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RIGHT HADAMARD FRACTIONAL DIFFERENCES AND SUMMATION BY PARTS

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ABSTRACT. This paper investigates the right Hadamard fractional differences. First, a Q–operator is defined for continuous Hadamard fractional calculus. It shows that the left and right Hadamard operators satisfy a dual identity. Then, the Q–operator is extended to define the right Hadamard fractional sum and difference, and present their properties. Initial value problem of right fractional difference equations is discussed. Finally the related summation by parts formulae are obtained. It can be concluded that the right Hadamard fractional sum and difference are well defined.

1. Introduction

In the past decades, since fractional derivatives can be used to describe long-term interactions or memory effects in various nonlinear phenomena [1, 2], it has been developed rapidly. However, it is challenging to provide an exact discretization tool for both theoretical analysis and discrete–time applications. The time scale theory can investigate dynamical systems in both continuous and discrete-time cases. This feature is particularly suitable for fractional modelings with computer implementations. Thus, fractional calculus on time scales was paid much attention such as Riemann and Caputo fractional differences [3, 4], the basics of discrete fractional calculus [5], initial value problems [6, 7], boundary value problems [8, 9], stability analysis [10, 11] etc.

Recently, a left Hadamard fractional calculus was proposed on time scales [12]. The logarithm function [13] was used as a discrete kernel function. Some useful propositions and exact solutions of linear fractional difference equations were obtained. Fractional chaotic dynamics were demonstrated [14]. Considering the right fractional calculus's important role in the variation approach [15], Riesz potential [16], diffusion problems [17] and right fractional Black–Scholes equations [18], it is crucial to define right fractional Hadamard differences and give their properties.

In fact, the left and right Riemann–Liouville fractional integrals and derivatives can be connected by a Q–operator [19]. This Q–operator can also associate the left and right fractional Riemann–Liouville fractional sums and differences (see [4, 20]). So we think that the left and right Hadamard fractional calculus also satisfies the dual identity through the Q–operator. However, Q–operators for different fractional calculus depend on the kernel functions. Thus this paper redefines a new Q–operator and gives definitions of right Hadamard discrete fractional calculus through left ones.

The paper is organized as follows: Section 2 introduces preliminaries of the Hadamard fractional calculus. A new Q–operator is given and the dual identities between left and right Hadamard fractional calculus are verified. Section 3 gives the definitions of right Hadamard fractional sum and difference,

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and some useful properties. Section 4 discusses the right Hadamard fractional initial value problem, and defines right discrete Mittag–Leffler function. Section 5 is devoted to the summation by parts for Hadamard fractional sums and differences. Section 6 arrives at the conclusion.

2. Hadamard fractional calculus

In this section, we first present some definitions of the Hadamard fractional calculus. And the Q–operator is defined to connect left and right Hadamard fractional calculus.

For the Hadamard fractional integral, define the space $X_c^p(a,b)$ $(c \in \mathbb{R},\ 1 \le p \le \infty)$ [1] of those complex–valued measurable functions x on $[a,b]$ for which $\|x\|_{X_c^p} < \infty$, where

$$\|x\|_{X_c^p} = \left(\int_a^b |t^c x(t)|^p \frac{\mathrm{d}t}{t} \right)^{1/p} \ (1 \le p < \infty)$$

and

$$\|x\|_{X_c^\infty} = ess\ sup_{a \le t \le b} [t^c\, |x(t)|].$$

Particularly, when $c = 1/p$ $(1 \le p \le \infty)$, the space $X_c^p(a,b)$ is reduced to the classical $L^p(a,b)$ space with

$$\|x\|_p = \left(\int_a^b |x(t)|^p \,\mathrm{d}t \right)^{1/p} \ (1 \le p < \infty)$$

and

$$\|x\|_\infty = ess\ sup_{a \le t \le b} |x(t)|\,.$$

Then, the space $AC_\delta^n[a,b]$ of Hadamard fractional derivative is given in [1]. Suppose function $x(t)$ have $\delta = xD$ $(D = \mathrm{d}/\mathrm{d}t)$ derivatives up to $n-1$ order on $[a,b]$ and $\delta^{n-1}x(t)$ is absolutely continuous on [a,b]:

$$AC_\delta^n[a,b] = \{x(t):\ [a,b] \to \mathbb{C}:\ \delta^{n-1}x(t) \in AC[a,b],\ \delta = t\frac{\mathrm{d}}{\mathrm{d}t}\}.$$

Here $AC[a,b]$ is the set of absolutely continuous functions on [a,b]. $AC[a,b]$ coincides with the space of primitives of Lebesgue measurable functions:

$$x(t) \in AC[a,b] \Leftrightarrow x(t) = x(a) + \int_a^t x'(s)\mathrm{d}s.$$

Definition 1. [21] *Let* $1 \le a < b < \infty$ *and* $x \in X_c^p(a,b)$. *The left and right Hadamard fractional integrals of order* $\alpha > 0$ *are defined by*

$$(1)\qquad {}_a^H I_t^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \Big(\log(t) - \log(s) \Big)^{\alpha-1} x(s) \frac{\mathrm{d}s}{s},\ t \in [a,\ b]$$

and

$$(2)\qquad {}_t^H I_b^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \Big(\log(s) - \log(t) \Big)^{\alpha-1} x(s) \frac{\mathrm{d}s}{s},\ t \in [a,\ b],$$

where Γ *is the Euler gamma function.*

Definition 2. [21] *Let $1 \le a < b < \infty$, $\alpha > 0$, $n = [\alpha]+1$ and $x \in AC^n_\delta[a,b]$. The left and right Hadamard fractional derivatives of order α are defined by*

$$ {}^H_aD^\alpha_t x(t) = \frac{1}{\Gamma(n-\alpha)}\delta^n \int_a^t \Big(\log(t)-\log(s)\Big)^{n-\alpha-1} x(s)\frac{\mathrm{d}s}{s},\ t \in [a,\ b] \tag{3}$$

and

$$ {}^H_tD^\alpha_b x(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\delta^n \int_t^b \Big(\log(s)-\log(t)\Big)^{n-\alpha-1} x(s)\frac{\mathrm{d}s}{s},\ t \in [a,\ b], \tag{4}$$

where $\delta = t\frac{d}{dt}$ and $\delta^n = \underbrace{t\frac{\mathrm{d}}{\mathrm{d}t}\, t\frac{\mathrm{d}}{\mathrm{d}t}\cdots t\frac{\mathrm{d}}{\mathrm{d}t}}_{n}$.

Theorem 3. [21] *Let $1 \le a < b < \infty$, $\alpha > 0$, $n = [\alpha]+1$ and $x \in AC^n_\delta[a,b]$. Then*

$$ {}^H_aI^\alpha_t\, {}^H_aD^\alpha_t x(t) = x(t) - \sum_{k=1}^{n} \frac{\delta^{n-k}\, {}^H_aI^{n-\alpha}_t x(a)}{\Gamma(\alpha-k+1)}\Big(\log(t)-\log(a)\Big)^{\alpha-k} \tag{5}$$

and

$$ {}^H_tI^\alpha_b\, {}^H_tD^\alpha_b x(t) = x(t) - \sum_{k=1}^{n} \frac{(-1)^{n-k}\delta^{n-k}\, {}^H_tI^\alpha_b x(b)}{\Gamma(\alpha-k+1)}\Big(\log(b)-\log(t)\Big)^{\alpha-k}. \tag{6}$$

We define a Q–operator to generate a dual identity, which shows that left and right Hadamard fractional integrals and derivatives are related.

Definition 4. *Let $a,b \in \mathbb{R}$ and x is defined on inteval $[a,b]$. The Q–operator is defined as*

$$ (Qx)(t) = x\Big(\frac{ab}{t}\Big). \tag{7}$$

We can verify that $Q^2x(t) = x(t)$. Next, the change of variable $u = \frac{ab}{s}$ is used. The following results are obtained.

Proposition 5. *Let $x \in X^p_c(a,b)$ and $\alpha > 0$. The left and right Hadamard fractional integrals satisfy the following equation*

$$ \big({}^H_aI^\alpha_t Qx\big)(t) = Q\big({}^H_tI^\alpha_b x\big)(t). \tag{8}$$

Proof. According to Definition 1, we get

$$\begin{aligned} \big({}^H_aI^\alpha_t Qx\big)(t) &= \frac{1}{\Gamma(\alpha)}\int_a^t \Big(\log(t)-\log(s)\Big)^{\alpha-1} x\Big(\frac{ab}{s}\Big)\frac{\mathrm{d}s}{s} \\ &= \frac{1}{\Gamma(\alpha)}\int_a^t \Big(\log\frac{ab}{s}-\log\frac{ab}{t}\Big)^{\alpha-1} x\Big(\frac{ab}{s}\Big)\frac{\mathrm{d}s}{s}. \end{aligned}$$

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Let $u=\frac{ab}{s}$. We have

$$\left({}_a^H I_t^\alpha Qx\right)(t)=\frac{1}{\Gamma(\alpha)}\int_b^{\frac{ab}{t}}\left(\log(u)-\log\frac{ab}{t}\right)^{\alpha-1}x(u)\frac{\mathrm{d}u}{-u}$$
$$=\frac{1}{\Gamma(\alpha)}\int_{\frac{ab}{t}}^{b}\left(\log(u)-\log\frac{ab}{t}\right)^{\alpha-1}x(u)\frac{\mathrm{d}u}{u}$$
$$=Q\left({}_t^H I_b^\alpha x\right)(t).$$

□

Proposition 6. *Let $\alpha>0$, $n=[\alpha]+1$ and $x\in AC_\delta^n[a,b]$. The left and right Hadamard fractional derivatives satisfy the following dual identity*

$$\left({}_a^H D_t^\alpha Qx\right)(t)=Q\left({}_t^H D_b^\alpha x\right)(t). \tag{9}$$

Proof. The proof process is similar to Proposition 5. By use of Definition 2, we obtain

$$\left({}_a^H D_t^\alpha Qx\right)(t)=\frac{1}{\Gamma(n-\alpha)}\delta^n\int_a^t\Big(\log(t)-\log(s)\Big)^{n-\alpha-1}x\left(\frac{ab}{s}\right)\frac{\mathrm{d}s}{s}$$
$$=\frac{1}{\Gamma(n-\alpha)}\delta^n\int_a^t\left(\log\frac{ab}{s}-\log\frac{ab}{t}\right)^{n-\alpha-1}x\left(\frac{ab}{s}\right)\frac{\mathrm{d}s}{s}.$$

For $n=1$, we have $t\frac{\mathrm{d}}{\mathrm{d}t}=-\frac{ab}{t}\frac{\mathrm{d}}{\mathrm{d}\frac{ab}{t}}$ and $\mathrm{d}s=-\frac{s^2}{ab}\mathrm{d}u$. So

$$\left({}_a^H D_t^\alpha Qx\right)(t)=\frac{1}{\Gamma(1-\alpha)}\left(-\frac{ab}{t}\frac{\mathrm{d}}{\mathrm{d}\frac{ab}{t}}\right)\int_a^t\left(\log\frac{ab}{s}-\log\frac{ab}{t}\right)^{-\alpha}x\left(\frac{ab}{s}\right)\frac{\mathrm{d}s}{s}$$
$$=\frac{1}{\Gamma(1-\alpha)}\left(-\frac{ab}{t}\frac{\mathrm{d}}{\mathrm{d}\frac{ab}{t}}\right)\int_b^{\frac{ab}{t}}\left(\log u-\log\frac{ab}{t}\right)^{-\alpha}x(u)\frac{\mathrm{d}u}{-u}$$
$$=\frac{1}{\Gamma(1-\alpha)}\left(-\frac{ab}{t}\frac{\mathrm{d}}{\mathrm{d}\frac{ab}{t}}\right)\int_{\frac{ab}{t}}^{b}\left(\log u-\log\frac{ab}{t}\right)^{-\alpha}x(u)\frac{\mathrm{d}u}{u}$$
$$=Q\left({}_t^H D_b^\alpha x\right)(t).$$

For $n=2$, we can obtain $\delta^2=t\frac{\mathrm{d}}{dt}t\frac{\mathrm{d}}{dt}=\left(-\frac{ab}{t}\frac{\mathrm{d}}{\mathrm{d}\frac{ab}{t}}\right)^2$ and

$$\left({}_a^H D_t^\alpha Qx\right)(t)=\frac{1}{\Gamma(2-\alpha)}\left(-\frac{ab}{t}\frac{\mathrm{d}}{\mathrm{d}\frac{ab}{t}}\right)^2\int_a^t\left(\log\frac{ab}{s}-\log\frac{ab}{t}\right)^{1-\alpha}x\left(\frac{ab}{s}\right)\frac{\mathrm{d}s}{s}$$
$$=\frac{1}{\Gamma(2-\alpha)}\left(-\frac{ab}{t}\frac{\mathrm{d}}{\mathrm{d}\frac{ab}{t}}\right)^2\int_{\frac{ab}{t}}^{b}\left(\log u-\log\frac{ab}{t}\right)^{1-\alpha}x(u)\frac{\mathrm{d}u}{u}$$
$$=Q\left({}_t^H D_b^\alpha x\right)(t).$$

Through induction, we can get

$$\begin{aligned}\left({}_a^H D_t^\alpha Qx\right)(t) &= \frac{1}{\Gamma(n-\alpha)}\left(-\frac{ab}{t}\frac{\mathrm{d}}{\mathrm{d}\frac{ab}{t}}\right)^n \int_b^{\frac{ab}{t}} \left(\log(u)-\log\frac{ab}{t}\right)^{n-\alpha-1} x(u)\frac{\mathrm{d}u}{-u} \\ &= \frac{(-1)^n}{\Gamma(n-\alpha)}\left(\frac{ab}{t}\frac{\mathrm{d}}{\mathrm{d}\frac{ab}{t}}\right)^n \int_{\frac{ab}{t}}^{b} \left(\log(u)-\log\frac{ab}{t}\right)^{n-\alpha-1} x(u)\frac{\mathrm{d}u}{u} \\ &= Q\left({}_t^H D_b^\alpha x\right)(t).\end{aligned}$$

Therefore, Eq. (6) holds. □

Proposition 7. *Let $\alpha > 0$, $n = [\alpha]+1$ and $x \in AC_\delta^n[a,b]$. Then*

$$\left({}_a^H I_t^\alpha {}_a^H D_t^\alpha Qx\right)(t) = Q\left({}_t^H I_b^\alpha {}_t^H D_b^\alpha x\right)(t). \tag{10}$$

Proof. According to Theorem 3, we can get

$$\begin{aligned}\left({}_a^H I_t^\alpha {}_a^H D_t^\alpha Qx\right)(t) &= x\left(\frac{ab}{t}\right) - \sum_{k=1}^n \frac{\left(\delta^{n-k}\left({}_a^H I_t^{n-\alpha} Qx\right)\right)(a)}{\Gamma(\alpha-k+1)}\Big(\log(t)-\log(a)\Big)^{\alpha-k} \\ &= x\left(\frac{ab}{t}\right) - \sum_{k=1}^n \frac{\left(\delta^{n-k}\left(Q\left({}_t^H I_b^\alpha x\right)\right)\right)(a)}{\Gamma(\alpha-k+1)}\left(\log\frac{ab}{a}-\log\frac{ab}{t}\right)^{\alpha-k}.\end{aligned}$$

For $n=1$, we have

$$t\frac{\mathrm{d}}{\mathrm{d}t}Qx(t) = t\frac{\mathrm{d}}{\mathrm{d}t}x\left(\frac{ab}{t}\right) = -\frac{ab}{t}\frac{\mathrm{d}}{\mathrm{d}\frac{ab}{t}}x\left(\frac{ab}{t}\right).$$

For $n=2$, we have

$$\begin{aligned}\delta^2 Qx(t) &= t\frac{\mathrm{d}}{\mathrm{d}t}t\frac{\mathrm{d}}{\mathrm{d}t}Qx(t) \\ &= t\frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{ab}{t}\frac{\mathrm{d}}{\mathrm{d}\frac{ab}{t}}x\left(\frac{ab}{t}\right)\right) \\ &= -\frac{ab}{t}\frac{\mathrm{d}}{\mathrm{d}\frac{ab}{t}}\left(-\frac{ab}{t}\frac{\mathrm{d}}{\mathrm{d}\frac{ab}{t}}x\left(\frac{ab}{t}\right)\right) \\ &= \left(-\frac{ab}{t}\frac{\mathrm{d}}{\mathrm{d}\frac{ab}{t}}\right)^2 x\left(\frac{ab}{t}\right).\end{aligned}$$

Suppose $\delta^{n-1}Qx(t) = \left(-\frac{ab}{t}\frac{\mathrm{d}}{\mathrm{d}\frac{ab}{t}}\right)^{n-1} x\left(\frac{ab}{t}\right)$ holds. We get

$$\begin{aligned}\delta^n Qx(t) &= t\frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{ab}{t}\frac{\mathrm{d}}{\mathrm{d}\frac{ab}{t}}\right)^{n-1} x\left(\frac{ab}{t}\right) \\ &= -\frac{ab}{t}\frac{\mathrm{d}}{\mathrm{d}\frac{ab}{t}}\left(-\frac{ab}{t}\frac{\mathrm{d}}{\mathrm{d}\frac{ab}{t}}\right)^{n-1} x\left(\frac{ab}{t}\right) \\ &= \left(-\frac{ab}{t}\frac{\mathrm{d}}{\mathrm{d}\frac{ab}{t}}\right)^n x\left(\frac{ab}{t}\right).\end{aligned}$$

So we arrive at

$$\left({}_a^H I_t^{\alpha}{}_a^H D_t^{\alpha} Qx\right)(t) = x\left(\frac{ab}{t}\right) - \sum_{k=1}^{n} \frac{(-1)^{n-k}\left(\left(\frac{ab}{t}\frac{\mathrm{d}}{\mathrm{d}\frac{ab}{t}}\right)^{n-k}\left({}_{\frac{ab}{t}}^H I_b^{\alpha} x\right)\right)(b)}{\Gamma(\alpha-k+1)}\left(\log(b) - \log\frac{ab}{t}\right)^{\alpha-k}$$
$$= Q\left({}_t^H I_b^{\alpha}{}_t^H D_b^{\alpha} x\right)(t).$$

□

The Q–operator holds dual identities and it is very useful to reflect the left fractional calculus to the right one. Now we turn to extend it to discrete cases and introduce how to obtain the right Hadamard fractional sum and difference as well as their propositions.

3. Right Hadamard fractional sum and difference

Before giving our results, we introduce some definitions and notations. For real numbers a and b $(1 \leq a < b < \infty)$, we denote

$$\mathbb{T}_a = e^{(h\mathbb{N})_{\log a}} = \{a, ae^h, ae^{2h}, \ldots, ae^{(N-1)h}, ae^{Nh}, \ldots\}$$

and

$${}_b\mathbb{T} = e^{{}_{\log b}(h\mathbb{N})} = \{\ldots, be^{-Nh}, be^{-(N-1)h}, \ldots, be^{-2h}, be^{-h}, b\}.$$

Notation 8.

I) The backward and forward jump operators are $\rho(t) = te^{-h}$ and $\sigma(t) = te^h$, respectively.

II) The graininess function is defined by $\mu(t) = \sigma(t) - t$. The backwards graininess function is defined by $\nu(t) = t - \rho(t)$.

III) Given that the forward and backward difference operators are defined by $\Delta f(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}$ and $\nabla f(t) = \frac{f(t) - f(\rho(t))}{\nu(t)}$, respectively.

IV) In this paper, the logarithm function is used as $\ell_p(t,s) = \log(\frac{t}{s})$. The Δ–differentiation and ∇–differentiation of $\ell_p(t,s)$ with respect to t are $\ell_p^{\Delta}(t,s) = \frac{1}{\mu(t)}\log\left(\frac{\sigma(t)}{t}\right)$ and $\ell_p^{\nabla}(t,s) = \frac{1}{\nu(t)}\log\left(\frac{t}{\rho(t)}\right)$, respectively.

V) We denote $\widetilde{\Delta} f(t) = \frac{1}{\ell_p^{\Delta}(t,r)} f^{\Delta}(t)$ and $\widetilde{\nabla} f(t) = \frac{1}{\ell_p^{\nabla}(t,r)} f^{\nabla}(t)$.

Definition 9. [22] *For arbitrary $\alpha \in \mathbb{R}$, the h-factorial function is defined by*

$$t_h^{\underline{\alpha}} := h^{\alpha}\frac{\Gamma(\frac{t}{h}+1)}{\Gamma(\frac{t}{h}+1-\alpha)},\ t \in (h\mathbb{N})_{\alpha h}. \tag{11}$$

Definition 10. [12] *Suppose $f : \mathbb{T}_a \to \mathbb{R}$ and $\alpha > 0$. The left Hadamard fractional sum of order α is defined by*

$$\widetilde{\Delta}_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\sum_{s=a}^{te^{-\alpha h}}\Big(\ell_p(t,r) - \ell_p(\sigma(s),r)\Big)_h^{\underline{\alpha-1}}\ell_p^{\Delta}(s,r)f(s)\mu(s),\ t \in e^{(h\mathbb{N})_{\log a + \alpha h}}. \tag{12}$$

Definition 11. [12] *Suppose $f:\mathbb{T}_a\to\mathbb{R}$, $M\in\mathbb{N}_1$ and $M-1<\alpha\le M$. The left Hadamard fractional difference of order α is given by*

$$(13)\qquad \widetilde{\Delta}_a^{\alpha} f(t)=\frac{1}{\Gamma(-\alpha)}\sum_{s=a}^{te^{\alpha h}}\Big(\ell_p(t,r)-\ell_p\big(\sigma(s),r\big)\Big)_h^{\underline{-\alpha-1}}\ell_p^{\Delta}(s,r)f(s)\mu(s),\ t\in e^{(h\mathbb{N})_{\log a+(M-\alpha)h}}.$$

Lemma 12. [12] *Let $\alpha,\ \beta>0$. Then*

$$(14)\qquad \widetilde{\Delta}_{ae^{\beta h}}^{-\alpha}\big(\ell_p(t,r)-\ell_p(a,r)\big)_h^{\underline{\beta}}=\frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\alpha)}\big(\ell_p(t,r)-\ell_p(a,r)\big)_h^{\underline{\beta+\alpha}},\ t\in e^{(h\mathbb{N})_{\log a+(\beta+\alpha)h}}$$

and

$$(15)\qquad \widetilde{\Delta}_{ae^{\beta h}}^{\alpha}\big(\ell_p(t,r)-\ell_p(a,r)\big)_h^{\underline{\beta}}=\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}\big(\ell_p(t,r)-\ell_p(a,r)\big)_h^{\underline{\beta-\alpha}},\ t\in e^{(h\mathbb{N})_{\log a+(\beta+M-\alpha)h}}.$$

Theorem 13. [12] *Suppose $f:\mathbb{T}_a\to\mathbb{R}$, α and $\beta>0$. Then*

$$(16)\qquad \widetilde{\Delta}_{ae^{\beta h}}^{-\alpha}\widetilde{\Delta}_a^{-\beta}f(t)=\widetilde{\Delta}_a^{-\alpha-\beta}f(t)=\widetilde{\Delta}_{ae^{\alpha h}}^{-\beta}\widetilde{\Delta}_a^{-\alpha}f(t),\ t\in e^{(h\mathbb{N})_{\log a+(\alpha+\beta)h}}.$$

Theorem 14. [12] *Suppose $f:\mathbb{T}_a\to\mathbb{R}$, α and $\beta>0$. Then*

$$(17)\qquad \widetilde{\Delta}_{ae^{\beta h}}^{\alpha}\widetilde{\Delta}_a^{-\beta}f(t)=\widetilde{\Delta}_a^{\alpha-\beta}f(t),\ t\in e^{(h\mathbb{N})_{\log a+(\beta+M-\alpha)h}}.$$

Theorem 15. [12] *Suppose $f:\mathbb{T}_a\to\mathbb{R}$, $L,M\in\mathbb{N}_1$ and $\alpha,\beta>0$ be given with $L-1<\alpha\le L$ and $M-1<\beta\le M$. Then*

$$(18)\qquad \begin{aligned}&\widetilde{\Delta}_{ae^{(M-\beta)h}}^{\alpha}\widetilde{\Delta}_a^{\beta}f(t)=\widetilde{\Delta}_a^{\alpha+\beta}f(t)\\&\qquad-\sum_{k=1}^{M}\frac{\big(\ell_p(t,r)-\ell_p(ae^{(M-\beta)h},r)\big)_h^{\underline{-\alpha-k}}}{\Gamma(-\alpha-k+1)}\widetilde{\Delta}^{M-k}\widetilde{\Delta}_a^{-(M-\beta)}f(ae^{(M-\beta)h}),\end{aligned}$$

where $t\in e^{(h\mathbb{N})_{\log a+(L-\alpha+M-\beta)h}}$.

Theorem 16. [12] *Let $f:\mathbb{T}_a\to\mathbb{R}$, $\alpha,\beta>0$, $M\in\mathbb{N}_1$ and $M-1<\beta\le M$. Then*

$$(19)\qquad \begin{aligned}&\widetilde{\Delta}_{ae^{(M-\beta)h}}^{-\alpha}\widetilde{\Delta}_a^{\beta}f(t)=\widetilde{\Delta}_a^{\beta-\alpha}f(t)\\&\qquad-\sum_{k=1}^{M}\frac{\big(\ell_p(t,r)-\ell_p(ae^{(M-\beta)h},r)\big)_h^{\underline{\alpha-k}}}{\Gamma(\alpha-k+1)}\widetilde{\Delta}^{M-k}\widetilde{\Delta}_a^{-(M-\beta)}f(ae^{(M-\beta)h}),\end{aligned}$$

where $t\in e^{(h\mathbb{N})_{\log a+(M-\beta+\alpha)h}}$.

From Hadamard fractional calculus, it is known that $\big({}_a^H I_t^{\alpha}Qx\big)(t)=Q\big({}_t^H I_b^{\alpha}x\big)(t)$ and $\big({}_a^H D_t^{\alpha}Qx\big)(t)=Q\big({}_t^H D_b^{\alpha}x\big)(t)$. We extend this idea to define the right Hadamard fractional sum and difference. We use the following notations:

1) ${}_b\widetilde{\Delta}^{-\alpha}$ represents the right Hadamard fractional sum. ${}_b\widetilde{\Delta}^{-\alpha}$ maps functions defined on ${}_b\mathbb{T}$ to those on $e^{\log b-\alpha h^{(h\mathbb{N})}}$.

2) ${}_b\widetilde{\Delta}^{\alpha}$ represents the right Hadamard fractional difference. ${}_b\widetilde{\Delta}^{\alpha}$ maps functions defined on ${}_b\mathbb{T}$ to those on $e^{\log b-(M-\alpha)h^{(h\mathbb{N})}}$, where $M\in\mathbb{N}_1$ and $M-1<\alpha\le M$.

We define Hadamard fractional sum as

(20)

$$
\begin{aligned}
{}_b\widetilde{\Delta}^{-\alpha} f(t) &:= Q\Big(\widetilde{\Delta}_a^{-\alpha} Qf\Big)(t) \\
&= Q\left(\frac{1}{\Gamma(\alpha)}\sum_{s=a}^{te^{-\alpha h}}\Big(\ell_p(t,r)-\ell_p\big(\sigma(s),r\big)\Big)_h^{\alpha-1}\ell_p^{\Delta}(s,r)(Qf)(s)\mu(s)\right) \\
&= Q\left(\frac{1}{\Gamma(\alpha)}\sum_{s=a}^{te^{-\alpha h}}\Big(\log\frac{t}{r}-\log\frac{\sigma(s)}{r}\Big)_h^{\alpha-1}\frac{\Big(\log\frac{\sigma(s)}{r}-\log\frac{s}{r}\Big)}{\mu(s)}f(\frac{ab}{s})\mu(s)\right) \\
&= Q\left(\frac{1}{\Gamma(\alpha)}\sum_{s=a}^{te^{-\alpha h}}\Big(\log\frac{ab}{\sigma(s)r}-\log\frac{ab}{tr}\Big)_h^{\alpha-1}\frac{\Big(\log\frac{ab}{sr}-\log\frac{ab}{\sigma(s)r}\Big)}{\mu(s)}f(\frac{ab}{s})\mu(s)\right) \\
&= Q\left(\frac{1}{\Gamma(\alpha)}\sum_{s=a}^{te^{-\alpha h}}\Big(\ell_p(\frac{ab}{\sigma(s)},r)-\ell_p(\frac{ab}{t},r)\Big)_h^{\alpha-1}\frac{\Big(\ell_p(\frac{ab}{s},r)-\ell_p\big(\frac{ab}{\sigma(s)},r\big)\Big)}{\mu(s)}f(\frac{ab}{s})\mu(s)\right) \\
&= Q\left(\frac{1}{\Gamma(\alpha)}\sum_{u=\frac{ab}{t}e^{\alpha h}}^{b}\Big(\ell_p(\rho(u),r)-\ell_p(\frac{ab}{t},r)\Big)_h^{\alpha-1}\Big(\ell_p(u,r)-\ell_p\big(\rho(u),r\big)\Big)f(u)\right) \\
&= Q\left(\frac{1}{\Gamma(\alpha)}\sum_{u=\frac{ab}{t}e^{\alpha h}}^{b}\Big(\ell_p(\rho(u),r)-\ell_p(\frac{ab}{t},r)\Big)_h^{\alpha-1}\frac{\Big(\ell_p(u,r)-\ell_p\big(\rho(u),r\big)\Big)}{\nu(u)}f(u)\nu(u)\right) \\
&= Q\left(\frac{1}{\Gamma(\alpha)}\sum_{u=\frac{ab}{t}e^{\alpha h}}^{b}\Big(\ell_p(\rho(u),r)-\ell_p(\frac{ab}{t},r)\Big)_h^{\alpha-1}\ell_p^{\nabla}(u,r)f(u)\nu(u)\right) \\
&= \frac{1}{\Gamma(\alpha)}\sum_{u=te^{\alpha h}}^{b}\Big(\ell_p(\rho(u),r)-\ell_p\big(t,r\big)\Big)_h^{\alpha-1}\ell_p^{\nabla}(u,r)f(u)\nu(u).
\end{aligned}
$$

Similarly, we can define

$$
\begin{aligned}
{}_b\widetilde{\Delta}^{\alpha} f(t) &:= Q\Big(\widetilde{\Delta}_a^{\alpha} Qf\Big)(t) \\
&= \frac{1}{\Gamma(-\alpha)}\sum_{u=te^{-\alpha h}}^{b}\Big(\ell_p(\rho(u),r)-\ell_p\big(t,r\big)\Big)_h^{-\alpha-1}\ell_p^{\nabla}(u,r)f(u)\nu(u).
\end{aligned}
\tag{21}
$$

We give the definitions of the right Hadamard fractional sum and difference explicitly.

Definition 17. *Suppose $f : {}_b\mathbb{T} \to \mathbb{R}$ and $\alpha > 0$. The right Hadamard fractional sum of order α is defined by*

$$
{}_b\widetilde{\Delta}^{-\alpha} f(t) := \frac{1}{\Gamma(\alpha)}\sum_{s=te^{\alpha h}}^{b}\Big(\ell_p(\rho(s),r)-\ell_p\big(t,r\big)\Big)_h^{\alpha-1}\ell_p^{\nabla}(s,r)f(s)\nu(s),\ t\in e^{\log b-\alpha h}(h\mathbb{N}). \tag{22}
$$

Definition 18. *Suppose $f : {}_b\mathbb{T} \to \mathbb{R}$, $M \in \mathbb{N}_1$ and $M-1 < \alpha \leq M$. The right Hadamard fractional difference of order α is given by*

$$(23)\quad {}_b\widetilde{\Delta}^{\alpha} f(t) := \frac{1}{\Gamma(-\alpha)} \sum_{s=te^{-\alpha h}}^{b} \Big(\ell_p(\rho(s),r) - \ell_p(t,r)\Big)_h^{\frac{-\alpha-1}{}} \ell_p^{\nabla}(s,r) f(s) \nu(s),\ t \in e^{\log b-(M-\alpha)h}(h\mathbb{N}).$$

Next, we first give the right Hadamard fractional power rule. Then some properties of the right Hadamard fractional sum and difference are given.

Lemma 19. *Let $\alpha,\ \beta > 0$. Then the following relations hold:*

$$(24)\quad {}_{be^{-\beta h}}\widetilde{\Delta}^{-\alpha}\big(\ell_p(b,r) - \ell_p(t,r)\big)_h^{\frac{\beta}{}} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\alpha)}\big(\ell_p(b,r) - \ell_p(t,r)\big)_h^{\frac{\beta+\alpha}{}},\ t \in e^{\log b-(\beta+\alpha)h}(h\mathbb{N})$$

and

$$(25)\quad {}_{be^{-\beta h}}\widetilde{\Delta}^{\alpha}\big(\ell_p(b,r) - \ell_p(t,r)\big)_h^{\frac{\beta}{}} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}\big(\ell_p(b,r) - \ell_p(t,r)\big)_h^{\frac{\beta-\alpha}{}},\ t \in e^{\log b-(\beta+M-\alpha)h}(h\mathbb{N}).$$

Proof. Due to

$$\begin{aligned} Q\Big(\big(\ell_p(b,r) - \ell_p(t,r)\big)_h^{\frac{\beta}{}}\Big) &= \big(\ell_p(b,r) - \ell_p(\tfrac{ab}{t},r)\big)_h^{\frac{\beta}{}} \\ &= \big(\ell_p(t,r) - \ell_p(a,r)\big)_h^{\frac{\beta}{}}, \end{aligned}$$

and by Eq. (14), we have

$$\begin{aligned} {}_{be^{-\beta h}}\widetilde{\Delta}^{-\alpha}\big(\ell_p(b,r) - \ell_p(t,r)\big)_h^{\frac{\beta}{}} &= Q\Big(\widetilde{\Delta}_{ae^{\beta h}}^{-\alpha} Q\big(\ell_p(b,r) - \ell_p(t,r)\big)_h^{\frac{\beta}{}}\Big) \\ &= Q\Big(\widetilde{\Delta}_{ae^{\beta h}}^{-\alpha}\big(\ell_p(t,r) - \ell_p(a,r)\big)_h^{\frac{\beta}{}}\Big) \\ &= Q\Big(\frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\alpha)}\big(\ell_p(t,r) - \ell_p(a,r)\big)_h^{\frac{\beta+\alpha}{}}\Big) \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\alpha)}\big(\ell_p(b,r) - \ell_p(t,r)\big)_h^{\frac{\beta+\alpha}{}}. \end{aligned}$$

So the formula (24) holds. Eq. (25) can be proved in a similar way. □

Theorem 20. *Suppose $f : {}_b\mathbb{T} \to \mathbb{R}$ and $\alpha,\ \beta > 0$. Then*

$$(26)\quad {}_{be^{-\beta h}}\widetilde{\Delta}^{-\alpha}\,{}_b\widetilde{\Delta}^{-\beta} f(t) = {}_b\widetilde{\Delta}^{-\alpha-\beta} f(t),\ t \in e^{\log b-(\alpha+\beta)h}(h\mathbb{N}).$$

Proof. Through Theorem 13, we get

$$\begin{aligned}
{}_{be^{-\beta h}}\widetilde{\Delta}^{-\alpha}{}_{b}\Delta^{-\beta}f(t) &= Q\left(\widetilde{\Delta}^{-\alpha}_{ae^{\beta h}}\widetilde{\Delta}^{-\beta}_{a}Qf\right)(t)\\
&= Q\left(\widetilde{\Delta}^{-\alpha-\beta}_{a}Qf\right)(t)\\
&= Q^{2}\left({}_{b}\widetilde{\Delta}^{-\alpha-\beta}f\right)(t)\\
&= {}_{b}\widetilde{\Delta}^{-\alpha-\beta}f(t).
\end{aligned}$$

So the relationship (26) holds. □

Theorem 21. *Suppose $f : {}_{b}\mathbb{T} \to \mathbb{R}$, $\beta > 0$, $M \in \mathbb{N}_1$ and $M-1 < \alpha \le M$. Then the following equality holds:*

$$
{}_{be^{-\beta h}}\widetilde{\Delta}^{\alpha}{}_{b}\widetilde{\Delta}^{-\beta}f(t) = {}_{b}\widetilde{\Delta}^{\alpha-\beta}f(t),\ t \in e^{\log b-(\beta+M-\alpha)h}(h\mathbb{N}). \tag{27}
$$

Proof. Using Theorem 14, the method of proving Eq. (27) is similar to Theorem 20. □

Theorem 22. *Suppose $f : {}_{b}\mathbb{T} \to \mathbb{R}$, $L,\ M \in \mathbb{N}_1$ and $\alpha,\ \beta > 0$ be given with $L-1 < \alpha \le L$ and $M-1 < \beta \le M$. Then*

$$
\begin{aligned}
{}_{be^{-(M-\beta)h}}\widetilde{\Delta}^{\alpha}{}_{b}\widetilde{\Delta}^{\beta}f(t) ={}& {}_{b}\widetilde{\Delta}^{\alpha+\beta}f(t) - \sum_{k=1}^{M}\frac{\left(\ell_p(be^{-(M-\beta)h},r)-\ell_p(t,r)\right)^{\frac{-\alpha-k}{h}}}{\Gamma(-\alpha-k+1)}\\
&\cdot(-1)^{M-k}\widetilde{\nabla}^{M-k}{}_{b}\widetilde{\Delta}^{-(M-\beta)}f(be^{-(M-\beta)h}),
\end{aligned} \tag{28}
$$

where $t \in e^{\log b-(L-\alpha+M-\beta)h}(h\mathbb{N})$.

Proof. According to Theorem 15 and the properties of Q–operator, we have

$$
\begin{aligned}
{}_{be^{-(M-\beta)h}}\widetilde{\Delta}^{\alpha}{}_{b}\widetilde{\Delta}^{\beta} f(t) &= Q\left(\widetilde{\Delta}^{\alpha}_{ae^{(M-\beta)h}}\widetilde{\Delta}^{\beta}_{a} Qf\right)(t)\\
&= Q\Bigg(\widetilde{\Delta}^{\alpha+\beta}_{a} Qf(t) - \sum_{k=1}^{M}\frac{\left(\ell_p(t,r)-\ell_p(ae^{(M-\beta)h},r)\right)_h^{\underline{-\alpha-k}}}{\Gamma(-\alpha-k+1)}\\
&\quad \cdot \widetilde{\Delta}^{M-k}\widetilde{\Delta}_a^{-(M-\beta)} Qf(ae^{(M-\beta)h})\Bigg)\\
&= Q\Bigg(Q\left({}_{b}\widetilde{\Delta}^{\alpha+\beta} f\right)(t) - \sum_{k=1}^{M}\frac{\left(\ell_p(\frac{ab}{ae^{(M-\beta)h}},r)-\ell_p(\frac{ab}{t},r)\right)_h^{\underline{-\alpha-k}}}{\Gamma(-\alpha-k+1)}\\
&\quad \cdot \widetilde{\Delta}^{M-k} Q\left({}_{b}\widetilde{\Delta}^{-(M-\beta)} f\right)(ae^{(M-\beta)h})\Bigg)\\
&= Q\Bigg(Q\left({}_{b}\widetilde{\Delta}^{\alpha+\beta} f\right)(t) - \sum_{k=1}^{M}\frac{\left(\ell_p(be^{-(M-\beta)h},r)-\ell_p(\frac{ab}{t},r)\right)_h^{\underline{-\alpha-k}}}{\Gamma(-\alpha-k+1)}\\
&\quad \cdot (-1)^{M-k}\widetilde{\nabla}^{M-k}{}_{b}\widetilde{\Delta}^{-(M-\beta)} f(be^{-(M-\beta)h})\Bigg)\\
&= {}_{b}\widetilde{\Delta}^{\alpha+\beta} f(t) - \sum_{k=1}^{M}\frac{\left(\ell_p(be^{-(M-\beta)h},r)-\ell_p(t,r)\right)_h^{\underline{-\alpha-k}}}{\Gamma(-\alpha-k+1)}\\
&\quad \cdot (-1)^{M-k}\widetilde{\nabla}^{M-k}{}_{b}\widetilde{\Delta}^{-(M-\beta)} f(be^{-(M-\beta)h}),
\end{aligned}
$$

which completes the proof. □

Theorem 23. *Let $f : {}_{b}\mathbb{T} \to \mathbb{R}$, $\alpha,\beta > 0$, $M \in \mathbb{N}_1$ and $M-1<\beta \le M$. Then*

$$
\begin{aligned}
{}_{be^{-(M-\beta)h}}\widetilde{\Delta}^{-\alpha}{}_{b}\widetilde{\Delta}^{\beta} f(t) &= {}_{b}\widetilde{\Delta}^{\beta-\alpha} f(t) - \sum_{k=1}^{M}\frac{\left(\ell_p(be^{-(M-\beta)h},r)-\ell_p(t,r)\right)_h^{\underline{\alpha-k}}}{\Gamma(\alpha-k+1)}\\
&\quad \cdot (-1)^{M-k}\widetilde{\nabla}^{M-k}{}_{b}\widetilde{\Delta}^{-(M-\beta)} f(be^{-(M-\beta)h}),
\end{aligned}
\tag{29}
$$

where $t \in e^{\log b-(M-\beta+\alpha)h}(h\mathbb{N})$.

Proof. The prove is similar to Theorem 22. □

4. Initial value problem of right fractional Hadamard difference equations

In this section, we use the above theorems to give an equivalent form of the right Hadamard fractional initial value problem.

Theorem 24. *Suppose $F(x(\cdot),\cdot): {}_b\mathbb{T} \to \mathbb{R}$ and $0 < \alpha \leq 1$. The $x(t)$ is a solution of the right Hadamard fractional difference equation*

$$\left\{ \begin{array}{l} {}_b\widetilde{\Delta}^{\alpha} x(t) = F(x(te^{(1-\alpha)h}), te^{(1-\alpha)h}), \\ {}_b\widetilde{\Delta}^{\alpha-1} x(be^{(\alpha-1)h}) = C,\ t \in e^{\log b-(1-\alpha)h}(h\mathbb{N}). \end{array} \right. \tag{30}$$

if and only if it is a solution of

$$x(t) = \frac{\left(\ell_p(be^{(\alpha-1)h}, r) - \ell_p(t,r)\right)^{\frac{\alpha-1}{h}}}{\Gamma(\alpha)} C + {}_{be^{(\alpha-1)h}}\widetilde{\Delta}^{-\alpha} F(x(te^{(1-\alpha)h}), te^{(1-\alpha)h}),\ t \in e^{\log b}(h\mathbb{N}). \tag{31}$$

Proof. Suppose $x(t) = \phi(t)$ is a solution of the initial value problem (30), then

$$\left\{ \begin{array}{l} {}_b\widetilde{\Delta}^{\alpha} \phi(t) = F(\phi(te^{(1-\alpha)h}), te^{(1-\alpha)h}), \\ {}_b\widetilde{\Delta}^{\alpha-1} \phi(be^{(\alpha-1)h}) = C,\ t \in e^{\log b-(1-\alpha)h}(h\mathbb{N}). \end{array} \right.$$

By Theorem 23, we obtain

$$\begin{aligned} \phi(t) = & \frac{\left(\ell_p(be^{(\alpha-1)h}, r) - \ell_p(t,r)\right)^{\frac{\alpha-1}{h}}}{\Gamma(\alpha)} {}_b\widetilde{\Delta}^{\alpha-1} \phi(be^{(\alpha-1)h}) \\ & + {}_{be^{(\alpha-1)h}}\widetilde{\Delta}^{-\alpha} F(\phi(te^{(1-\alpha)h}), te^{(1-\alpha)h}),\ t \in e^{\log b}(h\mathbb{N}). \end{aligned}$$

Due to the initial value condition

$${}_b\widetilde{\Delta}^{\alpha-1} x(be^{(\alpha-1)h}) = C,$$

we get

$$\phi(t) = \frac{\left(\ell_p(be^{(\alpha-1)h}, r) - \ell_p(t,r)\right)^{\frac{\alpha-1}{h}}}{\Gamma(\alpha)} C + {}_{be^{(\alpha-1)h}}\widetilde{\Delta}^{-\alpha} F(\phi(te^{(1-\alpha)h}), te^{(1-\alpha)h}).$$

Therefore, $x(t) = \phi(t)$ is also a solution of Eq. (31).

On the other hand, let $x(t) = \phi(t)$ be a solution of Eq. (31), then

$$\begin{aligned} \phi(t) = & \frac{\left(\ell_p(be^{(\alpha-1)h}, r) - \ell_p(t,r)\right)^{\frac{\alpha-1}{h}}}{\Gamma(\alpha)} C \\ & + {}_{be^{(\alpha-1)h}}\widetilde{\Delta}^{-\alpha} F(\phi(te^{(1-\alpha)h}), te^{(1-\alpha)h}),\ t \in e^{\log b}(h\mathbb{N}). \end{aligned} \tag{32}$$

We apply the operator ${}_b\widetilde{\Delta}^{\alpha}$ to obtain

$${}_b\widetilde{\Delta}^{\alpha} \phi(t) = \frac{{}_b\widetilde{\Delta}^{\alpha}\left(\ell_p(be^{(\alpha-1)h}, r) - \ell_p(t,r)\right)^{\frac{\alpha-1}{h}}}{\Gamma(\alpha)} C + {}_b\widetilde{\Delta}^{\alpha}{}_{be^{(\alpha-1)h}}\widetilde{\Delta}^{-\alpha} F(\phi(te^{(1-\alpha)h}), te^{(1-\alpha)h}).$$

From Lemma 19 and Theorem 21, we can give

$${}_b\widetilde{\Delta}^{\alpha}\left(\ell_p(be^{(\alpha-1)h}, r) - \ell_p(t,r)\right)^{\frac{\alpha-1}{h}} = 0$$

and

$$\begin{aligned} {}_b\widetilde{\Delta}^{\alpha}{}_{be^{(\alpha-1)h}}\widetilde{\Delta}^{-\alpha} F(\phi(te^{(1-\alpha)h}), te^{(1-\alpha)h}) &= {}_{be^{-h}}\widetilde{\Delta}^{\alpha}{}_{be^{(\alpha-1)h}}\widetilde{\Delta}^{-\alpha} F(\phi(te^{(1-\alpha)h}), te^{(1-\alpha)h}) \\ &= F(\phi(te^{(1-\alpha)h}), te^{(1-\alpha)h}), \end{aligned}$$

from which we arrive at

$$ {}_b\widetilde{\Delta}^{\alpha}\phi(t) = F(\phi(te^{(1-\alpha)h}), te^{(1-\alpha)h}). $$

$\phi(t)$ also satisfies the initial condition of (30). In fact, we apply the operator ${}_b\widetilde{\Delta}^{\alpha-1}$ to both sides of Eq. (32) to get

$$ \begin{aligned} {}_b\widetilde{\Delta}^{\alpha-1}\phi(t) &= \frac{{}_b\widetilde{\Delta}^{\alpha-1}\big(\ell_p(be^{(\alpha-1)h},r)-\ell_p(t,r)\big)^{\frac{\alpha-1}{h}}}{\Gamma(\alpha)}C \\ &\quad + {}_{be^{-h}}\widetilde{\Delta}^{\alpha-1}{}_{be^{(\alpha-1)h}}\widetilde{\Delta}^{-\alpha}F(\phi(te^{(1-\alpha)h}), te^{(1-\alpha)h}) \\ &= C + {}_{be^{(\alpha-1)h}}\widetilde{\Delta}^{-\alpha}F(\phi(te^{(1-\alpha)h}), te^{(1-\alpha)h}). \end{aligned} $$

It follows from

$$ {}_{be^{(\alpha-1)h}}\widetilde{\Delta}^{-\alpha}F(\phi(b),b) = 0 $$

that

$$ {}_b\widetilde{\Delta}^{\alpha-1}\phi(be^{(\alpha-1)h}) = C. $$

As a result, $\phi(t)$ is a solution of the initial value problem (30). The proof is completed. □

Definition 25. *For $\lambda\in\mathbb{R}$ and $\alpha>0$. The discrete Mittag–Leffler function of right Hadamard type is defined as*

$$ \varepsilon_{\alpha,\alpha}(\lambda,(\rho(b)-t)^{\underline{\alpha}}) = \sum_{m=0}^{\infty}\lambda^m\frac{\big(\ell_p(be^{(m+1)(\alpha-1)h},r)-\ell_p(t,r)\big)^{\frac{(m+1)\alpha-1}{h}}}{\Gamma(m\alpha+\alpha)}. \tag{33} $$

Example 26. *A right Hadamard fractional discrete–time equation is defined by*

$$ \begin{cases} {}_b\widetilde{\Delta}^{\alpha}x(t) = \lambda x(te^{(1-\alpha)h}), \\ {}_b\widetilde{\Delta}^{\alpha-1}x(be^{(\alpha-1)h}) = C, \end{cases} \tag{34} $$

where $t\in e^{\log b-(1-\alpha)h}(h\mathbb{N})$ and $0<\alpha\le 1$.

By Theorem 24, the solution of the initial value problem (34) is

$$ x(t) = \frac{\big(\ell_p(be^{(\alpha-1)h},r)-\ell_p(t,r)\big)^{\frac{\alpha-1}{h}}}{\Gamma(\alpha)}C + \lambda\,{}_{be^{(\alpha-1)h}}\widetilde{\Delta}^{-\alpha}x(te^{(1-\alpha)h}),\ t\in e^{\log b-h}(h\mathbb{N}). \tag{35} $$

And the iterative scheme is

$$ \begin{cases} x_{m+1}(t) = x_0(t) + \lambda\,{}_{be^{(\alpha-1)h}}\widetilde{\Delta}^{-\alpha}x_m(te^{(1-\alpha)h}), \\ x_0(t) = \frac{\big(\ell_p(be^{(\alpha-1)h},r)-\ell_p(t,r)\big)^{\frac{\alpha-1}{h}}}{\Gamma(\alpha)}C. \end{cases} \tag{36} $$

Apply Lemma 19 to show that

$$ \begin{aligned} x_1(t) &= x_0(t) + \lambda\,{}_{be^{(\alpha-1)h}}\widetilde{\Delta}^{-\alpha}x_0(te^{(1-\alpha)h}) \\ &= C\left(\frac{\big(\ell_p(be^{(\alpha-1)h},r)-\ell_p(t,r)\big)^{\frac{\alpha-1}{h}}}{\Gamma(\alpha)} + \lambda\frac{\big(\ell_p(be^{2(\alpha-1)h},r)-\ell_p(t,r)\big)^{\frac{2\alpha-1}{h}}}{\Gamma(2\alpha)}\right). \end{aligned} $$

Through induction, we obtain

$$x_{n+1}(t) = C\sum_{m=0}^{n+1}\lambda^m \frac{\left(\ell_p(be^{(m+1)(\alpha-1)h},r)-\ell_p(t,r)\right)^{\frac{(m+1)\alpha-1}{h}}}{\Gamma(m\alpha+\alpha)}.$$

When $n\to\infty$, we obtain the solution

$$x(t) = C\varepsilon_{\alpha,\alpha}(\lambda,(\rho(b)-t)^{\underline{\alpha}}),\ t\in e^{\log b-h}(h\mathbb{N}).$$

5. Summation by parts

In this section, we obtain the summation by parts of Hadamard fractional sum and difference.

Theorem 27. *Let $\alpha>0$. If $f:\mathbb{T}_a\to\mathbb{R}$ and $g:{}_b\mathbb{T}\to\mathbb{R}$, then we have*

$$\sum_{t=ae^{\alpha h}}^{b}\ell_p^{\nabla}(t,r)g(t)\widetilde{\Delta}_a^{-\alpha}f(t)\nu(t) = \sum_{t=a}^{be^{-\alpha h}}\ell_p^{\Delta}(t,r)f(t)_b\widetilde{\Delta}^{-\alpha}g(t)\mu(t). \tag{37}$$

Proof. By the Definition 17, we have

$$\begin{aligned}&\sum_{t=ae^{\alpha h}}^{b}\ell_p^{\nabla}(t,r)g(t)\widetilde{\Delta}_a^{-\alpha}f(t)\nu(t)\\ &\qquad=\sum_{t=ae^{\alpha h}}^{b}\ell_p^{\nabla}(t,r)g(t)\frac{1}{\Gamma(\alpha)}\sum_{s=a}^{te^{-\alpha h}}\left(\ell_p(t,r)-\ell_p(\sigma(s),r)\right)^{\frac{\alpha-1}{h}}\ell_p^{\Delta}(s,r)f(s)\mu(s)\nu(t).\end{aligned}$$

Then, we interchange the order of summation and arrive at

$$\begin{aligned}\sum_{t=ae^{\alpha h}}^{b}\ell_p^{\nabla}(t,r)g(t)\widetilde{\Delta}_a^{-\alpha}f(t)\nu(t) &= \sum_{s=a}^{be^{-\alpha h}}\ell_p^{\Delta}(s,r)f(s)\mu(s)\frac{1}{\Gamma(\alpha)}\sum_{t=se^{\alpha h}}^{b}\left(\ell_p(\rho(t),r)-\ell_p(s,r)\right)^{\frac{\alpha-1}{h}}\\ &\quad\cdot\ell_p^{\nabla}(t,r)g(t)\nu(t)\\ &=\sum_{t=a}^{be^{-\alpha h}}\ell_p^{\Delta}(t,r)f(t)_b\widetilde{\Delta}^{-\alpha}g(t)\mu(t).\end{aligned}$$

This concludes the proof. □

Theorem 28. *Let $\alpha>0$. If $f:\mathbb{T}_a\to\mathbb{R}$ and $g:{}_b\mathbb{T}\to\mathbb{R}$, then we have*

$$\sum_{t=ae^{-\alpha h}}^{b}\ell_p^{\nabla}(t,r)g(t)\widetilde{\Delta}_a^{\alpha}f(t)\nu(t) = \sum_{t=a}^{be^{\alpha h}}\ell_p^{\Delta}(t,r)f(t)_b\widetilde{\Delta}^{\alpha}g(t)\mu(t). \tag{38}$$

Proof. The proof process is similar to Theorem 27, so we do not give the proof here. □

6. Conclusions

In this paper, we define a Q–operator for Hadamard fractional calculus. It is verified by the Q–operator that left and right Hadamard fractional integrals and derivatives satisfy dual identities. By use of these dual identities, we extend the Q–operator to definitions of right Hadamard fractional order sum and difference are obtained. Then some useful properties are given such as right Hadamard fractional sum and difference of power-law functions, composition law, etc. A fractional sum equation is provided for the initial value problem. At the same time, the integration by parts formulas of Hadamard discrete fractional calculus is also given. This shows that the right Hadamard fractional sum and difference are well-defined. In the nearest future, other right fractional calculus on other time scales can be investigated such as the fractional calculus with general kernel functions [23, 24]. In addition, within the right Hadamard fractional differences, discrete fractional Laplacian and boundary value problems et al can be considered further by the results of this paper.

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