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HYERS-ULAM-RASSIAS STABILITY FOR NONAUTONOMOUS DYNAMICS

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ABSTRACT. We formulate sufficient conditions under which a nonautonomous dynamics exhibits Hyers–Ulam-Rassias stability. These conditions require that the linear part is exponentially stable and that the nonlinear part is Lipschitz small. We consider both the case of continuous and discrete time dynamics.

1. Introduction

In a recent years many works have been devoted to the study of the so-called Hyers-Ulam stability for various classes of differential and difference equations. Roughly speaking, we say that a given differential (or difference equation) exhibits Hyers-Ulam stability if in a neighborhood of its approximate solution we can find an exact solution. The importance of this concept stems from a simple observation that any numerical method for solving a given differential equation will result only with an approximate solution. Therefore, it is natural to ask whether in a vicinity of the constructed approximate solution of our equation we can find its true solution.

We stress that there is a vast literature devoted to the study of the Hyers-Ulam stability. The goal of those works is to formulate sufficient conditions under which a given class of differential (or difference) equations exhibits Hyers-Ulam stability. In particular, we mention the works of Anderson and Onitsuka [1], Barbu et al. [7, 8], Bernardes Jr. et al. [10], Buşe et al. [11, 12], Chen et al. [13], Fukutaka and Onitsuka [17, 18, 19], Popa and Raşa [21, 22] as well as Wang et al. [23, 24]. Furthermore, we point out to the works [2, 3, 4, 5, 6, 16] where the authors have formulated very general conditions under which a semilinear differential or difference equations exhibits Hyers-Ulam stability. These conditions require that the linear part is hyperbolic (i.e. that it admits exponential dichotomy or more generally exponential trichotomy), and that the nonlinear term is Lipschitz with a sufficiently small Lipschitz constant.

More recently, several authors (see [9, 14, 15, 20] and references therein) studied a more general concept of Hyers-Ulam-Rassias stability. In order to describe the difference between Hyers-Ulam and Hyers-Ulam-Rassias stability, let us consider a semilinear differential equation

$$x' = A(t)x + f(t,x) \quad t \in \mathbb{R}^+, \tag{1}$$

where $A(t)$ is a linear operator on $\mathbb{R}^n$ for each $t \in \mathbb{R}^+$ and $f\colon \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear map. We assume that $t \mapsto A(t)$ and f are continuous. In the classical notion of Hyers-Ulam stability, a

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pseudo-solution of (1) is any continuously differentiable map $y\colon \mathbb{R}^+ \to \mathbb{R}^n$ with the property that there is $\varepsilon > 0$ such that

$$\sup_{t\in\mathbb{R}^+} |y'(t) - A(t)y(t) - f(t,y(t))| \le \varepsilon. \tag{2}$$

Moreover, we say that (1) is Hyers-Ulam stable if there exists $L > 0$ such that for each pseudo-solution $y\colon \mathbb{R}^+ \to \mathbb{R}^n$ satisfying (2), there exists a solution $x\colon \mathbb{R}^+ \to \mathbb{R}^n$ of (1) with the property that

$$\sup_{t\in\mathbb{R}^+} |x(t) - y(t)| \le L\varepsilon. \tag{3}$$

On the other hand, in the notion of Hyers-Ulam-Rassias stability, the condition (2) is replaced by the following more general requirement:

$$|y'(t) - A(t)y(t) - f(t,y(t))| \le \varepsilon\varphi(t) \quad t \in \mathbb{R}^+, \tag{4}$$

where $\varphi\colon \mathbb{R}^+ \to (0,\infty)$ belongs to a certain class of functions. In addition, we say that (1) is Hyers-Ulam-Rassias stable (relative to φ) if there exists $L > 0$ such that for each pseudo-solution $y\colon \mathbb{R}^+ \to \mathbb{R}^n$ satisfying (4), there exists a solution $x\colon \mathbb{R}^+ \to \mathbb{R}^n$ of (1) with the property that

$$|x(t) - y(t)| \le L\varepsilon\varphi(t), \quad t \in \mathbb{R}^+. \tag{5}$$

Clearly, the notion of Hyers-Ulam stability is a particular case of the notion of Hyers-Ulam-Rassias stability and it corresponds to the case when $\varphi \equiv 1$. The motivation behind the study of Hyers-Ulam-Rassias stability stems from the following observation: when constructing an approximate solution of a given differential equation, we might have either a weaker or stronger control on the error at certain times. Consequently, it is not expected that we can find an exact solution satisfying a uniform estimate as in (3).

The main objective of the present paper is to formulate sufficient conditions under which (1) (and its discrete time counterpart) exhibits Hyers-Ulam-Rassias stability relative to a large class of functions φ. These conditions require that the linear part of (1) is exponentially stable and that the nonlinear term f is Lipschitz in x with a sufficiently small Lipschitz constant. We stress that we consider (1) (and its discrete time counterpart) posed on an arbitrary Banach space.

2. The case of continuous time

2.1. *Preliminaries.* Let $X = (X, |\cdot|)$ be an arbitrary Banach space. By $\mathscr{B}(X)$ we denote the space of all bounded linear operators on X equipped with the operator norm $\|\cdot\|$. Let $A\colon [0,\infty) \to \mathscr{B}(X)$ be a continuous map and consider the associated linear differential equation

$$x' = A(t)x, \quad t \ge 0. \tag{6}$$

We recall the notion of exponential stability.

Definition 1. *We say that* (6) *is* exponentially stable *if there exist constants $D, \lambda > 0$ such that:*

$$\|T(t,s)\| \le De^{-\lambda(t-s)} \quad \textit{for } t \ge s \ge 0, \tag{7}$$

where $T(t,s)$ denotes the evolution family of (6).

Remark 1. *We recall that when $A(t)=A$ for $t\geq 0$, we have that (6) is exponentially stable if and only if*

$$\sigma(A)\subset\{\lambda\in\mathbb{C}:\Re(\lambda)<0\},$$

where $\sigma(A)$ denotes the spectrum of A.

Let $f\colon[0,\infty)\times X\to X$ be a continuous map. We consider the semilinear differential equation given by

$$x'=A(t)x+f(t,x),\quad t\geq 0. \tag{8}$$

Observe that when $f\equiv 0$, (8) reduces to (6). Thus, we can view (8) as a perturbation of (6). We will assume that f is Lipschitz in x, i.e. that there exists $c\in[0,\infty)$ such that

$$|f(t,x)-f(t,y)|\leq c|x-y|,\quad \text{for } t\geq 0 \text{ and } x,y\in X. \tag{9}$$

2.2. *Hyers-Ulam-Rassias stability.*

We are now in a position to state our first result.

Theorem 1. *Assume that (6) is exponentially stable and that there exists $c\in[0,\infty)$ such that (9) holds. Furthermore, suppose that*

$$\frac{cD}{\lambda}<1, \tag{10}$$

where $D,\lambda>0$ are such that (7) holds. Then there exists $L>0$ such that for each continuous and increasing function $\varphi\colon[0,\infty)\to(0,\infty)$, $\varepsilon>0$, and a continuously differentiable function $y\colon[0,\infty)\to X$ satisfying

$$|y'(t)-A(t)y(t)-f(t,y(t))|\leq\varepsilon\varphi(t)\quad \textit{for } t\geq 0, \tag{11}$$

there exists a solution $x\colon[0,\infty)\to X$ of (8) such that $x(0)=y(0)$ and

$$|x(t)-y(t)|\leq L\varepsilon\varphi(t),\quad t\geq 0. \tag{12}$$

Proof. Set $L:=\frac{D}{\lambda-cD}$ so that

$$\frac{D}{\lambda}+\frac{cD}{\lambda}L=L. \tag{13}$$

Let $\mathscr{Y}$ denote the space of all continuous functions $z\colon[0,\infty)\to X$ such that $z(0)=0$ and

$$\|z\|_\varphi:=\sup_{t\geq 0}\left(\frac{1}{\varphi(t)}|z(t)|\right)<+\infty.$$

It is straighforward to verify that $(\mathscr{Y},\|\cdot\|_\varphi)$ is a Banach space. For $z\in\mathscr{Y}$, set

$$(\Gamma z)(t)=\int_0^t T(t,s)(f(s,y(s)+z(s))+A(s)y(s)-y'(s))\,ds,\quad t\geq 0.$$

Observe that it follows from (7) and (11) (recall also that φ is increasing) that

$$\begin{aligned}|(\Gamma 0)(t)| &\le \int_0^t |T(t,s)(f(s,y(s))+A(s)y(s)-y'(s))|\,ds \\ &\le \int_0^t \|T(t,s)\|\cdot|f(s,y(s))+A(s)y(s)-y'(s)|\,ds \\ &\le D\varepsilon\int_0^t e^{-\lambda(t-s)}\varphi(s)\,ds \\ &\le D\varepsilon\varphi(t)\int_0^t e^{-\lambda(t-s)}\,ds \\ &\le \frac{D\varepsilon}{\lambda}\varphi(t),\end{aligned}$$

for $t\ge 0$. Thus, $\Gamma 0\in\mathscr{Y}$ and

$$\|\Gamma 0\|_\varphi \le \frac{D\varepsilon}{\lambda}. \tag{14}$$

On the other hand, for $z\in\mathscr{Y}$, it follows from (9) and (11) that

$$\begin{aligned}|f(s,y(s)+z(s))+A(s)y(s)-y'(s)| &\le |f(s,y(s)+z(s))-f(s,y(s))| \\ &\quad +|f(s,y(s))+A(s)y(s)-y'(s)| \\ &\le c|z(s)|+\varepsilon\varphi(s),\end{aligned}$$

for $s\ge 0$. Hence,

$$\begin{aligned}|(\Gamma z)(t)| &\le \int_0^t |T(t,s)(f(s,y(s)+z(s))+A(s)y(s)-y'(s))|\,ds \\ &\le D\int_0^t e^{-\lambda(t-s)}(c|z(s)|+\varepsilon\varphi(s))\,ds \\ &\le cD\|z\|_\varphi\int_0^t e^{-\lambda(t-s)}\varphi(s)\,ds + D\varepsilon\int_0^t e^{-\lambda(t-s)}\varphi(s)\,ds \\ &\le D\varphi(t)\big(c\|z\|_\varphi+\varepsilon\big)\int_0^t e^{-\lambda(t-s)}\,ds \\ &\le \frac{D}{\lambda}\varphi(t)\big(c\|z\|_\varphi+\varepsilon\big),\end{aligned}$$

for $t\ge 0$. In particular,

$$\|\Gamma z\|_\varphi \le \frac{D}{\lambda}\big(c\|z\|_\varphi+\varepsilon\big) < +\infty.$$

Since $(\Gamma z)(0)=0$, we conclude that $\Gamma z\in\mathscr{Y}$. We have showed that $\Gamma(\mathscr{Y})\subset\mathscr{Y}$.

Take now $z_1, z_2 \in \mathscr{Y}$. For $t \geq 0$, we have that

$$\begin{aligned}
|(\Gamma z_1)(t) - (\Gamma z_2)(t)| &\leq \int_0^t |T(t,s)(f(s,y(s)+z_1(s)) - f(s,y(s)+z_2(s)))|\,ds \\
&\leq cD \int_0^t e^{-\lambda(t-s)}|z_1(s) - z_2(s)|\,ds \\
&\leq cD\|z_1 - z_2\|_\varphi \int_0^t e^{-\lambda(t-s)}\varphi(s)\,ds \\
&\leq \frac{cD}{\lambda}\varphi(t)\|z_1 - z_2\|_\varphi.
\end{aligned}$$

Therefore,

$$\|\Gamma z_1 - \Gamma z_2\|_\varphi \leq \frac{cD}{\lambda}\|z_1 - z_2\|_\varphi. \tag{15}$$

Set

$$\mathscr{Z} := \left\{ z \in \mathscr{Y} : \|z\|_\varphi \leq L\varepsilon \right\}.$$

For $z \in \mathscr{Z}$, by (13), (14) and (15) we have that

$$\|\Gamma z\|_\varphi \leq \|\Gamma 0\|_\varphi + \|\Gamma z - \Gamma 0\|_\varphi \leq \left(\frac{D}{\lambda} + \frac{cD}{\lambda}L\right)\varepsilon = L\varepsilon.$$

Consequently, $\Gamma(\mathscr{Z}) \subset \mathscr{Z}$. From (10) and (15), we conclude that Γ is a contraction on $\mathscr{Z}$. Hence, it has a unique fixed point $z \in \mathscr{Z}$. Thus,

$$z(t) = \int_0^t T(t,s)(f(s,y(s)+z(s)) + A(s)y(s) - y'(s))\,ds, \quad t \geq 0.$$

By differentiating, we obtain that

$$z'(t) = A(t)z(t) + f(t,y(t)+z(t)) + A(t)y(t) - y'(t), \quad t \geq 0. \tag{16}$$

Set $x := y + z$. It follows from (16) that x is a solution of (8). Since $z \in \mathscr{Y}$, we have that $z(0) = 0$ and thus $x(0) = y(0)$. Finally, note that

$$\frac{1}{\varphi(t)}|x(t) - y(t)| = \frac{1}{\varphi(t)}|z(t)| \leq \|z\|_\varphi \leq L\varepsilon \quad t \geq 0,$$

which yields (12). The proof of the theorem is completed. □

As a direct consequence of Theorem 1, we obtain the following corollary.

Corollary 1. *Assume that (6) is exponentially stable and that there exists $c \in [0,\infty)$ such that (9) and (10) hold. Then there exists $L > 0$ such that for each $\varepsilon > 0$, and a continuously differentiable function $y\colon [0,\infty) \to X$ satisfying*

$$|y'(t) - A(t)y(t) - f(t,y(t))| \leq \varepsilon \quad \textit{for } t \geq 0,$$

there exists a solution $x\colon [0,\infty) \to X$ of (8) such that $x(0) = y(0)$ and

$$|x(t) - y(t)| \leq L\varepsilon, \quad t \geq 0.$$

Proof. The desired conclusion follows readily from Theorem 1 applied to the case when $\varphi \equiv 1$. □

2.3. *Examples.* Let us illustrate Theorem 1 on some particular examples.

Example 1. *Let $X=\mathbb{R}$ and $\lambda>0$ or $X=\mathbb{C}$ and $\lambda\in\mathbb{C}$ such that $\Re(\lambda)>0$. Moreover, take a continuous function $r\colon[0,\infty)\to X$. We consider the equation:*

$$x'(t)+\lambda x(t)=r(t), \quad t\geq 0. \tag{17}$$

Observe that (17) is a particular case of (8) when $A(t)=-\lambda$ and $f(t,x)=r(t)$. Clearly, the corresponding linear equation (6) is exponentially stable (see Remark 1). Moreover, (9) is satisfied with $c=0$. In particular, (10) holds. We conclude that Theorem 1 is applicable to (17).

In this particular case, the result similar to Theorem 1 was obtained in [20, Theorem 8]. *We note that in* [20, Theorem 8] *it is required that y from the statement of Theorem 1 is the function of an exponential order, and it is proved that x also exhibits such property. On the other hand, our Theorem 1 does not require that y is of exponential order.*

Let us now consider (17) in the particular case when $\lambda=1$ and $r\equiv 0$. Moreover, let $y\colon[0,\infty)\to\mathbb{R}$ be given by $y(t)=t^2-2t+2+\sin t$, $t\geq 0$. Then, y is continuously differentiable and $|y'(t)+y(t)|=|t^2+\sin t+\cos t|\leq t^2+2$ for $t\geq 0$. Hence, y satisfies (11) with $\varepsilon=1$ and $\varphi(t)=t^2+2$ for $t\geq 0$. Observe that $\varphi\colon[0,\infty)\to(0,\infty)$ is continuous and increasing. It follows from Theorem 1 that there exists a differentiable function $x\colon[0,\infty)\to\mathbb{R}$ such that $x(0)=y(0)=2$, $x'(t)+x(t)=0$ and $|x(t)-y(t)|\leq t^2+2$ for $t\geq 0$ (note that $L=1$ in this case). Clearly, $x(t)=2e^{-t}$ for $t\geq 0$.

Example 2. *Let $X=\mathbb{R}$ and $\lambda>0$. We consider the equation:*

$$x'(t)+\lambda x(t)=\frac{\lambda}{2}\sin(x(t)), \quad t\geq 0. \tag{18}$$

Observe that (18) is a particular case of (8) when $A(t)=-\lambda$ and $f(t,x)=\frac{\lambda}{2}\sin(x)$. Then (6) is exponentially stable and we have that $D=1$ in (7). Moreover, f satisfies (9) with $c=\frac{\lambda}{2}$. Hence, (10) holds and Theorem 1 is applicable. We note that the results of [20] *are not applicable to (18) since f depends on x.*

Example 3. *Let us now illustrate how Theorem 1 can be applied to equations of higher order. Take $a_0,a_1\in\mathbb{R}$ and a continuous function $r\colon[0,\infty)\to\mathbb{R}$. We consider the equation:*

$$x''(t)+a_1x'(t)+a_0x(t)=r(t), \quad t\geq 0. \tag{19}$$

Set $X=\mathbb{R}^2$ equipped with the norm $|(x_1,x_2)|=\max\{|x_1|,|x_2|\}$ for $(x_1,x_2)\in X$. For $t\geq 0$, let

$$A(t)=A=\begin{pmatrix}-a_1 & -a_0\\ 1 & 0\end{pmatrix}.$$

Furthermore, we define $f\colon[0,\infty)\to\mathbb{R}^2$ by $f(t)=(r(t),0)$, $t\geq 0$. We consider the associated semilinear equation (8). Suppose that both solutions of the equation $\lambda^2+a_1\lambda+a_0=0$ belong to $\{\lambda\in\mathbb{C}:\Re(\lambda)<0\}$. Then (6) is exponentially stable. Moreover, (9) holds with $c=0$, and thus (10) is satisfied. Let $L>0$ be given by Theorem 1.

Take $\varphi\colon[0,\infty)\to(0,\infty)$ be an increasing function, $\varepsilon>0$ and a twice continuously differentiable $z\colon[0,\infty)\to\mathbb{R}$ such that

$$|z''(t)+a_1z'(t)+a_0z(t)-r(t)|\leq\varepsilon\varphi(t), \quad t\geq 0.$$

Set $y(t)=(z'(t),z(t))$, $t\ge 0$. Then (11) holds. Hence, there exists a solution $x\colon [0,\infty)\to \mathbb{R}^2$ of (8) such that

$$|x(t)-y(t)|\le L\varepsilon\varphi(t), \quad t\ge 0.$$

Write $x=(x_1,x_2)$. Then $x_2\colon [0,\infty)\to\mathbb{R}$ is a solution of (19) and

$$|x_2(t)-z(t)|\le L\varepsilon\varphi(t), \quad t\ge 0.$$

3. The case of discrete time

The goal of this section is to formulate the version of Theorem 1 for the case of discrete time. Let X and $\mathscr{B}(X)$ be as in the previous section. Let $(A_n)_{n\in\mathbb{Z}^+}$ be a sequence in $\mathscr{B}(X)$. We consider the associated linear difference equation given by

$$x_{n+1}=A_nx_n, \quad n\in\mathbb{Z}^+. \tag{20}$$

Definition 2. *We say that (20) is* exponentially stable *if there exist $D,\lambda>0$ such that*

$$\|\mathscr{A}(m,n)\|\le De^{-\lambda(m-n)} \quad m\ge n, \tag{21}$$

where

$$\mathscr{A}(m,n)=\begin{cases} A_{m-1}\cdots A_n & m>n;\\ Id & m=n.\end{cases}$$

Here, Id denotes the identity operator on X.

Remark 2. *When $A_n=A$ for $n\in\mathbb{Z}^+$, we have that (20) is exponentially stable if and only if*

$$\sigma(A)\subset\{\lambda\in\mathbb{C}:|\lambda|<1\}.$$

Let $f_n\colon X\to X$, $n\in\mathbb{Z}^+$ be a sequence of nonlinear maps. We consider the associated nonlinear difference equation given by

$$x_{n+1}=A_nx_n+f_n(x_n), \quad n\in\mathbb{Z}^+. \tag{22}$$

We will assume that there exists $c\in[0,\infty)$ such that

$$|f_n(x)-f_n(y)|\le c|x-y|, \quad \text{for } x,y\in X \text{ and } n\in\mathbb{Z}^+. \tag{23}$$

The following is a version of Theorem 1 for the case of discrete time.

Theorem 2. *Assume that (20) is exponentially stable and that there exists $c\in[0,\infty)$ such that (23) holds. Furthermore, suppose that*

$$\frac{cD}{1-e^{-\lambda}}<1, \tag{24}$$

where $D,\lambda>0$ are such that (21) holds. Then there exists $L>0$ such that for each increasing function $\varphi\colon\mathbb{N}\to(0,\infty)$, $\varepsilon>0$, and a sequence $(y_n)_{n\in\mathbb{Z}^+}\subset X$ satisfying

$$|y_{n+1}-A_ny_n-f_n(y_n)|\le\varepsilon\varphi(n+1) \quad \text{for } n\in\mathbb{Z}^+, \tag{25}$$

there exists a sequence $(x_n)_{n\in\mathbb{Z}^+}\subset X$ which solves (22) such that $x_0=y_0$ and

$$|x_n-y_n|\le L\varepsilon\varphi(n), \quad n\in\mathbb{N}. \tag{26}$$

Proof. Set $L := \frac{D}{1-e^{-\lambda}-cD} > 0$ so that

$$\frac{D}{1-e^{-\lambda}} + L\frac{cD}{1-e^{-\lambda}} = L. \tag{27}$$

Let $\mathscr{Y}'$ denote the space of all sequences $\mathbf{z} = (z_n)_{n\in\mathbb{Z}^+} \subset X$ such that $z_0 = 0$ and

$$\|\mathbf{z}\|'_\varphi := \sup_{n\in\mathbb{N}} \left(\frac{1}{\varphi(n)}|z_n|\right) < +\infty.$$

Then $(\mathscr{Y}', \|\cdot\|'_\varphi)$ is a Banach space. For $\mathbf{z} = (z_n)_{n\in\mathbb{Z}^+} \in \mathscr{Y}'$, set $(\Gamma'\mathbf{z})_0 = 0$ and

$$(\Gamma'\mathbf{z})_n := \sum_{k=1}^{n} \mathscr{A}(n,k)(f_{k-1}(y_{k-1}+z_{k-1}) + A_{k-1}y_{k-1} - y_k), \quad n \in \mathbb{N}.$$

It follows from (21) and (25) that

$$\begin{aligned}
|(\Gamma'\mathbf{0})_n| &\le \sum_{k=1}^{n} |\mathscr{A}(n,k)(f_{k-1}(y_{k-1}) + A_{k-1}y_{k-1} - y_k)| \\
&\le D\sum_{k=1}^{n} e^{-\lambda(n-k)}|f_{k-1}(y_{k-1}) + A_{k-1}y_{k-1} - y_k| \\
&\le D\sum_{k=1}^{n} e^{-\lambda(n-k)}\varphi(k) \\
&\le D\varphi(n)\sum_{k=1}^{n} e^{-\lambda(n-k)} \\
&\le \frac{D}{1-e^{-\lambda}}\varphi(n),
\end{aligned}$$

for $n \in \mathbb{N}$. Thus,

$$\|\Gamma'\mathbf{0}\|'_\varphi \le \frac{D}{1-e^{-\lambda}}. \tag{28}$$

On the other hand, for $\mathbf{z} = (z_n)_{n\in\mathbb{Z}^+} \in \mathscr{Y}'$, it follows from (23) and (25) that

$$\begin{aligned}
|f_{k-1}(y_{k-1}+z_{k-1}) + A_{k-1}y_{k-1} - y_k| &\le |f_{k-1}(y_{k-1}+z_{k-1}) - f_{k-1}(y_{k-1})| \\
&\quad + |f_{k-1}(y_{k-1}) + A_{k-1}y_{k-1} - y_k| \\
&\le c|z_{k-1}| + \varepsilon\varphi(k),
\end{aligned}$$

for $k \in \mathbb{N}$. Hence,

$$\begin{aligned}|(\Gamma'\mathbf{z})_n| &\le \sum_{k=1}^n |\mathscr{A}(n,k)(f_{k-1}(y_{k-1})+A_{k-1}y_{k-1}-y_k)| \\ &\le D\sum_{k=1}^n e^{-\lambda(n-k)}(c|z_{k-1}|+\varepsilon\varphi(k)) \\ &\le cD\|\mathbf{z}\|'_\varphi \sum_{k=1}^n e^{-\lambda(n-k)}\varphi(k-1)+D\varepsilon\sum_{k=1}^n e^{-\lambda(n-k)}\varphi(k) \\ &\le \frac{D}{1-e^{-\lambda}}\varphi(n)\left(c\|\mathbf{z}\|'_\varphi+\varepsilon\right),\end{aligned}$$

for $n \in \mathbb{N}$. In particular,

$$\|\Gamma'\mathbf{z}\|'_\varphi \le \frac{D}{1-e^{-\lambda}}\left(c\|\mathbf{z}\|'_\varphi+\varepsilon\right) < +\infty.$$

Thus, $\Gamma'\mathbf{z} \in \mathscr{Y}'$. We conclude that $\Gamma'(\mathscr{Y}') \subset \mathscr{Y}'$.

Take now $\mathbf{z}^1 = (z_n^1)_{n\in\mathbb{Z}^+}, \mathbf{z}^2 = (z_n^2)_{n\in\mathbb{Z}^+} \in \mathscr{Y}'$. For $n \in \mathbb{N}$, we have that

$$\begin{aligned}|(\Gamma'\mathbf{z}^1)_n-(\Gamma\mathbf{z}^2)_n| &\le \sum_{k=1}^n |\mathscr{A}(n,k)(f_{k-1}(y_{k-1}+z_{k-1}^1)-f_{k-1}(y_{k-1}+z_{k-1}^2))| \\ &\le cD\sum_{k=1}^n e^{-\lambda(n-k)}|z_{k-1}^1-z_{k-1}^2| \\ &\le cD\|\mathbf{z}^1-\mathbf{z}^2\|'_\varphi\sum_{k=1}^n e^{-\lambda(n-k)}\varphi(k-1) \\ &\le \frac{cD}{1-e^{-\lambda}}\varphi(n)\|\mathbf{z}^1-\mathbf{z}^2\|'_\varphi.\end{aligned}$$

Therefore,

$$\|\Gamma'\mathbf{z}^1-\Gamma'\mathbf{z}^2\|'_\varphi \le \frac{cD}{1-e^{-\lambda}}\|\mathbf{z}^1-\mathbf{z}^2\|'_\varphi. \tag{29}$$

Set

$$\mathscr{Z}' := \left\{\mathbf{z} \in \mathscr{Y}' : \|\mathbf{z}\|'_\varphi \le L\varepsilon\right\}.$$

It follows easily from (27), (28) and (29) that $\Gamma'(\mathscr{Z}') \subset \mathscr{Z}'$. Hence, Γ' has a unique fixed point $\mathbf{z} = (z_n)_{n\in\mathbb{Z}^+} \in \mathscr{Z}'$. Thus,

$$z_n = \sum_{k=1}^n \mathscr{A}(n,k)(f_{k-1}(y_{k-1}+z_{k-1})+A_{k-1}y_{k-1}-y_k), \quad n \in \mathbb{N}.$$

From this it follows that

$$\begin{aligned} z_{n+1}-A_nz_n &= \sum_{k=1}^{n+1}\mathscr{A}(n+1,k)(f_{k-1}(y_{k-1}+z_{k-1})+A_{k-1}y_{k-1}-y_k) \\ &\quad -\sum_{k=1}^{n}\mathscr{A}(n+1,k)(f_{k-1}(y_{k-1}+z_{k-1})+A_{k-1}y_{k-1}-y_k) \\ &= f_n(y_n+z_n)+A_ny_n-y_{n+1}, \end{aligned}$$

and therefore

$$y_{n+1}+z_{n+1}=A_n(y_n+z_n)+f_n(y_n+z_n), \quad n\in\mathbb{Z}^+. \tag{30}$$

Let $x_n:=y_n+z_n$, $n\in\mathbb{Z}^+$. By (30), we have that $(x_n)_{n\in\mathbb{Z}^+}$ is a solution of (22). Finally, we have that

$$\frac{1}{\varphi(n)}|x_n-y_n|\le\|\mathbf{z}\|'_\varphi\le L\varepsilon \quad \text{for } n\in\mathbb{N},$$

which implies (26). The proof of the theorem is completed. □

Corollary 2. *Assume that (20) is exponentially stable and that there exists $c\in[0,\infty)$ such that (23) holds. Furthermore, suppose that (24) holds. Then there exists $L>0$ such that for each $\varepsilon>0$ and a sequence $(y_n)_{n\in\mathbb{Z}^+}\subset X$ satisfying*

$$|y_{n+1}-A_ny_n-f_n(y_n)|\le\varepsilon \quad \textit{for } n\in\mathbb{Z}^+,$$

there exists a sequence $(x_n)_{n\in\mathbb{Z}^+}\subset X$ which solves (22) such that $x_0=y_0$ and

$$|x_n-y_n|\le L\varepsilon, \quad n\in\mathbb{N}.$$

Proof. We apply Theorem 2 for $\varphi\equiv 1$. □

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