Distribution of the primes represented by $\left\lfloor \frac{x}{n} \right\rfloor$ in short intervals

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Abstract. Let $\lfloor x \rfloor$ be the largest integer not exceeding x. For $0 < \theta \le 1$, let $\pi_\theta(x)$ denote the number of integers n with $1 \le n \le x^\theta$ such that $\lfloor x/n \rfloor$ is prime. Recently, Ma, Chen and Wu obtained the following interesting asymptotic formula

$$\pi_\theta(x) = \frac{x^\theta}{(1-\theta)\log x} + O\left(x^\theta (\log x)^{-2}\right),$$

provided that $\frac{23}{47} < \theta < 1$. They further conjectured that this asymptotic formula can be extended to all $0 < \theta < 1$. In this paper, we give an improvement of their result by showing that $\frac{9}{19} < \theta < 1$ is admissible.

1. Introduction

The investigations of the summations related to rounding up certain arithmetic functions are quite popular in recent years. It seems that this new term wave of enthusiasm starts from the paper of Bordellès, Dai, Heyman, Pan and Shparlinski [4], where the following asymptotic formula

$$\sum_{n \le x} f\left(\lfloor x/n \rfloor\right) = x \sum_{n=1}^{\infty} \frac{f(n)}{n(n+1)} + O_f(x^{1/2+\varepsilon})$$

is given, provided that f satisfies a broad condition involving the growth of the magnitude of it. Here, as usual ε denotes an arbitrary small positive number. As an application, Bordellès, Dai, Heyman, Pan and Shparlinski obtained that

$$\sum_{n \le x} \frac{\varphi(\lfloor x/n \rfloor)}{\lfloor x/n \rfloor} = \kappa x + O(x^{1/2}), \tag{1.1}$$

where $\varphi(n)$ is the Euler totient function and $\kappa = \sum_{n \le x} \frac{\varphi(n)}{n^2(n+1)}$. As Bordellès, Dai, Heyman, Pan and Shparlinski commented that the study of the above asymptotic formula is partially motivated by the Beatty $\lfloor \alpha n + \beta \rfloor$ sequences and Piatetski–Shapiro $\lfloor n^c \rfloor$ sequences which are surely well–known research objects in number theory (see for example [2, 6, 13, 14]). However, since the growth of φ is out of control regarding the magnitude condition assumed to f, they failed to offer the corresponding asymptotic formula of φ but only got a slightly weaker result, namely,

$$c_1 x \log x \le \sum_{n \le x} \varphi\left(\lfloor x/n \rfloor\right) \le c_2 x \log x$$

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with two explicitly given constants c_1 and c_2. Bordellès, Dai, Heyman, Pan and Shparlinski then conjectured that

$$\sum_{n\le x}\varphi\left(\lfloor x/n\rfloor\right)\sim\zeta(2)^{-1}x\log x,\quad \text{as}\quad x\to\infty.$$

Their conjecture was once considered to be out of reach at the present day but confirmed by Zhai [17] later. Another example of particular interest is the one for $f=\Lambda$. In [7], Liu, Wu and Zhang proved the following elaborate asymptotic formula

$$\sum_{n\le x}\Lambda\left(\lfloor x/n\rfloor\right)=x\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n(n+1)}+O(x^{9/19+\varepsilon}).$$

The growth conditions required by Bordellès, Dai, Heyman, Pan and Shparlinski or the corresponding error terms in their formulae are improved extensively in a large number of subsequent articles, see [1, 3, 7, 9, 10, 11, 15, 16, 17, 18].

In an other direction, the work of Bordellès, Luca, Moree and Shparlinski [5] about the Bernoulli polynomials has led to the consideration of truncated distribution of the primes represented by $\lfloor x/n\rfloor$. Let $0<\theta\le 1$ be a real number and $\pi_\theta(x)$ be the number of integers n with $1\le n\le x^\theta$ such that $\lfloor\frac{x}{n}\rfloor$ is prime. In [8], Ma, Chen and Wu proved

$$\pi_\theta(x)=\begin{cases}\displaystyle\sum_{k=1}^{L}\frac{(-1)^{k-1}(k-1)!}{(1-\theta)^k}\frac{x^\theta}{(\log x)^k}+O\left(\frac{x^\theta}{(\log x)^{L+1}}\right), & \dfrac{23}{47}<\theta<1,\\ \tau x+O\left(x^{\frac{26}{53}}(\log x)^{\frac{119}{53}}\right), & \theta=1,\end{cases}\tag{1.2}$$

where $\frac{23}{47}=0.489361\cdots$, $L\ge 1$ is any given integer and

$$\tau=\sum_{p\ \text{prime}}\frac{1}{p(p+1)}.$$

For $\theta=1$, their result reduces to an example of the theorem established by Bordellès, Dai, Heyman, Pan and Shparlinski (equation (1.1) with f being the character function of the primes). It is amazing that the leading term in the asymptotic formula of $\pi_\theta(x)$ is not continuous at the point $\theta=1$ when x is a given large number. Ma, Chen and Wu further proposed the following conjecture:

Conjecture 1.1. *For any $0<\theta<\frac{1}{2}$ and given integer $L\ge 1$, we have*

$$\pi_\theta(x)=\sum_{k=1}^{L}\frac{(-1)^{k-1}(k-1)!}{(1-\theta)^k}\frac{x^\theta}{(\log x)^k}+O\left(\frac{x^\theta}{(\log x)^{L+1}}\right).$$

In this article, we shall give an improvement of the result obtained by Ma, Chen and Wu by extending the scope of θ. Now, let's state our main result as the following theorem.

Theorem 1. *Let θ be a number with $\frac{9}{19}(=0.4736842\cdots)<\theta<1$ and $L\ge 1$ be a given integer. Then*

$$\pi_\theta(x)=\sum_{k=1}^{L}\frac{(-1)^{k-1}(k-1)!}{(1-\theta)^k}\frac{x^\theta}{(\log x)^k}+O\left(\frac{x^\theta}{(\log x)^{L+1}}\right),$$

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where the implied constant depends on θ, L and the real number $\varepsilon > 0$ which is contained in Lemma 1.

2. Proof of Theorem 1

From now on, let x be a large positive number. Let $\varepsilon > 0$ be an arbitrary small positive number which may not be the same throughout our paper. Let $\mathbb{N}$ and $\mathcal{P}$ be the set of positive integers and prime numbers, respectively. The notation p will always denote a prime number. Let $\pi(x)$ be the number of primes up to x and

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^\alpha, \\ 0 & \text{otherwise} \end{cases}$$

be the Mangoldt function. For any real number t, let

$$\rho(t) = t - \lfloor t \rfloor - 1/2.$$

For $0 < D \leq x$, $D < t \leq 2D$ and $\delta \notin -\mathbb{N}$, let

$$\Sigma_\delta(x, D, t) = \sum_{D < d \leq t} \Lambda(d) \rho\left(\frac{x}{d + \delta}\right)$$

We need some auxiliary results before the proof of Theorem 1.

Lemma 1. *Let $\delta \notin -\mathbb{N}$ be a fixed constant. For $D < t \leq 2D$ and $D < x^{2/3}$, we have*

$$\Sigma_\delta(x, D, t) \ll_\varepsilon \begin{cases} x^{1/2+\varepsilon} D^{-1/6} & \text{if } D < x^{3/7}, \\ x^{1/3+\varepsilon} D^{2/9} & \text{if } x^{3/7} \leq D < x^{6/13}, \\ x^{1/6+\varepsilon} D^{7/12} & \text{if } x^{6/13} \leq D < x^{2/3}. \end{cases}$$

Proof. From [7, Proposition 4.1] with $(\kappa, \lambda) = (\kappa', \lambda') = (1/2, 1/2)$, we have

$$\Sigma_\delta(x, D, 2D) \ll_\varepsilon x^\varepsilon \left(x^{1/6} D^{7/12} + D^{5/6} + x^{1/3} D^{2/9} + x^{1/2} D^{-1/6}\right)$$

for $D < x^{3/4}$. In fact, by carefully checking the proof of [7, Proposition 4.1], we still have

$$\Sigma_\delta(x, D, t) \ll_\varepsilon x^\varepsilon \left(x^{1/6} D^{7/12} + D^{5/6} + x^{1/3} D^{2/9} + x^{1/2} D^{-1/6}\right)$$

for $D < x^{3/4}$. The lemma then follows from direct discussions. □

For $D \leq x$ and $\delta \notin -\mathbb{N}$, let

$$\mathscr{S}_\delta(x, D) = \sum_{D < p \leq 2D} \rho\left(\frac{x}{p + \delta}\right).$$

Lemma 2. *Let $\delta \notin -\mathbb{N}$ be a fixed constant. For $D < x^{2/3}$, we have*

$$\mathscr{S}_\delta(x, D) \ll_\varepsilon \begin{cases} x^{1/2+\varepsilon} D^{-1/6} + D^{1/2} & \text{if } D < x^{3/7}, \\ x^{1/3+\varepsilon} D^{2/9} + D^{1/2} & \text{if } x^{3/7} \leq D < x^{6/13}, \\ x^{1/6+\varepsilon} D^{7/12} + D^{1/2} & \text{if } x^{6/13} \leq D < x^{2/3}. \end{cases}$$

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Proof. For $0 < D \le x$, $D < t \le 2D$ and $\delta \notin -\mathbb{N}$, let

$$\mathscr{G}_\delta(x, D, t) = \sum_{D<p\le t} \vartheta(d)\rho\left(\frac{x}{p+\delta}\right),$$

where

$$\vartheta(n) = \begin{cases} \log p & \text{if } n = p \text{ is a prime,} \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$\sum_{D<d\le t} \Lambda(d)\rho\left(\frac{x}{d+\delta}\right) = \sum_{D<d\le t} \vartheta(d)\rho\left(\frac{x}{d+\delta}\right) + O\left(t^{1/2}\right),$$

so we have

$$\mathscr{G}_\delta(x, D, t) = \Sigma_\delta(x, D, t) + O\left(D^{1/2}\right) \tag{2.1}$$

for any $D < t \le 2D$. Integrating by parts, we have

$$\mathscr{S}_\delta(x, D) = \frac{\mathscr{G}_\delta(x, D, 2D)}{\log 2D} + \int_D^{2D} \frac{\mathscr{G}_\delta(x, D, t)}{t(\log t)^2} dt.$$

Our lemma follows from equation (2.1) and Lemma 1 by routine computations. □

Lemma 3. [8, Lemma 2.3] *Let $\beta < -1$ be a real number and let L be a positive integer. For $x > 3$, we have*

$$\sum_{p>x} p^\beta = -\sum_{k=1}^{L} \frac{(k-1)!}{(\beta+1)^k} \frac{x^{\beta+1}}{(\log x)^k} + O\left(\frac{x^{\beta+1}}{(\log x)^L}\right),$$

where the implied constant depends only on L and β.

Lemma 4. *Let θ be a positive number with $0 < \theta < 1$ and $L \ge 1$ be a given integer. Then*

$$x \sum_{p\ge x^{1-\theta}} \frac{1}{p(p+1)} = \sum_{k=1}^{L} \frac{(-1)^{k-1}(k-1)!}{(1-\theta)^k} \frac{x^\theta}{(\log x)^k} + O\left(\frac{x^\theta}{(\log x)^{L+1}}\right),$$

where the implied constant depends only on L and θ.

Proof. It is plain that

$$\begin{aligned} \sum_{p\ge x^{1-\theta}} \frac{1}{p(p+1)} &= \sum_{p>x^{1-\theta}} \frac{1}{p(p+1)} + O\left(x^{2\theta-2}\right) \\ &= \sum_{p>x^{1-\theta}} \frac{1}{p^2} - \sum_{p>x^{1-\theta}} \frac{1}{p^2(p+1)} + O\left(x^{2\theta-2}\right) \\ &= \sum_{p>x^{1-\theta}} \frac{1}{p^2} + O\left(x^{2\theta-2}\right). \end{aligned}$$

The lemma follows from the former one via substitutions of x by $x^{1-\theta}$ and $\beta = -2$. □

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Lemma 5. [1, Proposition 3.1] *Let f be a positive-valued function on $\mathbb{N}$ and D a parameter with $D \le x$. Then,*

$$\sum_{D<n\le x} f\left(\lfloor x/n \rfloor\right) = \sum_{d\le x/D} f(d) \sum_{x/(d+1)<n\le x/d} 1 + O\left(f\left(\frac{x}{D}\right)\left(1+\frac{D^2}{x}\right)\right).$$

Let's turn back to the proof of Theorem 1.

Proof of Theorem 1. For $\theta > 9/19$, we spilt the sum $\pi_\theta(x)$ into the following two shorter sums

$$\pi_\theta(x) = S_1 + S_2,$$

where

$$S_1 = \sum_{\substack{n\le x^{9/19}\\ \lfloor x/n\rfloor\in\mathcal{P}}} 1 \quad \text{and} \quad S_2 = \sum_{\substack{x^{9/19}<n\le x^{\theta}\\ \lfloor x/n\rfloor\in\mathcal{P}}} 1.$$

Trivial estimate leads to the bound

$$S_1 \le \sum_{n\le x^{9/19}} 1 \le x^{9/19}.$$

By Lemma 5, we can rewrite S_2 as

$$\begin{aligned} S_2 = &\sum_{x^{1-\theta}\le p\le x^{10/19}} \sum_{x/(p+1)<n\le x/p} 1 + O(x^{2\theta-1}) \\ = &\sum_{x^{1-\theta}\le p\le x^{10/19}} \left\{\frac{x}{p} - \rho\left(\frac{x}{p}\right) - \frac{x}{p+1} + \rho\left(\frac{x}{p+1}\right)\right\} + O(x^{2\theta-1}) \\ =& x\sum_{p\ge x^{1-\theta}} \frac{1}{p(p+1)} - x\sum_{p>x^{10/19}} \frac{1}{p(p+1)} + R_1(x) - R_0(x) + O(x^{2\theta-1}), \end{aligned}$$

where

$$R_\delta(x) = \sum_{x^{1-\theta}\le p\le x^{10/19}} \rho\left(\frac{x}{p+\delta}\right) \quad (\delta = 0 \text{ or } 1).$$

It is easy to see that $x^{2\theta-1} \ll x^{\theta-\varepsilon}$ for any $\theta < 1$ and it is clear that

$$x\sum_{p>x^{10/19}} \frac{1}{p(p+1)} \le x\sum_{n\ge x^{10/19}} \frac{1}{n(n+1)} \ll x^{9/19}.$$

From Lemma 4, we have

$$x\sum_{p\ge x^{1-\theta}} \frac{1}{p(p+1)} = \sum_{k=1}^{L} \frac{(-1)^{k-1}(k-1)!}{(1-\theta)^k} \frac{x^\theta}{(\log x)^k} + O\left(\frac{x^\theta}{(\log x)^{L+1}}\right).$$

To complete the proof of our theorem, it remains to show that

$$R_\delta(x) \ll x^\theta(\log x)^{-(L+1)}$$

for $\delta = 0$ and 1. For any positive integer i, let $D_i = x^{10/19}2^{-i}$. Since $\theta > 9/19$, then $D_i \le x^{10/19} < x^{2/3}$ for all $1 \le i \le \left\lfloor \frac{\theta-9/19}{\log 2}\log x\right\rfloor + 1$. By Lemma 2,

$$|R_\delta(x)| \le \sum_{1\le i\le \lfloor \frac{\theta-9/19}{\log 2}\log x\rfloor+1} \mathscr{S}_\delta(x, D_i)$$

$$\ll_\varepsilon \sum_{1\le i\le \lfloor \frac{\theta-9/19}{\log 2}\log x\rfloor+1} \left(x^{1/2+\varepsilon}D_i^{-1/6}+x^{1/3+\varepsilon}D_i^{2/9}+x^{1/6+\varepsilon}D_i^{7/12}+D_i^{1/2}\right)$$
$$\ll_\varepsilon x^{(\theta+2/6)}+x^{9/19+\varepsilon}\ll_{\varepsilon,\theta} x^{9/19+\varepsilon}\ll_{\varepsilon,\theta} x^\theta(\log x)^{-(L+1)},$$

valid for $9/19<\theta<16/19$. Combined with the effective range $\frac{23}{47}<\theta<1$ of Ma–Chen–Wu, we get the theorem. □

3. A weighted summation

In this section, we shall give a weighted version of the results obtained by Ma, Chen and Wu. The asymptotic formulae of the sum below is investigated:

$$\Lambda_\theta(x):=\sum_{n\le x^\theta}\Lambda(\lfloor x/n\rfloor).$$

Theorem 2. *Let θ be a number with $\frac{9}{19}<\theta<1$. For any any integer $A\ge 1$, We have*

$$\Lambda_\theta(x)=x^\theta+O\left(x^\theta(\log x)^{-A}\right),$$

where the implied constant depends on θ, L and the real number $\varepsilon>0$ which is contained in Lemma 1.

It is worth mentioning that Theorem 2 cannot be derived by Theorem 1 directly via integration by parts. Parallel to Lemma 4, we have the following weighted one.

Lemma 6. *Let θ be a positive number with $0<\theta<1$ and A is any given positive number, we have*

$$x\sum_{d\ge x^{1-\theta}}\frac{\Lambda(d)}{d(d+1)}=x^\theta+O\left(x^\theta(\log x)^{-A}\right),$$

where the implied constant depends only on A and θ.

Proof. Let $\psi(x)=\sum_{n\le x}\Lambda(n)$. Then for any $A>0$ we have

$$\psi(x)=x+O\left(x(\log x)^{-A}\right)$$

via the prime number theorem. It is plain that

$$\begin{aligned}x\sum_{d\ge x^{1-\theta}}\frac{\Lambda(d)}{d(d+1)}&=x\sum_{d>x^{1-\theta}}\frac{\Lambda(d)}{d(d+1)}+O\left(\frac{\log x}{x^{1-2\theta}}\right),\\&=x\sum_{d>x^{1-\theta}}\frac{\Lambda(d)}{d^2}-x\sum_{d>x^{1-\theta}}\frac{\Lambda(d)}{d^2(d+1)}+O\left(x^{2\theta-1}\log x\right)\\&=x\sum_{d>x^{1-\theta}}\frac{\Lambda(d)}{d^2}+O\left(x^{2\theta-1}\log x\right).\end{aligned}\tag{3.1}$$

We need to deal with the summation $\sum_{d>x^{1-\theta}}\frac{\Lambda(d)}{d^2}$. Integration by parts gives

$$\begin{aligned}\sum_{d>x^{1-\theta}}\frac{\Lambda(d)}{d^2}&=\frac{\psi(t)}{t^2}\Big|_{x^{1-\theta}}^{\infty}+2\int_{x^{1-\theta}}^\infty\frac{\psi(t)}{t^3}dt\\&=-x^{\theta-1}+O\left(x^{\theta-1}(\log x)^{-A}\right)+2\int_{x^{1-\theta}}^\infty\frac{1}{t^2}dt+O\left(\int_{x^{1-\theta}}^\infty\frac{1}{t^2(\log t)^A}dt\right)\end{aligned}$$

$$= x^{\theta-1} + O\left(x^{\theta-1}(\log x)^{-A} + (\log x)^{-A}\int_{x^{1-\theta}}^{\infty}\frac{1}{t^2}dt\right)$$
$$= x^{\theta-1} + O\left(x^{\theta-1}(\log x)^{-A}\right). \tag{3.2}$$

Now the lemma follows from equations (3.1) and (3.2) immediately. □

Proof of Theorem 2. The proof of

$$\Lambda_\theta(x) = x^\theta + O\left(x^\theta(\log x)^{-A}\right) \tag{3.3}$$

is similar to the proof of Theorem 1 by replacing Lemma 4 with Lemma 6. □

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