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SELBERG-TYPE INTEGRALS AND THE VARIANCE CONJECTURE FOR THE OPERATOR NORM

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ABSTRACT. The variance conjecture in Asymptotic Convex Geometry stipulates that the Euclidean norm $\|\mathscr{X}_K\|_2$ of a random vector $\mathscr{X}_K$ uniformly distributed in a (properly normalised) high-dimensional convex body $K \subset \mathbb{R}^n$ satisfies a Poincaré-type inequality (which will imply that the variance of $\|\mathscr{X}_K\|_2$ is much smaller than its expectation). We settle the conjecture for the cases when K is the unit ball of the operator norm in classical subspaces of square matrices, which include the subspaces of self-adjoint matrices. Through the estimates we establish, we are also able to show that the unit ball of the operator norm in the subspace of real symmetric matrices or in the subspace of Hermitian matrices is not isotropic, yet is in almost isotropic position (i.e. its covariance matrix has small condition number).

1. Introduction

This note is a follow-up on [41], in which we were concerned with the question whether the variance (or thin-shell) conjecture holds true for unit balls of the p-Schatten norms. Given a convex body K in $\mathbb{R}^m$, that is, a convex, compact set with non-empty interior, its covariance matrix $\mathrm{Cov}(K)$ is given by

$$\mathrm{Cov}(K)_{i,j} := \frac{\int_K x_i x_j\,dx}{\int_K \mathbf{1}\,dx} - \frac{\int_K x_i\,dx}{\int_K \mathbf{1}\,dx}\,\frac{\int_K x_j\,dx}{\int_K \mathbf{1}\,dx} \qquad \text{for } 1 \leqslant i, j \leqslant m. \tag{1}$$

If $\mathrm{Cov}(K)$ has small condition number (the ratio of the largest singular value to the smallest one), then the variance conjecture states that most of the mass of K will be found in an annulus of width much smaller than its average radius, a "thin shell" (see the ε-*Concentration Hypothesis* of Anttila, Ball and Perissinaki [4], or the quantitatively stronger statement (2) suggested by Bobkov and Koldobsky [11]). Supposing first for simplicity that K has Lebesgue volume 1, barycentre at the origin, and that K is *isotropic*, that is, $\mathrm{Cov}(K)$ is a multiple of the identity matrix, the conjecture can be stated as asking that

$$\mathrm{Var}_K\big(\|x\|_2^2\big) := \int_K \|x\|_2^4\,dx - \left(\int_K \|x\|_2^2\,dx\right)^2 \lesssim \frac{1}{m}\left(\int_K \|x\|_2^2\,dx\right)^2, \tag{2}$$

where $\|\cdot\|_2$ stands for the Euclidean norm on $\mathbb{R}^m$, and '$\lesssim$' implies a multiplicative constant that should not depend on the dimension m or the body K. To motivate (2) further, it is known that it is equivalent to

$$\mathrm{Var}_K\big(\|x\|_2\big) \lesssim \big(\det[\mathrm{Cov}(K)]\big)^{\frac{1}{m}} \simeq \big(\det[\mathrm{Cov}(K)]\big)^{\frac{1}{2m}} \cdot \frac{\int_K \|x\|_2\,dx}{\sqrt{m}}. \tag{2a}$$

Although stated separately and with different motivations initially, inequality (2) is a special case of the KLS conjecture (put forth by Kannan, Lovász and Simonovits [30]) when the latter is equivalently reformulated as a Poincaré inequality for convex bodies (the equivalence following by works of Maz'ya,

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Cheeger, Buser and Ledoux): given a convex body $K \subset \mathbb{R}^m$ of volume 1 with barycentre at the origin (not necessarily isotropic), and any (locally) Lipschitz function $f : \mathbb{R}^m \to \mathbb{R}$, we should have

$$\mathrm{Var}_K(f) \lesssim s_{\max}\big[\mathrm{Cov}(K)\big] \cdot \int_K \|\nabla f(x)\|_2^2\, dx, \tag{3}$$

where $s_{\max}\big[\mathrm{Cov}(K)\big]$ denotes the largest singular value of the covariance matrix of K. To see how (3) gives (2) (and (2a)) immediately, observe that, when $\mathrm{Cov}(K)$ is a multiple of the identity matrix, we have

$$s_{\max}\big[\mathrm{Cov}(K)\big] = \big(\det\big[\mathrm{Cov}(K)\big]\big)^{\frac{1}{m}} \quad \text{and also} \; = \frac{1}{m}\mathrm{tr}\big[\mathrm{Cov}(K)\big] = \frac{1}{m}\int_K \|x\|_2^2\, dx. \tag{4}$$

Of course, with the KLS conjecture in mind, it makes sense to ask about the validity of a suitably modified inequality (2) even when $\mathrm{Cov}(K)$ is not a multiple of the identity, and when (4) is not true even approximately (or we don't know a priori whether it is).

Conjecture 1. ("Generalised Variance Conjecture") There is an absolute constant $C > 0$ such that, given any convex body $K \subset \mathbb{R}^m$ of volume 1 with barycentre at the origin, one has

$$\mathrm{Var}_K\big(\|x\|_2^2\big) \leqslant C \cdot s_{\max}\big[\mathrm{Cov}(K)\big] \int_K \|x\|_2^2\, dx. \tag{5}$$

The assumption that K has volume 1 is merely for convenience: if instead we don't specify the volume of K, integration above is understood with respect to the density $\mathbf{1}_K(x)/(\int_K \mathbf{1}\, dx)$.

In this note we verify this conjecture for the unit ball of the operator norm on several classical subspaces of square matrices.

Before we turn to particulars, let us recall that, despite the fact that Conjecture 1, or its more restricted version for isotropic convex bodies only, seem like very special cases of the KLS conjecture, they are in fact almost equivalent reformulations of it: according to a breakthrough result by Eldan [19], whatever estimates one obtains for the constant C appearing in (5) (*for all* centred convex bodies), or even just for inequality (2) (*for all* isotropic convex bodies), the same estimates (up to some multiplicative logarithmic factors in the dimension m) will also be valid for the implied constant in (3). The best known estimates for the constant $C = C(m)$ in (2) follow from recent remarkable developments for the KLS conjecture: in a breakthrough result which builds on Eldan's seminal stochastic localisation method from [19] and further analysis of it by Lee and Vempala [37], Yuansi Chen [13] obtained bounds which are asymptotically smaller than any power of m. More recently, Klartag and Lehec [33], and subsequently Jambulapati, Lee and Vempala [24], refined the technique even further and combined it with other closely related methods to obtain improvements which were polylogarithmic in the dimension m. Finally, in March 2023 Klartag [32] obtained the best known estimate for the KLS conjecture (and its special cases, the classical Variance Conjecture and its generalised version), by showing that $C(m) \lesssim \sqrt{\log(m)}$.

The abovementioned methods are very powerful and have had far-reaching applications. Still, for decades, and in parallel, the above conjectures have also been studied for special families of convex bodies via methods which are more specific to said special families. Inequality (2) was (optimally) established early on for the unit balls of the ℓ_p norms by Ball and Perissinaki [7]. Then Conjecture 1 was verified by Klartag [31] for all unconditional convex bodies, and soon thereafter, via extending Klartag's method in [31], Barthe and Cordero-Erausquin [8] showed it for all convex bodies that have many symmetries (maybe fewer than those of an unconditional body, but still enough; one such example is the simplex, or any other convex body which has the symmetries of the simplex). Conjecture 1 has also been verified by Alonso-Gutiérrez and Bastero [1] for hyperplane projections of the unit balls of the ℓ_p norms. Obviously, it is also true for all classes of convex bodies for which the even stronger KLS conjecture (equivalently, inequality (3)) has been optimally established: e.g. Kolesnikov and Milman [34] have done so for certain Orlicz balls (see also [9, Section 5] by Barthe and Wolff). We refer the reader to [34, p. 4 (3581)] for a comprehensive list of other such results.

Finally, of most relevance here is a work independent of this present note, which also appeared chronologically after this note was first posted on arXiv: Dadoun, Fradelizi, Guédon and Zitt [17] established Conjecture 1 for the unit balls of p-Schatten norms on subspaces of self-adjoint matrices when $p \in (3,\infty)$ is fixed (see next couple of paragraphs for definitions and terminology). Even though there is a common starting point in both their work and the present note (namely the invariances of p-Schatten norms), the crucial ingredients in their work are very different, and their approach utilises beautifully a connection to the theory of logarithmic potentials which was developed in prior works [25]-[27] (again independent of this note). We will elaborate a little more on these works after giving the definitions for the Schatten classes.

We now state the main result of this note. Let $\mathcal{M}_n(\mathbb{F})$ denote the space of all $n\times n$ matrices with entries from the division algebra $\mathbb{F}$, which stands either for $\mathbb{R}$ or $\mathbb{C}$ or the skew field $\mathbb{H}$ of quaternions (note that in all cases we view $\mathcal{M}_n(\mathbb{F})$ as a real vector space, which can thus be thought of as $\mathbb{R}^m$ where $m=\beta n^2$ with $\beta = 1, 2$ or 4 respectively). For a matrix $T\in\mathcal{M}_n(\mathbb{F})$ and $p\geqslant 1$, the p-Schatten norm of T is given by

$$\|T\|_{S_p^n} := \|s(T)\|_p = \left(\sum_{i=1}^n s_i(T)^p\right)^{1/p},$$

where $s(T) = (s_1(T),\dots,s_n(T))$ is the non-increasing rearrangement of the singular values of T, that is, of the eigenvalues of $(T^*T)^{1/2}$. The limiting case of $p=\infty$ is defined in the usual way: $\|T\|_{S_\infty^n} := \|s(T)\|_\infty = s_{\max}(T)$ is the *operator* or *spectral* norm of T. Also, the Euclidean norm $\|\cdot\|_2$ on $\mathcal{M}_n(\mathbb{F})$ coincides with the 2-Schatten norm $\|\cdot\|_{S_2^n}$, also known as the *Hilbert-Schmidt* or *Frobenius* norm.

We will focus on establishing Conjecture 1 when K is the unit ball of S_∞^n on either of the spaces $\mathcal{M}_n(\mathbb{F})$, or moreover on its classical subspace of $\mathbb{F}$-self-adjoint matrices.

Theorem 2. *Let $\mathbb{F}$ stand for either $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$, and let $E=\mathcal{M}_n(\mathbb{F})$ or the subspace of $\mathbb{F}$-self-adjoint matrices. Set $d_n=\dim(E)$, and write B_E for the unit ball of $\|\cdot\|_{S_\infty^n}$ on E, and $\overline{B_E}$ for its homothetic copy of volume 1, that is, $\overline{B_E} := \frac{B_E}{[\mathrm{vol}(B_E)]^{1/d_n}}$. Then there are absolute constants $C_1, C_2>0$ so that*

$$C_1 \leqslant \sigma_{B_E}^2 := d_n \frac{\mathrm{Var}_{\overline{B_E}}\big(\|T\|_{S_2^n}^2\big)}{\left(\int_{\overline{B_E}} \|T\|_{S_2^n}^2\, dT\right)^2} \leqslant C_2. \tag{6}$$

Remark 3. Obviously this implies Conjecture 1 for the (normalised) unit ball $\overline{B_E}$ of the operator norm on E since we always have $\frac{1}{d_n}\int_{\overline{B_E}}\|T\|_{S_2^n}^2\,dT = \frac{1}{d_n}\mathrm{tr}\big[\mathrm{Cov}\big(\overline{B_E}\big)\big] \leqslant s_{\max}\big[\mathrm{Cov}\big(\overline{B_E}\big)\big]$.

For most of the cases of E mentioned above these estimates were also established in [41] by J. Radke and the author (with somewhat similar methods as we will see): these are the cases of the whole spaces, and the subspace of Hermitian matrices. For the subspaces of symmetric (or real self-adjoint) matrices, and of quaternionic self-adjoint matrices, the result is new.

It is worth noting that the unit balls $B_{E,p}$ in $\mathcal{M}_n(\mathbb{F})$ of all p-Schatten norms (thus including the operator norm) have enough symmetries/invariances (and are also isotropic; see the next paragraphs

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for details) that the method of Barthe and Cordero-Erausquin in [8] could give the estimate $\sigma^2_{B_{E,p}} = O(n) = O\big(\sqrt{\dim(E)}\big)$ (this was the uniform estimate known before the recent developments for the KLS conjecture). On the other hand, it is unclear whether either this method, or any of the general results we now know, could imply the exact same estimates that they give for $\sigma_{B_{\mathcal{M}_n(\mathbb{F})}}$ also in the case of subspaces of self-adjoint matrices. This is because it is not known (to the best of our knowledge) if the condition number of the covariance matrix of B_E in such a subspace E is small (similarly this appears not to be known for any other p-Schatten norm besides $p = 2$). In this note we also show that this condition number is small in the cases where E consists of the real or complex self-adjoint matrices (see Theorem 4 below). Observe that the estimates in (6) are established regardless of that.

The starting point here, as well as for the arguments in [41], is a key idea and strategy which, in the context of problems on volumetric properties of the Schatten classes, appeared first in the paper [43] by Saint Raymond. It was further developed by König, Meyer and Pajor [35], and by Guédon and Paouris [22]. We start with the key observation/fact that the uniform distribution on $\overline{B_E}$ defines an *invariant* ensemble of 'random' matrices from E: the distribution remains the same under multiplication by an $\mathbb{F}$-unitary matrix (by which we understand either multiplication from left or from right when $E = \mathcal{M}_n(\mathbb{F})$, or conjugation by the matrix when $\overline{B_E}$ contains only $\mathbb{F}$-self-adjoint matrices). Equivalently, the distribution depends only on the non-increasing rearrangement of the singular values $s_i(T)$ of $T \in E$ when $E = \mathcal{M}_n(\mathbb{F})$, or of the eigenvalues $e_i(T)$ of $T \in E$ when E consists of the $\mathbb{F}$-self-adjoint matrices. As a consequence of this, and also of the fact that the integrands we care about depend only on the singular values of T, the integrals in (6) which we wish to estimate can be reduced to integrals of highly symmetric distributions over $\mathbb{R}^n$ (see Lemma 6 and Proposition 7).

It is worth noting here that, in [25]-[27], Kabluchko, Prochno and Thäle refined this strategy and reduced the estimation of the latter type of integrals to the study of the empirical distribution of n particles/'unit charges' on the real line which have pairwise repulsive logarithmic interaction and are also confined by an external field. This allowed them to invoke results from the log-potential theory, a theory which, in many cases of external fields, provides concrete information about the equilibrium density of such an ensemble of particles. Dadoun, Fradelizi, Guédon and Zitt [17] build further on this approach, and reduce questions about different moments of the Euclidean norm or about its variance to the convergence or the fluctuations of linear statistics of these empirical measures.

In this paper, in contrast, the initial reduction is used in a more direct way: to estimate $\sigma^2_{B_E}$ (recall that, here, B_E is the unit ball of $\|\cdot\|_{S^n_\infty}$ on E), it is equivalent to obtain estimates for the variance of the Euclidean norm with respect to the density

$$x = (x_1, \dots, x_n) \in \mathbb{R}^n \quad \longmapsto \quad \mathbf{1}_{[-1,1]^n}(x) \prod_{1 \leqslant i < j \leqslant n} \left| x_i^a - x_j^a \right|^b \cdot \prod_{1 \leqslant i \leqslant n} |x_i|^c \, dx,$$

where a, b, c are integers depending only on E ($a \in \{1, 2\}$, $b = \beta = \dim_{\mathbb{R}}(\mathbb{F})$, and $c \in \{0, \beta - 1\}$). This requires us to study integrals of the form

$$(7) \qquad \int_{-1}^{1} \int_{-1}^{1} \cdots \int_{-1}^{1} s(x) \prod_{1 \leqslant i < j \leqslant n} \left| x_i^a - x_j^a \right|^b \cdot \prod_{1 \leqslant i \leqslant n} |x_i|^c \, dx_n \dots dx_2 dx_1$$

where $a = 1$ or 2, and where the integrand $s(x)$ is a symmetric polynomial (in this case we will have $s(x) = \sum_i x_i^k$ with $k = 2$ or 4, or $s(x) = \sum_{i<j} x_i^2 x_j^2$).

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With suitable changes of variables, all such integrals can be related to integrals of a similar form:

$$\int_0^1\int_0^1\cdots\int_0^1 \tilde{s}(t)\prod_{1\leqslant i\leqslant n} t_i^{u-1}(1-t_i)^{w-1}\prod_{1\leqslant i<j\leqslant n}\left|t_i-t_j\right|^{2\kappa}dt_n\dots dt_2dt_1 \tag{8}$$

where again $\tilde{s}(t)$ is a symmetric polynomial, and where $u>0$, $w>0$ and $\kappa\geqslant 0$ (we can even think of u,w,κ as complex numbers, with the inequalities-constraints then holding for their real part). Selberg [44] was the first to study such a family of integrals in the case where $\tilde{s}(t)=\mathbf{1}$ (using crucially the fact that the change of variables $t_i\mapsto 1-t_i$ leaves the integrals in this family unchanged), and he showed that each of them equals a certain product of Gamma factors (that is, of values of the Gamma function) whose inputs depend only linearly on u, w and κ in a pre-specified manner:

$$\begin{aligned} I_0(n;u,w,\kappa) &:= \int_0^1\int_0^1\cdots\int_0^1 \prod_{1\leqslant i\leqslant n} t_i^{u-1}(1-t_i)^{w-1}\prod_{1\leqslant i<j\leqslant n}\left|t_i-t_j\right|^{2\kappa}dt_n\dots dt_2dt_1 \\ &= \prod_{1\leqslant i\leqslant n}\frac{\Gamma\big(1+(n-i+1)\kappa\big)}{\Gamma(1+\kappa)}\prod_{1\leqslant i\leqslant n}\frac{\Gamma\big(u+(n-i)\kappa\big)\Gamma\big(w+(n-i)\kappa\big)}{\Gamma\big(u+w+(2n-i-1)\kappa\big)}. \end{aligned} \tag{9}$$

Aomoto [5], and then Kadell [28], the latter confirming a conjecture by Macdonald [38, Conjecture (C5)], have generalised this result by establishing completely analogous 'closed-form' expressions for the corresponding integrals when $\tilde{s}(t)$ ranges in different families of non-constant symmetric polynomials. In fact, Kadell's result encompasses all the previous results since the family of polynomials $\tilde{s}(t)$ which one can consider according to his result contains the family of Jack symmetric polynomials (under a standard normalisation) and therefore spans the space of symmetric polynomials (see Subsection 2.2 for definitions and specifics; also, for other proofs of Kadell's result, see Kaneko [29], Baker and Forrester [6] (see also [20] for a streamlined sketch of this proof), and Warnaar [48]).

In Section 3 we show how to use Aomoto's result (as well as an immediate extension of it) in order to reestablish the conclusion in Theorem 2 when $E=\mathcal{M}_n(\mathbb{F})$, and furthermore how to use Kadell's more general result to obtain Theorem 2 for the subspaces of self-adjoint matrices too.

The estimates we obtain for integrals of the form (7) allow us to also deal with the question of what the covariance matrix of $\overline{B_E}$ is when E is one of the subspaces of self-adjoint matrices. Note that in the case of the spaces $\mathcal{M}_n(\mathbb{F})$ it is not difficult to see that simply the symmetries/invariances of the respective unit balls $B_{\mathcal{M}_n(\mathbb{F})}$ (and similarly of the unit balls of all other p-Schatten norms) guarantee these bodies are isotropic (see e.g. [41, Proposition 26]); however in the case of the subspaces of self-adjoint matrices the symmetries are no longer enough for a similar conclusion.

Let us observe that, since $\overline{B_E}$ has volume 1 and the origin as a centre of symmetry, computing the entries of the covariance matrix as in (1) reduces essentially to computing integrals of the form

$$\int_{\overline{B_E}}|T_{i,j}|^2\,dT,\quad 1\leqslant i,j\leqslant n,\qquad \text{as well as}\quad \int_{\overline{B_E}}T_{i,j}T_{l,k}\,dT\quad \text{for } (i,j)\neq(l,k). \tag{10}$$

This is made possible through the Weingarten calculus which allows to estimate integrals of polynomial functions of the entries of a random matrix (in the case of several important types of matrix ensembles) via relating them to integrals of symmetric functions of the eigenvalues: for our setting we need a result of Collins, Matsumoto and Saad [15] for conjugate invariant ensembles of self-adjoint matrices with real or complex entries (see Subsection 2.3 for details).

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The estimates we obtain are summarised in the following theorem, and show that $\overline{B_E}$ is almost isotropic when E is the subspace of symmetric matrices, or the subspace of Hermitian matrices (see Section 4 for the details and more precise estimates including constants).

Theorem 4. *Let E be the subspace of $\mathbb{F}$-self-adjoint matrices with $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$. Then all integrals of the first form in* (10) *are of the order of 1, while all integrals of the second form are zero except when $i=j\neq l=k$. In fact, when $\mathbb{F}=\mathbb{C}$ and, say, $i\neq j$, we also have*

$$\int_{\overline{B_E}} \mathrm{Re}(T_{i,j})\,\mathrm{Re}(T_{l,k})\,dT = \int_{\overline{B_E}} \mathrm{Im}(T_{i,j})\,\mathrm{Im}(T_{l,k})\,dT = \int_{\overline{B_E}} \mathrm{Re}(T_{i,j})\,\mathrm{Im}(T_{l,k})\,dT = 0,$$

$$\text{as well as} \quad \int_{\overline{B_E}} \mathrm{Re}(T_{i,j})\,\mathrm{Im}(T_{i,j})\,dT = 0.$$

On the other hand, when $i=j\neq l=k$, we have

$$\int_{\overline{B_E}} T_{i,i}\,T_{k,k}\,dT \simeq -\frac{1}{n}.$$

Remark 5. As we will see, the precise conclusions of Theorem 4 show that, in the case that $\mathbb{F}=\mathbb{C}$, the condition number of $\mathrm{Cov}\big(\overline{B_E}\big)$ is equal to $4+o(1)$, while in the case of $\mathbb{F}=\mathbb{R}$ it is equal to $2+o(1)$.

To the best of our knowledge, the almost isotropicity (or lack thereof) of the unit balls of p-Schatten norms in subspaces of self-adjoint matrices has not been examined for any other values of p except for $p=2$ (in which case we get the Euclidean ball in the corresponding subspaces).

It is also worth noting that, in the case $p=2$, the joint distribution of the eigenvalues of these matrix ensembles is closely linked (see Lemma 6, (14)) to the joint eigenvalue distributions of the well-known Gaussian Orthogonal, Unitary and Symplectic Ensembles (these are central among matrix ensembles and are extensively reviewed in the literature, see e.g. [40] and [2], and further references there; see also the mostly expository note [12], where the GOE, GUE and GSE are studied as part of another family of matrix ensembles). In [12, Subsection 10.2.2] it is observed that the joint eigenvalue distributions of the GOE, GUE and GSE, symmetrised so that they are invariant under permutation of the coordinates, are asymptotically isotropic. From our estimates in Section 3, the same can be concluded about the symmetrised joint eigenvalue distributions of the matrix ensembles we study here (see Proposition 16, (37) and (38), Proposition 17, (43) and (44), and Proposition 18, (49) and (50)).

The rest of the paper is organised as follows. In Section 2 we give exact statements for all the abovementioned results that we need. Theorem 2 and Theorem 4 are proven in Sections 3 and 4 respectively.

Finally, we make use of the fact that, in [15], Collins, Matsumoto and Saad deal also with the case of left-right invariant ensembles (which covers e.g. integration of polynomial functions over $B_{\mathcal{M}_n(\mathbb{F})}$). In Section 5 we exploit this to add to and complete the conclusions from [41] concerning the question whether the entries of $T\sim \mathrm{Unif}\big(B_{\mathcal{M}_n(\mathbb{F})}\big)$ are negatively correlated in a certain sense (for the precise definitions and statements see Section 5).

2. Preliminaries and overview of key prior results

We will denote by $\|\cdot\|_p$ the ℓ_p norm on $\mathbb{R}^n$ and by B_p^n its unit ball, namely $B_p^n=\big\{x\in\mathbb{R}^n : \|x\|_p := \big(\sum_{i=1}^n |x_i|^p\big)^{1/p}\leqslant 1\big\}$.

Let S_n be the symmetric group of permutations of the elements of $[n] := \{1, 2, \ldots, n\}$. We will say a function $F : \mathbb{R}^n \to \mathbb{R}$ is symmetric if $F(x_1, x_2, \ldots, x_n) = F(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$ for every $\sigma \in S_n$. Given $s \in \mathbb{R}$, we will say F is s-homogeneous if, for every $t > 0$, we have $F(tx) = t^s F(x)$.

Let n be a positive integer. A *partition* λ of n is a sequence of positive integers $(\lambda_1, \ldots, \lambda_m)$ such that $\lambda_1 \geqslant \cdots \geqslant \lambda_m$ and $\sum_{i=1}^m \lambda_i = n$; in such a case we write $\lambda \vdash n$ or $|\lambda| = n$. The integers λ_i are called the *parts* of λ, and their total number is the *length* of λ and is denoted by $l(\lambda)$. Sometimes we may need to consider sequences with a fixed number of terms, say m_0, in which case we will think of all partitions λ with $l(\lambda) \leqslant m_0$ as giving such sequences once we annex to them a finite number of 0's as appropriate (in this case $l(\lambda)$ will just be the number of non-zero parts, and we can also speak of partitions of 0 all of whose parts are necessarily 0).

Given a partition λ, the *monomial symmetric function* $m_\lambda(\boldsymbol{t})$ in n variables, where $n \geqslant l(\lambda)$, is given by

$$m_\lambda(t_1, \ldots, t_n) = \frac{1}{|\mathrm{Stab}(\lambda)|} \sum_{\sigma \in S_n} t_{\sigma(1)}^{\lambda_1} \cdots t_{\sigma(n)}^{\lambda_n},$$

where $|\mathrm{Stab}(\lambda)|$ denotes the order of the stabiliser of any monomial of type λ under the action of S_n (and dividing by it ensures we add each monomial only once). By convention, $m_\lambda(t_1, \ldots, t_n) = 0$ if $n < l(\lambda)$. Moreover, when $\lambda = (1, 1, \ldots, 1) = (1^k)$ for some $k \geqslant 1$, then we may also write $e_k(\boldsymbol{t})$ instead of $m_{(1^k)}(\boldsymbol{t})$ and call this the *k-th elementary symmetric function*.

The letters c, c', c_1, c_2 etc. denote absolute positive constants (which do not depend on the dimension of the Euclidean space we're in, or moreover on any of the other parameters unless specifically stated); their value may change from line to line. We will use the notation $A \simeq B$ (or $A \lesssim B$) to mean there exist absolute constants $c_1, c_2 > 0$ such that $c_1 A \le B \le c_2 A$ (or $A \leqslant c_1 B$). We will also use the Landau notation: $A = O(B)$ has the same meaning as $A \lesssim B$, whereas $A = o(B)$ will mean the ratio A/B tends to 0 as the dimension grows to infinity.

Recall that the uniform distribution over the unit ball of any p-Schatten norm in $\mathcal{M}_n(\mathbb{F})$ or its subspace of self-adjoint matrices defines an invariant ensemble of random matrices: we will call this *left-right invariant* ensemble if the distribution remains unchanged under multiplication either from the left or from the right by a fixed $\mathbb{F}$-unitary matrix (this is true in the case of $\mathcal{M}_n(\mathbb{F})$), and we will call it *conjugate invariant* if the distribution remains unchanged under conjugation by an $\mathbb{F}$-unitary matrix (this is true in the case of $\mathbb{F}$-self-adjoint matrices). Equivalently, the underlying distribution of a left-right invariant ensemble depends only on the distribution of the non-increasing rearrangement of the singular values of the matrices, whereas that of a conjugate invariant ensemble depends only on (the non-increasing rearrangement of) the eigenvalues.

2.1. *Reduction to Selberg-type integrals.* A consequence of left-right or conjugate invariance is that estimating integrals of functions that would also only depend on the singular values or eigenvalues of a matrix T in the ensemble (as is the case for the implied integrals in Theorem 2) can be reduced to computing integrals of highly symmetric distributions over $\mathbb{R}^n$ (and then we can examine whether there are more, analytic or combinatorial, tools to use). Moreover, when we consider the same question for any other p-Schatten norm, then (given that the integrands we are interested in, which are powers of the Euclidean norm, are also homogeneous functions) we can equivalently try to estimate the corresponding integrals with respect to densities of the form $\exp\big(-\|T\|_{S_p^n}^p\big)dT$. Proposition 7 below was proven in [41] based on the following key fact from Random Matrix Theory which makes what was just described precise (see for example [40] or [2, Propositions 4.1.3 and 4.1.1] for proofs).

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Lemma 6. *Let $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$, and let $F:\mathbb{R}^n\to\mathbb{R}$ be a measurable and symmetric function. Let us write $K_{p,E}$ for the unit ball of the p-Schatten norm on a subspace E of $\mathscr{M}_n(\mathbb{F})$, d_n for the dimension of E, and $f_{a,b,c}$ for the function*

$$x\in\mathbb{R}^n\mapsto\prod_{1\leqslant i<j\leqslant n}\left|x_i^a-x_j^a\right|^b\cdot\prod_{1\leqslant i\leqslant n}|x_i|^c.$$

Then:

(I) if $E=\mathscr{M}_n(\mathbb{F})$, there is a constant c_n depending only on E, such that

$$\int_{K_{p,E}}F\big(s_1(T),\cdots,s_n(T)\big)\,dT=c_n\int_{B_p^n}F\big(|x_1|,\cdots,|x_n|\big)\cdot f_{2,\beta,\beta-1}\,dx,\tag{11}$$

where $\beta=\dim_{\mathbb{R}}(\mathbb{F})$; furthermore, if $p<\infty$, and if F is also s-homogeneous for some $s>-d_n$, then

$$\int_{K_{p,E}}F\big(s_1(T),\cdots,s_n(T)\big)\,dT=\frac{c_n}{\Gamma\Big(1+\frac{d_n+s}{p}\Big)}\int_{\mathbb{R}^n}F\big(|x_1|,\cdots,|x_n|\big)e^{-\|x\|_p^p}f_{2,\beta,\beta-1}(x)\,dx.\tag{12}$$

(II) if E is the subspace of $\mathbb{F}$-self-adjoint matrices, there is a constant c_n depending only on E, such that

$$\int_{K_{p,E}}F\big(e_1(T),\cdots,e_n(T)\big)\,dT=c_n\int_{B_p^n}F(x)\cdot f_{1,\beta,0}\,dx;\tag{13}$$

similarly, if $p<\infty$ and F is s-homogeneous for some $s>-d_n$, then

$$\int_{K_{p,E}}F\big(e_1(T),\cdots,e_n(T)\big)\,dT=\frac{c_n}{\Gamma\Big(1+\frac{d_n+s}{p}\Big)}\int_{\mathbb{R}^n}F(x)e^{-\|x\|_p^p}f_{1,\beta,0}(x)\,dx.\tag{14}$$

Denote by $M_p(f)$ the integral of a function $f:\mathbb{R}^n\to\mathbb{R}$ with respect to the density $f_{a,b,c}(x)\cdot e^{-\|x\|_p^p}\,dx$, where a,b,c are going to depend appropriately on the subspace E that we consider. Furthermore, denote by $N_p(f)$ the corresponding integral with respect to the density $f_{a,b,c}(x)\cdot\mathbf{1}_{B_p^n}(x)\,dx$. The following proposition, following from Lemma 6, appears in [41]. (Note that one of the facts it relies on is that

$$\frac{N_p\big(\|x\|_2^2\big)}{N_p(1)}\simeq n^{1-\frac{2}{p}}\simeq d_n\,[\mathrm{vol}(K_{p,E})]^{2/d_n}\quad\text{and}\quad\frac{M_p\big(\|x\|_2^2\big)}{M_p(1)}\simeq n^{1+\frac{2}{p}};$$

these estimates follow by the main results of [43] and [35] and by [22, Proposition 3].)

Proposition 7. *For every $p\geqslant 1$, we have*

$$\sigma_{K_{p,E}}^2:=d_n\frac{\mathrm{Var}_{\overline{K_{p,E}}}\big(\|T\|_{S_2^n}^2\big)}{\left(\int_{\overline{K_{p,E}}}\|T\|_{S_2^n}^2\,dT\right)^2}\simeq n^{4/p}\,\mathrm{Var}_{N_p}\big(\|x\|_2^2\big):=n^{4/p}\,\frac{N_p\big(\|x\|_2^4\big)}{N_p(1)}-\left(\frac{N_p\big(\|x\|_2^2\big)}{N_p(1)}\right)^2,$$

while, if $p<\infty$ too, then

$$\mathrm{Var}_{M_p}\big(\|x\|_2^2\big):=\frac{M_p\big(\|x\|_2^4\big)}{M_p(1)}-\left(\frac{M_p\big(\|x\|_2^2\big)}{M_p(1)}\right)^2\simeq\max\Big\{\sigma_{K_{p,E}}^2,\frac{1}{p}\Big\}\cdot n^{4/p}.$$

Focusing on $p=\infty$ now, we see that, to accurately estimate $\sigma_{K_{\infty,E}}^2\equiv\sigma_{B_E}^2$, we should study integrals of the form

$$\int_{-1}^1\int_{-1}^1\cdots\int_{-1}^1 s(x)\prod_{1\leqslant i<j\leqslant n}\left|x_i^a-x_j^a\right|^b\cdot\prod_{1\leqslant i\leqslant n}|x_i|^c\,dx_n\ldots dx_2dx_1$$

where $a=1$ or 2, and where the integrand $s(x)$ is a symmetric polynomial (here of degree at most 4).

2.2. *Selberg's, Aomoto's, and Kadell's results.* Recall the formula for the value of the Euler beta integral:

$$\int_0^1 x^{u-1}(1-x)^{w-1}\,dx = \frac{\Gamma(u)\Gamma(w)}{\Gamma(u+w)},$$

where $\mathrm{Re}(u), \mathrm{Re}(w) > 0$. Selberg [44] (see also [40, Chapter 17] for a presentation of his original proof) discovered a high-dimensional generalisation of this formula: for every triple of complex numbers u, w, κ with

$$\mathrm{Re}(u) > 0, \quad \mathrm{Re}(w) > 0, \quad \mathrm{Re}(\kappa) > -\min\left(\frac{1}{n}, \frac{\mathrm{Re}(u)}{n-1}, \frac{\mathrm{Re}(w)}{n-1}\right),$$

if we set

$$h(\boldsymbol{t}; u, w, \kappa) := \prod_{1\leqslant i\leqslant n} t_i^{u-1}(1-t_i)^{w-1} \cdot \prod_{1\leqslant i<j\leqslant n} \left|t_i - t_j\right|^{2\kappa}$$

we have

$$I_0(n; u, w, \kappa) := \int\limits_{[0,1]^n} h(\boldsymbol{t}; u, w, \kappa)\,d\boldsymbol{t} \;=\; \prod_{1\leqslant i\leqslant n} \frac{\Gamma\big(1+(n-i+1)\kappa\big)}{\Gamma(1+\kappa)} \prod_{1\leqslant i\leqslant n} \frac{\Gamma\big(u+(n-i)\kappa\big)\Gamma\big(w+(n-i)\kappa\big)}{\Gamma\big(u+w+(2n-i-1)\kappa\big)}.$$

Aomoto [5] extended Selberg's result to more general integrals, where the integrand could be $h(\boldsymbol{t}; u, w, \kappa)$ multiplied by an elementary symmetric function $e_m(\boldsymbol{t})$:

$$e_m(\boldsymbol{t}) \;:=\; \sum_{1\leqslant i_1<\cdots<i_m\leqslant n} t_{i_1}\cdots t_{i_m} \qquad \text{with } 1\leqslant m<n.$$

We observe that by symmetry we have

$$\text{(15)}\qquad \int\limits_{[0,1]^n} e_m(\boldsymbol{t})\cdot h(\boldsymbol{t}; u, w, \kappa)\,d\boldsymbol{t} \;=\binom{n}{m}\int\limits_{[0,1]^n}\prod_{1\leqslant i\leqslant m} t_i\cdot h(\boldsymbol{t}; u, w, \kappa)\,d\boldsymbol{t}$$

$$\text{which Aomoto showed} \qquad =\binom{n}{m}\prod_{i=1}^{m}\frac{u+(n-i)\kappa}{u+w+(2n-i-1)\kappa}\, I_0(n; u, w, \kappa)$$

(recall that $I_0(n; u, w, \kappa)$ is Selberg's integral, and we can naturally extend this notation by writing $I_m = I_m(n; u, w, \kappa)$ for the right-hand-side integral in (15)). In fact, Aomoto used these expressions to conclude that the ratio

$$\frac{1}{I_0(n; u, w, \kappa)}\int\limits_{[0,1]^n}\prod_{1\leqslant i\leqslant n}(t_i - y)\cdot h(\boldsymbol{t}; u, w, \kappa)\,d\boldsymbol{t}$$

is equal to a certain Jacobi polynomial:

$$\frac{1}{I_0(n; u, w, \kappa)}\int\limits_{[0,1]^n}\prod_{1\leqslant i\leqslant n}(t_i - y)\cdot h(\boldsymbol{t}; u, w, \kappa)\,d\boldsymbol{t} = \frac{n!}{\prod_i(\alpha+\beta+n+i)} P_n^{(\alpha,\beta)}(1-2y),$$

where $\alpha = -1+2u/\kappa,\ \beta = -1+2w/\kappa$ and $P_n^{(\alpha,\beta)}$ is the Jacobi polynomial of degree n.

Aomoto's approach relied on finding recurrence relations between the different I_m which would follow from integration by parts. It should be mentioned that our main argument in [41] was along very similar lines.

With only a little more effort (see [3, Chapter 8]), Aomoto's proof method can also give similar formulas when the integrand involves slightly more general symmetric polynomials having terms of

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the form

$$\prod_{i=1}^{m_1} t_i \cdot \prod_{j=m_1+1-m_3}^{m_1+m_2-m_3} (1-t_j)$$

where $m_1, m_2, m_3 \geqslant 0$ and $m_3 \leqslant m_1$, $m_1 + m_2 - m_3 \leqslant n$: we have

$$(16) \quad I_{m_1,m_2,m_3} := \int_{[0,1]^n} \prod_{i=1}^{m_1} t_i \cdot \prod_{j=m_1+1-m_3}^{m_1+m_2-m_3} (1-t_j) \cdot h(\boldsymbol{t}; u, w, \kappa)\, d\boldsymbol{t}$$

$$= \prod_{i=1}^{m_3} \frac{\big(u+w+(n-i-1)\kappa\big)}{\big(u+w+1+(2n-i-1)\kappa\big)} \cdot \frac{\prod_{i=1}^{m_1}\big(u+(n-i)\kappa\big)\prod_{i=1}^{m_2}\big(w+(n-i)\kappa\big)}{\prod_{i=1}^{m_1+m_2}\big(u+w+(2n-i-1)\kappa\big)} I_0(n; u, w, \kappa).$$

Note that if $m_3 > 0$, then there is some overlap in factors of the two products, something which allows us to get additional factors of the form $t_i(1-t_i)$ for some i only (and will allow us, for instance, to exactly compute $\int_{B_E} \|T\|^4_{S_4^n}\, dT$ when $E = \mathscr{M}_n(\mathbb{F})$).

Kadell [28] (see also Kaneko [29], as well as later proofs in [6] and [48]) has extended these results in the most general way: he has shown that, for each $\kappa \geqslant 0$, there is an infinite family of homogeneous symmetric polynomials $\{s^\kappa_\lambda(\boldsymbol{t})\}$ indexed by the partitions, which spans the space of symmetric polynomials, and such that the polynomial corresponding to the partition λ has the following properties:

- $s^\kappa_\lambda(t_1, \dots, t_n) = m_\lambda(\boldsymbol{t}) + \sum_{\substack{\mu \neq \lambda \\ |\mu| = |\lambda|}} a^\kappa_{\lambda,\mu,n}\, m_\mu(\boldsymbol{t})$ where $n \geqslant l(\lambda)$, and where $a^\kappa_{\lambda,\mu,n}$ are coefficients which depend on κ, λ and μ, and which might also depend on the number of variables n (but, as we will shortly see, don't).
- For every $n \geqslant l(\lambda)$ we have $s^\kappa_\lambda(1^n) = \dfrac{f^\kappa_n[\lambda]}{f^\kappa_n[(0)]}$ where

$$f^\kappa_n[\lambda] := \prod_{\substack{i<j \\ \lambda_i - \lambda_j > 0}} \big(\lambda_i - \lambda_j + (j-i)\kappa\big)_\kappa \cdot \prod_{\substack{i<j \\ \lambda_i - \lambda_j = 0}} \frac{j-i}{j-i+1} \cdot \big(1 + (j-i)\kappa\big)_\kappa$$

  and where $(x)_m := \frac{\Gamma(x+m)}{\Gamma(x)}$ stands for the *Pochhammer function* or *rising factorial* (here m can take non-integer values too), and moreover we have

$$\int_{[0,1]^n} s^\kappa_\lambda(\boldsymbol{t}) \cdot \prod_{1\leqslant i \leqslant n} t_i^{u-1}(1-t_i)^{w-1} \prod_{1 \leqslant i < j \leqslant n} \big|t_i - t_j\big|^{2\kappa}\, d\boldsymbol{t} \equiv \int_{[0,1]^n} s^\kappa_\lambda(\boldsymbol{t}) \cdot h(\boldsymbol{t}; u, w, \kappa)\, d\boldsymbol{t}$$

$$(17) \quad = I_0(n; u, w, \kappa) \cdot s^\kappa_\lambda(1^n) \prod_{i=1}^{n} \frac{\big(u+(n-i)\kappa\big)_{\lambda_i}}{\big(u+w+(2n-i-1)\kappa\big)_{\lambda_i}}$$

$$= n!\, f^\kappa_n[\lambda] \prod_{1 \leqslant i \leqslant n} \frac{\Gamma\big(u+(n-i)\kappa+\lambda_i\big)\Gamma\big(w+(n-i)\kappa\big)}{\Gamma\big(u+w+(2n-i-1)\kappa+\lambda_i\big)}.$$

This family can in fact be taken to be the family of (monic) Jack polynomials corresponding to the parameter $1/\kappa$, that is, $s^k_\lambda(\boldsymbol{t}) = P_\lambda(\boldsymbol{t}; 1/\kappa)$ for every partition λ.

Although we will not need this in the sequel, let us recall for the sake of completeness that one way of defining the family of Jack polynomials $\big\{P_\lambda(\boldsymbol{t};\xi)\big\}$ corresponding to a parameter ξ is as follows (see e.g. [39, Chapter VI]). Recall that, for any non-negative integer b, we can define the power-sum function

$p_b(t_1,\dots,t_n) := \sum_{i=1}^n t_i^b$; we then extend this notion by defining for every partition $\lambda = (\lambda_1,\dots,\lambda_m)$ a power-sum function $p_\lambda(\boldsymbol{t}) := \prod_{j=1}^m p_{\lambda_j}(\boldsymbol{t})$. We can also define a (partial) ordering of the partitions, called the *dominance* ordering, by setting $\mu \preccurlyeq \lambda$ if and only if $|\mu| = |\lambda|$ and $\mu_1 + \cdots + \mu_i \leqslant \lambda_1 + \cdots + \lambda_i$ for every $i \geqslant 1$. Finally, consider the field $\mathbb{Q}(\xi)$ of all rational functions of ξ (seen as an indeterminate) with coefficients in $\mathbb{Q}$ and also the vector space $\mathbb{Q}(\xi)\big[\{m_\lambda(t_1,\dots,t_n) : \lambda \text{ partition}, l(\lambda) \leqslant n\}\big]$ of all symmetric polynomials in n variables with coefficients from $\mathbb{Q}(\xi)$. We can define a scalar product $\langle\cdot,\cdot\rangle_\xi$ on this vector space by setting

$$\big\langle p_\lambda, p_\mu \big\rangle_\xi := z_\lambda \xi^{l(\lambda)} \cdot \mathbf{1}_{\lambda=\mu}, \tag{18}$$

where $z_\lambda = \prod_{i=1}^{l(\lambda)} a_i! \cdot i^{a_i}$ with a_i being the number of parts of λ equal to i. Then the family of Jack polynomials $\{P_\lambda(\boldsymbol{t};\xi) : \lambda \text{ partition}\}$ in n variables is the unique family of functions in $\mathbb{Q}(\xi)\big[\{m_\lambda(\boldsymbol{t})\}\big]$ satisfying the following two properties:

- <u>Orthogonality</u> $\big\langle P_\lambda(\boldsymbol{t};\xi), P_\mu(\boldsymbol{t};\xi)\big\rangle_\xi = 0$ if $\mu \neq \lambda$.
- <u>Triangularity</u> If we write

$$P_\lambda(\boldsymbol{t};\xi) = \sum_{\mu : l(\mu) \leqslant n} c(\lambda,\mu,n;\xi)\, m_\mu(\boldsymbol{t})$$

  for some coefficients $c(\lambda,\mu,n;\xi) \in \mathbb{Q}(\xi)$, then $c(\lambda,\mu,n;\xi) \neq 0$ only if $\mu \preccurlyeq \lambda$ and $c(\lambda,\lambda,n;\xi) = 1$.

Actually this definition overdetermines the family of Jack polynomials, which means that a priori it is not clear that there exists any family from $\mathbb{Q}(\xi)\big[\{m_\lambda(\boldsymbol{t})\}\big]$ which has these two properties. However it can be shown that such a family exists, and then necessarily it is unique.

Moreover, it can be shown that the coefficients $c(\lambda,\mu,n;\xi)$ do not depend on n, and therefore the Jack polynomials have the following stability property: for every $n_1 \geqslant n_2 \geqslant l(\lambda)$,

$$P_\lambda\big((t_1,\dots,t_{n_2},\mathbf{0}_{n_1-n_2});\xi\big) \equiv P_\lambda\big((t_1,\dots,t_{n_2});\xi\big).$$

For convenience we also set $P_\lambda\big((t_1,\dots,t_m);\xi\big) \equiv 0$ if $m < l(\lambda)$.

Alternatively, we can obtain the Jack polynomials corresponding to ξ by considering the eigenfunctions of the following operator arising in the Calogero-Sutherland model, which aims to describe a system of n identical quantum particles on a circle (see e.g. [47], [45]):

$$D_\xi^* = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}\left(x_i \frac{\partial}{\partial x_i}\right) + \frac{1}{\xi}\sum_{i<j} \frac{x_i + x_j}{x_i - x_j}\left(x_i \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_j}\right).$$

The Jack polynomial $P_\lambda(t_1,\dots,t_n;\xi)$ is the unique homogeneous and symmetric polynomial eigenfunction with eigenvalue $\sum_{i=1}^n \big(\lambda_i^2 + \frac{1}{\xi}(n-1-2i)\lambda_i\big)$ which is monic and whose leading terms are of type λ (in other words, we choose the normalisation $P_\lambda(\boldsymbol{t};\xi) = m_\lambda(\boldsymbol{t}) + \sum_{\mu<\lambda} c(\lambda,\mu;\xi) m_\mu(\boldsymbol{t})$).

Setting ξ equal to different non-zero real values (although it has to be noted that the orthogonalising inner product defined above will be positive definite only for positive real values), we obtain different families of symmetric polynomials. With $\xi = 1$ the corresponding family is the Schur polynomials $\big\{P_\lambda(\boldsymbol{t};1)\big\}$, which are intimately connected with the representation theory of the symmetric groups S_n and of the (complex) general linear groups. Other important values, and essentially the only ones we care about for the main applications in this paper, are $\xi = 2$, which gives the zonal polynomials $\big\{P_\lambda(\boldsymbol{t};2)\big\}$ associated with real symmetric matrices, and $\xi = \frac{1}{2}$, which gives the quaternion zonal polynomials $\big\{P_\lambda(\cdot;1/2)\big\}$ associated with the quaternionic self-adjoint matrices.

What is important to us in this note is having transition matrices from the basis $\{s_\lambda^\kappa(\boldsymbol{t})\} = \{P_\lambda^{1/\kappa}(\boldsymbol{t})\}$ to the basis of monomial functions of degree up to 4 and vice versa. These can be found via the determinantal expressions for the Jack polynomials in terms of the monomial functions which were established by Lapointe, Lascoux and Morse [36]. They are given in the following tables (and actually, in the specific cases of the special families of the Schur or zonal polynomials ($\kappa = 1, 1/2$ or 2), such tables were known even before [36]).

(19)

$$P^{1/\kappa}_{(1)} = m_{(1)}$$

	$m_{(2)}$	$m_{(1^2)}$
$P^{1/\kappa}_{(2)}$	1	$\frac{2\kappa}{\kappa+1}$
$P^{1/\kappa}_{(1^2)}$	0	1

	$m_{(3)}$	$m_{(2,1)}$	$m_{(1^3)}$
$P^{1/\kappa}_{(3)}$	1	$\frac{3\kappa}{\kappa+2}$	$\frac{6\kappa^2}{(\kappa+1)(\kappa+2)}$
$P^{1/\kappa}_{(2,1)}$	0	1	$\frac{6\kappa}{2\kappa+1}$
$P^{1/\kappa}_{(1^3)}$	0	0	1

	$m_{(4)}$	$m_{(3,1)}$	$m_{(2^2)}$	$m_{(2,1^2)}$	$m_{(1^4)}$
$P^{1/\kappa}_{(4)}$	1	$\frac{4\kappa}{\kappa+3}$	$\frac{6\kappa(\kappa+1)}{(\kappa+2)(\kappa+3)}$	$\frac{12\kappa^2}{(\kappa+2)(\kappa+3)}$	$\frac{24\kappa^3}{(\kappa+1)(\kappa+2)(\kappa+3)}$
$P^{1/\kappa}_{(3,1)}$	0	1	$\frac{2\kappa}{\kappa+1}$	$\frac{(5\kappa+3)\kappa}{(\kappa+1)^2}$	$\frac{12\kappa^2}{(\kappa+1)^2}$
$P^{1/\kappa}_{(2^2)}$	0	0	1	$\frac{2\kappa}{\kappa+1}$	$\frac{12\kappa^2}{(\kappa+1)(2\kappa+1)}$
$P^{1/\kappa}_{(2,1^2)}$	0	0	0	1	$\frac{12\kappa}{3\kappa+1}$
$P^{1/\kappa}_{(1^4)}$	0	0	0	0	1

(20)

	$P^{1/\kappa}_{(2)}$	$P^{1/\kappa}_{(1^2)}$
$m_{(2)}$	1	$-\frac{2\kappa}{\kappa+1}$
$m_{(1^2)}$	0	1

	$P^{1/\kappa}_{(3)}$	$P^{1/\kappa}_{(2,1)}$	$P^{1/\kappa}_{(1^3)}$
$m_{(3)}$	1	$-\frac{3\kappa}{\kappa+2}$	$\frac{6\kappa^2}{(\kappa+1)(2\kappa+1)}$
$m_{(2,1)}$	0	1	$-\frac{6\kappa}{2\kappa+1}$
$m_{(1^3)}$	0	0	1

	$P^{1/\kappa}_{(4)}$	$P^{1/\kappa}_{(3,1)}$	$P^{1/\kappa}_{(2^2)}$	$P^{1/\kappa}_{(2,1^2)}$	$P^{1/\kappa}_{(1^4)}$
$m_{(4)}$	1	$-\frac{4\kappa}{\kappa+3}$	$\frac{2\kappa(\kappa-1)}{(\kappa+1)(\kappa+2)}$	$\frac{4\kappa^2}{(\kappa+1)^2}$	$-\frac{24\kappa^3}{(\kappa+1)(2\kappa+1)(3\kappa+1)}$
$m_{(3,1)}$	0	1	$-\frac{2\kappa}{\kappa+1}$	$-\frac{\kappa(\kappa+3)}{(\kappa+1)^2}$	$\frac{24\kappa^2}{(2\kappa+1)(3\kappa+1)}$
$m_{(2^2)}$	0	0	1	$-\frac{2\kappa}{\kappa+1}$	$\frac{12\kappa^2}{(2\kappa+1)(3\kappa+1)}$
$m_{(2,1^2)}$	0	0	0	1	$-\frac{12\kappa}{3\kappa+1}$
$m_{(1^4)}$	0	0	0	0	1

2.3. *Weingarten calculus for invariant ensembles.* A permutation $\sigma \in S_k$ can be decomposed into cycles. If the numbers of lengths of cycles are $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_l$, then the sequence $\mu = (\mu_1, \mu_2, \dots, \mu_l)$ is a partition of k. We will refer to μ as the cycle-type of σ. Recall that the different cycle-types correspond to the different conjugacy classes of S_k. Recall also that characters of S_k are class functions, that is, they take the same value at permutations belonging to the same conjugacy class or, in other words, having the same cycle-type.

For the (pairwise non-isomorphic) irreducible representations of S_k, there is a canonical way of identifying each one of them with a unique partition of k and vice-versa (see e.g. [42, Section 2.3] or [21, Chapter 4]). This also gives a natural one-to-one and onto correspondence between the irreducible characters of S_k and partitions of k, which allows us to write the character table of S_k in terms of partitions (in fact, to find $\chi^\lambda(\mu)$, the value of the character correspoding to λ at a permutation with cycle-type μ, one can use the Frobenius formula, see e.g. [21, Proposition 4.37]). In our computations in Sections 4 and 5 we will need to plug in values of characters of S_2, S_3 and S_4, so the character tables for these are recalled here:

(21)

	(1^2)	(2)
$\chi^{(2)}$	1	1
$\chi^{(1^2)}$	1	-1

	(1^3)	$(2,1)$	(3)
$\chi^{(3)}$	1	1	1
$\chi^{(2,1)}$	2	0	-1
$\chi^{(1^3)}$	1	-1	1

(22)

	(1^4)	$(2,1^2)$	(2^2)	$(3,1)$	(4)
$\chi^{(4)}$	1	1	1	1	1
$\chi^{(3,1)}$	3	1	-1	0	-1
$\chi^{(2^2)}$	2	0	2	-1	0
$\chi^{(2,1^2)}$	3	-1	-1	0	1
$\chi^{(1^4)}$	1	-1	1	1	-1

2.3.1. *The unitary case.* For two sequences $\boldsymbol{i} = (i_1, \dots, i_k)$ and $\boldsymbol{i}' = (i'_1, \dots, i'_k)$ of positive integers and for a permutation $\pi \in S_k$, set

$$\delta_\pi(\boldsymbol{i}, \boldsymbol{i}') = \prod_{s=1}^{n} \delta_{i_{\pi(s)}, i'_s}, \tag{23}$$

where $\delta_{i,j} = \mathbf{1}_{\{i=j\}}$.

Given a square matrix A and a permutation $\pi \in S_k$ of cycle-type $\mu = (\mu_1, \mu_2, \dots, \mu_l)$, set

$$\mathrm{Tr}_\pi(A) = \prod_{j=1}^{l} \mathrm{Tr}(A^{\mu_j}). \tag{24}$$

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Finally, given a partition λ of k and a number $z \in \mathbb{C}$, define

$$C_\lambda(z) = \prod_{i=1}^{l(\lambda)} \prod_{j=1}^{\lambda_i} (z + j - i) \tag{25}$$

(in the applications below we are going to evaluate $C_\lambda(z)$ at $z = n$; in this case, this is just the value at $1^n = (1, \dots, 1)$ of the Jack polynomial $J^1_\lambda(\boldsymbol{t}) \equiv c_\lambda \cdot P^1_\lambda(\boldsymbol{t})$ under a different normalisation, see e.g. [46, Theorem 5.4]).

One of the equivalent ways of defining the *unitary Weingarten function* on S_k with one complex parameter $z \in \mathbb{C}$ (see [16] or [15]) is the following: it is the complex-valued function on S_k given by

$$\pi \in S_k \quad \mapsto \quad \mathrm{Wg}^U(\pi; z) := \frac{1}{k!} \sum_{\substack{\lambda \vdash k \\ C_\lambda(z) \neq 0}} \frac{\chi^\lambda(e)}{C_\lambda(z)} \chi^\lambda(\pi), \tag{26}$$

where e is the identity permutation in S_k. Note that, unless $z \in \{0, \pm 1, \dots, \pm(k-1)\}$, $C_\lambda(z) \neq 0$ for all partitions $\lambda \vdash k$. Note also that $\mathrm{Wg}^U(\pi; z)$ depends only on the cycle-type of π.

It is convenient to also consider the convolution of two Weingarten functions. Recall that, for two complex-valued functions f_1, f_2 on S_k,

$$(f_1 * f_2)(\pi) := \sum_{\tau \in S_k} f_1(\pi\tau) f_2(\tau^{-1}) = \sum_{\tau \in S_k} f_1(\tau) f_2(\tau^{-1}\pi).$$

We set

$$\pi \in S_k \quad \mapsto \quad \mathrm{Wg}^U(\pi; z, w) := \left(\mathrm{Wg}^U(\cdot; z) * \mathrm{Wg}^U(\cdot; w)\right)(\pi), \tag{27}$$

where $z, w \in \mathbb{C}$.

By Schur's lemma and the orthogonality relations it entails (see also [23, Theorem 2.13] for a different derivation), we can also write

$$\mathrm{Wg}^U(\pi; z, w) = \frac{1}{k!} \sum_{\substack{\lambda \vdash k \\ C_\lambda(z) C_\lambda(w) \neq 0}} \frac{\chi^\lambda(e)}{C_\lambda(z) C_\lambda(w)} \chi^\lambda(\pi). \tag{28}$$

Theorem 8. (Conjugacy invariance, [15, Theorem 3.1]) *Let $T = (T_{ij})$ be an $n \times n$ Hermitian random matrix whose distribution has the property that UTU^* is distributed in the same way as T for any unitary matrix U. For two sequences $\boldsymbol{i} = (i_1, \dots, i_k)$ and $\boldsymbol{j} = (j_1, \dots, j_k)$, we have*

$$\mathbb{E}[T_{i_1 j_1} T_{i_2 j_2} \cdots T_{i_k j_k}] = \sum_{\sigma, \tau \in S_k} \delta_\sigma(\boldsymbol{i}, \boldsymbol{j}) \mathrm{Wg}^U(\sigma^{-1}\tau; n)\, \mathbb{E}[\mathrm{Tr}_\tau(T)].$$

Theorem 9. (Left-right invariance, [15, Theorem 3.4]) *Let X be a complex $n \times p$ random matrix which has the same distribution as UXV for any unitary matrices U, V. Then, for four sequences $\boldsymbol{i} = (i_1, \dots, i_k)$, $\boldsymbol{i}' = (i'_1, \dots, i'_k)$, $\boldsymbol{j} = (j_1, \dots, j_k)$ and $\boldsymbol{j}' = (j'_1, \dots, j'_k)$, we have*

$$\mathbb{E}\big[X_{i_1 j_1} X_{i_2 j_2} \cdots X_{i_k j_k} \overline{X_{i'_1 j'_1} X_{i'_2 j'_2} \cdots X_{i'_k j'_k}}\big] = \sum_{\sigma_1, \sigma_2, \tau \in S_k} \delta_{\sigma_1}(\boldsymbol{i}, \boldsymbol{i}') \delta_{\sigma_2}(\boldsymbol{j}, \boldsymbol{j}') \mathrm{Wg}^U(\tau \sigma_1^{-1} \sigma_2; n, p)\, \mathbb{E}[\mathrm{Tr}_\tau(XX^*)].$$

Remark 10. The proof of either theorem proceeds along very similar lines: one notes that T or X has the same distribution as UDU^* or UDV^* respectively, where D is a diagonal matrix (with the same distribution of eigenvalues or singular values as T or X respectively), U, V are Haar-distributed random unitary matrices, and D, U and V are all independent. Then, once the integrals we are interested in are rewritten using these decompositions, one invokes the following pivotal result in Weingarten calculus (see e.g. [16, Corollary 3.4]).

Theorem 11. *Let $U = (U_{ij})_{1\leqslant i,j\leqslant n}$ be an $n\times n$ Haar-distributed unitary matrix. For four sequences $\boldsymbol{i} = (i_1,\dots,i_k)$, $\boldsymbol{i}' = (i'_1,\dots,i'_k)$, $\boldsymbol{j} = (j_1,\dots,j_k)$ and $\boldsymbol{j}' = (j'_1,\dots,j'_k)$ of positive integers in $[n]$, we have*

$$\int_{U(n)} U_{i_1j_1}U_{i_2j_2}\cdots U_{i_kj_k}\overline{U_{i'_1j'_1}U_{i'_2j'_2}\cdots U_{i'_kj'_k}}\,dU = \sum_{\sigma,\tau\in S_k}\delta_\sigma(\boldsymbol{i},\boldsymbol{i}')\delta_\tau(\boldsymbol{j},\boldsymbol{j}')\,\mathrm{Wg}^U(\sigma^{-1}\tau;n).$$

2.3.2. *The orthogonal case.* For every $\sigma\in S_{2k}$ we can consider an undirected graph $G(\sigma)$ with vertices $1,2,\dots,2k$ and edge set consisting of

$$\big\{\{2i-1,2i\} : i=1,2,\dots,k\big\}\cup\big\{\{\sigma(2i-1),\sigma(2i)\} : i=1,2,\dots,k\big\}$$

(note that we consider as different every two edges of the form $\{2i-1,2i\}$ and $\{\sigma(2j-1),\sigma(2j)\}$ even if the sets coincide). Then each vertex lies on exactly two edges, and the number of vertices in each connected component is even. If the numbers of vertices in the connected components are $2\mu_1\geqslant 2\mu_2\geqslant\cdots\geqslant 2\mu_l$, then the sequence $\mu=(\mu_1,\mu_2,\dots,\mu_l)$ is a partition of k which is called the coset-type of σ.

Let M_{2k} be the set of all pair partitions of the set $[2k]=\{1,\dots,2k\}$. A pair partition $\sigma\in M_{2k}$ can be uniquely expressed in the form

$$\sigma = \big\{\{\sigma(1),\sigma(2)\},\{\sigma(3),\sigma(4)\},\dots,\{\sigma(2k-1),\sigma(2k)\}\big\}$$

where $1=\sigma(1)<\sigma(3)<\cdots<\sigma(2i-1)<\cdots<\sigma(2k-1)$ and $\sigma(2i-1)<\sigma(2i)$ for every $1\leqslant i\leqslant k$. Then σ can also be regarded as a permutation $\begin{pmatrix}1 & 2 & \cdots & 2k\\ \sigma(1) & \sigma(2) & \cdots & \sigma(2k)\end{pmatrix}$ in S_{2k}. In this way we can embed M_{2k} into S_{2k} (in particular, we can talk about the coset-type of a pair partition $\sigma\in M_{2k}$).

For a permutation $\sigma\in S_{2k}$ and a $2k$-tuple $\boldsymbol{i}=(i_1,i_2,\dots,i_{2k})$ of positive integers, set

$$\delta'_\sigma(\boldsymbol{i}) = \prod_{s=1}^{k}\delta_{i_{\sigma(2s-1)},i_{\sigma(2s)}}. \tag{29}$$

In particular, if $\sigma\in M_{2k}$, then we can more simply write $\delta'_\sigma(\boldsymbol{i}) = \prod_{\{a,b\}\in\sigma}\delta_{i_a,i_b}$.

Given a square matrix A and $\sigma\in S_{2k}$ with coset-type $\mu=(\mu_1,\mu_2,\dots,\mu_l)$, set

$$\mathrm{Tr}'_\sigma(A) = \prod_{j=1}^{l}\mathrm{Tr}(A^{\mu_j}). \tag{30}$$

Finally, given a partition λ of k and a number $z\in\mathbb{C}$, define

$$C'_\lambda(z) = \prod_{i=1}^{l(\lambda)}\prod_{j=1}^{\lambda_i}(z+2j-i-1) \tag{31}$$

(again $C'_\lambda(n) = J^2_\lambda(1^n)$, see [46, Theorem 5.4]).

To be able to give the definition for the *orthogonal Weingarten function* that is analogous to the one we gave above in the unitary case, we first need to recall how the *zonal spherical functions* on S_{2k} are defined. Let H_k be the hyperoctahedral group of order $2^k k!$; this can be realised as the subgroup of S_{2k} generated by adjacent tranpositions $(2i-1\ \ 2i)$ for any $1\leqslant i\leqslant k$ and double transpositions of the form $(2i-1\ \ 2j-1)(2i\ \ 2j)$ for any $1\leqslant i<j\leqslant k$. Then for each partition λ of k, consider the

partition $2\lambda = (2\lambda_1, 2\lambda_2, \ldots, 2\lambda_{l(\lambda)})$ of $2k$ and the corresponding character $\chi^{2\lambda}$ of S_{2k}, and define the zonal spherical function ω^λ corresponding to λ by

$$\sigma \in S_{2k} \quad \mapsto \quad \omega^\lambda(\sigma) := \frac{1}{2^k k!}\big(\chi^{2\lambda} * \mathbf{1}_{H_k}\big)(\sigma) = \frac{1}{2^k k!}\sum_{\pi \in S_{2k}} \chi^{2\lambda}(\sigma\pi)\, \mathbf{1}_{H_k}\big(\pi^{-1}\big). \tag{32}$$

Given that H_k is a subgroup of S_{2k} and that M_{2k} contains a unique representative of each **left** coset σH_k of H_k in S_{2k}, this definition can be rewritten in a somewhat simpler way:

$$\omega^\lambda(\sigma) = \frac{1}{2^k k!}\sum_{\tau \in M_{2k}}\sum_{\zeta \in H_k} \chi^{2\lambda}\big(\sigma\tau\zeta\big)\, \mathbf{1}_{H_k}\big((\tau\zeta)^{-1}\big) = \frac{1}{2^k k!}\sum_{\zeta \in H_k} \chi^{2\lambda}(\sigma\zeta). \tag{33}$$

Recall finally that the zonal sperical functions ω^λ corresponding to partitions λ of k form a linear basis of $L(S_{2k}, H_k)$, the space of all complex-valued functions on S_{2k} which are H_k-bi-invariant, that is, the set

$$\big\{f : S_{2k} \to \mathbb{C} \mid f(\zeta\sigma) = f(\sigma\zeta) = f(\sigma) \text{ for every } \sigma \in S_{2k}, \zeta \in H_k\big\}.$$

We now define the orthogonal Weingarten function on S_{2k} with one complex parameter $z \in \mathbb{C}$ (see [14] or [15]):

$$\sigma \in S_{2k} \quad \mapsto \quad \mathrm{Wg}^O(\sigma; z) := \frac{2^k k!}{(2k)!}\sum_{\substack{\lambda \vdash k \\ C'_\lambda(z) \neq 0}} \frac{\chi^{2\lambda}(e)}{C'_\lambda(z)}\, \omega^\lambda(\sigma). \tag{34}$$

Note that all ω^λ, and therefore also $Wg^O(\cdot; z)$, take the same value at permutations σ_1, σ_2 with the same coset-type (where equivalently σ_1 has the same coset-type as σ_2 if and only if $\sigma_1 \in H_k \sigma_2 H_k$).

Theorem 12. (Conjugacy invariance, [15, Theorem 3.3]) *Let $T = (T_{ij})$ be an $n \times n$ real symmetric random matrix with the invariance property that OTO^t has the same distribution as T for any orthogonal matrix O. For any sequence $\boldsymbol{i} = (i_1, \ldots, i_{2k})$, we have*

$$\mathbb{E}\big[T_{i_1 i_2} T_{i_3 i_4} \cdots T_{i_{2k-1} i_{2k}}\big] = \sum_{\sigma, \tau \in M_{2k}} \delta'_\sigma(\boldsymbol{i})\, \mathrm{Wg}^O(\sigma^{-1}\tau; n)\, \mathbb{E}\big[\mathrm{Tr}'_\tau(T)\big].$$

Theorem 13. (Left-right invariance, [15, Theorem 3.5]) *Let X be a real $n \times p$ random matrix which has the same distribution as OXQ for any orthogonal matrices O, Q. Then, for two sequences $\boldsymbol{i} = (i_1, \ldots, i_{2k})$ and $\boldsymbol{j} = (j_1, \ldots, j_{2k})$, we have*

$$\mathbb{E}\big[X_{i_1 j_1} X_{i_2 j_2} \cdots X_{i_{2k} j_{2k}}\big] = \sum_{\sigma_1, \sigma_2, \tau_1, \tau_2 \in M_{2k}} \delta'_{\sigma_1}(\boldsymbol{i})\delta'_{\sigma_2}(\boldsymbol{j})\, \mathrm{Wg}^O(\sigma_1^{-1}\tau_1; n)\, \mathrm{Wg}^O(\sigma_2^{-1}\tau_2; p)\, \mathbb{E}[\mathrm{Tr}'_{\tau_1^{-1}\tau_2}(XX^t)].$$

Remark 14. Again the proof of the theorems follows from a decomposition of T or X as ODO^t or ODQ^t respectively (with D diagonal with the same distribution of eigenvalues or singular values as T or X respectively, O and Q Haar-distributed random orthogonal matrices, and D, O and Q independent), combined with the use of the following result (see [16, Corollary 3.4] and [14]).

Theorem 15. *Let $O = (O_{ij})_{1 \le i, j \le n}$ be an $n \times n$ Haar-distributed orthogonal matrix. For sequences $\boldsymbol{i} = (i_1, \ldots, i_{2k})$, $\boldsymbol{j} = (j_1, \ldots, j_{2k})$ of positive integers in $[n]$, we have*

$$\int_{O(n)} O_{i_1 j_1} O_{i_2 j_2} \cdots O_{i_{2k} j_{2k}}\, dO = \sum_{\sigma, \tau \in M_{2k}} \delta'_\sigma(\boldsymbol{i})\delta'_\tau(\boldsymbol{j})\, \mathrm{Wg}^O(\sigma^{-1}\tau; n).$$

Note that the statement of Theorem 13 above is slightly different from that in [15], the conclusion following from the proof on [15, p. 9], and being compatible with the invariances of ensembles such as $X \sim \mathrm{Unif}(K_{p, \mathcal{M}_n(\mathbb{R})})$ under taking transpose.

3. Proof of Theorem 2

Let us start with the case where $E = \mathscr{M}_n(\mathbb{F})$. By Proposition 7 it suffices to show that

$$\mathrm{Var}_{N_\infty}\big(\|x\|_2^2\big) = \frac{N_\infty\big(\|x\|_2^4\big)}{N_\infty(1)} - \left(\frac{N_\infty\big(\|x\|_2^2\big)}{N_\infty(1)}\right)^2 \simeq 1,$$

where in this case

$$N_\infty(f) = \int_{[-1,1]^n} f(x) \cdot \prod_{1\leqslant i<j\leqslant n} \big|x_i^2 - x_j^2\big|^\beta \cdot \prod_{1\leqslant i\leqslant n} |x_i|^{\beta-1}\,dx$$

with $\beta = \dim_{\mathbb{R}}(\mathbb{F})$. Since all the functions f we need to consider are symmetric and in addition their values only depend on what the absolute values of the coordinates of their input are, we have

$$\mathrm{Var}_{N_\infty}\big(\|x\|_2^2\big) = \frac{\tilde{N}_\infty\big(\|x\|_2^4\big)}{\tilde{N}_\infty(1)} - \left(\frac{\tilde{N}_\infty\big(\|x\|_2^2\big)}{\tilde{N}_\infty(1)}\right)^2$$

where

$$\tilde{N}_\infty(f) := \int_{[0,1]^n} f(x) \cdot \prod_{1\leqslant i<j\leqslant n} \big|x_i^2 - x_j^2\big|^\beta \cdot \prod_{1\leqslant i\leqslant n} |x_i|^{\beta-1}\,dx = \frac{1}{2^n} N_\infty(f)$$

for all the functions considered. Furthermore, by symmetry again,

$$\mathrm{Var}_{N_\infty}\big(\|x\|_2^2\big) = n\frac{\tilde{N}_\infty(x_1^4)}{\tilde{N}_\infty(1)} + n(n-1)\frac{\tilde{N}_\infty(x_1^2x_2^2)}{\tilde{N}_\infty(1)} - n^2\left(\frac{\tilde{N}_\infty(x_1^2)}{\tilde{N}_\infty(1)}\right)^2. \tag{35}$$

Employing now the transformation $x = (x_1, x_2, \ldots, x_n) \in [0,1]^n \mapsto \big(\sqrt{x_1}, \sqrt{x_2}, \ldots, \sqrt{x_n}\big)$ which has Jacobian $x \in (0,1)^n \mapsto 2^{-n}\prod_i x_i^{-1/2}$, we can obtain the following:

$$\tilde{N}_\infty(1) = 2^{-n}\int_{[0,1]^n} \prod_{1\leqslant i\leqslant n} x_i^{\frac{\beta}{2}-1} \cdot \prod_{1\leqslant i<j\leqslant n} \big|x_i - x_j\big|^\beta dx = 2^{-n} I_0\Big(n; \frac{\beta}{2}, 1, \frac{\beta}{2}\Big),$$

$$\tilde{N}_\infty(x_1^2) = 2^{-n}\int_{[0,1]^n} x_1 \prod_{1\leqslant i\leqslant n} x_i^{\frac{\beta}{2}-1} \cdot \prod_{1\leqslant i<j\leqslant n} \big|x_i - x_j\big|^\beta dx = 2^{-n} I_1\Big(n; \frac{\beta}{2}, 1, \frac{\beta}{2}\Big),$$

$$\tilde{N}_\infty\big(x_1^2x_2^2\big) = 2^{-n}\int_{[0,1]^n} x_1x_2 \prod_{1\leqslant i\leqslant n} x_i^{\frac{\beta}{2}-1} \cdot \prod_{1\leqslant i<j\leqslant n} \big|x_i - x_j\big|^\beta dx = 2^{-n} I_2\Big(n; \frac{\beta}{2}, 1, \frac{\beta}{2}\Big),$$

and finally

$$\tilde{N}_\infty\big(x_1^4\big) = 2^{-n}\int_{[0,1]^n} x_1^2 \prod_{1\leqslant i\leqslant n} x_i^{\frac{\beta}{2}-1} \cdot \prod_{1\leqslant i<j\leqslant n} \big|x_i - x_j\big|^\beta dx = 2^{-n}\Big(I_1\Big(n; \frac{\beta}{2}, 1, \frac{\beta}{2}\Big) - I_{1,1,1}\Big(n; \frac{\beta}{2}, 1, \frac{\beta}{2}\Big)\Big)$$

(recall the notation in Subsection 2.2). Using the formulas in (15) and (16), we see that

$$\frac{\tilde{N}_\infty\big(x_1^2\big)}{\tilde{N}_\infty(1)} = \frac{n\beta/2}{1 + (2n-1)\beta/2},$$

$$\frac{\tilde{N}_\infty\big(x_1^2x_2^2\big)}{\tilde{N}_\infty(1)} = \frac{n(n-1)\beta^2/4}{(1 + (2n-1)\beta/2)(1 + (n-1)\beta)},$$

$$\begin{aligned}\frac{\tilde N_\infty(x_1^4)}{\tilde N_\infty(1)} &= \frac{n\beta/2}{1+(2n-1)\beta/2} - \frac{1+(n-1)\beta/2}{2+(2n-1)\beta/2}\cdot\frac{n\beta/2(1+(n-1)\beta/2)}{(1+(2n-1)\beta/2)(1+(n-1)\beta)}\\ &= \frac{n\beta/2(1/2+3(n-1)\beta/4)}{(1+(2n-1)\beta/2)(1+(n-1)\beta)} + \frac{n\beta^2/8(1+(n-1)\beta/2)}{(2+(2n-1)\beta/2)(1+(2n-1)\beta/2)(1+(n-1)\beta)}.\end{aligned}$$

Plugging these into (35), we deduce that

$$\begin{aligned}&n\frac{\tilde N_\infty(x_1^4)}{\tilde N_\infty(1)} + n(n-1)\frac{\tilde N_\infty(x_1^2x_2^2)}{\tilde N_\infty(1)} - n^2\left(\frac{\tilde N_\infty(x_1^2)}{\tilde N_\infty(1)}\right)^2\\ &= \frac{n^4\beta^2/4 - n^3\beta^2/8 + n^2\beta/2(1/2-\beta/4)}{(1+(2n-1)\beta/2)(1+(n-1)\beta)} + \frac{n^3\beta^3/16 + n^2\beta^2/8(1-\beta/2)}{(2+(2n-1)\beta/2)(1+(2n-1)\beta/2)(1+(n-1)\beta)}\\ &\quad - \frac{n^4\beta^2/4}{(1+(2n-1)\beta/2)(1+(n-1)\beta)}\left(1-\frac{\beta/2}{1+(2n-1)\beta/2}\right)\\ &= \frac{n^2\beta/2(1/2-\beta/4)}{(1+(2n-1)\beta/2)(1+(n-1)\beta)} + \frac{n^3\beta^3/16 + n^2\beta^2/8(1-\beta/2)}{(2+(2n-1)\beta/2)(1+(2n-1)\beta/2)(1+(n-1)\beta)}\\ &\quad + \frac{n^3\beta^2/8(\beta/2-1)}{(1+(2n-1)\beta/2)^2(1+(n-1)\beta)}\\ &= \frac{n^3\beta^2/8 + n^2\beta/2\big((1/2-\beta/4)(2-\beta/2)+\beta/4(1-\beta/2)\big)}{(2+(2n-1)\beta/2)(1+(2n-1)\beta/2)(1+(n-1)\beta)} + \frac{n^3\beta^2/8(\beta/2-1)}{(2+(2n-1)\beta/2)(1+(2n-1)\beta/2)^2(1+(n-1)\beta)}\\ &= \frac{1}{8\beta} + O\Big(\frac{1}{n}\Big).\end{aligned}$$

This agrees with the conclusion of [41, Theorem 1] (see more specifically the end of Section 4 in [41]).

We now turn to the cases of the subspaces of $\mathbb{F}$-self-adjoint matrices. Recall that by Proposition 7 it suffices to show

$$\begin{aligned}\mathrm{Var}_{N_\infty}\big(\|x\|_2^2\big) &= \frac{N_\infty\big(\|x\|_2^4\big)}{N_\infty(1)} - \left(\frac{N_\infty\big(\|x\|_2^2\big)}{N_\infty(1)}\right)^2\\ &= n\frac{N_\infty(x_1^4)}{N_\infty(1)} + n(n-1)\frac{N_\infty(x_1^2x_2^2)}{N_\infty(1)} - n^2\left(\frac{N_\infty(x_1^2)}{N_\infty(1)}\right)^2 \simeq 1,\end{aligned}\tag{36}$$

where now

$$N_\infty(f) = \int_{[-1,1]^n} f(x)\cdot\prod_{1\leqslant i<j\leqslant n}\big|x_i-x_j\big|^\beta\,dx$$

with $\beta=\dim_{\mathbb{R}}(\mathbb{F})$. For each of the functions f in (36) we can write

$$\begin{aligned}N_\infty(f) &= \int_{[-\frac12,\frac12]^n} 2^n f(2x_1,\dots,2x_n)\cdot\prod_{1\leqslant i<j\leqslant n}\big|2x_i-2x_j\big|^\beta\,dx\\ &= 2^{n+\beta n(n-1)/2+s}\int_{[-\frac12,\frac12]^n} f(x_1,\dots,x_n)\cdot\prod_{1\leqslant i<j\leqslant n}\big|x_i-x_j\big|^\beta\,dx\\ &= 2^{n+\beta n(n-1)/2+s}\int_{[0,1]^n} f\Big(t_1-\frac12,\dots,t_n-\frac12\Big)\cdot\prod_{1\leqslant i<j\leqslant n}\big|t_i-t_j\big|^\beta\,d\boldsymbol{t},\end{aligned}$$

where s is the degree of homogeneity of f. Thus, upon writing

$$J_\infty(g) = \int_{[0,1]^n} g(\boldsymbol{t}) \cdot \prod_{1\leq i<j\leq n} \left| t_i - t_j \right|^\beta d\boldsymbol{t},$$

we see that, to verify (36), we need to estimate

$$J_\infty\left(\left(t_1 - \frac{1}{2}\right)^2\right) = J_\infty(t_1^2) - J_\infty(t_1) + \frac{1}{4} J_\infty(1) = \frac{1}{n} J_\infty(m_{(2)}) - \frac{1}{n} J_\infty(m_{(1)}) + \frac{1}{4} J_\infty(1),$$

$$\begin{aligned} J_\infty\left(\left(t_1 - \frac{1}{2}\right)^2\left(t_2 - \frac{1}{2}\right)^2\right) &= J_\infty(t_1^2 t_2^2) - J_\infty(t_1^2 t_2 + t_1 t_2^2) + \frac{1}{4} J_\infty(t_1^2 + t_2^2) + J_\infty(t_1 t_2) - \frac{1}{4} J_\infty(t_1 + t_2) + \frac{1}{16} J_\infty(1) \\ &= \frac{2}{n(n-1)} J_\infty(m_{(2^2)}) - \frac{2}{n(n-1)} J_\infty(m_{(2,1)}) + \frac{1}{2n} J_\infty(m_{(2)}) \\ &\quad + \frac{2}{n(n-1)} J_\infty(m_{(1^2)}) - \frac{1}{2n} J_\infty(m_{(1)}) + \frac{1}{16} J_\infty(1), \end{aligned}$$

$$\begin{aligned} J_\infty\left(\left(t_1 - \frac{1}{2}\right)^4\right) &= J_\infty(t_1^4) - 2J_\infty(t_1^3) + \frac{3}{2} J_\infty(t_1^2) - \frac{1}{2} J_\infty(t_1) + \frac{1}{16} J_\infty(1) \\ &= \frac{1}{n} J_\infty(m_{(4)}) - \frac{2}{n} J_\infty(m_{(3)}) + \frac{3}{2n} J_\infty(m_{(2)}) - \frac{1}{2n} J_\infty(m_{(1)}) + \frac{1}{16} J_\infty(1). \end{aligned}$$

We will do so by recalling the decompositions of the monomial symmetric functions in the bases of the Schur or the zonal or the quaternionic zonal polynomials (see tables (19) and (20)), and by using integration formula (17). Denote by $I_n^\kappa(\lambda)$ the integral

$$\int_{[0,1]^n} P_\lambda^{1/\kappa}(\boldsymbol{t}) \prod_{1\leq i<j\leq n} \left| t_i - t_j \right|^{2\kappa} d\boldsymbol{t} = \int_{[0,1]^n} s_\lambda^\kappa(\boldsymbol{t}) \prod_{1\leq i<j\leq n} \left| t_i - t_j \right|^{2\kappa} d\boldsymbol{t}.$$

For simplicity and to make it easier to check the tedious computations, in what follows we treat the cases of $\mathbb{C}, \mathbb{R}$ and $\mathbb{H}$ separately (note moreover that, even though the below computations could be done for more general values of β (see Remark 19), and would still have an interpretation via a random matrix model (see [18]), this interpretation would not correspond to the same type of variance problem as the one we are interested in here).

Proposition 16. *(Case of $\beta = 2, \kappa = 1$; Hermitian matrices) The following estimates are true:*

(37)
$$\begin{aligned} \frac{N_\infty(x_1^2)}{N_\infty(1)} &= 2\frac{J_\infty\left(\left(t_1 - \frac{1}{2}\right)^2\right)}{J_\infty(1)} = \frac{1}{4} - \frac{1}{16n^2} + O\left(\frac{1}{n^3}\right), \\ \frac{N_\infty(x_1^2 x_2^2)}{N_\infty(1)} &= 4\frac{J_\infty\left(\left(t_1 - \frac{1}{2}\right)^2\left(t_2 - \frac{1}{2}\right)^2\right)}{J_\infty(1)} = \frac{1}{16} - \frac{1}{32n} - \frac{1}{32n^2} + O\left(\frac{1}{n^3}\right) \end{aligned}$$

and

$$\frac{N_\infty(x_1^4)}{N_\infty(1)} = 4\frac{J_\infty\left(\left(t_1 - \frac{1}{2}\right)^4\right)}{J_\infty(1)} = \frac{3}{32} + O\left(\frac{1}{n^2}\right).$$

As a consequence,

$$\mathrm{Var}_{N_\infty}\left(\|x\|_2^2\right) = n\frac{N_\infty(x_1^4)}{N_\infty(1)} + n(n-1)\frac{N_\infty(x_1^2 x_2^2)}{N_\infty(1)} - n^2\left(\frac{N_\infty(x_1^2)}{N_\infty(1)}\right)^2 = \frac{1}{32} + O\left(\frac{1}{n}\right).$$

Moreover,

$$\frac{N_\infty(x_1x_2)}{N_\infty(1)} = 2\frac{J_\infty\left(\left(t_1-\frac{1}{2}\right)\left(t_2-\frac{1}{2}\right)\right)}{J_\infty(1)} = -\frac{1}{4n}-\frac{1}{8n^2}-\frac{1}{16n^3}+O\left(\frac{1}{n^4}\right) \tag{38}$$

(this is an estimate we will need in the following section).

Proof. We begin with the simple observation that for all κ we have

$$J_\infty(1) = I_n^\kappa((0)) \qquad \text{and} \qquad J_\infty(m_{(1)}) = I_n^\kappa((1)) = \frac{n}{2}I_n^\kappa((0)).$$

Furthermore, when $\kappa = 1$,

$$J_\infty(m_{(1^2)}) = I_n^1((1^2)) = I_n^1((0))\frac{n(n-1)}{4}\frac{n-1}{2n-1} = I_n^1((0))\,\frac{n(n-1)}{2}\left(\frac{1}{4}-\frac{1}{8n}-\frac{1}{16n^2}-\frac{1}{32n^3}+O\left(\frac{1}{n^4}\right)\right)$$
$$= I_n^1((0))\,n\left(\frac{n}{8}-\frac{3}{16}+\frac{1}{32n}+\frac{1}{64n^2}+O\left(\frac{1}{n^3}\right)\right)$$
$$\text{and} \qquad I_n^1((2)) = I_n^1((0))\frac{n(n+1)}{4}\frac{n+1}{2n+1} = I_n^1((0))\,n\left(\frac{n}{8}+\frac{3}{16}+\frac{1}{32n}-\frac{1}{64n^2}+O\left(\frac{1}{n^3}\right)\right).$$

Therefore,

$$J_\infty(m_{(2)}) = I_n^1((2)) - I_n^1((1^2)) = I_n^1((0))\,n\left(\frac{3}{8}-\frac{1}{32n^2}+O\left(\frac{1}{n^3}\right)\right),$$

which also gives

$$J_\infty\left(\left(t_1-\frac{1}{2}\right)^2\right) = \frac{1}{n}J_\infty(m_{(2)}) - \frac{1}{n}J_\infty(m_{(1)}) + \frac{1}{4}J_\infty(1) = I_n^1((0))\left(\frac{1}{8}-\frac{1}{32n^2}+O\left(\frac{1}{n^3}\right)\right). \tag{39}$$

Note also that

$$J_\infty\left(\left(t_1-\frac{1}{2}\right)\left(t_2-\frac{1}{2}\right)\right) = \frac{2}{n(n-1)}J_\infty(m_{(1^2)}) - \frac{1}{n}J_\infty(m_{(1)}) + \frac{1}{4}J_\infty(1) = I_n^1((0))\left(-\frac{1}{8n}-\frac{1}{16n^2}-\frac{1}{32n^3}+O\left(\frac{1}{n^4}\right)\right). \tag{40}$$

Next observe that

$$I_n^1((1^3)) = I_n^1((0))\frac{n(n-1)(n-2)}{24}\frac{n-2}{2n-1} = I_n^1((0))\,\frac{n(n-1)}{2}\left(\frac{n}{24}-\frac{7}{48}+\frac{3}{32n}+\frac{3}{64n^2}+O\left(\frac{1}{n^3}\right)\right)$$
$$= I_n^1((0))\,n\left(\frac{n^2}{48}-\frac{3n}{32}+\frac{23}{192}-\frac{3}{128n}+O\left(\frac{1}{n^2}\right)\right),$$

$$I_n^1((2,1)) = I_n^1((0))\frac{n(n-1)(n+1)}{6}\frac{n+1}{2n+1}\frac{n-1}{2n-1} = I_n^1((0))\,\frac{n(n-1)}{2}\left(\frac{n}{12}+\frac{1}{12}-\frac{1}{16n}-\frac{1}{16n^2}+O\left(\frac{1}{n^3}\right)\right)$$
$$= I_n^1((0))\,n\left(\frac{n^2}{24}-\frac{7}{96}+O\left(\frac{1}{n^2}\right)\right)$$

and

$$I_n^1((3)) = I_n^1((0))\frac{(n+2)(n+1)n}{24}\frac{n+2}{2n+1} = I_n^1((0))\,n\left(\frac{n^2}{48}+\frac{3n}{32}+\frac{23}{192}+\frac{3}{128n}+O\left(\frac{1}{n^2}\right)\right).$$

It follows that

$$J_\infty(m_{(2,1)}) = I_n^1((2,1)) - 2I_n^1((1^3)) = I_n^1((0))\,\frac{n(n-1)}{2}\left(\frac{3}{8}-\frac{1}{4n}-\frac{5}{32n^2}+O\left(\frac{1}{n^3}\right)\right)$$

and

$$J_\infty(m_{(3)}) = I_n^1((3)) - I_n^1((2,1)) + I_n^1((1^3)) = I_n^1((0))\,n\left(\frac{5}{16}+O\left(\frac{1}{n^2}\right)\right).$$

Moreover,

$$I_n^1((1^4)) = I_n^1((0))\frac{n(n-1)(n-2)(n-3)}{96}\frac{n-2}{2n-1}\frac{n-3}{2n-3} = I_n^1((0))\frac{n(n-1)}{2}\Big(\frac{n^2}{192}-\frac{n}{24}+\frac{27}{256}-\frac{9}{128n}-\frac{33}{1024n^2}+O\Big(\frac{1}{n^3}\Big)\Big)$$
$$= I_n^1((0))\,n\Big(\frac{n^3}{384}-\frac{3n^2}{128}+\frac{113n}{1536}-\frac{45}{1536}+\frac{39}{2048n}+O\Big(\frac{1}{n^2}\Big)\Big),$$

$$I_n^1((2,1^2)) = I_n^1((0))\frac{(n+1)n(n-1)(n-2)}{32}\frac{n+1}{2n+1}\frac{n-2}{2n-1} = I_n^1((0))\frac{n(n-1)}{2}\Big(\frac{n^2}{64}-\frac{n}{32}-\frac{11}{256}+\frac{7}{128n}+\frac{53}{1024n^2}+O\Big(\frac{1}{n^3}\Big)\Big)$$
$$= I_n^1((0))\,n\Big(\frac{n^3}{128}-\frac{3n^2}{128}-\frac{3n}{512}+\frac{25}{512}-\frac{3}{2048n}+O\Big(\frac{1}{n^2}\Big)\Big),$$

$$I_n^1((2^2)) = I_n^1((0))\frac{n(n-1)(n+1)n}{48}\frac{n+1}{2n+1}\frac{n-1}{2n-1} = I_n^1((0))\frac{n(n-1)}{2}\Big(\frac{n^2}{96}+\frac{n}{96}-\frac{1}{128}-\frac{1}{128n}-\frac{1}{512n^2}+O\Big(\frac{1}{n^3}\Big)\Big)$$
$$= I_n^1((0))\,n\Big(\frac{n^3}{192}-\frac{7n}{768}+\frac{3}{1024n}+O\Big(\frac{1}{n^2}\Big)\Big),$$

while

$$I_n^1((3,1)) = I_n^1((0))\frac{(n+2)(n+1)n(n-1)}{32}\frac{n+2}{2n+1}\frac{n-1}{2n-1} = I_n^1((0))\,n\Big(\frac{n^3}{128}+\frac{3n^2}{128}-\frac{3n}{512}-\frac{25}{512}-\frac{3}{2048n}+O\Big(\frac{1}{n^2}\Big)\Big)$$

and

$$I_n^1((4)) = I_n^1((0))\frac{(n+3)(n+2)(n+1)n}{96}\frac{n+3}{2n+3}\frac{n+2}{2n+1} = I_n^1((0))\,n\Big(\frac{n^3}{384}+\frac{3n^2}{128}+\frac{113n}{1536}+\frac{45}{512}+\frac{39}{2048n}+O\Big(\frac{1}{n^2}\Big)\Big).$$

It follows that

$$J_\infty(m_{(2^2)}) = I_n^1((2^2)) - I_n^1((2,1^2)) + I_n^1((1^4)) = I_n^1((0))\frac{n(n-1)}{2}\Big(\frac{9}{64}-\frac{17}{128n}-\frac{11}{128n^2}+O\Big(\frac{1}{n^3}\Big)\Big)$$

and

$$J_\infty(m_{(4)}) = I_n^1((4)) - I_n^1((3,1)) + I_n^1((2,1^2)) + I_n^1((1^4)) = I_n^1((0))\,n\Big(\frac{35}{128}+O\Big(\frac{1}{n^2}\Big)\Big).$$

We conclude that

$$J_\infty\Big(\Big(t_1-\frac{1}{2}\Big)^2\Big(t_2-\frac{1}{2}\Big)^2\Big) = \frac{2}{n(n-1)}J_\infty(m_{(2^2)}) - \frac{2}{n(n-1)}J_\infty(m_{(2,1)}) + \frac{1}{2n}J_\infty(m_{(2)})$$
$$+\frac{2}{n(n-1)}J_\infty(m_{(1^2)}) - \frac{1}{2n}J_\infty(m_{(1)}) + \frac{1}{16}J_\infty(1)$$
$$= I_n^1((0))\Big(\frac{1}{64}-\frac{1}{128n}-\frac{1}{128n^2}+O\Big(\frac{1}{n^3}\Big)\Big), \tag{41}$$

while

$$J_\infty\Big(\Big(t_1-\frac{1}{2}\Big)^4\Big) = \frac{1}{n}J_\infty(m_{(4)}) - \frac{2}{n}J_\infty(m_{(3)}) + \frac{3}{2n}J_\infty(m_{(2)}) - \frac{1}{2n}J_\infty(m_{(1)}) + \frac{1}{16}J_\infty(1)$$
$$= I_n^1((0))\Big(\frac{3}{128}+O\Big(\frac{1}{n^2}\Big)\Big). \tag{42}$$

This completes the proof of Theorem 2 when $\mathbb{F}=\mathbb{C}$. □

Proposition 17. *(Case of $\beta = 1$, $\kappa = \frac{1}{2}$; $\mathbb{R}$-self-adjoint matrices) The following estimates are true:*

$$\frac{N_\infty(x_1^2)}{N_\infty(1)} = 2\frac{J_\infty\Big(\big(t_1-\frac{1}{2}\big)^2\Big)}{J_\infty(1)} = \frac{1}{4} - \frac{1}{8n} + \frac{1}{16n^2} + O\Big(\frac{1}{n^3}\Big), \tag{43}$$

$$\frac{N_\infty(x_1^2x_2^2)}{N_\infty(1)} = 4\frac{J_\infty\Big(\big(t_1-\frac{1}{2}\big)^2\big(t_2-\frac{1}{2}\big)^2\Big)}{J_\infty(1)} = \frac{1}{16} - \frac{3}{32n} + \frac{3}{32n^2} + O\Big(\frac{1}{n^3}\Big)$$

and

$$\frac{N_\infty(x_1^4)}{N_\infty(1)} = 4\frac{J_\infty\Big(\big(t_1-\frac{1}{2}\big)^4\Big)}{J_\infty(1)} = \frac{3}{32} - \frac{5}{64n} + O\Big(\frac{1}{n^2}\Big).$$

As a consequence,

$$\mathrm{Var}_{N_\infty}\big(\|x\|_2^2\big) = \frac{1}{16} + O\Big(\frac{1}{n}\Big).$$

Moreover,

$$\frac{N_\infty(x_1x_2)}{N_\infty(1)} = 2\frac{J_\infty\Big(\big(t_1-\frac{1}{2}\big)\big(t_2-\frac{1}{2}\big)\Big)}{J_\infty(1)} = -\frac{1}{4n} + \frac{1}{8n^2} - \frac{1}{16n^3} + O\Big(\frac{1}{n^4}\Big). \tag{44}$$

Proof. When $\kappa = \frac{1}{2}$,

$$J_\infty(m_{(1^2)}) = I_n^{1/2}((1^2)) = I_n^{1/2}((0))\frac{n(n-1)}{4}\frac{n}{2n+1} = I_n^{1/2}((0))\,\frac{n(n-1)}{2}\Big(\frac{1}{4} - \frac{1}{8n} + \frac{1}{16n^2} - \frac{1}{32n^3} + O\Big(\frac{1}{n^4}\Big)\Big)$$
$$= I_n^{1/2}((0))\,n\Big(\frac{n}{8} - \frac{3}{16} + \frac{3}{32n} - \frac{3}{64n^2} + O\Big(\frac{1}{n^3}\Big)\Big)$$

and $$I_n^{1/2}((2)) = I_n^{1/2}((0))\frac{n(n+2)}{12}\frac{n+3}{n+2} = I_n^{1/2}((0))\,n\Big(\frac{n}{12} + \frac{1}{4}\Big).$$

Therefore,

$$J_\infty(m_{(2)}) = I_n^{1/2}((2)) - \frac{2}{3}I_n^{1/2}((1^2)) = I_n^{1/2}((0))\,n\Big(\frac{3}{8} - \frac{1}{16n} + \frac{1}{32n^2} + O\Big(\frac{1}{n^3}\Big)\Big),$$

which also gives

$$J_\infty\Big(\Big(t_1-\frac{1}{2}\Big)^2\Big) = \frac{1}{n}J_\infty(m_{(2)}) - \frac{1}{n}J_\infty(m_{(1)}) + \frac{1}{4}J_\infty(1) = I_n^{1/2}((0))\Big(\frac{1}{8} - \frac{1}{16n} + \frac{1}{32n^2} + O\Big(\frac{1}{n^3}\Big)\Big). \tag{45}$$

Note also that

$$J_\infty\Big(\Big(t_1-\frac{1}{2}\Big)\Big(t_2-\frac{1}{2}\Big)\Big) = \frac{2}{n(n-1)}J_\infty(m_{(1^2)}) - \frac{1}{n}J_\infty(m_{(1)}) + \frac{1}{4}J_\infty(1) = I_n^{1/2}((0))\Big(-\frac{1}{8n} + \frac{1}{16n^2} - \frac{1}{32n^3} + O\Big(\frac{1}{n^4}\Big)\Big). \tag{46}$$

Next observe that

$$I_n^{1/2}((1^3)) = I_n^{1/2}((0))\frac{n(n-1)(n-2)}{24}\frac{n-1}{2n+1} = I_n^{1/2}((0))\,\frac{n(n-1)}{2}\Big(\frac{n}{24} - \frac{7}{48} + \frac{5}{32n} - \frac{5}{64n^2} + O\Big(\frac{1}{n^3}\Big)\Big)$$
$$= I_n^{1/2}((0))\,n\Big(\frac{n^2}{48} - \frac{3n}{32} + \frac{29}{192} - \frac{15}{128n} + O\Big(\frac{1}{n^2}\Big)\Big),$$

$$I_n^{1/2}((2,1)) = I_n^{1/2}((0))\frac{n(n-1)(n+2)}{16}\frac{n+3}{n+2}\frac{n}{2n+1} = I_n^{1/2}((0))\,\frac{n(n-1)}{2}\Big(\frac{n}{16} + \frac{5}{32} - \frac{5}{64n} + \frac{5}{128n^2} + O\Big(\frac{1}{n^3}\Big)\Big)$$
$$= I_n^{1/2}((0))\,n\Big(\frac{n^2}{32} + \frac{3n}{64} - \frac{15}{128} + \frac{15}{256n} + O\Big(\frac{1}{n^2}\Big)\Big)$$

and

$$I_n^{1/2}((3)) = I_n^{1/2}((0))\frac{(n+4)(n+2)n}{120}\frac{n+5}{n+2} = I_n^{1/2}((0))\, n\Big(\frac{n^2}{120}+\frac{3n}{40}+\frac{1}{6}\Big).$$

It follows that

$$J_\infty(m_{(2,1)}) = I_n^{1/2}((2,1)) - \frac{3}{2}I_n^{1/2}((1^3)) = I_n^{1/2}((0))\,\frac{n(n-1)}{2}\Big(\frac{3}{8}-\frac{5}{16n}+\frac{5}{32n^2}+O\Big(\frac{1}{n^3}\Big)\Big)$$

and

$$J_\infty(m_{(3)}) = I_n^{1/2}((3)) - \frac{3}{5}I_n^{1/2}((2,1)) + \frac{1}{2}I_n^{1/2}((1^3)) = I_n^{1/2}((0))\, n\Big(\frac{5}{16}-\frac{3}{32n}+O\Big(\frac{1}{n^2}\Big)\Big).$$

Moreover,

$$\begin{aligned}I_n^{1/2}((1^4)) = I_n^{1/2}((0))\frac{n(n-1)(n-2)(n-3)}{96}\frac{n-1}{2n+1}\frac{n-2}{2n-1} &= I_n^{1/2}((0))\,\frac{n(n-1)}{2}\Big(\frac{n^2}{192}-\frac{n}{24}+\frac{31}{256}-\frac{5}{32n}+\frac{95}{1024n^2}+O\Big(\frac{1}{n^3}\Big)\Big)\\ &= I_n^{1/2}((0))\, n\Big(\frac{n^3}{384}-\frac{3n^2}{128}+\frac{125n}{1536}-\frac{71}{512}+\frac{255}{2048n}+O\Big(\frac{1}{n^2}\Big)\Big),\end{aligned}$$

$$\begin{aligned}I_n^{1/2}((2,1^2)) &= I_n^{1/2}((0))\frac{(n+2)n(n-1)(n-2)}{80}\frac{n+3}{n+2}\frac{n-1}{2n+1}\\ &= I_n^{1/2}((0))\,\frac{n(n-1)}{2}\Big(\frac{n^2}{80}-\frac{n}{160}-\frac{27}{320}+\frac{15}{128n}-\frac{15}{256n^2}+O\Big(\frac{1}{n^3}\Big)\Big)\\ &= I_n^{1/2}((0))\, n\Big(\frac{n^3}{160}-\frac{3n^2}{320}-\frac{5n}{128}+\frac{129}{1280}-\frac{45}{512n}+O\Big(\frac{1}{n^2}\Big)\Big),\end{aligned}$$

$$\begin{aligned}I_n^{1/2}((2^2)) &= I_n^{1/2}((0))\frac{n(n-1)(n+2)(n+1)}{96}\frac{n+3}{n+2}\frac{n+2}{2n+3}\frac{n}{2n+1}\\ &= I_n^{1/2}((0))\,\frac{n(n-1)}{2}\Big(\frac{n^2}{192}+\frac{n}{96}+\frac{3}{256}-\frac{1}{128n}+\frac{7}{1024n^2}+O\Big(\frac{1}{n^3}\Big)\Big)\\ &= I_n^{1/2}((0))\, n\Big(\frac{n^3}{384}+\frac{n^2}{128}-\frac{7n}{1536}-\frac{5}{512}\frac{15}{2048n}+O\Big(\frac{1}{n^2}\Big)\Big),\end{aligned}$$

while

$$I_n^{1/2}((3,1)) = I_n^{1/2}((0))\frac{(n+4)(n+2)n(n-1)}{144}\frac{n+5}{n+2}\frac{n}{2n+1} = I_n^{1/2}((0))\, n\Big(\frac{n^3}{288}+\frac{5n^2}{192}+\frac{29n}{1152}-\frac{21}{256}+\frac{21}{512n}+O\Big(\frac{1}{n^2}\Big)\Big)$$

and

$$I_n^{1/2}((4)) = I_n^{1/2}((0))\frac{(n+6)(n+4)(n+2)n}{1680}\frac{n+7}{n+4}\frac{n+5}{n+2} = I_n^{1/2}((0))\, n\Big(\frac{n^3}{1680}+\frac{3n^2}{280}+\frac{107n}{1680}+\frac{1}{8}\Big).$$

It follows that

$$J_\infty(m_{(2^2)}) = I_n^{1/2}((2^2)) - \frac{2}{3}I_n^{1/2}((2,1^2)) + \frac{3}{5}I_n^{1/2}((1^4)) = I_n^{1/2}((0))\,\frac{n(n-1)}{2}\Big(\frac{9}{64}-\frac{23}{128n}+\frac{13}{128n^2}+O\Big(\frac{1}{n^3}\Big)\Big)$$

and

$$J_\infty(m_{(4)}) = I_n^{1/2}((4)) - \frac{4}{7}I_n^{1/2}((3,1)) - \frac{2}{15}I_n^{1/2}((2^2)) + \frac{4}{9}I_n^{1/2}((2,1^2)) - \frac{2}{5}I_n^{1/2}((1^4)) = I_n^{1/2}((0))\, n\Big(\frac{35}{128}-\frac{29}{256n}+O\Big(\frac{1}{n^2}\Big)\Big).$$

We conclude that

$$J_\infty\Big(\Big(t_1-\frac{1}{2}\Big)^2\Big(t_2-\frac{1}{2}\Big)^2\Big)=\frac{2}{n(n-1)}J_\infty(m_{(2^2)})-\frac{2}{n(n-1)}J_\infty(m_{(2,1)})+\frac{1}{2n}J_\infty(m_{(2)})$$
$$+\frac{2}{n(n-1)}J_\infty(m_{(1^2)})-\frac{1}{2n}J_\infty(m_{(1)})+\frac{1}{16}J_\infty(1)$$
$$=I_n^{1/2}((0))\Big(\frac{1}{64}-\frac{3}{128n}+\frac{3}{128n^2}+O\Big(\frac{1}{n^3}\Big)\Big),\tag{47}$$

while

$$J_\infty\Big(\Big(t_1-\frac{1}{2}\Big)^4\Big)=\frac{1}{n}J_\infty(m_{(4)})-\frac{2}{n}J_\infty(m_{(3)})+\frac{3}{2n}J_\infty(m_{(2)})-\frac{1}{2n}J_\infty(m_{(1)})+\frac{1}{16}J_\infty(1)$$
$$=I_n^{1/2}((0))\Big(\frac{3}{128}-\frac{5}{256n}+O\Big(\frac{1}{n^2}\Big)\Big).\tag{48}$$

This completes the proof of Theorem 2 when $\mathbb{F}=\mathbb{R}$. □

Proposition 18. *(Case of $\beta=4,\kappa=2$; $\mathbb{H}$-self-adjoint matrices) The following estimates are true:*

$$\frac{N_\infty(x_1^2)}{N_\infty(1)}=2\frac{J_\infty\Big(\big(t_1-\frac{1}{2}\big)^2\Big)}{J_\infty(1)}=\frac{1}{4}+\frac{1}{16n}+\frac{1}{64n^2}+O\Big(\frac{1}{n^3}\Big),\tag{49}$$

$$\frac{N_\infty(x_1^2x_2^2)}{N_\infty(1)}=4\frac{J_\infty\Big(\big(t_1-\frac{1}{2}\big)^2\big(t_2-\frac{1}{2}\big)^2\Big)}{J_\infty(1)}=\frac{1}{16}-\frac{3}{256n^2}+O\Big(\frac{1}{n^3}\Big)$$

and

$$\frac{N_\infty(x_1^4)}{N_\infty(1)}=4\frac{J_\infty\Big(\big(t_1-\frac{1}{2}\big)^4\Big)}{J_\infty(1)}=\frac{3}{32}+\frac{5}{128n}+O\Big(\frac{1}{n^2}\Big).$$

As a consequence,

$$\mathrm{Var}_{N_\infty}(\|x\|_2^2)=\frac{1}{64}+O\Big(\frac{1}{n}\Big).$$

Moreover,

$$\frac{N_\infty(x_1x_2)}{N_\infty(1)}=2\frac{J_\infty\Big(\big(t_1-\frac{1}{2}\big)\big(t_2-\frac{1}{2}\big)\Big)}{J_\infty(1)}=-\frac{1}{4(n-1)}.\tag{50}$$

Proof. When $\kappa=2$,

$$J_\infty(m_{(1^2)})=I_n^2((1^2))=I_n^2((0))\frac{n(n-1)}{16}\frac{2n-3}{n-1}=I_n^2((0))\frac{n(n-1)}{2}\Big(\frac{1}{4}-\frac{1}{8n}-\frac{1}{8n^2}-\frac{1}{8n^3}+O\Big(\frac{1}{n^4}\Big)\Big)$$
$$=I_n^2((0))\,n\Big(\frac{n}{8}-\frac{3}{16}\Big)$$

$$\text{and}\qquad I_n^2((2))=I_n^2((0))\frac{n(2n+1)}{6}\frac{2n}{4n-1}=I_n^2((0))\,n\Big(\frac{n}{6}+\frac{1}{8}+\frac{1}{32n}+\frac{1}{128n^2}+O\Big(\frac{1}{n^3}\Big)\Big).$$

Therefore,

$$J_\infty(m_{(2)})=I_n^2((2))-\frac{4}{3}I_n^2((1^2))=I_n^2((0))\,n\Big(\frac{3}{8}+\frac{1}{32n}+\frac{1}{128n^2}+O\Big(\frac{1}{n^3}\Big)\Big),$$

which also gives

$$J_\infty\Big(\Big(t_1-\frac{1}{2}\Big)^2\Big)=\frac{1}{n}J_\infty(m_{(2)})-\frac{1}{n}J_\infty(m_{(1)})+\frac{1}{4}J_\infty(1)=I_n^2((0))\Big(\frac{1}{8}+\frac{1}{32n}+\frac{1}{128n^2}+O\Big(\frac{1}{n^3}\Big)\Big).\tag{51}$$

Note also that

$$J_\infty\Big(\Big(t_1-\frac{1}{2}\Big)\Big(t_2-\frac{1}{2}\Big)\Big) = \frac{2}{n(n-1)}J_\infty(m_{(1^2)})-\frac{1}{n}J_\infty(m_{(1)})+\frac{1}{4}J_\infty(1) = -I_n^2((0))\frac{1}{8(n-1)} = I_n^2((0))\Big(-\frac{1}{8n}-\sum_{i=2}^{\infty}\frac{1}{8n^i}\Big).$$

Next observe that

$$I_n^2((1^3)) = I_n^2((0))\frac{n(n-1)(n-2)}{96}\frac{2n-5}{n-1} = I_n^2((0))\,\frac{n(n-1)}{2}\Big(\frac{n}{24}-\frac{7}{48}+\frac{1}{16n}+\frac{1}{16n^2}+O\Big(\frac{1}{n^3}\Big)\Big)$$
$$= I_n^2((0))\,n\Big(\frac{n^2}{48}-\frac{3n}{32}+\frac{5}{48}+O\Big(\frac{1}{n^2}\Big)\Big),$$

$$I_n^2((2,1)) = I_n^2((0))\frac{n(n-1)(2n+1)}{20}\frac{n}{4n-1}\frac{2n-3}{n-1} = I_n^2((0))\,\frac{n(n-1)}{2}\Big(\frac{n}{10}+\frac{1}{40}-\frac{11}{160n}-\frac{59}{640n^2}+O\Big(\frac{1}{n^3}\Big)\Big)$$
$$= I_n^2((0))\,n\Big(\frac{n^2}{20}-\frac{3n}{80}-\frac{3}{64}-\frac{3}{256n}+O\Big(\frac{1}{n^2}\Big)\Big)$$

and

$$I_n^2((3)) = I_n^2((0))\frac{(n+1)(2n+1)n}{24}\frac{2n+1}{4n-1} = I_n^2((0))\,n\Big(\frac{n^2}{24}+\frac{3n}{32}+\frac{29}{384}+\frac{15}{512n}+O\Big(\frac{1}{n^2}\Big)\Big).$$

It follows that

$$J_\infty(m_{(2,1)}) = I_n^2((2,1))-\frac{12}{5}I_n^2((1^3)) = I_n^2((0))\,\frac{n(n-1)}{2}\Big(\frac{3}{8}-\frac{7}{32n}-\frac{31}{128n^2}+O\Big(\frac{1}{n^3}\Big)\Big)$$

and

$$J_\infty(m_{(3)}) = I_n^2((3))-\frac{3}{2}I_n^2((2,1))+\frac{8}{5}I_n^2((1^3)) = I_n^2((0))\,n\Big(\frac{5}{16}+\frac{3}{64n}+O\Big(\frac{1}{n^2}\Big)\Big).$$

Moreover,

$$I_n^2((1^4)) = I_n^2((0))\frac{n(n-1)(n-2)(n-3)}{1536}\frac{2n-5}{n-1}\frac{2n-7}{n-2} = I_n^2((0))\,\frac{n(n-1)}{2}\Big(\frac{n^2}{192}-\frac{n}{24}+\frac{25}{256}-\frac{5}{128n}-\frac{5}{128n^2}+O\Big(\frac{1}{n^3}\Big)\Big)$$
$$= I_n^2((0))\,n\Big(\frac{n^3}{384}-\frac{3n^2}{128}+\frac{107n}{1536}-\frac{35}{1536}+O\Big(\frac{1}{n^2}\Big)\Big),$$

$$I_n^2((2,1^2)) = I_n^2((0))\frac{(2n+1)n(n-1)(n-2)}{112}\frac{n}{4n-1}\frac{2n-5}{n-1}$$
$$= I_n^2((0))\,\frac{n(n-1)}{2}\Big(\frac{n^2}{56}-\frac{11n}{224}-\frac{15}{896}+\frac{129}{3584n}+\frac{705}{14336n^2}+O\Big(\frac{1}{n^3}\Big)\Big)$$
$$= I_n^2((0))\,n\Big(\frac{n^3}{112}-\frac{15n^2}{448}+\frac{29n}{1792}+\frac{27}{1024}+\frac{27}{4096n}+O\Big(\frac{1}{n^2}\Big)\Big),$$

$$I_n^2((2^2)) = I_n^2((0))\frac{n(n-1)(2n+1)(2n-1)}{60}\frac{2n}{4n-1}\frac{2n-2}{4n-3}\frac{2n-3}{4n-4}$$
$$= I_n^2((0))\,\frac{n(n-1)}{2}\Big(\frac{n^2}{60}-\frac{n}{120}-\frac{1}{64}-\frac{1}{128n}-\frac{5}{1024n^2}+O\Big(\frac{1}{n^3}\Big)\Big)$$
$$= I_n^2((0))\,n\Big(\frac{n^3}{120}-\frac{n^2}{80}-\frac{7n}{1920}+\frac{1}{256}+\frac{3}{2048n}+O\Big(\frac{1}{n^2}\Big)\Big),$$

while

$$I_n^2((3,1)) = I_n^2((0))\frac{(2n+2)(2n+1)n(n-1)}{72}\frac{2n+1}{4n-1}\frac{2n-3}{4n-4} = I_n^2((0))\,n\Big(\frac{n^3}{72}+\frac{n^2}{96}-\frac{25n}{1152}-\frac{43}{1536}-\frac{25}{2048n}+O\Big(\frac{1}{n^2}\Big)\Big)$$

and

$$I_n^2((4)) = I_n^2((0))\frac{(2n+3)(2n+2)(2n+1)n}{240}\frac{2n+2}{4n+1}\frac{2n+1}{4n-1} = I_n^2((0))\,n\left(\frac{n^3}{120}+\frac{3n^2}{80}+\frac{25n}{384}+\frac{71}{1280}+\frac{51}{2048n}+O\left(\frac{1}{n^2}\right)\right).$$

It follows that

$$J_\infty(m_{(2^2)}) = I_n^2((2^2)) - \frac{4}{3}I_n^2((2,1^2)) + \frac{48}{35}I_n^2((1^4)) = I_n^2((0))\,\frac{n(n-1)}{2}\left(\frac{9}{64}-\frac{7}{64n}-\frac{127}{1024n^2}+O\left(\frac{1}{n^3}\right)\right)$$

and

$$J_\infty(m_{(4)}) = I_n^2((4)) - \frac{8}{5}I_n^2((3,1)) + \frac{1}{3}I_n^2((2^2)) + \frac{16}{9}I_n^2((2,1^2)) - \frac{64}{35}I_n^2((1^4)) = I_n^2((0))\,n\left(\frac{35}{128}+\frac{29}{512n}+O\left(\frac{1}{n^2}\right)\right).$$

We conclude that

$$\begin{aligned} J_\infty\left(\left(t_1-\frac{1}{2}\right)^2\left(t_2-\frac{1}{2}\right)^2\right) &= \frac{2}{n(n-1)}J_\infty(m_{(2^2)}) - \frac{2}{n(n-1)}J_\infty(m_{(2,1)}) + \frac{1}{2n}J_\infty(m_{(2)}) \\ &\quad + \frac{2}{n(n-1)}J_\infty(m_{(1^2)}) - \frac{1}{2n}J_\infty(m_{(1)}) + \frac{1}{16}J_\infty(1) \\ &= I_n^2((0))\left(\frac{1}{64}-\frac{3}{1024n^2}+O\left(\frac{1}{n^3}\right)\right), \end{aligned} \tag{52}$$

while

$$\begin{aligned} J_\infty\left(\left(t_1-\frac{1}{2}\right)^4\right) &= \frac{1}{n}J_\infty(m_{(4)}) - \frac{2}{n}J_\infty(m_{(3)}) + \frac{3}{2n}J_\infty(m_{(2)}) - \frac{1}{2n}J_\infty(m_{(1)}) + \frac{1}{16}J_\infty(1) \\ &= I_n^2((0))\left(\frac{3}{128}+\frac{5}{512n}+O\left(\frac{1}{n^2}\right)\right). \end{aligned} \tag{53}$$

This completes the proof of Theorem 2 in all cases. □

Remark 19. We can unify the above computations, which can be made for all large enough β, as follows: as long as $\beta = 2\kappa$ is bounded away from zero, i.e. $\beta \geqslant \beta_0$ for some fixed $\beta_0 > 0$, we have

$$\begin{aligned} &\frac{1}{I_n^{\beta/2}((0))}\cdot\left(\int_{[-\frac{1}{2},\frac{1}{2}]^n} m_{(4)}(\boldsymbol{x})\,|\Delta_n(\boldsymbol{x})|^\beta\,d\boldsymbol{x} \;+\; 2\int_{[-\frac{1}{2},\frac{1}{2}]^n} m_{(2^2)}(\boldsymbol{x})\,|\Delta_n(\boldsymbol{x})|^\beta\,d\boldsymbol{x}\right) \\ &\qquad\qquad - \left(\frac{1}{I_n^{\beta/2}((0))}\int_{[-\frac{1}{2},\frac{1}{2}]^n} m_{(2)}(\boldsymbol{x})\,|\Delta_n(\boldsymbol{x})|^\beta\,d\boldsymbol{x}\right)^2 \\ &= \left(\frac{3}{128}n+\frac{5(\beta-2)}{256\beta}\right) + \left(\frac{1}{64}n^2-\frac{\beta+4}{128\beta}n+\frac{\beta^2-9\beta+14}{128\beta^2}\right) - \left(\frac{1}{64}n^2+\frac{\beta-2}{64\beta}n+\frac{7\beta^2-32\beta+28}{256\beta^2}\right) + O_{\beta_0}\left(\frac{1}{n}\right) \\ &= \frac{1}{64\beta} + O_{\beta_0}\left(\frac{1}{n}\right). \end{aligned}$$

4. Almost isotropicity of B_E in the subspaces of self-adjoint matrices

Here we establish Theorem 4.

Proof in the case where E is the subspace of Hermitian matrices. The orthonormal basis that we fix is the following:

$$\{J^{kk} : 1 \leqslant k \leqslant n\}\bigcup\{\tfrac{1}{\sqrt{2}}(J^{kl}+J^{lk}) : k<l\}\bigcup\{\tfrac{i}{\sqrt{2}}(J^{kl}-J^{lk}) : k<l\}$$

where J^{kl} is the single-entry matrix whose only non-zero entry is the (k,l)-th one and is equal to 1. According to Theorem 8, we have

$$\frac{1}{\mathrm{vol}(B_E)}\int_{B_E} T_{k_1l_1}T_{k_2l_2}\,dT = 0$$

whenever $\{k_1,k_2\}\neq\{l_1,l_2\}$. This immediately shows that any pair of marginals of the distribution which correspond to one diagonal and (either the real or the imaginary part of) one non-diagonal entry is linearly uncorrelated. Similarly, if they correspond to two non-diagonal entries (k_1,l_1), (k_2,l_2) with $(k_2,l_2)\notin\{(k_1,l_1),(l_1,k_1)\}$ we can observe the following:

$$\begin{aligned}0 &= \frac{1}{\mathrm{vol}(B_E)}\int_{B_E} T_{k_1l_1}T_{k_2l_2}\,dT\\ &= \frac{1}{\mathrm{vol}(B_E)}\int_{B_E}\big(\mathrm{Re}(T_{k_1l_1})\,\mathrm{Re}(T_{k_2l_2})-\mathrm{Im}(T_{k_1l_1})\,\mathrm{Im}(T_{k_2l_2})\big)\,dT\\ &\quad+\frac{i}{\mathrm{vol}(B_E)}\int_{B_E}\big(\mathrm{Re}(T_{k_1l_1})\,\mathrm{Im}(T_{k_2l_2})+\mathrm{Im}(T_{k_1l_1})\,\mathrm{Re}(T_{k_2l_2})\big)\,dT,\end{aligned}$$

while

$$\begin{aligned}0 &= \frac{1}{\mathrm{vol}(B_E)}\int_{B_E} T_{k_1l_1}T_{l_2k_2}\,dT\\ &= \frac{1}{\mathrm{vol}(B_E)}\int_{B_E}\big(\mathrm{Re}(T_{k_1l_1})\,\mathrm{Re}(T_{l_2k_2})-\mathrm{Im}(T_{k_1l_1})\,\mathrm{Im}(T_{l_2k_2})\big)\,dT\\ &\quad+\frac{i}{\mathrm{vol}(B_E)}\int_{B_E}\big(\mathrm{Re}(T_{k_1l_1})\,\mathrm{Im}(T_{l_2k_2})+\mathrm{Im}(T_{k_1l_1})\,\mathrm{Re}(T_{l_2k_2})\big)\,dT\\ &= \frac{1}{\mathrm{vol}(B_E)}\int_{B_E}\big(\mathrm{Re}(T_{k_1l_1})\,\mathrm{Re}(T_{k_2l_2})+\mathrm{Im}(T_{k_1l_1})\,\mathrm{Im}(T_{k_1l_2})\big)\,dT\\ &\quad+\frac{i}{\mathrm{vol}(B_E)}\int_{B_E}\big(-\mathrm{Re}(T_{k_1l_1})\,\mathrm{Im}(T_{k_2l_2})+\mathrm{Im}(T_{k_1l_1})\,\mathrm{Re}(T_{k_2l_2})\big)\,dT.\end{aligned}$$

Combined, these show that all the above integrals are equal to 0.

Let us examine the remaining cases, where the marginals correspond to two different diagonal entries $(k,k),(l,l)$, or to the real and to the imaginary part of the same non-diagonal entry (k,l), $k\neq l$.

In the latter case, we can write

$$\begin{aligned}0 &= \frac{1}{\mathrm{vol}(B_E)}\int_{B_E} T_{kl}T_{kl}\,dT\\ &= \frac{1}{\mathrm{vol}(B_E)}\int_{B_E}\big(\mathrm{Re}(T_{kl})^2-\mathrm{Im}(T_{kl})^2\big)\,dT+\frac{2i}{\mathrm{vol}(B_E)}\int_{B_E}\mathrm{Re}(T_{kl})\,\mathrm{Im}(T_{kl})\,dT,\end{aligned}\tag{54}$$

which shows that the marginals are uncorrelated.

In the former case, we have from Theorem 8 and from Proposition 16 that

$$\begin{aligned}\frac{1}{\mathrm{vol}(B_E)}\int_{B_E} T_{kk}T_{ll}\,dT &= \mathrm{Wg}^U(e;n)\,\frac{1}{\mathrm{vol}(B_E)}\int_{B_E}\mathrm{Tr}_e(T)\,dT+\mathrm{Wg}^U((12);n)\frac{1}{\mathrm{vol}(B_E)}\int_{B_E}\mathrm{Tr}_{(12)}(T)\,dT\\ &= \frac{1}{(n-1)(n+1)}\,\frac{1}{\mathrm{vol}(B_E)}\int_{B_E}\big(\mathrm{Tr}(T)\big)^2\,dT-\frac{1}{n(n-1)(n+1)}\,\frac{1}{\mathrm{vol}(B_E)}\int_{B_E}\mathrm{Tr}(T^2)\,dT\\ &= \frac{1}{(n-1)(n+1)}\left(n\frac{N_\infty(x_1^2)}{N_\infty(1)}+n(n-1)\frac{N_\infty(x_1x_2)}{N_\infty(1)}\right)-\frac{1}{(n-1)(n+1)}\,\frac{N_\infty(x_1^2)}{N_\infty(1)}\\ &= \frac{1}{n+1}\left(\frac{N_\infty(x_1^2)}{N_\infty(1)}+n\frac{N_\infty(x_1x_2)}{N_\infty(1)}\right)\\ &= -\frac{1}{8n(n+1)}+O\Big(\frac{1}{n^3}\Big).\end{aligned}$$

Moreover, turning to second moments of the marginals, we see that

$$\begin{aligned}\frac{1}{\mathrm{vol}(B_E)}\int_{B_E} T_{kk}^2\,dT &= \mathrm{Wg}^U(e;n)\,\frac{1}{\mathrm{vol}(B_E)}\int_{B_E}\mathrm{Tr}_e(T)\,dT+\mathrm{Wg}^U((12);n)\frac{1}{\mathrm{vol}(B_E)}\int_{B_E}\mathrm{Tr}_{(12)}(T)\,dT\\ &\quad+\mathrm{Wg}^U((12);n)\,\frac{1}{\mathrm{vol}(B_E)}\int_{B_E}\mathrm{Tr}_e(T)\,dT+\mathrm{Wg}^U(e;n)\frac{1}{\mathrm{vol}(B_E)}\int_{B_E}\mathrm{Tr}_{(12)}(T)\,dT\\ &= -\frac{1}{8n(n+1)}+O\Big(\frac{1}{n^3}\Big)\\ &\quad-\frac{1}{n(n-1)(n+1)}\,\frac{1}{\mathrm{vol}(B_E)}\int_{B_E}\big(\mathrm{Tr}(T)\big)^2\,dT+\frac{1}{(n-1)(n+1)}\,\frac{1}{\mathrm{vol}(B_E)}\int_{B_E}\mathrm{Tr}(T^2)\,dT\\ &= -\frac{1}{8n(n+1)}+O\Big(\frac{1}{n^3}\Big)\\ &\quad-\frac{1}{n(n-1)(n+1)}\left(n\frac{N_\infty(x_1^2)}{N_\infty(1)}+n(n-1)\frac{N_\infty(x_1x_2)}{N_\infty(1)}\right)+\frac{n}{(n-1)(n+1)}\,\frac{N_\infty(x_1^2)}{N_\infty(1)}\\ &= -\frac{1}{8n(n+1)}+O\Big(\frac{1}{n^3}\Big)\\ &\quad-\frac{1}{n(n-1)(n+1)}\left(\frac{1}{8}+O\Big(\frac{1}{n^2}\Big)\right)+\frac{n}{(n-1)(n+1)}\left(\frac{1}{4}-\frac{1}{16n^2}+O\Big(\frac{1}{n^3}\Big)\right)\\ &= \frac{n}{4(n-1)(n+1)}+O\Big(\frac{1}{n^2}\Big).\end{aligned}$$

On the other hand, when we consider a non-diagonal entry (k,l), (54) shows that

$$\frac{1}{\mathrm{vol}(B_E)}\int_{B_E}\mathrm{Re}(T_{kl})^2\,dT=\frac{1}{\mathrm{vol}(B_E)}\int_{B_E}\mathrm{Im}(T_{kl})^2\,dT.$$

To compute this integral, we note that

$$\frac{1}{\mathrm{vol}(B_E)}\int_{B_E} 2\,\mathrm{Re}(T_{kl})^2\,dT = \frac{1}{\mathrm{vol}(B_E)}\int_{B_E}\left(\mathrm{Re}(T_{kl})^2+\mathrm{Im}(T_{kl})^2\right)dT = \frac{1}{\mathrm{vol}(B_E)}\int_{B_E} T_{kl}T_{lk}\,dT$$
$$= \mathrm{Wg}^U((12);n)\,\frac{1}{\mathrm{vol}(B_E)}\int_{B_E}\mathrm{Tr}_e(T)\,dT+\mathrm{Wg}^U(e;n)\frac{1}{\mathrm{vol}(B_E)}\int_{B_E}\mathrm{Tr}_{(12)}(T)\,dT$$
$$= -\frac{1}{n(n-1)(n+1)}\left(\frac{1}{8}+O\Big(\frac{1}{n^2}\Big)\right)+\frac{n}{(n-1)(n+1)}\left(\frac{1}{4}-\frac{1}{16n^2}+O\Big(\frac{1}{n^3}\Big)\right)$$
$$= \frac{n}{4(n-1)(n+1)}+O\Big(\frac{1}{n^3}\Big).$$

We conclude that the covariance matrix $\mathrm{Cov}(B_E)$ of B_E has the following form: all its diagonal entries are $=\frac{n}{4(n-1)(n+1)}+O\Big(\frac{1}{n^2}\Big)$, while the only non-zero non-diagonal entries are those giving the correlation between marginals corresponding to two different diagonal entries of $T\in B_E$, and these are $=-\frac{1}{8n(n+1)}+O\Big(\frac{1}{n^3}\Big)$. It follows that, in order to find all eigenvalues of $\mathrm{Cov}(B_E)$, it suffices to find the eigenvalues of the $n\times n$ submatrix D_{B_E} which involves only the marginals corresponding to diagonal entries of $T\in B_E$ (since the remaining eigenvalues are all $=\frac{n}{4(n-1)(n+1)}+O\Big(\frac{1}{n^2}\Big)$ as immediately seen from the form of $\mathrm{Cov}(B_E)$).

The submatrix D_{B_E} is of the form

$$(a-b)I_n+bJ_n$$

where J_n is the matrix with all entries equal to 1 and $a=\frac{n}{4(n-1)(n+1)}+O\Big(\frac{1}{n^2}\Big)$, $b=-\frac{1}{8n(n+1)}+O\Big(\frac{1}{n^3}\Big)$. It is not difficult to see that such a matrix can only have two eigenvalues: the eigenvalue $a+(n-1)b$ (corresponding to the vector $(1,1,\dots,1)$) and the eigenvalue $a-b$ (which will have mutliplicity $n-1$). In our case, these eigenvalues are $=\frac{1}{8(n+1)}+O\Big(\frac{1}{n^2}\Big)$ and $=\frac{n}{4(n-1)(n+1)}+O\Big(\frac{1}{n^2}\Big)$ respectively. This shows that all eigenvalues of D_{B_E}, and thus of $\mathrm{Cov}(B_E)$ too, are approximately equal.

Finally, the covariance matrix $\mathrm{Cov}(\overline{B_E})$ of the volume-normalised unit ball $\overline{B_E}$ can be found by multiplying $\mathrm{Cov}(B_E)$ by $[\mathrm{vol}(B_E)]^{-2/n^2}\simeq n$. □

Proof in the case where E is the subspace of $\mathbb{R}$-self-adjoint matrices. Our aim is to compute integrals of the form

$$\frac{1}{\mathrm{vol}(B_E)}\int_{B_E} T_{j_1l_1}T_{j_2l_2}\,dT, \qquad 1\leqslant j_1,j_2,l_1,l_2\leqslant n,$$

so we apply Theorem 12 with $k=2$. Here

$$M_{2k}=M_4=\Big\{\big\{\{1,2\},\{3,4\}\big\},\big\{\{1,3\},\{2,4\}\big\},\big\{\{1,4\},\{2,3\}\big\}\Big\}$$

and if we express the pair partitions as permutations in S_{2k} per our convention

$$=\big\{e,(23),(243)\big\}.$$

Therefore,

$$M_4^{-1}M_4:=\big\{\sigma^{-1}\tau:\sigma,\tau\in M_4\big\}=\big\{e,(23),(24),(243),(234)\big\},$$

and all these permutations have coset-type (2) except for the trivial permutation e which has coset-type (1^2).

Moreover,

$$H_2=\big\langle(12),(34),(13)(24)\big\rangle=\big\{e,(12),(34),(13)(24),(12)(34),(14)(23),(1324),(1423)\big\}.$$

To compute the orthogonal Weingarten function on S_4, we first find the zonal spherical functions $\omega^{(2)}$ and $\omega^{(1^2)}$. It is easily seen that

$$\omega^{(2)}(\sigma) = \frac{1}{8}\sum_{\zeta\in H_2}\chi^{(4)}(\sigma\zeta) = 1 \qquad \text{for every } \sigma\in S_4.$$

On the other hand,

$$\omega^{(1^2)}(e) = \frac{1}{8}\sum_{\zeta\in H_2}\chi^{(2^2)}(\zeta) = 1,$$

while

$$\omega^{(1^2)}(\sigma) = \omega^{(1^2)}((23)) = \frac{1}{8}\sum_{\zeta\in H_2}\chi^{(2^2)}\big((23)\zeta\big) = -\frac{1}{2} \qquad \text{for every } \sigma\in S_4 \text{ with coset-type } (2)$$

(in particular for every permutation $\sigma\in M_4^{-1}M_4\setminus\{e\}$).

We can now compute:

$$\begin{aligned}\mathrm{Wg}^O(\sigma;n) &= \frac{8}{24}\sum_{\lambda\vdash 2}\frac{\chi^{2\lambda}(e)}{C'_\lambda(n)}\,\omega^\lambda(\sigma)\\ &= \frac{1}{3}\left(\frac{\chi^{(4)}(e)\omega^{(2)}(\sigma)}{C'_{(2)}(n)} + \frac{\chi^{(2^2)}(e)\omega^{(1^2)}(\sigma)}{C'_{(1^2)}(n)}\right) = \begin{cases}\frac{n+1}{n(n-1)(n+2)} & \text{if } \sigma = e\\ -\frac{1}{n(n-1)(n+2)} & \text{if } \sigma\in M_4^{-1}M_4\setminus\{e\}\end{cases}.\end{aligned}$$

The orthonormal basis that we have fixed is the following:

$$\{J^{kk} : 1\leqslant k\leqslant n\}\bigcup\{\tfrac{1}{\sqrt{2}}\big(J^{kl}+J^{lk}\big) : k<l\}.$$

According to Theorem 12, we have

$$\frac{1}{\mathrm{vol}(B_E)}\int_{B_E} T_{i_1i_2}T_{i_3i_4}\,dT = 0$$

if there is at least one index that appears an odd number of times among the i_j, $j=1,\dots,4$. This immediately shows that marginals of the distribution which correspond to two different non-diagonal entries or to one non-diagonal and one diagonal entry are linearly uncorrelated.

The only other case, where we have correlation, is when $i_1=i_2=j\neq k=i_3=i_4$. In this case

$$\begin{aligned}\frac{1}{\mathrm{vol}(B_E)}\int_{B_E} T_{jj}T_{kk}\,dT &= \mathrm{Wg}^O(e;n)\,\frac{1}{\mathrm{vol}(B_E)}\int_{B_E}\mathrm{Tr}'_e(T)\,dT + \mathrm{Wg}^O((23);n)\,\frac{1}{\mathrm{vol}(B_E)}\int_{B_E}\mathrm{Tr}'_{(23)}(T)\,dT\\ &\quad + \mathrm{Wg}^O((243);n)\,\frac{1}{\mathrm{vol}(B_E)}\int_{B_E}\mathrm{Tr}'_{(243)}(T)\,dT\\ &= \frac{n+1}{n(n-1)(n+2)}\,\frac{1}{\mathrm{vol}(B_E)}\int_{B_E}\big(\mathrm{Tr}(T)\big)^2\,dT - \frac{2}{n(n-1)(n+2)}\,\frac{1}{\mathrm{vol}(B_E)}\int_{B_E}\mathrm{Tr}(T^2)\,dT\\ &= \frac{n+1}{n(n-1)(n+2)}\left(n\frac{N_\infty(x_1^2)}{N_\infty(1)} + n(n-1)\frac{N_\infty(x_1x_2)}{N_\infty(1)}\right) - \frac{2}{(n-1)(n+2)}\,\frac{N_\infty(x_1^2)}{N_\infty(1)}\\ &= \frac{1}{n+2}\,\frac{N_\infty(x_1^2)}{N_\infty(1)} + \frac{n+1}{n+2}\,\frac{N_\infty(x_1x_2)}{N_\infty(1)}\\ &= -\frac{1}{4n(n+2)} + O\Big(\frac{1}{n^3}\Big).\end{aligned}$$

Turning to second moments, we first handle the case $i_1 = i_3 = j \neq k = i_2 = i_4$:

$$\begin{aligned}\frac{1}{\mathrm{vol}(B_E)}\int_{B_E}\left(\tfrac{1}{\sqrt{2}}(T_{jk}+T_{kj})\right)^2 dT &= \frac{1}{\mathrm{vol}(B_E)}\int_{B_E} 2T_{jk}^2\,dT \\ &= 2\Big(\mathrm{Wg}^O((23);n)\,\frac{1}{\mathrm{vol}(B_E)}\int_{B_E}\mathrm{Tr}'_e(T)\,dT + \mathrm{Wg}^O(e;n)\,\frac{1}{\mathrm{vol}(B_E)}\int_{B_E}\mathrm{Tr}'_{(23)}(T)\,dT \\ &\qquad + \mathrm{Wg}^O((24);n)\,\frac{1}{\mathrm{vol}(B_E)}\int_{B_E}\mathrm{Tr}'_{(243)}(T)\,dT\Big) \\ &= 2\Big(-\frac{1}{n(n-1)(n+2)}\,\frac{1}{\mathrm{vol}(B_E)}\int_{B_E}\big(\mathrm{Tr}(T)\big)^2\,dT + \frac{n+1}{n(n-1)(n+2)}\,\frac{1}{\mathrm{vol}(B_E)}\int_{B_E}\mathrm{Tr}(T^2)\,dT \\ &\qquad - \frac{1}{n(n-1)(n+2)}\,\frac{1}{\mathrm{vol}(B_E)}\int_{B_E}\mathrm{Tr}(T^2)\,dT\Big) \\ &= -\frac{2}{n(n-1)(n+2)}\left(n\frac{N_\infty(x_1^2)}{N_\infty(1)} + n(n-1)\frac{N_\infty(x_1x_2)}{N_\infty(1)}\right) + \frac{2}{(n-1)(n+2)}\,n\frac{N_\infty(x_1^2)}{N_\infty(1)} \\ &= \frac{2}{n+2}\left(\frac{N_\infty(x_1^2)}{N_\infty(1)} - \frac{N_\infty(x_1x_2)}{N_\infty(1)}\right) \\ &= \frac{1}{2(n+2)} + O\Big(\frac{1}{n^2}\Big).\end{aligned}$$

Finally,

$$\begin{aligned}\frac{1}{\mathrm{vol}(B_E)}\int_{B_E} T_{jj}^2\,dT = \sum_{\sigma\in M_4}\Big(&\mathrm{Wg}^O(\sigma^{-1}e;n)\,\frac{1}{\mathrm{vol}(B_E)}\int_{B_E}\mathrm{Tr}'_e(T)\,dT + \mathrm{Wg}^O(\sigma^{-1}(23);n)\,\frac{1}{\mathrm{vol}(B_E)}\int_{B_E}\mathrm{Tr}'_{(23)}(T)\,dT \\ &+ \mathrm{Wg}^O(\sigma^{-1}(243);n)\,\frac{1}{\mathrm{vol}(B_E)}\int_{B_E}\mathrm{Tr}'_{(243)}(T)\,dT\Big) \\ = \frac{1}{\mathrm{vol}(B_E)}&\int_{B_E} T_{jj}T_{kk}\,dT + 2\cdot\frac{1}{\mathrm{vol}(B_E)}\int_{B_E} T_{jk}^2\,dT \\ = \frac{1}{2(n+2)}& + O\Big(\frac{1}{n^2}\Big).\end{aligned}$$

We conclude that the covariance matrix $\mathrm{Cov}(B_E)$ of B_E has the following form: all its diagonal entries are $= \frac{1}{2(n+2)} + O\big(\frac{1}{n^2}\big)$, while the only non-zero non-diagonal entries are those giving the correlation between marginals corresponding to two different diagonal entries of $T \in B_E$, and these are $= -\frac{1}{4n(n+2)} + O\big(\frac{1}{n^3}\big)$.

As before, it follows that the volume-normalised unit ball $\overline{B_E}$ is in almost isotropic position. This completes the proof of Theorem 4 in the orthogonal case too. □

5. Entrywise negative correlation property of $B_{\mathcal{M}_n(\mathbb{R})}$ or $B_{\mathcal{M}_n(\mathbb{C})}$

According to one of the main results in [41], a necessary condition for the variance conjecture to be true for the unit ball of any p-Schatten norm on $\mathcal{M}_n(\mathbb{F})$ is that the corresponding density $f_{a,b,c}(x)\cdot e^{-\|x\|_p^p}\,dx$ appearing in Lemma 6 and Proposition 7 satisfies a certain negative correlation property: more specifically, we need to have

$$\frac{M_p(x_i^2x_j^2)}{M_p(1)} = \frac{M_p(x_1^2x_2^2)}{M_p(1)} < \left(\frac{M_p(x_1^2)}{M_p(1)}\right)^2 = \frac{M_p(x_i^2)}{M_p(1)}\,\frac{M_p(x_j^2)}{M_p(1)} \tag{55}$$

for any $i \neq j$. This could be used to deduce similar inequalities for the original uniform densities on the unit balls of the p-Schatten norms which satisfy the conjecture: in [41] we showed that, if p is large enough (and, as a limiting case, if $p = \infty$ as well), then (55) holds true and, combined with the invariances of $K_{p,\mathcal{M}_n(\mathbb{F})}$, implies that

$$\int_{\overline{K}_{p,\mathcal{M}_n(\mathbb{F})}} |T_{i,j}|^2 |T_{i,r}|^2 \, dT = \int_{\overline{K}_{p,\mathcal{M}_n(\mathbb{F})}} |T_{j,i}|^2 |T_{r,i}|^2 \, dT < \left(\int_{\overline{K}_{p,\mathcal{M}_n(\mathbb{F})}} |T_{i,j}|^2 \, dT\right)\left(\int_{\overline{K}_{p,\mathcal{M}_n(\mathbb{F})}} |T_{i,r}|^2 \, dT\right)$$

for all $i, j, r, \; j \neq r$. However, it was unclear from our method whether a similar negative correlation property is true for the remaining pairs of entries, that is, when we consider the integrals $\int_{\overline{K}_{p,\mathcal{M}_n(\mathbb{F})}} |T_{i,j}|^2 |T_{l,r}|^2 \, dT$ with $i \neq l, \; j \neq r$.

We can now check that this fails to be true and that we do not have negative correlation for the remaining pairs of entries of $T \sim \mathrm{Unif}\big(K_{\infty,\mathcal{M}_n(\mathbb{F})}\big)$ when $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$ (of course it doesn't fail by much since the variance conjecture is correct in these cases). The key ingredients we will use to check this are the relevant tools in the Weingarten calculus coming from [15] and the estimates we obtained in Section 3 (which also allow us to verify again the negative correlation property for pairs of entries coming from the same row or the same column).

Proof when $\mathbb{F} = \mathbb{C}$. To compute and compare the integrals

$$\frac{1}{\mathrm{vol}(K_\infty)} \int_{K_\infty} |T_{i,j}|^2 |T_{l,r}|^2 \, dT, \qquad \left(\frac{1}{\mathrm{vol}(K_\infty)} \int_{K_\infty} |T_{1,1}|^2 \, dT\right)^2,$$

we apply Theorem 9 with $k = 2$ or 1 respectively. Starting with the latter, we see that

$$\begin{aligned}\frac{1}{\mathrm{vol}(K_\infty)} \int_{K_\infty} |T_{1,1}|^2 \, dT &= \frac{2}{\mathrm{vol}(K_\infty)} \int_{K_\infty} \mathrm{Re}^2(T_{1,1}) \, dT = \frac{2}{\mathrm{vol}(K_\infty)} \int_{K_\infty} \mathrm{Im}^2(T_{1,1}) \, dT \\ &= \frac{1}{n^2} \cdot \frac{1}{\mathrm{vol}(K_\infty)} \int_{K_\infty} \mathrm{Tr}(TT^*) \, dT\end{aligned}$$

as expected from the isotropicity of $\overline{K}_{\infty,\mathcal{M}_n(\mathbb{C})}$,

$$= \frac{1}{n^2} \cdot \frac{N_\infty\big(\|x\|_2^2\big)}{N_\infty(1)} = \frac{1}{2n}.$$

Moreover,

$$\begin{aligned}\frac{4}{\mathrm{vol}(K_\infty)} \int_{K_\infty} \mathrm{Re}^2(T_{i,j}) \mathrm{Re}^2(T_{l,r}) \, dT &= \frac{4}{\mathrm{vol}(K_\infty)} \int_{K_\infty} \mathrm{Im}^2(T_{i,j}) \mathrm{Im}^2(T_{l,r}) \, dT = \frac{4}{\mathrm{vol}(K_\infty)} \int_{K_\infty} \mathrm{Re}^2(T_{i,j}) \mathrm{Im}^2(T_{l,r}) \, dT \\ &= \frac{1}{\mathrm{vol}(K_\infty)} \int_{K_\infty} |T_{i,j}|^2 |T_{l,r}|^2 \, dT = \frac{1}{\mathrm{vol}(K_\infty)} \int_{K_\infty} T_{i,j} T_{l,r} \overline{T_{i,j} T_{l,r}} \, dT\end{aligned}$$

and when $i \neq l$, $j \neq r$

$$= \mathrm{Wg}^U(e;n,n)\,\frac{1}{\mathrm{vol}(K_\infty)}\int_{K_\infty}\big(\mathrm{Tr}(TT^*)\big)^2\,dT$$
$$+\,\mathrm{Wg}^U((12);n,n)\,\frac{1}{\mathrm{vol}(K_\infty)}\int_{K_\infty}\mathrm{Tr}\big((TT^*)^2\big)\,dT$$
$$= \frac{n^2+1}{(n(n^2-1))^2}\,\frac{N_\infty\big(\|x\|_2^4\big)}{N_\infty(1)} - \frac{2}{n(n^2-1)^2}\,\frac{N_\infty\big(\|x\|_4^4\big)}{N_\infty(1)}$$
$$= \frac{n^2+1}{(n(n^2-1))^2}\,\frac{n^4}{4n^2-1} - \frac{2}{n(n^2-1)^2}\,\frac{3n^3-n}{2(4n^2-1)}$$
$$= \frac{n^6-2n^4+n^2}{n^2(n^2-1)^2(4n^2-1)} = \frac{1}{4n^2-1}.$$

We thus see that

$$\frac{1}{\mathrm{vol}(K_\infty)}\int_{K_\infty}|T_{i,j}|^2|T_{l,r}|^2\,dT > \left(\frac{1}{\mathrm{vol}(K_\infty)}\int_{K_\infty}|T_{i,j}|^2\,dT\right)\left(\frac{1}{\mathrm{vol}(K_\infty)}\int_{K_\infty}|T_{l,r}|^2\,dT\right)$$
$$> \big(1-O(1/n^2)\big)\,\frac{1}{\mathrm{vol}(K_\infty)}\int_{K_\infty}|T_{i,j}|^2|T_{l,r}|^2\,dT$$

(the latter inequality being a necessary consequence of the variance conjecture holding true).

On the other hand,

$$\frac{1}{\mathrm{vol}(K_\infty)}\int_{K_\infty}|T_{i,j}|^2|T_{i,r}|^2\,dT = \frac{1}{\mathrm{vol}(K_\infty)}\int_{K_\infty}|T_{j,i}|^2|T_{r,i}|^2\,dT$$
$$= \big(\mathrm{Wg}^U(e;n,n)+\mathrm{Wg}^U((12);n,n)\big)\cdot\frac{1}{\mathrm{vol}(K_\infty)}\int_{K_\infty}\Big(\big(\mathrm{Tr}(TT^*)\big)^2+\mathrm{Tr}\big((TT^*)^2\big)\Big)\,dT$$
$$= \frac{1}{n^2(n+1)^2}\,\frac{2n^4+3n^3-n}{2(4n^2-1)} = \frac{1}{2n(2n+1)} < \left(\frac{1}{\mathrm{vol}(K_\infty)}\int_{K_\infty}|T_{1,1}|^2\,dT\right)^2$$

in accordance with the conclusions from [41]. □

Proof when $\mathbb{F}=\mathbb{R}$. Applying Theorem 13 with $k=1$ or 2, we can obtain:

$$\frac{1}{\mathrm{vol}(K_\infty)}\int_{K_\infty}|T_{1,1}|^2\,dT = \frac{1}{n^2}\cdot\frac{1}{\mathrm{vol}(K_\infty)}\int_{K_\infty}\mathrm{Tr}(TT^t)\,dT = \frac{1}{n^2}\cdot\frac{N_\infty\big(\|x\|_2^2\big)}{N_\infty(1)} = \frac{1}{2n+1};$$

$$\frac{1}{\mathrm{vol}(K_\infty)}\int_{K_\infty}|T_{i,j}|^2|T_{l,r}|^2\,dT = \sum_{\tau_1,\tau_2\in M_4}\mathrm{Wg}^O(\tau_1;n)\mathrm{Wg}^O(\tau_2;n)\,\frac{1}{\mathrm{vol}(K_\infty)}\int_{K_\infty}\mathrm{Tr}'_{\tau_1^{-1}\tau_2}(TT^t)\,dT$$
$$= \Big(\big(\mathrm{Wg}^O(e;n)\big)^2+2\big(\mathrm{Wg}^O((23);n)\big)^2\Big)\,\frac{1}{\mathrm{vol}(K_\infty)}\int_{K_\infty}\big(\mathrm{Tr}(TT^t)\big)^2\,dT$$
$$+\Big(4\mathrm{Wg}^O(e;n)\mathrm{Wg}^O((23);n)+2\big(\mathrm{Wg}^O((23);n)\big)^2\Big)\,\frac{1}{\mathrm{vol}(K_\infty)}\int_{K_\infty}\mathrm{Tr}\big((TT^t)^2\big)\,dT$$
$$= \frac{n^2+2n+3}{(n(n-1)(n+2))^2}\,\frac{N_\infty\big(\|x\|_2^4\big)}{N_\infty(1)} - \frac{4n+2}{(n(n-1)(n+2))^2}\,\frac{N_\infty\big(\|x\|_4^4\big)}{N_\infty(1)}$$
$$= \frac{n+1}{n(2n+1)(2n+3)}$$

when $i \neq l$, $j \neq r$, while

$$\begin{aligned}
\frac{1}{\mathrm{vol}(K_\infty)} \int_{K_\infty} |T_{i,j}|^2 |T_{i,r}|^2 \, dT &= \frac{1}{\mathrm{vol}(K_\infty)} \int_{K_\infty} |T_{j,i}|^2 |T_{r,i}|^2 \, dT \\
&= \sum_{\tau_1,\tau_2,\sigma_2 \in M_4} \mathrm{Wg}^O(\tau_1; n) \mathrm{Wg}^O(\sigma_2^{-1}\tau_2; n) \, \frac{1}{\mathrm{vol}(K_\infty)} \int_{K_\infty} \mathrm{Tr}'_{\tau_1^{-1}\tau_2}(TT^t) \, dT \\
&= \left(\sum_{\sigma_2 \in M_4} \mathrm{Wg}^O(\sigma_2^{-1}; n) \right)^2 \frac{N_\infty(\|x\|_2^4)}{N_\infty(1)} + 2 \left(\sum_{\sigma_2 \in M_4} \mathrm{Wg}^O(\sigma_2^{-1}; n) \right)^2 \frac{N_\infty(\|x\|_4^4)}{N_\infty(1)} \\
&= \frac{1}{(n(n+2))^2} \left(\frac{n^4+n^3+n}{(2n+1)(2n+3)} + \frac{3n^3+4n^2-n}{(2n+1)(2n+3)} \right) \\
&= \frac{1}{(2n+1)(2n+3)}.
\end{aligned}$$

These show that we have analogous conclusions as in the unitary case. □

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