THREE SOLUTIONS TO A ROBIN PROBLEM WITH TWO WEIGHTED GENERALIZED VARIABLE EXPONENT SOBOLEV SPACES

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ABSTRACT. By applying Ricceri's variational principle, we demonstrate the existence of solutions for a double weighted Robin problem in weighted variable exponent Sobolev spaces under some appropriate conditions.

1. Introduction. Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded smooth domain. Assume that ω_1 and ω_2 are weight functions. The aim of this study is to discuss the three solutions for the following Robin problem

$$(1) \qquad \begin{cases} -\operatorname{div}\left(\omega_1(x)\left|\nabla u\right|^{p(x)-2}\nabla u\right) = \lambda\omega_2(x)f(x,u), & x \in \Omega \\ \omega_1(x)\left|\nabla u\right|^{p(x)-2}\frac{\partial u}{\partial v} + \beta(x)\left|u\right|^{p(x)-2}u = 0, & x \in \partial\Omega, \end{cases}$$

where $\frac{\partial u}{\partial v}$ is the outer unit normal derivative of u with respect to $\partial\Omega$, $\lambda > 0$, $p, q \in C\left(\overline{\Omega}\right)$ with $\inf_{x\in\overline{\Omega}} p(x) > 1$, and $\beta \in L^\infty\left(\partial\Omega\right)$ such that $\beta^- = \inf_{x\in\partial\Omega}\beta(x) > 0$.

Discrete nonlinear equations with parameter dependence play an important role in describing many physical problems, such as nonlinear elasticity theory or mechanics and engineering topics. In recent years, the investigating of the existence of weak solutions of partial differential equations involving weighted $p(.)$-Laplacian in variable exponent (weighted or unweighted) Sobolev spaces have been very popular, see **[4]**, **[6]**, **[12]**, **[16]**, **[17]**, **[26]**, **[35]**. Because some such type of equations can explain several physical problems such as electrorheological fluids, image processing, elastic mechanics, fluid dynamics and calculus of variations, see **[19]**, **[27]**, **[29]**, **[36]**, **[37]**.

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In many applications, we might encounter boundary value elliptic problems whose ellipticity is "disturbed" in the sense that some degeneration or singularity appears. This "unpleasant" or "undesirable" behaviour can be caused by the coefficients (weights) of the corresponding differential operator. For degenerate partial differential equations, in other words, equations with various types of singularities in the coefficients (weights), it should be look for solutions in weighted Sobolev spaces, see **[8]**, **[9]**, **[20]**, **[34]**.

The Robin problem involving $p(.)$-Laplacian was studied by several authors, see **[1]**, **[10]**, **[13]**, **[21]**, **[32]**. In 2013, Tsouli et al. **[33]** obtained some results about weak solutions of the following Robin problem

$$\begin{array}{rl} -\mathrm{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)=\lambda f(x,u), & x\in\Omega \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial v}+\beta(x)\,|u|^{p(x)-2}\,u=0, & x\in\partial\Omega \end{array} \tag{2}$$

using the variational methods under some suitable conditions for the function f. In addition, they showed that the problem (2) has at least three solutions.

In **[7]**, the authors obtain some new compact embedding theorems in a generalized variable exponent Sobolev spaces with two weights, and show the existence of several different weak solutions of the problem (1) in these spaces.

In the light of the articles mentioned above, we discuss the existence of multiplicity solutions of the problem (1) in the variable exponent Sobolev spaces $W^{1,p(.)}_{\omega_1,\omega_2}(\Omega)$ with respect to two different weight functions ω_1 and ω_2. Moreover, we introduce a more general norm compared to the norm given by Deng **[13]**. Finally, we find more general results than **[33]** using the technical approach, which is mainly based on Ricceri's theorem.

2. Notation and preliminaries. Let Ω be a bounded open subset of $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$. Then, the set is defined by

$$C_+\left(\overline{\Omega}\right)=\left\{p\in C\left(\overline{\Omega}\right):\inf_{x\in\overline{\Omega}}p(x)>1\right\},$$

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where $C\left(\overline{\Omega}\right)$ consists of all continuous functions on $\overline{\Omega}$. For any $p \in C_{+}\left(\overline{\Omega}\right)$, we indicate

$$p^{-}=\inf_{x\in\Omega} p(x) \text{ and } p^{+}=\sup_{x\in\Omega} p(x).$$

Let $p \in C_{+}\left(\overline{\Omega}\right)$ and $1<p^{-}\leq p(.)\leq p^{+}<\infty$. The space $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega)=\left\{u \,\middle|\, u:\Omega\rightarrow\mathbb{R} \text{ is measurable and } \int_{\Omega}|u(x)|^{p(x)}\,dx<\infty\right\}$$

with the (Luxemburg) norm

$$\|u\|_{p(.)}=\inf\left\{\lambda>0:\varrho_{p(.)}\left(\frac{u}{\lambda}\right)\leq 1\right\}$$

where

$$\varrho_{p(.)}(u)=\int_{\Omega}|u(x)|^{p(x)}\,dx$$

see [**24**].

The weighted Lebesgue space $L_{\omega}^{p(\cdot)}(\Omega)$ is defined by

$$L_{\omega}^{p(\cdot)}(\Omega)=\left\{u \,\middle|\, u:\Omega\longrightarrow\mathbb{R} \text{ measurable and } \int_{\Omega}|u(x)|^{p(x)}\,\omega(x)dx<\infty\right\}$$

such that

$$\|u\|_{p(.),\omega}=\left\|u\omega^{\frac{1}{p(\cdot)}}\right\|_{p(.)}<\infty$$

for $u\in L_{\omega}^{p(\cdot)}(\Omega)$, where ω is a weight function from Ω to $(0,\infty)$. It is known that the space $\left(L_{\omega}^{p(\cdot)}(\Omega),\|.\|_{p(.),\omega}\right)$ is a Banach space. Moreover, the dual space of $L_{\omega}^{p(\cdot)}(\Omega)$ is $L_{\omega^{*}}^{r(\cdot)}(\Omega)$ where $\frac{1}{p(.)}+\frac{1}{r(.)}=1$ and $\omega^{*}=\omega^{1-r(.)}=\omega^{-\frac{1}{p(.)-1}}$. If $\omega\in L^{\infty}(\Omega)$, then $L_{\omega}^{p(\cdot)}=L^{p(\cdot)}$, see [**3**], [**5**].

Moreover, we can define the space $L_{\omega}^{p(\cdot)}(\partial\Omega)$ similarly by

$$L_{\omega}^{p(\cdot)}(\partial\Omega)=\left\{u \,\middle|\, u:\partial\Omega\longrightarrow\mathbb{R} \text{ measurable and } \int_{\partial\Omega}|u(x)|^{p(x)}\,\omega(x)d\sigma<+\infty\right\}$$

with the norm

$$\|u\|_{p(.),\omega,\partial\Omega}=\inf\left\{\tau>0:\int\limits_{\partial\Omega}\left|\frac{u(x)}{\tau}\right|^{p(x)}\omega(x)d\sigma\leq 1\right\}$$

for $u\in L_{\omega}^{p(.)}(\partial\Omega)$ where $d\sigma$ is the measure on the boundary of Ω. Then $\left(L_{\omega}^{p(.)}(\partial\Omega),\|.\|_{p(.),\omega,\partial\Omega}\right)$ is a Banach space, see **[11]**.

Proposition 1. *(see* **[3]***,* **[17]***,* **[22]***,* **[25]***) For all $u,v\in L_{\omega}^{p(.)}(\Omega)$, we have*

(i) $\|u\|_{p(.),\omega}<1$ *(resp.*$=1,>1$*) if and only if* $\varrho_{p(.),\omega}(u)<1$ *(resp.*$=1,>1$*),*
(ii) $\|u\|_{p(.),\omega}^{p^-}\leq\varrho_{p(.),\omega}(u)\leq\|u\|_{p(.),\omega}^{p^+}$ *with* $\|u\|_{p(.),\omega}>1$,
(iii) $\|u\|_{p(.),\omega}^{p^+}\leq\varrho_{p(.),\omega}(u)\leq\|u\|_{p(.),\omega}^{p^-}$ *with* $\|u\|_{p(.),\omega}<1$,
(iv) $\min\left\{\|u\|_{p(.),\omega}^{p^-},\|u\|_{p(.),\omega}^{p^+}\right\}\leq\varrho_{p(.),\omega}(u)\leq\max\left\{\|u\|_{p(.),\omega}^{p^-},\|u\|_{p(.),\omega}^{p^+}\right\}$,
(v) $\min\left\{\varrho_{p(.),\omega}(u)^{\frac{1}{p^-}},\varrho_{p(.),\omega}(u)^{\frac{1}{p^+}}\right\}\leq\|u\|_{p(.),\omega}\leq\max\left\{\varrho_{p(.),\omega}(u)^{\frac{1}{p^-}},\varrho_{p(.),\omega}(u)^{\frac{1}{p^+}}\right\}$,
(vi) $\varrho_{p(.),\omega}(u-v)\to 0$ *if and only if* $\|u-v\|_{p(.),\omega}\to 0$.

Here, the functional $\varrho_{p(.),\omega}(u)$ *is defined by the integral* $\int\limits_{\Omega}|u(x)|^{p(x)}\omega(x)dx$.

Definition 1. Let $\omega^{-\frac{1}{p(.)-1}}\in L_{loc}^{1}(\Omega)$. The space $W_{\omega}^{k,p(.)}(\Omega)$ is defined by

$$W_{\omega}^{k,p(.)}(\Omega)=\left\{u\in L_{\omega}^{p(.)}(\Omega):D^{\alpha}u\in L_{\omega}^{p(.)}(\Omega),0\leq|\alpha|\leq k\right\}$$

equipped with the norm

$$\|u\|_{\omega}^{k,p(.)}=\sum_{0\leq|\alpha|\leq k}\|D^{\alpha}u\|_{p(.),\omega}$$

where $\alpha\in\mathbb{N}_0^N$ is a multi-index, $|\alpha|=\alpha_1+\alpha_2+...+\alpha_N$ and $D^{\alpha}=\frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1}...\partial_{x_N}^{\alpha_N}}$. In particular, the space $W_{\omega}^{1,p(.)}(\Omega)$ is defined by

$$W_{\omega}^{1,p(.)}(\Omega)=\left\{u\in L_{\omega}^{p(.)}(\Omega):|\nabla u|\in L_{\omega}^{p(.)}(\Omega)\right\}$$

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equipped with the norm

$$\|u\|_{\omega}^{1,p(.)} = \|u\|_{p(.),\omega} + \|\nabla u\|_{p(.),\omega}.$$

The space $W_{\omega^*}^{-1,r(.)}(\Omega)$ is the topological dual for $W_{\omega}^{1,p(.)}(\Omega)$ where $\frac{1}{p(.)} + \frac{1}{r(.)} = 1$ and $\omega^* = \omega^{1-r(.)} = \omega^{-\frac{1}{p(.)-1}}$. Moreover, the space $W_{\omega}^{1,p(.)}(\Omega)$ is a separable and reflexive Banach space, see **[5]**.

Let $\omega_1^{-\frac{1}{p(.)-1}}, \omega_2^{-\frac{1}{p(.)-1}} \in L_{loc}^1(\Omega)$. The space $W_{\omega_1,\omega_2}^{1,p(.)}(\Omega)$ is defined by

$$W_{\omega_1,\omega_2}^{1,p(.)}(\Omega) = \left\{u \in L_{\omega_2}^{p(.)}(\Omega) : |\nabla u| \in L_{\omega_1}^{p(.)}(\Omega)\right\}$$

equipped with the norm

$$\|u\|_{\omega_1,\omega_2}^{1,p(.)} = \|\nabla u\|_{p(.),\omega_1} + \|u\|_{p(.),\omega_2}.$$

It is clear that the space $W_{\omega_1,\omega_2}^{1,p(.)}(\Omega)$ is separable and reflexive Banach space, see **[25]**, **[35]**.

Theorem 2. *(see* **[6]***) Let $p(.) > N$ and $\omega_1^{-\alpha(.)} \in L^1(\Omega)$ with $\alpha(.) \in \left(\frac{N}{p(.)}, \infty\right) \cap \left[\frac{1}{p(.)-1}, \infty\right)$. If we define the variable exponent*

$$p_*(.) = \frac{\alpha(.)\,p(.)}{\alpha(.) + 1}$$

with $N < p_*^-$, then we have a continuous embedding $W_{\omega_1,\omega_2}^{1,p(.)}(\Omega) \hookrightarrow W^{1,p_*(.)}(\Omega)$ and a compact embedding $W_{\omega_1,\omega_2}^{1,p(.)}(\Omega) \hookrightarrow C\left(\overline{\Omega}\right)$.

Corollary 1. *Since the compact embedding $W_{\omega_1,\omega_2}^{1,p(.)}(\Omega) \hookrightarrow C\left(\overline{\Omega}\right)$ is satisfied, there exists a $c_1 > 0$ such that*

$$\|u\|_{\infty} \leq c_1 \|u\|_{\omega_1,\omega_2}^{1,p(.)}$$

for any $u \in W_{\omega_1,\omega_2}^{1,p(.)}(\Omega)$ where $\|u\|_{\infty} = \sup_{x\in\overline{\Omega}} u(x)$ for $u \in C\left(\overline{\Omega}\right)$.

For $A \subset \overline{\Omega}$, denote by $p^{-}(A) = \inf\limits_{x \in A} p(x)$ and $p^{+}(A) = \sup\limits_{x \in A} p(x)$. We define

$$p^{\partial}(x) = (p(x))^{\partial} = \begin{cases} \frac{(N-1)p(x)}{N-p(x)}, & \text{if } p(x) < N, \\ +\infty, & \text{if } p(x) \geq N, \end{cases}$$

and

$$p^{\partial}_{r(x)}(x) = \frac{r(x)-1}{r(x)} p^{\partial}(x)$$

for any $x \in \partial\Omega$, where $r \in C(\partial\Omega)$ with $r^{-} = \inf\limits_{x \in \partial\Omega} r(x) > 1$.

Theorem 3. *(see* **[11]***) Assume that the boundary of Ω possesses the cone property and $p \in C(\overline{\Omega})$ with $p^{-} > 1$. Suppose that $\omega \in L^{r(.)}(\partial\Omega)$, $r \in C(\partial\Omega)$ with $r(x) > \frac{p^{\partial}(x)}{p^{\partial}(x)-1}$ for all $x \in \partial\Omega$. If $q \in C(\partial\Omega)$ and $1 \leq q(x) < p^{\partial}_{r(x)}(x)$ for all $x \in \partial\Omega$, then there is a compact embedding $W^{1,p(.)}(\Omega) \hookrightarrow L^{q(.)}_{\omega}(\partial\Omega)$. In particular, there is a compact embedding $W^{1,p(.)}(\Omega) \hookrightarrow L^{q(.)}(\partial\Omega)$ where $1 \leq q(x) < p^{\partial}(x)$ for all $x \in \partial\Omega$.*

Corollary 2. *(see* **[11]***)*

(i) There is a compact embedding $W^{1,p(.)}(\Omega) \hookrightarrow L^{p(.)}(\partial\Omega)$ where $1 \leq p(x) < p^{\partial}(x)$ for all $x \in \partial\Omega$.
(ii) There is a compact embedding $W^{1,p(.)}(\Omega) \hookrightarrow L^{p(.)}_{\omega}(\partial\Omega)$ where $1 \leq p(x) < p^{\partial}_{r(x)}(x) < p^{\partial}(x)$ for all $x \in \partial\Omega$.

Corollary 3. *Assume that all assumptions of Theorem 2 and Theorem 3 are satisfied. If $p(x) < p^{\partial}_{*,r(x)}(x) < p^{\partial}_{*}(x)$ for all $x \in \partial\Omega$, then we have the following compact embeddings $W^{1,p(.)}_{\omega_1,\omega_2}(\Omega) \hookrightarrow W^{1,p_*(.)}(\Omega) \hookrightarrow L^{p(.)}_{\omega_1}(\partial\Omega)$.*

Theorem 4. *(see* **[13]***) Assume that the boundary of Ω possesses the cone property and $p \in C(\overline{\Omega})$ with $p^{-} > 1$. If $q(.) \in C(\overline{\Omega})$ and $1 \leq q(x) < p^{\gamma}(x)$ for all $x \in \overline{\Omega}$, then there is a compact embedding $W^{1,p(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega)$ where*

$$p^{\gamma}(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if } p(x) < N \\ +\infty, & \text{if } p(x) \geq N. \end{cases}$$

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Corollary 4. *Assume that all assumptions of Theorem 2 and Theorem 4 are satisfied. Let* $N < p_*^-$ *and* $1 \leq q(x) < (p_*)^{\gamma}(x)$ *for all* $x \in \overline{\Omega}$. *Then we have the compact embedding* $W^{1,p(.)}_{\omega_1,\omega_2}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$.

Proof. By Theorem 2 and Theorem 4, we have the continuous embedding $W^{1,p(.)}_{\omega_1,\omega_2}(\Omega) \hookrightarrow W^{1,p_*(\cdot)}(\Omega)$ and the compact embedding $W^{1,p_*(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$. Thus it is easy to see that the compact embedding $W^{1,p(.)}_{\omega_1,\omega_2}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is valid. □

If we apply the technique in [**13,** Theorem 2.1], then we prove the following theorem similarly. Moreover, due to this theorem we can find out the existence of weak solutions of the problem (1).

Theorem 5. *Let* $u \in W^{1,p(.)}_{\omega_1,\omega_2}(\Omega)$. *Then the norms* $\|u\|_{\partial}$ *and* $\|u\|^{1,p(.)}_{\omega_1,\omega_2}$ *are equivalent on* $W^{1,p(.)}_{\omega_1,\omega_2}(\Omega)$ *where*

$$\|u\|_{\partial} = \|\nabla u\|_{p(.),\omega_1} + \|u\|_{p(.),\omega_1,\partial\Omega}\,.$$

Let $\beta \in L^{\infty}(\partial\Omega)$ such that $\beta^- = \inf_{x\in\partial\Omega} \beta(x) > 0$. Then, the norm $\|u\|_{\beta(x)}$ is defined by

$$\|u\|_{\beta(x)} = \inf\left\{\tau > 0 : \int_{\Omega} \omega_1(x)\left|\frac{\nabla u(x)}{\tau}\right|^{p(x)} dx + \int_{\partial\Omega} \beta(x)\left|\frac{u(x)}{\tau}\right|^{p(x)} d\sigma \leq 1\right\}$$

for any $u \in W^{1,p(.)}_{\omega_1,\omega_2}(\Omega)$. Moreover, $\|.\|_{\beta(x)}$ and $\|.\|^{1,p(.)}_{\omega_1,\omega_2}$ are equivalent on $W^{1,p(.)}_{\omega_1,\omega_2}(\Omega)$ by Theorem 5.

Proposition 6. *(see* [**13**]*) Let*

$$I_{\beta(x)}(u) = \int_{\Omega} \omega_1(x)\,|\nabla u(x)|^{p(x)}\,dx + \int_{\partial\Omega} \beta(x)\,|u(x)|^{p(x)}\,d\sigma$$

with $\beta^- > 0$. *For any* $u, u_k \in W^{1,p(.)}_{\omega_1,\omega_2}(\Omega)$ $(k = 1, 2, ...)$, *we have*

(i) $\|u\|^{p^-}_{\beta(x)} \leq I_{\beta(x)}(u) \leq \|u\|^{p^+}_{\beta(x)}$ *with* $\|u\|_{\beta(x)} \geq 1$,
(ii) $\|u\|^{p^+}_{\beta(x)} \leq I_{\beta(x)}(u) \leq \|u\|^{p^-}_{\beta(x)}$ *with* $\|u\|_{\beta(x)} \leq 1$,

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(iii) $\min\left\{\|u\|_{\beta(x)}^{p^-},\|u\|_{\beta(x)}^{p^+}\right\}\leq I_{\beta(x)}(u)\leq\max\left\{\|u\|_{\beta(x)}^{p^-},\|u\|_{\beta(x)}^{p^+}\right\}$,
(iv) $\|u-u_k\|_{\beta(x)}\to 0$ *if and only if* $I_{\beta(x)}(u-u_k)\to 0$ *as* $k\to\infty$,
(v) $\|u_k\|_{\beta(x)}\to\infty$ *if and only if* $I_{\beta(x)}(u_k)\to\infty$ *as* $k\to\infty$.

The following Proposition can be proved by Proposition 2.2 in **[18]**.

Proposition 7. *Let us define the functional* $L_{\beta(x)}:W_{\omega_1,\omega_2}^{1,p(.)}(\Omega)\to\mathbb{R}$ *by*

$$L_{\beta(x)}(u)=\int\limits_{\Omega}\frac{\omega_1(x)}{p(x)}|\nabla u|^{p(x)}dx+\int\limits_{\partial\Omega}\frac{\beta(x)}{p(x)}|u(x)|^{p(x)}d\sigma$$

for all $u\in W_{\omega_1,\omega_2}^{1,p(.)}(\Omega)$. *Then, we obtain* $L_{\beta(x)}\in C^1\left(W_{\omega_1,\omega_2}^{1,p(.)}(\Omega),\mathbb{R}\right)$ *and*

$$L'_{\beta(x)}(u)(v)=\left\langle L'_{\beta(x)}(u),v\right\rangle=\int\limits_{\Omega}\omega_1(x)|\nabla u|^{p(x)-2}\nabla u\nabla v dx+\int\limits_{\partial\Omega}\beta(x)|u(x)|^{p(x)-2}uvd\sigma$$

for any $u,v\in W_{\omega_1,\omega_2}^{1,p(.)}(\Omega)$. *In addition, we have the following properties*

(i) $L'_{\beta(x)}:W_{\omega_1,\omega_2}^{1,p(.)}(\Omega)\longrightarrow W_{\omega_1^*,\omega_2^*}^{-1,p'(.)}(\Omega)$ *is continuous, bounded and strictly monotone operator,*
(ii) $L'_{\beta(x)}:W_{\omega_1,\omega_2}^{1,p(.)}(\Omega)\longrightarrow W_{\omega_1^*,\omega_2^*}^{-1,p'(.)}(\Omega)$ *is a mapping of type* (S_+), *i.e., if*

$$u_n\rightharpoonup u$$

in $W_{\omega_1,\omega_2}^{1,p(.)}(\Omega)$ *and*

$$\limsup_{n\longrightarrow\infty}L'_{\beta(x)}(u_n)(u_n-u)\leq 0,$$

then $u_n\longrightarrow u$ *in* $W_{\omega_1,\omega_2}^{1,p(.)}(\Omega)$
(iii) $L'_{\beta(x)}:W_{\omega_1,\omega_2}^{1,p(.)}(\Omega)\longrightarrow W_{\omega_1^*,\omega_2^*}^{-1,p'(.)}(\Omega)$ *is a homeomorphism.*

Theorem 8. *(see* **[28]***) Let* X *be a separable and reflexive real Banach space;* $\Phi:X\to\mathbb{R}$ *a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on* X^**;* $\Psi:X\to\mathbb{R}$ *a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Moreover, assume that* *(i)* $\lim\limits_{\|u\|\to\infty} (\Phi(u) + \lambda\Psi(u)) = \infty$ *for all* $\lambda > 0$,

(ii) *there are* $r \in \mathbb{R}$ *and* $u_0, u_1 \in X$ *such that* $\Phi(u_0) < r < \Phi(u_1)$,

(iii) $\inf\limits_{u\in\Phi^{-1}((-\infty,r])} \Psi(u) > \frac{(\Phi(u_1)-r)\Psi(u_0)+(r-\Phi(u_0))\Psi(u_1)}{\Phi(u_1)-\Phi(u_0)}$.

Then there exist an open interval $\Lambda \subset (0,\infty)$ *and a positive constant* $\rho > 0$ *such that for any* $\lambda \in \Lambda$ *the equation*

$$\Phi'(u) + \lambda\Psi'(u) = 0$$

has at least three solutions in X *whose norms are less than* ρ.

3. The Main Result. Throughout the paper, we assume that the following conditions:

(I) $|f(x,t)| \leq h(x) + c_2 |t|^{s(x)-1}$ for any $(x,t) \in \Omega \times \mathbb{R}$, $c_2 > 0$ where the function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $h(x) \in L^{\frac{s(x)}{s(x)-1}}(\Omega)$, $h(x) \geq 0$ and $s(x) \in C_+(\Omega)$,

$$1 < s^- = \inf_{x\in\overline{\Omega}} s(x) \leq s^+ = \sup_{x\in\overline{\Omega}} s(x) < p^-$$

with $s(x) < (p_*)^\gamma(x)$ for all $x \in \overline{\Omega}$.

(II) *(i)* $f(x,t) < 0$ for all $(x,t) \in \Omega \times \mathbb{R}$, and $|t| \in (0,1)$,
(ii) $f(x,t) \geq k > 0$, when $|t| \in (t_0,\infty)$, $t_0 > 1$.

Let $u \in W^{1,p(.)}_{\omega_1,\omega_2}(\Omega)$. Then the functional $\Phi_\lambda(u)$ is defined by

$$\Phi_\lambda(u) = L_{\beta(x)}(u) + \lambda\Psi(u),$$

where

$$\Psi(u) = -\int_\Omega F(x,u)dx \text{ and } F(x,t) = \int_0^t f(x,y)dy.$$

Moreover, $\Phi_\lambda(u)$ is called energy functional of the problem (1).

It is obvious that $\left(L'_{\beta(x)}\right)^{-1} : W^{-1,q(.)}_{\omega_1^*,\omega_2^*}(\Omega) \longrightarrow W^{1,p(.)}_{\omega_1,\omega_2}(\Omega)$ exists and continuous, because $L'_{\beta(x)} : W^{1,p(.)}_{\omega_1,\omega_2}(\Omega) \longrightarrow W^{-1,q(.)}_{\omega_1^*,\omega_2^*}(\Omega)$ is a homeomorphism by Proposition 7. Moreover, due to the assumption (I) it is well known that $\Psi \in C^1\left(W^{1,p(.)}_{\omega_1,\omega_2}(\Omega), \mathbb{R}\right)$ with the derivatives given by

$$\langle \Psi'(u), v\rangle = -\int_{\Omega} f(x,u)v dx$$

for any $u, v \in W^{1,p(.)}_{\omega_1,\omega_2}(\Omega)$, and $\Psi' : W^{1,p(.)}_{\omega_1,\omega_2}(\Omega) \longrightarrow W^{-1,q(.)}_{\omega_1^*,\omega_2^*}(\Omega)$ is completely continuous by [**2,** Theorem 2.9]. Therefore, $\Psi' : W^{1,p(.)}_{\omega_1,\omega_2}(\Omega) \longrightarrow W^{-1,q(.)}_{\omega_1^*,\omega_2^*}(\Omega)$ is compact.

Definition 2. We call that $u \in W^{1,p(.)}_{\omega_1,\omega_2}(\Omega)$ is a weak solution of the problem (1) if

$$\int_{\Omega} \omega_1(x) |\nabla u|^{p(x)-2} \nabla u \nabla v dx + \int_{\partial\Omega} \beta(x) |u(x)|^{p(x)-2} uv d\sigma - \lambda \int_{\Omega} \omega_2(x) f(x,u) v dx = 0$$

for all $v \in W^{1,p(.)}_{\omega_1,\omega_2}(\Omega)$. We point out that if $\lambda \in \mathbb{R}$ is an eigenvalue of the problem (1), then the corresponding $u \in W^{1,p(.)}_{\omega_1,\omega_2}(\Omega) - \{0\}$ is a weak solution of (1).

Zhikov and Surnachev [**37**] prove that a sufficient condition for the density of smooth functions in the weighted Sobolev space with variable exponent is obtained. Moreover, it is known that the space $C_0^\infty(\Omega)$ is a subspace of $W^{1,p(.)}_{\omega_1,\omega_2}(\Omega)$, and $W^{1,p(.)}_{\omega_1,\omega_2}(\Omega) = W^{1,p(.)}(\Omega)$ for $0 < a_1 \le \omega_1(x) \le a_2$, $0 < b_1 \le \omega_2(x) \le b_2$ and a_1, a_2, b_1, $b_2 \in \mathbb{R}$. Thus we obtain that the spaces $C_0^\infty(\Omega)$ and $C^\infty(\Omega) \cap W^{1,p(.)}_{\omega_1,\omega_2}(\Omega)$ is dense in $W^{1,p(.)}_{\omega_1,\omega_2}(\Omega)$ since log-Hölder continuity of the exponent is sufficient for the density of smooth functions, see [**14,** Theorem 9.1.8], [**15**], [**30**], [**31**].

The class $A_{p(.)}$ consists of those weights ω such that

$$\|\omega\|_{A_{p(.)}} = \sup_{B \in ß} |B|^{-p_B} \|\omega\|_{L^1(B)} \left\| \frac{1}{\omega} \right\|_{L^{\frac{p'(.)}{p(.)}}(B)} < \infty$$

where ß denotes the set of all balls in Ω, $p_B = \left(\frac{1}{|B|} \int_B \frac{1}{p(x)} dx \right)^{-1}$ and $\frac{1}{p(.)} + \frac{1}{p'(.)} = 1$. If $\omega_1 \in A_{p(.)}$, $0 < b_1 \le \omega_2(x) \le b_2$ and $p(.) \in P^{\log}(\mathbb{R}^n)$, i.e. $p(.)$ satisfy log-Hölder continuity condition, then $C_0^\infty(\Omega)$ is dense in $W^{1,p(.)}_{\omega_1,\omega_2}(\Omega)$, see [**3**], [**23**]. Therefore, the weak solution of the problem (1) is well defined. In the paper, we assume that $\omega_1 \in A_{p(.)}$, $0 < b_1 \leq \omega_2(x) \leq b_2$ and $p(.) \in P^{\log}(\mathbb{R}^n)$.

Theorem 9. *There exist an open interval $\Lambda \subset (0, \infty)$ and a positive constant $\rho > 0$ such that for any $\lambda \in \Lambda$, the problem (1) has at least three solutions in $W^{1,p(.)}_{\omega_1,\omega_2}(\Omega)$ whose norms are less than ρ.*

Proof. We only need to prove the conditions *(i)*, *(ii)* and *(iii)* in Theorem 8. Using Proposition 7, we get

$$\begin{aligned} L_{\beta(x)}(u) &= \int_{\Omega} \frac{\omega_1(x)}{p(x)} |\nabla u|^{p(x)} dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)} |u(x)|^{p(x)} d\sigma \\ &\geq \frac{1}{p^+} I_{\beta(x)}(u) \\ &\geq \frac{1}{p^+} \|u\|^{p^-}_{\beta(x)} \end{aligned} \tag{3}$$

for any $u \in W^{1,p(.)}_{\omega_1,\omega_2}(\Omega)$ with $\|u\|_{\beta(x)} > 1$.

In addition, due to (I) and Hölder inequality, we have

$$\begin{aligned} -\Psi(u) &= \int_{\Omega} F(x,u)dx = \int_{\Omega} \left(\int_0^{u(x)} f(x,t)dt \right) dx \\ &\leq \int_{\Omega} \left(h(x) |u(x)| + \frac{c_2}{s(x)} |u(x)|^{s(x)} \right) dx \\ &\leq 2 \|h\|_{\frac{s(.)}{s(.)-1},\Omega} \|u\|_{s(.),\Omega} + \frac{c_2}{s^-} \int_{\Omega} |u(x)|^{s(x)} dx. \end{aligned} \tag{4}$$

By Corollary 4, there exist the continuous embedding $W^{1,p(.)}_{\omega_1,\omega_2}(\Omega) \hookrightarrow L^{s(.)}(\Omega)$ and the inequality

$$\int_{\Omega} |u(x)|^{s(x)} dx \leq \max \left\{ \|u\|^{s^-}_{s(.),\Omega}, \|u\|^{s^+}_{s(.),\Omega} \right\} \leq c_3 \|u\|^{s^+}_{\beta(x)}. \tag{5}$$

If we use (4) and (5), then we get

$$-\Psi(u) \leq 2C_7 \|h\|_{\frac{s(.)}{s(.)-1},\Omega} \|u\|_{\beta(x)} + \frac{c_4}{s^-} \|u\|^{s^+}_{\beta(x)}. \tag{6}$$

For any $\lambda > 0$ we can write

$$L_{\beta(x)}(u)+\lambda\Psi(u) \geq \frac{1}{p^+}\|u\|_{\beta(x)}^{p^-}-2\lambda C_7\|h\|_{\frac{s(.)}{s(.)-1},\Omega}\|u\|_{\beta(x)}-\frac{c_4}{s^-}\lambda\|u\|_{\beta(x)}^{s^+}$$

by (3) and (6). Since $1 < s^+ < p^-$ and

$$\lim_{\|u\|_{\beta(x)}\to\infty}\left(L_{\beta(x)}(u)+\lambda\Psi(u)\right)=\infty$$

for all $\lambda > 0$, the proof of *(i)* is completed.

Due to $\frac{\partial F(x,t)}{\partial t} = f(x,t)$ and (II), it is easy to see that $F(x,t)$ is increasing and decreasing for $t \in (t_0,\infty)$ and $(0,1)$ with respect to $x \in \Omega$, respectively. Since $F(x,t) \geq kt$ uniformly for x, we have $F(x,t) \to \infty$ as $t \to \infty$. Then for a real number $\delta > t_0$, we can obtain

$$F(x,t) \geq 0 = F(x,0) \geq F(x,\tau) \text{ for all } x \in \Omega,\ t > \delta,\ \tau \in (0,1)\,. \tag{7}$$

Let β, γ be two real numbers such that $0 < \beta < \min\{1, c_1\}$ where c_1 is given in Corollary 1, and $\gamma > \delta$ $(\gamma > 1)$ satisfies $\gamma^{p^-}\|\beta\|_{1,\partial\Omega} > 1$. If we use relation (7), then we have $F(x,t) \leq F(x,0) = 0$ for $t \in [0,\beta]$, and

$$\int_{\Omega}\sup_{0\leq t\leq\beta}F(x,t)dx \leq \int_{\Omega}F(x,0)dx = 0. \tag{8}$$

Using $\gamma > \delta$ and (7), we have $\int_{\Omega}F(x,\delta)dx > 0$ and

$$\frac{1}{c_1^{p^+}}\frac{\beta^+}{\gamma^{p^-}}\int_{\Omega}F(x,\delta)dx > 0. \tag{9}$$

If we use the inequalities in (8) and (9), then we get

$$\int_{\Omega}\sup_{0\leq t\leq a}F(x,t)dx \leq 0 < \frac{1}{c_1^{p^+}}\frac{\beta^+}{\gamma^{p^-}}\int_{\Omega}F(x,\delta)dx.$$

Define $u_0, u_1 \in W_{\omega_1,\omega_2}^{1,p(.)}(\Omega)$ with $u_0(x) = 0$ and $u_1(x) = \gamma$ for any $x \in \Omega$.

If we take $r = \frac{1}{p^+}\left(\frac{\beta}{c_1}\right)^{p^+}$, then $r \in (0,1)$, $L_{\beta(x)}(u_0) = \Psi(u_0) = 0$ and

$$\begin{aligned} L_{\beta(x)}(u_1) &= \int_{\partial\Omega} \frac{\beta(x)}{p(x)} \gamma^{p(x)} d\sigma \geq \frac{\gamma^{p^-}}{p^+} \int_{\partial\Omega} \beta(x) d\sigma = \frac{1}{p^+} \gamma^{p^-} \|\beta\|_{1,\partial\Omega} \\ &\geq \frac{1}{p^+} > r. \end{aligned}$$

Thus we have $L_{\beta(x)}(u_0) < r < L_{\beta(x)}(u_1)$ and

$$\Psi(u_1) = -\int_{\Omega} F(x, u_1) dx = -\int_{\Omega} F(x,\gamma) dx < 0.$$

Then the proof of *(ii)* is obtained.

On the other hand, we have

$$\begin{aligned} -\frac{\left(L_{\beta(x)}(u_1) - r\right)\Psi(u_0) + \left(r - L_{\beta(x)}(u_0)\right)\Psi(u_1)}{L_{\beta(x)}(u_1) - L_{\beta(x)}(u_0)} &= -r\frac{\Psi(u_1)}{L_{\beta(x)}(u_1)} \\ &= r\frac{\int_{\Omega} F(x,\gamma)dx}{\int_{\partial\Omega} \frac{\beta(x)}{p(x)}\gamma^{p(x)} d\sigma} > 0. \end{aligned}$$

Now, let $u \in W^{1,p(.)}_{\omega_1,\omega_2}(\Omega)$ with $L_{\beta(x)}(u) \leq r < 1$. Since

$$\frac{1}{p^+} I_{\beta(x)}(u) \leq L_{\beta(x)}(u) \leq r,$$

we obtain

$$I_{\beta(x)}(u) \leq p^+ r = \left(\frac{\beta}{c_1}\right)^{p^+} < 1.$$

Due to Proposition 6, we see that $\|u\|_{\beta(x)} < 1$ and

$$\frac{1}{p^+} \|u\|^{p^+}_{\beta(x)} \leq \frac{1}{p^+} I_{\beta(x)}(u) \leq L_{\beta(x)}(u) \leq r.$$

Then using Corollary 1, we can get

$$|u(x)| \leq c_1 \|u\|_{\beta(x)} \leq c_1 \left(p^+ r\right)^{\frac{1}{p^+}} = \beta$$

for all $u \in W^{1,p(.)}_{\omega_1,\omega_2}(\Omega)$ and $x \in \Omega$ with $\Phi(u) \leq r$.

The last inequality implies that

$$-\inf_{u\in\Phi^{-1}((-\infty,r])}\Psi(u)=\sup_{u\in\Phi^{-1}((-\infty,r])}-\Psi(u)\leq\int_{\Omega}\sup_{0\leq t\leq\beta}F(x,t)dx\leq 0.$$

Then we have

$$-\inf_{u\in\Phi^{-1}((-\infty,r])}\Psi(u)<r\frac{\int_{\Omega}F(x,\gamma)dx}{\int_{\partial\Omega}\frac{\beta(x)}{p(x)}\gamma^{p(x)}d\sigma}$$

and

$$\inf_{u\in\Phi^{-1}((-\infty,r])}\Psi(u)>\frac{(\Phi(u_1)-r)\,\Psi(u_0)+(r-\Phi(u_0))\,\Psi(u_1)}{\Phi(u_1)-\Phi(u_0)}.$$

This completes the proof. □

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