta_1}}{(1-q_3^*)\Gamma(\zeta_1+1)} < 1,$$

then the implicit fractional problem (1)*-*(3) *has at least one solution on* Θ.

Proof. Transform problem (1)-(3) into a fixed point problem.

Step 1: The operator $A : \Upsilon \longrightarrow \Upsilon$ is continuous.
Let $\{\xi_n\}_{n\in\mathbb{N}}$ be a sequence such that $\xi_n \longrightarrow \xi$ in Υ. If $\theta \in [\varkappa, \varkappa+\delta]$, then

$$|A\xi_n(\theta) - A\xi(\theta)| = 0.$$

If $\theta \in \Theta$, we have

$$\begin{aligned}|A\xi_n(\theta) - A\xi(\theta)| \leq & \frac{1}{\Gamma(\zeta_2)}\int_0^{\varkappa} s^{\zeta_2-1}|\wp_n(s) - \wp(s)|ds \\ & + \frac{1}{\Gamma(\zeta_1)}\int_0^{\theta}(\theta-s)^{\zeta_1-1}|\wp_n(s) - \wp(s)|ds \\ & + \frac{1}{\Gamma(\zeta_2)}\int_\theta^{\varkappa}(s-\theta)^{\zeta_2-1}|\wp_n(s) - \wp(s)|ds.\end{aligned}$$

By $(B2)$, we have

$$|\wp_n(\theta) - \wp(\theta)| \leq \lambda_1\|\xi_n^\theta - \xi^\theta\|_{[0,\delta]} + \lambda_2|\wp_n(\theta) - \wp(\theta)|.$$

Then,

$$|\wp_n(\theta) - \wp(\theta)| \leq \frac{\lambda_1}{1-\lambda_2}\|\xi_n^\theta - \xi^\theta\|_{[0,\delta]}.$$

Thus,

$$\begin{aligned}|A\xi_n(\theta) - A\xi(\theta)| \leq & \frac{\lambda_1}{(1-\lambda_2)\Gamma(\zeta_2)}\int_0^{\varkappa} s^{\zeta_2-1}\|\xi_n^s - \xi^s\|_{[0,\delta]}ds \\ & + \frac{\lambda_1}{(1-\lambda_2)\Gamma(\zeta_1)}\int_0^{\theta}(\theta-s)^{\zeta_1-1}\|\xi_n^s - \xi^s\|_{[0,\delta]}ds \\ & + \frac{\lambda_1}{(1-\lambda_2)\Gamma(\zeta_2)}\int_\theta^{\varkappa}(s-\theta)^{\zeta_2-1}\|\xi_n^s - \xi^s\|_{[0,\delta]}ds.\end{aligned}$$

By applying the Lebesgue dominated convergence theorem, we get

$$|A\xi_n(\theta) - A\xi(\theta)| \longrightarrow 0 \quad as\ n \longrightarrow \infty.$$

Hence,

$$\|A\xi_n - A\xi\|_{\Upsilon} \longrightarrow 0 \quad as\ n \longrightarrow \infty,$$

which implies that A is continuous.

Let $R > 0$ and define the ball

$$D_R = \{\xi \in \Upsilon : \|\xi\|_{\Upsilon} \leq R\}.$$

It is clear that D_R is a bounded, closed and convex.

Step 2: $A(D_R)$ is bounded.
Let $\xi \in D_R$. If $\theta \in [\varkappa, \varkappa+\delta]$, then

$$|A\xi(\theta)| = |\psi(\theta)| \leq \|\psi\|_{[\varkappa,\varkappa+\delta]}.$$

If $\theta \in \Theta$, we have

$$|A\xi(\theta)| \leq |\xi_0| + \frac{1}{\Gamma(\zeta_2)}\int_0^{\varkappa} s^{\zeta_2-1}|\wp(s)|ds + \frac{1}{\Gamma(\zeta_1)}\int_0^{\theta}(\theta-s)^{\zeta_1-1}|\wp(s)|ds$$
$$+\frac{1}{\Gamma(\zeta_2)}\int_{\theta}^{\varkappa}(s-\theta)^{\zeta_2-1}|\wp(s)|ds.$$

From hypothesis $(B2)$, we have

$$\begin{aligned}
|\wp(\theta)| &= |f(\theta,\xi(\theta),\wp(\theta))| \\
&\leq q_1(\theta) + q_2^*\|\xi^{\theta}\|_{[0,\delta]} + q_3^*|\wp(\theta)| \\
&\leq q_1^* + q_2^*\|\xi\|_{\Upsilon} + q_3^*|\wp(\theta)| \\
&\leq q_1^* + q_2^* R + q_3^*|\wp(\theta)|.
\end{aligned}$$

Then,

$$|\wp(\theta)| \leq \frac{q_1^* + q_2^* R}{1-q_3^*}.$$

Thus,

$$\begin{aligned}
|A\xi(\theta)| &\leq |\xi_0| + \frac{(q_1^*+q_2^*R)}{(1-q_3^*)\Gamma(\zeta_2)}\int_0^{\varkappa} s^{\zeta_2-1}ds \\
&\quad + \frac{(q_1^*+q_2^*R)}{(1-q_3^*)\Gamma(\zeta_1)}\int_0^{\theta}(\theta-s)^{\zeta_1-1}ds \\
&\quad + \frac{(q_1^*+q_2^*R)}{(1-q_3^*)\Gamma(\zeta_2)}\int_{\theta}^{\varkappa}(s-\theta)^{\zeta_2-1}ds \\
&\leq |\xi_0| + \frac{(q_1^*+q_2^*R)\varkappa^{\zeta_2}}{(1-q_3^*)\Gamma(\zeta_2+1)} + \frac{(q_1^*+q_2^*R)\varkappa^{\zeta_1}}{(1-q_3^*)\Gamma(\zeta_1+1)} \\
&\quad + \frac{(q_1^*+q_2^*R)\varkappa^{\zeta_2}}{(1-q_3^*)\Gamma(\zeta_2+1)}
\end{aligned}$$

$$\begin{aligned}&\leq |\xi_0| + \frac{2(q_1^*+q_2^*R)\varkappa^{\zeta_2}}{(1-q_3^*)\Gamma(\zeta_2+1)} + \frac{(q_1^*+q_2^*R)\varkappa^{\zeta_1}}{(1-q_3^*)\Gamma(\zeta_1+1)}\\ &:= K.\end{aligned}$$

Then,

$$\|A\xi\|_\Upsilon \leq max\{\|\psi\|_{[\varkappa,\varkappa+\delta]}, K\}.$$

Hence, $A(D_R)$ is bounded.

Step 3: $A(D_R)$ is equicontinuous.
Let $\theta_1, \theta_2 \in \Theta$, where $\theta_1 < \theta_2$ and $\xi \in D_R$. Then,

$$\begin{aligned}|A\xi(\theta_2) - A\xi(\theta_1)| &= \Bigg| -\frac{1}{\Gamma(\zeta_2)}\int_0^{\theta_2} s^{\zeta_2-1}\wp(s)ds - \frac{1}{\Gamma(\zeta_2)}\int_{\theta_2}^{\varkappa} s^{\zeta_2-1}\wp(s)ds\\ &\quad + \frac{1}{\Gamma(\zeta_1)}\int_0^{\theta_2}(\theta_2-s)^{\zeta_1-1}\wp(s)ds + \frac{1}{\Gamma(\zeta_2)}\int_{\theta_2}^{\varkappa}(s-\theta_2)^{\zeta_2-1}\wp(s)ds\\ &\quad + \frac{1}{\Gamma(\zeta_2)}\int_0^{\theta_1} s^{\zeta_2-1}\wp(s)ds + \frac{1}{\Gamma(\zeta_2)}\int_{\theta_1}^{\varkappa} s^{\zeta_2-1}\wp(s)ds\\ &\quad - \frac{1}{\Gamma(\zeta_1)}\int_0^{\theta_1}(\theta_1-s)^{\zeta_1-1}\wp(s)ds - \frac{1}{\Gamma(\zeta_2)}\int_{\theta_1}^{\varkappa}(s-\theta_1)^{\zeta_2-1}\wp(s)ds\Bigg|\\ &\leq \frac{2}{\Gamma(\zeta_2)}\int_{\theta_1}^{\theta_2} s^{\zeta_2-1}|\wp(s)|ds\\ &\quad + \frac{1}{\Gamma(\zeta_1)}\int_0^{\theta_1}[(\theta_2-s)^{\zeta_1-1} - (\theta_1-s)^{\zeta_1-1}]|\wp(s)|ds\\ &\quad + \frac{1}{\Gamma(\zeta_1)}\int_{\theta_1}^{\theta_2}(\theta_2-s)^{\zeta_1-1}|\wp(s)|ds\\ &\quad + \frac{1}{\Gamma(\zeta_2)}\int_{\theta_2}^{\varkappa}[(s-\theta_2)^{\zeta_2-1} - (s-\theta_1)^{\zeta_2-1}]|\wp(s)|ds\\ &\quad + \frac{1}{\Gamma(\zeta_2)}\int_{\theta_1}^{\theta_2}(s-\theta_1)^{\zeta_2-1}|\wp(s)|ds\\ &\leq \frac{2(q_1^*+q_2^*R)}{(1-q_3^*)\Gamma(\zeta_2+1)}(\theta_2^{\zeta_2} - \theta_1^{\zeta_2}) + \frac{(q_1^*+q_2^*R)}{(1-q_3^*)\Gamma(\zeta_1+1)}(\theta_2^{\zeta_1} - \theta_1^{\zeta_1})\\ &\quad + \frac{(q_1^*+q_2^*R)}{(1-q_3^*)\Gamma(\zeta_1+1)}(\theta_2-\theta_1)^{\zeta_1}\\ &\quad + \frac{(q_1^*+q_2^*R)}{(1-q_3^*)\Gamma(\zeta_2+1)}[(\varkappa-\theta_2)^{\zeta_2} - (\varkappa-\theta_1)^{\zeta_2}]\\ &\quad + \frac{(q_1^*+q_2^*R)}{(1-q_3^*)\Gamma(\zeta_2+1)}(\theta_2-\theta_1)^{\zeta_2}.\end{aligned}$$

Then, when $\theta_1 \longrightarrow \theta_2$, the right-hand side of the inequality above tend to zero, therefore the operator A is equicontinuous. According to the Arzela-Ascoli theorem, the operator A is compact.

Step 4: A priori bounds.
We now show that there exists an open set $U \subseteq \Upsilon$, with $\xi \neq \varpi A\xi$, for $\varpi \in (0,1)$ and $\xi \in \partial U$. Let $\xi \in \Upsilon$ and $\xi = \varpi A\xi$ for some $0 < \varpi < 1$. Thus, for each $\theta \in [\varkappa, \varkappa+\delta]$, we have

$$\begin{aligned}|\xi(\theta)| &= |\varpi A\xi(\theta)| \\ &\leq \|\psi\|_{[\varkappa,\varkappa+\delta]}.\end{aligned}$$

If $\theta \in \Theta$, we have

$$\begin{aligned}\xi(\theta) &= \varpi A\xi(\theta) \\ &= \varpi\xi_0 + \frac{\varpi}{\Gamma(\zeta_2)}\int_0^{\varkappa} s^{\zeta_2-1}\wp(s)ds + \frac{\varpi}{\Gamma(\zeta_1)}\int_0^{\theta}(\theta-s)^{\zeta_1-1}\wp(s)ds \\ &\quad + \frac{\varpi}{\Gamma(\zeta_2)}\int_{\theta}^{\varkappa}(s-\theta)^{\zeta_2-1}\wp(s)ds.\end{aligned}$$

Then,

$$|\xi(\theta)| \leq |\xi_0| + \frac{2(q_1^*+q_2^*\|\xi\|_\Upsilon)\varkappa^{\zeta_2}}{(1-q_3^*)\Gamma(\zeta_2+1)} + \frac{(q_1^*+q_2^*\|\xi\|_\Upsilon)\varkappa^{\zeta_1}}{(1-q_3^*)\Gamma(\zeta_1+1)}.$$

Thus, for each $\theta \in [0, \varkappa+\delta]$, we have

$$\begin{aligned}\|\xi\|_\Upsilon &\leq \frac{\|\psi\|_{[\varkappa,\varkappa+\delta]} + |\xi_0| + \dfrac{2q_1^*\varkappa^{\zeta_2}}{(1-q_3^*)\Gamma(\zeta_2+1)} + \dfrac{q_1^*\varkappa^{\zeta_1}}{(1-q_3^*)\Gamma(\zeta_1+1)}}{1-\left[\dfrac{2q_2^*\varkappa^{\zeta_2}}{(1-q_3^*)\Gamma(\zeta_2+1)} + \dfrac{q_2^*\varkappa^{\zeta_1}}{(1-q_3^*)\Gamma(\zeta_1+1)}\right]} \\ &:= \kappa.\end{aligned}$$

Let

$$U = \{\xi \in \Upsilon, \|\xi\|_\Upsilon < \kappa+1\}.$$

Thus, by our choice of U, there is no $\xi \in \partial U$ such that $\xi = \varpi A\xi$ for $0 < \varpi < 1$. As consequence, from Leray-Schauder's fixed point theorem, we deduce that the operator A has at least one fixed point which is a solution of the problem (1)-(3). □

4. Ulam-Hyers Stability

In this section, we will establish the Ulam stability for the problem (1)-(3).

Definition 4.1 ([27, 22, 1])**.** *Problem (1)-(3) is Ulam-Hyers stable if there exists a real number $C_f > 0$ such that for each $\varepsilon > 0$ and for each solution $\xi \in \Upsilon$ of the inequality*

$$\left|{}_0^C D_{\varkappa}^{\zeta_1,\zeta_2;\gamma}\xi(\theta) - f(\theta,\xi(\theta), {}_0^C D_{\varkappa}^{\zeta_1,\zeta_2;\gamma}\xi(\theta))\right| < \varepsilon, \quad \theta \in \Theta, \tag{9}$$

there exists a solution $\bar{\xi} \in \Upsilon$ *of the problem* (1)-(3) *with*

$$|\xi(\theta) - \bar{\xi}(\theta))| < C_f \varepsilon, \quad \theta \in \Theta.$$

Definition 4.2 ([27, 22, 1]). *Problem* (1)-(3) *is generalized Ulam-Hyers stable if there exists* $\phi_f \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\phi_f(0) = 0$ *such that for each solution* $\xi \in \Upsilon$ *of the inequality* (9) *there exists a solution* $\bar{\xi} \in \Upsilon$ *of the problem* (1)-(3) *with*

$$|\xi(\theta) - \bar{\xi}(\theta))| < \phi_f \varepsilon, \quad \theta \in \Theta.$$

Remark 4.3. *A function* $\xi \in \Upsilon$ *is a solution of the inequality* (9) *if and only if there exists a function* $\ell \in C^1(\Theta, \mathbb{R})$ *(which depend on* ξ*) such that*

(1) $|\ell(\theta)| \leq \varepsilon, \quad for\ each\ \theta \in \Theta.$

(2) ${}_0^C D_{\varkappa}^{\zeta_1,\zeta_2;\gamma} \xi(\theta) = f(\theta, \xi^\theta, {}_0^C D_{\varkappa}^{\zeta_1,\zeta_2;\gamma} \xi(\theta)) + \ell(\theta), \quad for\ each\ \theta \in \Theta.$

Lemma 4.4. *The solution of the following perturbed problem*

$$\begin{aligned} {}_0^C D_{\varkappa}^{\zeta_1,\zeta_2;\gamma} \xi(\theta) &= f(\theta, \xi^\theta, {}_0^C D_{\varkappa}^{\zeta_1,\zeta_2;\gamma} \xi(\theta)) + \ell(\theta), \quad \theta \in \Theta, \\ \xi(0) &= \xi_0, \\ \xi(\theta) &= \psi(\theta), \quad \theta \in [\varkappa, \varkappa + \delta], \end{aligned}$$

is given by

$$\xi(\theta) = \begin{cases} \xi_0 - \dfrac{1}{\Gamma(\zeta_2)} \displaystyle\int_0^{\varkappa} s^{\zeta_2 - 1} \wp(s) ds + \dfrac{1}{\Gamma(\zeta_1)} \displaystyle\int_0^{\theta} (\theta - s)^{\zeta_1 - 1} \wp(s) ds \\ + \dfrac{1}{\Gamma(\zeta_2)} \displaystyle\int_\theta^{\varkappa} (s - \theta)^{\zeta_2 - 1} \wp(s) ds - \dfrac{1}{\Gamma(\zeta_2)} \displaystyle\int_0^{\varkappa} s^{\zeta_2 - 1} \ell(s) ds \\ + \dfrac{1}{\Gamma(\zeta_1)} \displaystyle\int_0^{\theta} (\theta - s)^{\zeta_1 - 1} \ell(s) ds + \dfrac{1}{\Gamma(\zeta_2)} \displaystyle\int_\theta^{\varkappa} (s - \theta)^{\zeta_2 - 1} \ell(s) ds, & if\ \theta \in \Theta, \\ \psi(\theta), \quad if\ \theta \in [\varkappa, \varkappa + \delta]. \end{cases}$$

Moreover, the solution satisfies the following inequality

$$\begin{aligned} &\left| \xi(\theta) - \left[\xi_0 - \frac{1}{\Gamma(\zeta_2)} \int_0^{\varkappa} s^{\zeta_2 - 1} \wp(s) ds + \frac{1}{\Gamma(\zeta_1)} \int_0^{\theta} (\theta - s)^{\zeta_1 - 1} \wp(s) ds \right.\right. \\ &\left.\left. + \frac{1}{\Gamma(\zeta_2)} \int_\theta^{\varkappa} (s - \theta)^{\zeta_2 - 1} \wp(s) ds \right] \right| \\ &\leq \left[\frac{2\varkappa^{\zeta_2}}{\Gamma(\zeta_2 + 1)} + \frac{\varkappa^{\zeta_1}}{\Gamma(\zeta_1 + 1)} \right] \varepsilon, \quad for\ each\ \theta \in \Theta, \end{aligned}$$

Theorem 4.5. *Assume that* $(B1)$-$(B2)$ *hold and that the condition (8) is verified. Then the problem* (1)-(3) *is Ulam-Hyers stable.*

Proof. Let $\xi \in \Upsilon$ be a solution of the inequality (9) and $\bar{\xi} \in \Upsilon$ a solution of the problem (1)-(3), then

$$\begin{aligned}
|\xi(\theta)-\bar{\xi}(\theta)| &= \Big|\xi(\theta)-\Big[\xi_0-\frac{1}{\Gamma(\zeta_2)}\int_0^{\varkappa}s^{\zeta_2-1}h(s)ds \\
&\quad +\frac{1}{\Gamma(\zeta_1)}\int_0^{\theta}(\theta-s)^{\zeta_1-1}h(s)ds+\frac{1}{\Gamma(\zeta_2)}\int_\theta^{\varkappa}(s-\theta)^{\zeta_2-1}h(s)ds\Big]\Big| \\
&\leq \Big|\xi(\theta)-\Big[\xi_0-\frac{1}{\Gamma(\zeta_2)}\int_0^{\varkappa}s^{\zeta_2-1}\wp(s)ds \\
&\quad +\frac{1}{\Gamma(\zeta_1)}\int_0^{\theta}(\theta-s)^{\zeta_1-1}\wp(s)ds+\frac{1}{\Gamma(\zeta_2)}\int_\theta^{\varkappa}(s-\theta)^{\zeta_2-1}\wp(s)ds\Big]\Big| \\
&\quad +\frac{1}{\Gamma(\zeta_2)}\int_0^{\varkappa}s^{\zeta_2-1}|\wp(s)-h(s)|ds \\
&\quad +\frac{1}{\Gamma(\zeta_1)}\int_0^{\theta}(\theta-s)^{\zeta_1-1}|\wp(s)-h(s)|ds \\
&\quad +\frac{1}{\Gamma(\zeta_2)}\int_\theta^{\varkappa}(s-\theta)^{\zeta_2-1}|\wp(s)-h(s)|ds.
\end{aligned}$$

By hypothesis $(B2)$, we have

$$|\wp(\theta)-h(s)| \leq \lambda_1\|\xi^\theta-\bar{\xi}^\theta\|_{[0,\delta]}+\lambda_2|\wp(\theta)-h(\theta)|.$$

Then,

$$|\wp(\theta)-h(\theta)| \leq \frac{\lambda_1}{1-\lambda_2}\|\xi^\theta-\bar{\xi}^\theta\|_{[0,\delta]}.$$

Thus,

$$\begin{aligned}
|\xi(\theta)-\bar{\xi}(\theta)| &\leq \left[\frac{2\varkappa^{\zeta_2}}{\Gamma(\zeta_2+1)}+\frac{\varkappa^{\zeta_1}}{\Gamma(\zeta_1+1)}\right]\varepsilon+\frac{\lambda_1\varkappa^{\zeta_2}}{(1-\lambda_2)\Gamma(\zeta_2+1)}\|\xi-\bar{\xi}\|_\Upsilon \\
&\quad +\frac{\lambda_1\varkappa^{\zeta_1}}{(1-\lambda_2)\Gamma(\zeta_1+1)}\|\xi-\bar{\xi}\|_\Upsilon+\frac{\lambda_1\varkappa^{\zeta_2}}{(1-\lambda_2)\Gamma(\zeta_2+1)}\|\xi-\bar{\xi}\|_\Upsilon \\
&\leq \left[\frac{2\varkappa^{\zeta_2}}{\Gamma(\zeta_2+1)}+\frac{\varkappa^{\zeta_1}}{\Gamma(\zeta_1+1)}\right]\varepsilon \\
&\quad +\left[\frac{2\lambda_1\varkappa^{\zeta_2}}{(1-\lambda_2)\Gamma(\zeta_2+1)}+\frac{\lambda_1\varkappa^{\zeta_1}}{(1-\lambda_2)\Gamma(\zeta_1+1)}\right]\|\xi-\bar{\xi}\|_\Upsilon.
\end{aligned}$$

Then,

$$\|\xi-\bar{\xi}\|_\Upsilon \leq \frac{\dfrac{2\varkappa^{\zeta_2}}{\Gamma(\zeta_2+1)}+\dfrac{\varkappa^{\zeta_1}}{\Gamma(\zeta_1+1)}}{1-\dfrac{2\lambda_1\varkappa^{\zeta_2}}{(1-\lambda_2)\Gamma(\zeta_2+1)}-\dfrac{\lambda_1\varkappa^{\zeta_1}}{(1-\lambda_2)\Gamma(\zeta_1+1)}}\varepsilon := C_f\varepsilon.$$

Consequently, the problem (1)-(3) is Ulam-Hyers stable. If we take $\phi_f(\varepsilon) = C_f \varepsilon$ and $\phi_f(0) = 0$, then we get the generalized Ulam-Hyers stability of the problem (1)-(3). □

5. Examples

Example 5.1. *Consider the following implicit problem which is an example of our problem* (1)-(3) *with Riesz-Caputo fractional derivative:*

$$ {}_0^C D_1^{\frac{1}{2},\frac{1}{2};\frac{1}{2}} \xi(\theta) = \frac{\|\xi^\theta\|_{[0,\delta]} + \left| {}_0^C D_1^{\frac{1}{2},\frac{1}{2};\frac{1}{2}} \xi(\theta) \right|}{100 e^{\sin(\theta)+1}}, \quad \theta \in [0,1], \tag{10}$$

$$ \xi(0) = 1, \tag{11}$$

$$ \xi(\theta) = \psi(\theta), \quad \theta \in [1,2], \tag{12}$$

where $\psi \in C([1,2],\mathbb{R})$.

Set

$$ f(\theta,\chi,\beta) = \frac{\|\chi\|_{[0,\delta]} + |\beta|}{100 e^{\sin(\theta)+1}}, \quad \theta \in [0,1],\ \chi \in C([0,\delta],\mathbb{R}), \beta \in \mathbb{R}.$$

Clearly, f is a continuous function, then the hypothesis $(B1)$ *is satisfied.*

For any $\chi, \bar{\chi} \in C([0,\delta],\mathbb{R})$, $\beta, \bar{\beta} \in \mathbb{R}$ *and* $\theta \in [0,1]$, *we have*

$$ |f(\theta,\chi,\beta) - f(\theta,\bar{\chi},\bar{\beta})| \leq \frac{1}{100e}[\|\chi - \bar{\chi}\|_{[0,\delta]} + |\beta - \bar{\beta}|],$$

then the assumption $(B2)$ *is satisfied with* $\lambda_1 = \lambda_2 = \frac{1}{100e}$. *Also we have*

$$\begin{aligned} \frac{2\lambda_1 \varkappa^{\zeta_2}}{(1-\lambda_2)\Gamma(\zeta_2+1)} + \frac{\lambda_1 \varkappa^{\zeta_1}}{(1-\lambda_2)\Gamma(\zeta_1+1)} &= \frac{2}{(100e-1)\frac{\sqrt{\pi}}{2}} + \frac{1}{(100e-1)\frac{\sqrt{\pi}}{2}} \\ &= \frac{3}{(100e-1)\frac{\sqrt{\pi}}{2}} \\ &\approx 0.0124992069352421 \\ &< 1, \end{aligned}$$

for $\varkappa = 1, \zeta_1 = \zeta_2 = \frac{1}{2}$ *and* $\gamma = \frac{1}{2}$. *It follows from Theorem 3.3 that the problem* (10)-(12) *has a unique solution on* $[0,1]$. *Moreover the conditions of Theorem 4.5 are verified then the problem* (10)-(12) *is Ulam-Hyers stable.*

Example 5.2. *Consider the following problem:*

$$ {}_0^C D_1^{\frac{1}{3},\frac{1}{4};\frac{1}{6}} \xi(\theta) = \frac{\frac{\pi}{2}\cos(\theta) + \frac{1}{2}\|\xi^\theta\|_{[0,\delta]} + 2\left| {}_0^C D_1^{\frac{1}{3},\frac{1}{4};\frac{1}{6}} \xi(\theta) \right|}{300\left(1 + \|\xi^\theta\|_{[0,\delta]} + \left| {}_0^C D_1^{\frac{1}{3},\frac{1}{4};\frac{1}{6}} \xi(\theta) \right|\right)}, \quad \theta \in [0,1], \tag{13}$$

$$\xi(0) = 1, \tag{14}$$

$$\xi(\theta) = \psi(\theta), \quad \theta \in [1,2], \tag{15}$$

where $\psi \in C([1,2],\mathbb{R})$.

Set

$$f(\theta,\chi,\beta) = \frac{\frac{\pi}{2}\cos(\theta) + \frac{1}{2}\|\chi\|_{[0,\delta]} + 2|\beta|}{300(1+\|\chi\|_{[0,\delta]} + |\beta|)}, \quad \theta \in [0,1],\ \chi \in C([0,\delta],\mathbb{R}), \beta \in \mathbb{R}.$$

Obviously, f is a continuous function, then the hypothesis $(B1)$ *is met.*

For any $\chi, \bar{\chi} \in C([0,\delta],\mathbb{R}), \beta, \bar{\beta} \in \mathbb{R}$ *and* $\theta \in [0,1]$, *we have*

$$|f(\theta,\chi,\beta) - f(\theta,\bar{\chi},\bar{\beta})| \leq \frac{1}{300}\left[\frac{1}{2}\|\chi - \bar{\chi}\|_{[0,\delta]} + 2|\beta - \bar{\beta}|\right].$$

Then, the hypothesis $(B2)$ *is verified with* $\lambda_1 = \frac{1}{600}$ *and* $\lambda_2 = \frac{2}{300}$. *Also we have*

$$|f(\theta,\chi,\beta)| \leq \frac{\frac{\pi}{2}cos(\theta)}{300} + \frac{1}{300}\left(\frac{1}{2}|\chi| + 2|\beta|\right).$$

So $q_1(\theta) = \frac{\frac{\pi}{2}cos(\theta)}{300}$, $q_2^* = \frac{1}{600}$ *and* $q_3^* = \frac{2}{300}$.

And as

$$\frac{2q_2^*\varkappa^{\zeta_2}}{(1-q_3^*)\Gamma(\zeta_2+1)} + \frac{q_2^*\varkappa^{\zeta_1}}{(1-q_3^*)\Gamma(\zeta_1+1)} \approx 0.00339834257128931 < 1,$$

for $\varkappa = 1, \zeta_1 = \frac{1}{3}, \zeta_2 = \frac{1}{4}$ *and* $\gamma = \frac{1}{6}$. *Then, Theorem 3.5 assures that the problem* (13)-(15) *has at least one solution on* $[0,1]$. *Moreover*

$$\begin{aligned}\frac{2\lambda_1\varkappa^{\zeta_2}}{(1-\lambda_2)\Gamma(\zeta_2+1)} + \frac{\lambda_1\varkappa^{\zeta_1}}{(1-\lambda_2)\Gamma(\zeta_1+1)} &= \frac{1}{298\Gamma(\frac{5}{4})} + \frac{\frac{1}{2}}{298\Gamma(\frac{4}{3})} \\ &\approx 0.02039005542 \\ &< 1,\end{aligned}$$

then by Theorem 4.5, we can deduce that our problem is Ulam-Hyers stable.

Conclusion

In the present research, we have investigated existence and uniqueness results for a class of initial value problems for implicit nonlinear fractional differential equations and combined Caputo fractional derivative with advanced arguments. The fixed-point technique, namely the Banach contraction principle and nonlinear alternative of Leray-Schauder fixed point theorem, was employed to reach the necessary outcomes for the given problem. Also, we dedicated a section to the study of the Ulam stability for problem (1)-(3). Illustrations are presented to show how the primary findings may be implemented. Our results in the provided context are novel and add significantly to the literature on this emerging topic of research. Due to the small amount of publications on implicit combined Caputo fractional differential equations, we believe there are several possible study paths such as coupled systems, problems with infinite delays, and many more.

Declarations

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

Competing interests It is declared that authors have no competing interests.

Author's contributions The study was carried out in collaboration of all authors. All authors read and approved the final manuscript.

Funding Not available.

Availability of data and materials Data sharing not applicable to this paper as no data sets were generated or analyzed during the current study.

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