INDECOMPOSABLE INVERSE LIMITS ON GRAPHS

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ABSTRACT. We obtain results, involving easily observable properties of bonding mappings, that ensure indecomposability of inverse limits on graphs. An allowable collection of n arcs in a graph is defined, and two related properties of collections of arcs in a graph are introduced. In an inverse sequence on graphs, if compositions of the bonding mappings are n-pass maps on certain allowable collections of n arcs, then the inverse limit space will be indecomposable. We provide examples that illustrate the use of our results.

1. Introduction. A *compactum* is a compact metric space. A *continuum* is a nonempty connected compactum. A continuum X is *decomposable* if there exist two non-empty proper subcontinua A and B of X such that $A \cup B = X$. A continuum X is *indecomposable* if it is not decomposable. A continuous function will be referred to as a *map* or *mapping*. A bijective mapping $f\colon X \to Y$ is a *homeomorphism* if $f^{-1}\colon Y \to X$ is also a mapping. A continuum X is an *arc continuum* if each proper nondegenerate subcontinuum of X is an arc.

We define three properties of collections of arcs in a graph that can produce indecomposable inverse limits on graphs if compositions of the bonding mappings throw each of the arcs onto previous factor spaces. The first property, which we call an allowable collection of arcs in a graph, is generally easy to determine, and in many cases, it is sufficient for the collection of arcs to have the other two properties as well. As we will see, the properties are fundamentally related to results for inverse limits of D.P. Kuykendall [**27**], A. van Heemert [**14**], J. Segal [**39**], and W.T. Ingram [**19**]. We clarify the relationships as the paper proceeds. We extend Ingram's notion of a two-pass map to that of an n-pass map for $n \geq 2$.

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The importance of indecomposable continua cannot be overstated. The first indecomposable continuum was constructed by L.E.J. Brouwer [**8**] in 1910 to disprove a conjecture of A. Schoenflies. A diagram of Brouwer's construction adorned the cover of the June/July, 2014 issue of the Notices of the American Mathematical Society. A brief discussion of the history related to Brouwer's example is given therein on page 610. K. Kuratowski also discusses the history related to Brouwer's example, and to indecomposable examples of Z. Janiszewski and of B. Knaster in [**26,** pages 71–73]. In [**20**], Ingram provides a description of Brouwer's example as an inverse limit on a circle with a single bonding mapping. Other descriptions of early examples of indecomposable continua can be found in [**21**] and [**23**]. In the 112 years after Brouwer's example, there has been an explosion of interest in indecomposable continua.

Topological groups can be indecomposable continua. In 1930, D. van Dantzig [**11, 12**] defined the n-adic solenoids, which are indecomposable continua. These continua can be realized as inverse limits on the unit circle with bonding mappings that are covering maps. The inverse limit structure, together with complex multiplication on the unit circle, induces a group structure on the inverse limit space. The n-adic solenoids are also homogeneous continua, a property they share with closed, connected manifolds, although the local structures of the continua in these two classes are very different. Specifically, the local structure of solenoids is topologically the product of a Cantor set and an open interval, while the local structure of the closed, connected manifolds is Euclidean. A topological space X is *homogeneous* if for each two points x and y in X, there exists a homeomorphism $f\colon X \to X$ where $f(x) = y$. The study of homogeneous spaces is also well-represented in the literature.

Regarding homogeneous continua, rather remarkably, the pseudo-arc P [**28, 29**] is both homogeneous and hereditarily indecomposable, meaning that each subcontinuum of P is indecomposable. Furthermore, P can be realized as an inverse limit on $[0,1]$; P is homeomorphic to each of its subcontinua; and P is a non-separating planar continuum. The earliest hereditarily indecomposable continua were constructed by Knaster, E.E. Moise, and Bing. W. Lewis gives an interesting account of this history in [**28,** (1.1), page 26].

In 1938, O.H. Hamilton [**13,** Theorems II, III, and IV] established a connection between indecomposable continua and fixed point free homeomorphisms on tree-like continua and on non-separating planar continua. Specifically, for a non-separating continuum X in the Euclidean plane, he showed that if $f\colon X \to X$ is a fixed point free homeomorphism, then X contains an indecomposable subcontinuum in its boundary. H. Bell [**4,** Theorem 1] and K. Sieklucki [**41,** Theorem 1.1] generalized Hamilton's result to fixed point free mappings on non-separating planar continua. The classical planar fixed point problem, "Does every mapping on a non-separating planar continuum have a fixed point?" remains unanswered. R.H. Bing [**7**] states that this question has been called the most interesting outstanding problem in planar topology. From Bell's and Sieklucki's results, we see that if one wishes to construct an example in the plane that provides a negative answer to this early 20th century problem, the example must contain an indecomposable continuum. Further discussion related to this classical problem can be found in [**9,** Section 1] and [**24,** Section 12, pages 66–70]. The first example of a tree-like continuum admitting a fixed point free mapping was not discovered until 1979. Not surprisingly, this example of D. Bellamy's [**5**], is indecomposable. Bellamy modifies the 6-adic solenoid Σ by replacing an arc in Σ with the suspension of a totally disconnected set. He then uses the group operations on Σ to complete a clever, insightful construction of the example. A number of other noteworthy examples of tree-like continua admitting fixed point free mappings followed. See the references in [**15**] for a list of some of them. Also provided in [**15**] is the simplest inverse limit description of such a tree-like continuum.

Indecomposable continua often appear as attractors or invariant sets of diffeomorphisms on the plane and on manifolds. For example, the global attractor of the S. Smale [**42, 43**] horseshoe map on a planar disk is indecomposable. In [**1**], M. Barge establishes connections between attracting sets of horseshoe maps and inverse limits on $[0,1]$ that are indecomposable Knaster continua. *Knaster continua* are inverse limits on $[0,1]$ with piecewise linear, open bonding mappings. Barge's results were generalized by S.E. Holte in [**16**]. R. F. Williams [**44**] showed that all hyperbolic one-dimensional attractors are inverse limits of maps on branched one-manifolds (graphs). Discussion of these results and an extensive list of references related to indecomposable continua can be found in [**23**]. For a detailed and thorough discussion of the history of indecomposable continua and their importance in continuum theory

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and in dynamical systems, we refer the reader to the two excellent articles by J. Kennedy [**22, 23**].

There are many characterizations and sufficient sets of conditions in the literature for indecomposability of continua. An early, and well-known characterization by S. Mazurkiewicz [**32**] says that a continuum X is indecomposable if and only if there exist three points in X for which X is irreducible between each pair of them. This characterization of indecomposability involves the structure of the continuum. Some other conditions that are sufficient for indecomposability, and that involve the structure of the continuum can be found in [**25**], [**30,** Theorem 34], [**31**], [**33**], [**34**], and [**35**]. The first reference is concerned with planar continua. The second reference is concerned with chainable continua, and the other four references are related to graph-like continua. Many important examples of indecomposable continua are arc continua. Showing that each proper subcontinuum of a continuum X is an arc, and observing that X is neither an arc nor a simple closed curve establishes that X is indecomposable. This method of proof has been used frequently, even though it is often non-trivial.

For inverse limits X on arcs, trees, and graphs with a single bonding mapping, there are a number of results where dynamical properties of the bonding mappings ensure that X contains an indecomposable subcontinuum, see for example, [**2**], [**3**], [**17,** Section 7], and [**46,** Prop. 1]. The results in the first two references are about positive entropy of the bonding mapping. The third reference mostly discusses inverse limits on arcs, and the results are related to the existence of certain periodic points of the bonding mapping. In Proposition 1 of the fourth reference, it is shown that a bonding mapping that is mixing produces an indecomposable inverse limit. Also, in the fourth reference, Ye establishes an equivalence for the existence of an indecomposable subcontinuum of an inverse limit on a finite set of graphs if the composition bonding mappings in some subsequence on a single graph have horseshoes. A mapping $f\colon G \to G$ *has a horseshoe* if there exist two non-overlapping arcs J_1 and J_2, lying in an edge of G, such that $J_1 \cup J_2 \subset f(J_1) \cap f(J_2)$.

Our results are related to conditions on the bonding mappings in an inverse sequence on graphs. The conditions are generally easy to check, they apply to inverse sequences where the factor spaces and bonding mappings may be different, and they determine indecomposability of the inverse limit space itself, as opposed to only guaranteeing the existence of an indecomposable subcontinuum. Since each subcontinuum of an inverse limit space is, itself, the inverse limit on its projections to the factor spaces with the bonding mappings restricted to these projections, the techniques can also be applied to subcontinua of an inverse limit on graphs.

Although our results involve inverse sequences and their limits, a reader unfamiliar with these notions may simply think of an inverse sequence as a method of describing or constructing a complicated, perhaps pathological, continuum by a limiting or approximating process. In our case, all inverse limits are approximated by inverse sequences of topological graphs, where the bonding mappings in the inverse sequence fold the graphs, one onto the other, in a manner where compositions of the bonding mappings indicate the approximations. If a reader wishes to have an introduction to inverse limits, see **[17]** or **[18]**. Ingram also discusses indecomposable inverse limits in both references. The more technical, or complicated proofs in the paper, Theorems 8 and 9 for example, are results about collections of arcs in a graph. Drawing pictures of graphs with appropriate properties should help to follow the various cases in the proofs.

2. Definitions and preliminaries. An *arc* is a homeomorphic image of the interval $[0,1]$. If L is an arc contained in a compactum X, and $h\colon [0,1] \to L$ is a homeomorphism, we refer to $h(0)$ and $h(1)$ as the *endpoints* of L, and to $h([0,1]) \setminus \{h(0), h(1)\}$ as the set of *interior points of* L, which we denote by L°. A *simple closed curve* is a homeomorphic image of the unit circle S^1 in $\mathbb{R}^2$. A *topological graph* G, or simply a *graph* G, is a continuum that is a union of finitely many arcs, each two of which are either disjoint or meet only at one or both of their endpoints. A *tree* is a graph containing no simple closed curves.

Let G be a graph. If arcs A and B in G meet only at a common endpoint p, we say that A and B are *abutting arcs* (at p). For $x \in G$, the *order of x in G*, denoted $o(x)$, is the largest number of arcs in G each two of which are abutting at x. The set of *endpoints* of G, and the set of *branchpoints* of G are, respectivley, defined as $E(G) = \{x \in G \mid o(x) = 1\}$ and $B(G) = \{x \in G \mid o(x) \geq 3\}$. For a tree T, we let $V(T) = E(T) \cup B(T)$ be the *the set of vertices of* T. For a graph G, *a set of vertices for* G is any finite subset $V(G)$ of G such that $E(T) \cup B(T) \subset V(G)$, and each simple closed curve in G contains at least two points of $V(G)$.

An *edge* of G is an arc L in G such that $L \cap V(G)$ is the set of endpoints of L. A *cycle* in G is a simple closed curve in G. Note that each cycle in G contains at least two edges of G. An arc A with endpoints p and q in a graph G is a *unique arc in G* if whenever L is an arc in G with endpoints p and q, then $L = A$. A cycle S in a graph G is a *unique cycle* in G provided that if S' is a cycle in G and S and S' share an edge of G, then $S' = S$.

In our setting of connected graphs, the *cyclomatic number* of a graph G is $e - v + 1$, where e is the number of edges of G and v is the number of vertices of G. By choosing an orientation for each cycle in G, it is possible to associate each cycle with a vector in some Euclidean space $\mathbb{R}^m$, and by doing so, one can define a set of cycles in G to be *independent* if the associated vectors in $\mathbb{R}^m$ are independent. If n is the cyclomatic number of G, then G has n, and not more than n independent cycles. The number of independent cycles in G may be less than the total number of distinct cycles in G. As an example, the theta-curve defined immediately before Corollary 4 has three distinct cycles, but only two independent cycles (cyclomatic number is 2). The cyclomatic number of a graph G is zero if and only if G is a tree, and the cyclomatic number of a graph G is one if and only if G has (exactly) one cycle. For $n \geq 2$, we say that *G has n independent cycles* if and only if the cyclomatic number of G is n. For our purposes, it is not necessary to have the precise definition of sets of indenpendent cycles, but it may be helpful to think of the cyclomatic number as the minimum number of edges of G that must be removed to obtain a graph with no cycles. Discussion and precise definitions of these terms can be found in **[6,** Chapter 2].

If L is an arc with endpoints u and v in a graph G, we sometimes denote L by $[u, v]$, even though there may be other arcs in G with endpoints u and v. If $a, b \in [u, v]$ with $a \neq b$, we write $[a, b]$ for the subarc of L with endpoints a and b. If each of H and K is a subcontinuum of a graph G, and L is an arc in G such that $L \cap (H \cup K) = \{u, v\}$, where $u \in H$, $v \in K$, and u and v are the endpoints of L, we call L *an arc from H to K* or *an arc joining H to K*. If the intersection of two subsets A and B of a graph G contains an arc, we say that A and B *overlap*. Otherwise, A and B are *non-*

overlapping.

As previously mentioned, the focus of the paper is to determine properties of collections of arcs in graphs, where compositions of the bonding mappings in an inverse sequence on graphs throw the arcs onto previous factor spaces, thereby ensuring an indecomposable inverse limit space. The properties are listed below. The first property is, in general, the easiest to check for a given collection of arcs in a graph.

Let $\mathcal{J} = \{J_1, \dots, J_n\}$ be a finite collection of arcs in a graph G.

(1) $\mathcal{J}$ is an *allowable collection of arcs in* G if for each $1 \leq i < k \leq n$, J_i and J_k have disjoint interiors, and no arc in G joining J_i to J_k meets each of J_i and J_k at an interior point.

(2) $\mathcal{J}$ has the 3-*endpoint property in* G if there exist three points, each of which is an endpoint of some member of $\mathcal{J}$, and so that if a and b are any two of the three points, then each arc in G with endpoints a and b contains some member of $\mathcal{J}$. We note that if a subcollection of $\mathcal{J}$ has the 3-endpoint property in G, then $\mathcal{J}$ also has the 3-endpoint property in G.

(3) $\mathcal{J}$ is *decomposition saturated in* G provided that whenever G is the union of two subcontinua A and B, there exists $1 \leq i \leq n$ such that either $J_i \subset A$ or $J_i \subset B$.

A collection of two unique arcs in a graph G is allowable if and only if it has the 3-endpoint property if and only if it is decomposition saturated, as we see in Theorem 5. Hence, the three properties above are equivalent for each collection of two arcs in a tree. A collection of three arcs contained in a unique cycle in a graph G is allowable if and only if it has the 3-endpoint property if and only if it is decomposition saturated, see Theorem 6. Based on these two cases, one might conjecture that the three properties are equivalent for collections of $n+2$ arcs, each contained in one of n unique cycles in a graph. Unfortunately, this is not the case in general, see Example 1 at the end of Section 3. Our focus is not to determine when we have equivalence of the three properties, but to find conditions on allowable collections of arcs that imply either the 3-endpoint property or decomposition saturated. We establish a number of simple conditions on allowable collections of arcs in graphs which allow for easy determination of indecomposable inverse limits on graphs. We also pose several questions that should provide further avenues of investigation.

Also of importance to us are the notions of a wrapping defined by Segal [**39**], and a two-pass map defined by Ingram [**19**]. Let X and Y be continua, and let G be a graph. A mapping $f\colon X \to Y$ is a *wrapping* if for any subcontinua A and B of X such that $X = A \cup B$, we have either $f(A) = X$ or $f(B) = X$. A mapping $f\colon G \to G$ is a *two-pass map* if there exists two non-overlapping subgraphs G_1 and G_2 of G such that $f(G_i) = G$ for each $i = 1, 2$.

We define a related property for mappings between graphs H and G. If $f\colon H \to G$ is a mapping, and there exist a collection of arcs $\mathcal{J} = \{J_1, \ldots, J_n\}$ in H such that $f(J_i) = G$ for each $1 \leq i \leq n$, then we say that f is an *n-pass map for the collection $\mathcal{J}$*.

We state theorems of van Heemert [**14**] and Kuykendall [**27**] that are central to our techniques. Van Heemert's result can also be found in [**39,** Remark, page 602] and [**36,** Theorem 2.7]. Kuykendall's result can also be found in [**19,** Theorem 2.1]. For an inverse sequence $\{X_i, g_i^{i+1}\}$, and each pair of integers n and m with $1 \leq n < m - 1$, we let $g_n^m\colon X_m \to X_n$ denote the composition mapping $g_n^{n+1} \circ \ldots \circ g_{m-1}^m$.

Theorem 1. (van Heemert) *Let $X = \varprojlim\{X_i, g_i^{i+1}\}$, where, for each $i \geq 1$, X_i is a nondegenerate continuum. If, for each $i \geq 1$, $g_i^{i+1}\colon X_{i+1} \to X_i$ is a wrapping, then X is indecomposable.*

Theorem 2. (Kuykendall) *Let $X = \varprojlim\{X_i, g_i^{i+1}\}$, where, for each $i \geq 1$, X_i is a nondegenerate continuum, and g_i^{i+1} is a surjective mapping. Then the following statements are equivalent.*

(1) *X is indecomposable.*
(2) *For $\epsilon > 0$ and $n \in \mathbb{N}$, there exists a positive integer $m > n$ and three points of X_m such that if K is a subcontinuum of X_m containing two of them, then the distance of x to $g_n^m(K)$ is less than ϵ for each point $x \in X_n$.*

For inverse sequences, we call statement (2) in Kuykendall's theorem the 3-*point criterion, or the Kuykendall criterion* for indecomposability of the inverse limit. Theorems 3 and 4 below demonstrate a connection between collections of n arcs with the properties defined in (2) and (3), compositions of bonding mappings that are n-pass maps, and Theorems

1 and 2. We show in Observation 1 that the property defined in (2) is sufficient for the property in (3). That the property in (1) is often sufficient for the property in (2) is established throughout the paper.

Theorem 3. *Let $X = \varprojlim\{G_i, g_i^{i+1}\}$, where for each $i \geq 1$, G_i is a graph. Suppose for each $n \geq 1$, there exists $m > n$ and a decomposition saturated collection of arcs $\mathcal{J}_m = \{J_1, \ldots, J_{k_m}\}$ in G_m such that $g_n^m \colon G_m \to G_n$ is a k_m-pass map for $\mathcal{J}_m$. Then X is indecomposable.*

Proof. We wish to apply Theorem 1. By hypothesis, we can choose an inverse sequence $\{G_{u_i}, g_{u_i}^{u_{i+1}}\}$, where for each $i \geq 1$, $g_{u_i}^{u_{i+1}} \colon G_{u_{i+1}} \to G_{u_i}$ is a $k_{u_{i+1}}$-pass map for some decomposition saturated collection of arcs $\mathcal{J}_{u_{i+1}}$ in $G_{u_{i+1}}$. It is well-known that the limit of such an inverse sequence is homeomorphic to X. To see that each $g_{u_i}^{u_{i+1}}$ is a wrapping, let $A_{u_{i+1}}$ and $B_{u_{i+1}}$ be a decomposition of $G_{u_{i+1}}$. By hypothesis, there exists a member J of $\mathcal{J}_{u_{i+1}}$ such that either $J \subset A_{u_{i+1}}$ or $J \subset B_{u_{i+1}}$. We assume, without loss of generality, that $J \subset A_{u_{i+1}}$. Since $g_{u_i}^{u_{i+1}}$ is a $k_{u_{i+1}}$-pass map for $\mathcal{J}_{u_{i+1}}$, we have $G_{u_i} = g_{u_i}^{u_{i+1}}(J) \subset g_{u_i}^{u_{i+1}}(A_{u_{i+1}})$. So, $g_{u_i}^{u_{i+1}}$ is a wrapping, and by Theorem 1, X is indecomposable. □

Theorem 4. *Let $X = \varprojlim\{G_i, g_i^{i+1}\}$, where for each $i \geq 1$, G_i is a graph. Suppose for each $n \geq 1$, there exists $m > n$ and a collection of arcs $\mathcal{J}_m = \{J_1, \ldots, J_{k_m}\}$ in G_m that has the 3-endpoint property, and for which $g_n^m \colon G_m \to G_n$ is a k_m-pass map for $\mathcal{J}_m$. Then X is indecomposable.*

Proof. We show that $\{G_i, g_i^{i+1}\}$ satisfies the Kuykendall criterion. Let $\epsilon > 0$ and $n \in \mathbb{N}$. Pick $m > n$, and a collection of arcs $\mathcal{J}_m = \{J_1, \ldots, J_{k_m}\}$ in G_m that has the 3-endpoint property, and where g_n^m is a k_m-pass map for $\mathcal{J}_m$. Let p, v, and q be the three points among the endpoints of the arcs $J_1, \ldots, J_{k_m}$ that have the 3-endpoint property. Suppose K is a subcontinuum of G_m containing two of the points p, v, and q; say $p, q \in K$. Since K is arcwise connected, there exists an arc L in K with endpoints p and q. By the 3-endpoint property, for some $1 \leq i \leq k_m$, we have that $J_i \subset L$. Since g_n^m is a k_m-pass map for $\mathcal{J}_m$, we have $G_n = g_n^m(J_i) \subset g_n^m(L) \subset g_n^m(K)$. Clearly, for $x \in G_n$, the distance from x to $g_n^m(K) = G_n$ is zero, which is less than ϵ. So, by Theorem 2, X is indecomposable. □

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3. Cases for which the three properties are equivalent. Observation 1 and Lemma 1 will be useful tools throughout.

Observation 1. If $\mathcal{J}$ is a collection of arcs in a graph G that has the 3-endpoint property, then $\mathcal{J}$ is decomposition saturated.

Proof. Let $G = A \cup B$, where A and B are subcontinua of G. Let a, b, and c be endpoints of arcs in $\mathcal{J}$ that are guaranteed by the 3-endpoint property. Assume, without loss of generality, that $a, c \in A$. Since A is arcwise connected, there exists an arc $L \subset A$ with endpoints a and c. By the 3-endpoint property, some $J \in \mathcal{J}$ is a subset of L. So, $J \subset L \subset A$. We have that $\mathcal{J}$ is decomposition saturated. $\square$

A point v of a graph G is a *separating point* of G if $G \setminus \{v\}$ is disconnected. The point v *separates the points* p and q in G if p and q are in different components of $G \setminus \{v\}$.

Lemma 1. Let $J = [a, b]$ be an arc in G with endpoints a and b.

(a) These statements are equivalent.
   (i) J is a unique arc in G.
   (ii) Each interior point of J separates a from b in G.
   (iii) J overlaps no cycle in G.
(b) If J is a unique arc in G, then each subarc of J is a unique arc in G.
(c) If $u, v \in J$ with $a < u \leq v < b$ in the order on J from a to b, and the subarcs $J_1 = [a, u]$ and $J_2 = [v, b]$ of J are unique arcs in G, then J_1 and J_2 are subarcs of each arc in G with endpoints a and b. Additionally, if $u = v$, then J is a unique arc in G.
(d) If v is in $J \setminus \{a\}$, and the subarc $[a, v]$ of J is a unique arc in G, then $[a, v]$ is a subarc of each arc in G with endpoints a and b.

Proof. (a) (i)$\Rightarrow$(ii): This implication is clear.

(ii)$\Rightarrow$(iii): We prove the contrapositive statement. Suppose J overlaps a cycle S in G. Let A be an arc in $J \cap S$, and let p be an interior point of A that is not a branchpoint of G. Now p does not separate S, and since p is not a branchpoint of G, it follows that p does not separate G.

(iii)⇒(i): We prove the contrapositive. Suppose J is not a unique arc in G. Let L be an arc, distinct from J, with endpoints a and b. So, $J \not\subset L$. Let J' be the closure of a component of $J \setminus L$. Then J' is an arc whose endpoints u and v are in L. Let L' denote the subarc of L with endpoints u and v. We have that $J' \cup L'$ is a cycle in G. By definition, J overlaps $J' \cup L'$.

(b) Let $[u, v]$ be a subarc of J. Suppose that $[u, v]$ is not a unique arc in G. Let L be an arc in G, distinct from $[u, v]$, with endpoints u and v. Clearly, $[u, v] \not\subset L$, so let p be a point of $[u, v] \setminus L$. Clearly, p is an interior point of $[u, v]$. It follows that $(J \setminus [u, v]) \cup L$ is a continuum in $G \setminus \{p\}$ containing the points a and b, contradicting (a).

(c) By way of contradiction, suppose L is an arc in G with endpoints a and b, and $J_1 \not\subset L$. As in the proof of (b), picking a point p in $J_1 \setminus L$, we have that $(J \setminus J_1) \cup L$ is a continuum in $G \setminus \{p\}$ containing a and u, which is a contradiction of (a) for the unique arc J_1. That J is unique if $u = v$ follows immediately.

(d) The proof is similar to the proof of (c). □

Theorem 5 and Corollary 1 generalize the results in Section 4 of [**30**]. A *simple triod* is a tree T where $B(T) = \{v\}$, and $o(v) = 3$. So, a simple triod is homeomorphic to the symbol $\perp$.

Theorem 5. *Let J_1 and J_2 be two unique arcs in a graph G. The following statements are equivalent.*

(i) *$\{J_1, J_2\}$ is an allowable collection of arcs in G.*
(ii) *J_1 and J_2 are non-overlapping, and $J_1 \cup J_2$ is contained in either an arc or a simple triod in G.*
(iii) *$\{J_1, J_2\}$ has the 3-endpoint property in G.*
(iv) *$\{J_1, J_2\}$ is decomposition saturated in G.*

Proof. *(i)* ⇒ *(ii)*: Since J_1 and J_2 have disjoint interiors, they are non-overlapping. We note that if $J_1 \cap J_2 \neq \emptyset$, then $J_1 \cap J_2$ must be a continuum since J_1 and J_2 are unique arcs. So, $J_1 \cap J_2$ is a singleton, say $J_1 \cap J_2 = \{p\}$. Furthermore, since $\{J_1, J_2\}$ is an allowable collection of arcs, p must be an endpoint of one of J_1 or J_2. Assume p is an endpoint of J_1. If p is also an endpoint of J_2, then $J_1 \cup J_2$ is an arc. If p is an interior point of J_2, then $J_1 \cup J_2$ is a simple triod. In either case, the proof is complete. So, we assume that $J_1 \cap J_2 = \emptyset$.

Let $[u, v]$ denote an arc in G joining J_1 to J_2. Let $C = J_1 \cup [u, v] \cup J_2$. Either u is not an interior point of J_1, or v is not an interior point of J_2. So, assume u is an endpoint of J_1. Analogously as in the previous paragraph, C is either an arc or a simple triod containing $J_1 \cup J_2$.

$(ii) \Rightarrow (iii)$: Suppose J_1 and J_2 are non-overlapping, and $J_1 \cup J_2$ is contained in an arc A. Let $[u, v]$ denote the minimal subarc of A, with respect to inclusion, that contains $J_1 \cup J_2$. We may assume, without loss of generality, that u is an endpoint of J_1 and v is an endpoint of J_2. Let a be the endpoint of J_1 that is neither u nor v. The points a, u, and v are the desired three endpoints. We consider each two of them. By uniqueness of J_1, J_1 is the only arc in G with endpoints a and u. For u and v, by Lemma 1(c), if L is an arc in G with endpoints u and v, then L contains each of J_1 and J_2. For a and v, by Lemma 1(d), if L is an arc in G with endpoints a and v, then L contains J_2.

Suppose $J_1 \cup J_2$ is not contained in an arc in G, but is contained in a simple triod K' with branchpoint v. Then one of J_1 or J_2 contains v in its interior, say v is in the interior of J_2. Let K be the minimal triod in K', with respect to inclusion, that contains $J_1 \cup J_2$. Then the endpoints b and c of the two edges of K that contain J_2 are also endpoints of J_2. Since J_1 and J_2 are non-overlapping, J_1 is contained in the remaining edge of K, whose endpoint a is also an endpoint of J_1. The points a, b, and c are the desired three endpoints. Let $[a, v]$, $[b, v]$, and $[c, v]$ denote the three subarcs of K whose union is K. We note that, by Lemma 1(b), the subarcs $[b, v]$ and $[c, v]$ of J_2 are unique arcs in G. We consider each two of the three points. For b and c, J_2 is the unique arc in G with endpoints b and c. For a and b, the arc $[b, v] \cup [v, a]$ is an arc with two unique arcs containing its endpoints, so, by Lemma 1(c), each arc in G with endpoints a and b contains J_1. For a and c, the situation is analogous to that of a and b.

$(iii) \Rightarrow (iv)$: This implication follows from Observation 1.

$(iv) \Rightarrow (i)$: We prove the contrapositive statement. Suppose $\{J_1, J_2\}$ is not an allowable collection of arcs. Then either the interiors of J_1 and J_2 are not disjoint, or there exists an arc in G joining J_1 to J_2 whose endpoints are interior points of J_1 and J_2. Let a and b be the endpoints of J_1, and let c and d be the endpoints of J_2.

Suppose the interiors of J_1 and J_2 are not disjoint. Suppose J_1 and J_2 overlap. Then $J_1 \cap J_2$ is an arc. Pick a point p in the interior of $J_1 \cap J_2$ that is not a branchpoint of G. Since $o(p) = 2$, by Lemma 1(a), $G \setminus \{p\}$ has two components, each one containing exactly one endpoint from each of J_1 and J_2. Let A and B be the closures of these two components. Clearly, $G = A \cup B$, and for $i = 1, 2$, $J_i \not\subset A$ and $J_i \not\subset B$, contradicting that $\{J_1, J_2\}$ is decomposition saturated. So, we have that $J_1 \cap J_2 = \{v\}$, where v is an interior point of each of J_1 and J_2. So, $v \in B(G)$ and $o(v) \geq 4$. Combining several parts of Lemma 1, we see that v separates each pair of points in $\{a, b, c, d\}$. For $r \in \{a, b, c, d\}$, let G_r be the closure of the component of $G \setminus \{v\}$ that contains r. Let F be the union of the closures of the remaining components of $G \setminus \{v\}$. Let $A = G_a \cup G_c$, $B = G_b \cup G_d \cup F$. Clearly, $A \cup B$ is a decomposition of G where neither J_1 nor J_2 is contained in either A or B. So, $\{J_1, J_2\}$ is not decomposition saturated.

Suppose the interiors of J_1 and J_2 are disjoint, and L is an arc in G joining J_1 to J_2. Assume u and v are the endpoints of L lying, respectively, in the interiors of J_1 and J_2. If J_1 and J_2 are not disjoint, there is an arc J' in $J_1 \cup J_2$ with endpoints u and v. By Lemma 1, J' is a unique arc in G, which is a contradiction since $L \neq J'$. So, we assume that J_1 and J_2 are disjoint. By uniqueness of J_1 and J_2, each arc in G joining J_1 to J_2 has endpoints u and v. Let H be the union of all arcs in G joining J_1 to J_2. For $r \in \{a, b, c, d\}$, let G_r be the closure of the component of $G \setminus H$ that contains r. Let F be the union of the closures of the remaining components of $G \setminus H$. Let $A = G_a \cup H \cup G_c$, and $B = G_b \cup H \cup G_d \cup F$. As in the previous paragraph, $A \cup B$ is a decomposition of G where neither J_1 nor J_2 is contained in either A or B, giving us that $\{J_1, J_2\}$ is not decomposition saturated. □

Corollary 1. Let $X = \varprojlim\{G_i, g_i^{i+1}\}$, where for each $i \geq 1$, G_i is a graph, and g_i^{i+1} is a surjective mapping. Suppose for $n \geq 1$, there exist $m > n$ and an allowable collection of unique arcs $\{J_1, J_2\}$ in G_m where g_n^m is a 2-pass map for $\{J_1, J_2\}$. Then X is indecomposable.

Proof. Since, by Theorem 5, each allowable pair of unique arcs in a graph has the 3-endpoint property, it follows from Theorem 4 that X is indecomposable. □

Corollary 2. Let G be a graph, $g\colon G \to G$ be a surjective mapping, and $X = \varprojlim\{G_i, g_i^{i+1}\}$ where for each $i \geq 1$, $G_i = G$ and $g_i^{i+1} = g$. Suppose there exist an allowable pair of unique arcs $\{J_1, J_2\}$ in G, and $k \geq 1$ such that $g^k(J_1) = G = g^k(J_2)$. Then X is indecomposable.

Proof. For $n \geq 1$, let $m = n + k$. We have that, for $i = 1, 2$, $g_n^m(J_i) = g_n^{n+k}(J_i) = g^k(J_i) = G = G_n$. Hence, g^k is a 2-pass map for $\{J_1, J_2\}$, and by Corollary 1, X is indecomposable. □

Remark 1. In Theorem 5, and in Corollaries 1 and 2, if the graphs are trees, the results hold for any allowable collection of two arcs, since all arcs in a tree are unique arcs.

Remark 2. As in Theorem 5, there will be theorems throughout that show certain allowable collections of arcs in graphs have the 3-endpoint property, and hence, there will be corollaries analogous to Corollaries 1 and 2 that establish indecomposability of inverse limits on graphs. We will not continue to display them, as they are obvious from Theorem 4.

Corollary 2 generalizes Ingram's Theorems 3.3 and 3.4 in [**19**]. To see this, we verify Observation 2 below, and note that Ingram's Theorem 3.4 follows directly from his Theorem 3.3. We provide a statement of Ingram's Theorem 3.3 for convenience to the reader.

Ingram's Theorem. *Suppose T is a tree and $f\colon T \to T$ is a mapping. Suppose H and I are non-overlapping subtrees of T, and I is an arc such that if p is a branchpoint of T that belongs to I, then p is an endpoint of I. If $f(H) = f(I) = T$, then $\varprojlim\{T, f\}$ is indecomposable.*

Observation 2. Suppose we have the hypothesis of Ingram's Theorem. Then there exists an arc $J \subset H$ such that $f^2(J) = T$. Hence, $f^2(J) = T = f^2(I)$, and Corollary 2 applies.

Proof. By Ingram's Lemma 3.2 in [**19**] (or see Theorem 6 in [**37**] for a more general result in graphs), there exists a subcontinuum K of H such that $f(K) = I$. Let $a, b \in K$ where $f(a)$ and $f(b)$ are the endpoints of I. Let J be the arc in K with endpoints a and b. Then $f(J) = I$. By hypothesis, $f^2(J) = f(I) = T$.

Since $H \cap I = \emptyset$, we have that $J \cap I = \emptyset$. Also, since I is contained in an edge of T, the arc in T from J to I must meet I at one of its endpoints. Hence, $\{I, J\}$ is an allowable collection of arcs in T, and Corollary 2 is satisfied for $\{I, J\}$ with $k = 2$. □

Remark 3. It is possible to have a decomposable inverse limit on a single tree T with a single bonding mapping f, where f is a 2-pass map on two non-overlapping arcs in T. Ingram provides such an example in [**19,** Example 5.1]. In Ingram's example, the tree T is a simple 4-od, and the two arcs α and β meet only at the branchpoint of T, which is an interior point of each of α and β. So, $\{\alpha, \beta\}$ is not an allowable collection of arcs in T.

We need a few definitions and lemmas before our next equivalence theorem. Given a simple closed curve S, we endow S with a counter-clockwise orientation via some homeomorphism of S^1 onto S. For $n \geq 3$ and any collection $\alpha_1, \ldots, \alpha_n$ of non-overlapping subcontinua of S, we write $\alpha_1 < \alpha_2 < \ldots < \alpha_n$ to indicate their orientation relative to movement in a counter-clockwise direction. If, for some $1 \leq i < n$, α_i and α_{i+1} are points that may, or may not, be equal, we write $\alpha_i \leq \alpha_{i+1}$. If a and b are the endpoints of an arc J in S, $t \in J^\circ$, and $a < t < b$, we say that a is *the least point of* J and b is *the largest point of* J.

Lemma 2 is straightforward to verify. We provide a proof for Lemma 3.

Lemma 2. Let S be a unique cycle in a graph G.

(a) If $p \notin S$, then there exists a vertex v_p in S such that if L is an arc in G joining p to S, then v_p is the endpoint of L in S. Furthermore, if L is an arc in G joining p to a point q of $S \setminus \{v_p\}$, then L is the union of an arc from p to v_p and one of the two arcs in S from v_p to q.

(b) If p and q are two points in S, then if L is an arc in G with endpoints p and q, then $L \subset S$. So, there are exactly two arcs in G with endpoints p and q, and their union is S.

(c) If L is an arc in G, then $L \cap S$ is either empty, a point, or an arc.

Lemma 3. Suppose J_1 and J_2 are arcs in G, and S is a unique cycle in G, where J_2 overlaps S and J_1 is disjoint from S. Suppose $[u, v]$ is an arc in G joining J_1 to S where u is an interior point of J_1. If $\{J_1, J_2\}$ is an allowable collection of arcs in G, then either J_2 has an endpoint in $[u, v]$ or $J_2 \subset S \setminus \{v\}$.

Proof. Suppose J_2 has no endpoint in $[u, v]$. If $J_2 \cap [u, v] \neq \emptyset$, we consider the natural order on the arc $[u, v]$ from u to v, and we let u' be the least point of the closed set $J_2 \cap [u, v]$. Since, by assumption, u' is not an endpoint of J_2, $u \neq u'$, for otherwise, the interiors of J_1 and J_2 meet. So, the subarc of $[u, v]$ from u' to u joins the interiors of J_1 and J_2, contradicting that $\{J_1, J_2\}$ is an allowable collection. Hence, if J_2 has no endpoint in $[u, v]$, then $J_2 \cap [u, v]$ must be empty. Also, in this case, J_2 does not meet J_1, for otherwise, we violate Lemma 2(a).

If $J_2 \not\subset S$, then by Lemma 2(b), not both endpoints of J_2 lie in S. Suppose b is an endpoint of J_2 that is not in S. Let $[b, w]$ denote the subarc of J_2 that joins b to S. So, w is an interior point of J_2 since J_2 overlaps S. One of the two arcs in S with endpoints w and v only meets J_2 at the point w. Let K denote this arc. Then $K \cup [u, v]$ joins the interiors of J_1 and J_2, a contradiction. So, $J_2 \subset S \setminus \{v\}$. □

Theorem 6. *Let $\{J_1, J_2, J_3\}$ be a collection of three arcs contained in a unique cycle S in a graph G. Then $\{J_1, J_2, J_3\}$ is allowable if and only if $\{J_1, J_2, J_3\}$ has the* 3*-endpoint property if and only if $\{J_1, J_2, J_3\}$ is decomposition saturated.*

Proof. Suppose $\mathcal{J} = \{J_1, J_2, J_3\}$ is an allowable collection of arcs in G. According to our orientation convention, assume $J_1 < J_2 < J_3$. Let a_i be the least point of J_i for $i = 1, 2, 3$. These are the desired three points. Consider two of them, say a_1 and a_3. By Lemma 2(b), there are two arcs in G with endpoints a_1 and a_3. One contains both J_1 and J_2, and the other contains J_3. Analogously, arcs in G with endpoints among other pairs of a_1, a_2, and a_3 contain members of $\mathcal{J}$.

The second implication follows from Observation 1.

We show, by way of contradiction, that if $\mathcal{J}$ is decomposition saturated, then it is an allowable collection in G. Assume $\mathcal{J}$ is decomposition saturated, but is not an allowable collection. Then we may suppose, without loss of generality, that the interior of J_1 meets the interior of J_2. So, J_1 and J_2 overlap. Pick a point p in the interior of $J_1 \cap J_2$, and a point $q \neq p$ in the interior of J_3. Let α and β be the closures of the two components of $S \setminus \{p, q\}$. Let A be the union of α and each component of $G \setminus S$ whose closure meets α. Let B be the union of β and each component of $G \setminus S$ whose closure meets β. Clearly, $A \cup B = G$, and no one of J_1, J_2, and J_3 is contained in either A or B, a contradiction. $\square$

Remark 4. We note that if a collection of arcs $\mathcal{J}$ in a graph G has the 3-endpoint property, the three points exhibiting the property need not be unique. For example, in the first paragraph of the proof of Theorem 6, a_1 and a_2 could be chosen to be the endpoints of J_1, and a_3 chosen to be the greatest point of J_2.

Example 1 shows that, in general, the three properties are not equivalent for collections of $n + 2$ arcs in a graph with n independent cycles, even when all arcs are contained in unique cycles.

Example 1. The graph G shown in Figure 1 is a graph with two unique cycles and one branchpoint. Let $\mathcal{J} = \{J_1, J_2, J_3, J_4\}$ be the collection of four arcs shown in the figure. It is easy to check that $\mathcal{J}$ is decomposition satuared, but is not an allowable collection, and does not have the 3-endpoint property.

Removing J_1 from the collection, we note that $\{J_2, J_3, J_4\}$ is allowable, and decomposition saturated, but does not have the 3-endpoint property.

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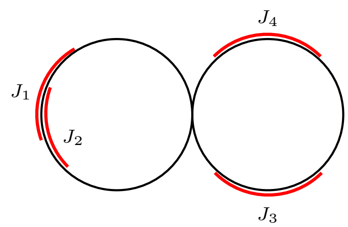


Figure 1. Non-equivalence of the three properties

4. Cases for which allowable collections of arcs are sufficient for the 3-endpoint property. In this section, we establish many special-case results that have practical use, which we demonstrate in the examples in Section 5. All results, except for Theorems 7 and 10, involve only a small number of arcs in the allowable collections.

We begin the section with two questions.

Question 1. Let $n \geq 2$, let G be a graph having n independent cycles, and let $\mathcal{J}$ be an allowable collection of arcs in G with $n+2$ members.

(a) Does $\mathcal{J}$ have the 3-endpoint property?
(b) Is $\mathcal{J}$ decomposition saturated?

Question 2. Do parts (a) and (b) of Question 1 have affirmative answers if we assume that each cycle in G is unique?

By Observation 1, an affirmative answer to Question 1(a) provides an affirmative answer to Question 1(b). For graphs with no cycles, Theorem 5 gives us that an allowable collection of two arcs has the 3-endpoint property. For graphs with one cycle, Corollary 3 in this section gives us that an allowable collection of three arcs has the 3-endpoint property. Question 1(a) is natural based on these two cases.

We are unable to answer either Question 1(a) or 1(b), but we establish a number of useful partial results showing that if the arcs in an allowable collection $\mathcal{J}$ are arranged in a nice way in a graph, then $\mathcal{J}$ will have the 3-endpoint property. In practice, these results can be more useful than general results that require locating a large allowable collection of arcs. By Theorems 3 and 4, all results that establish either the 3-endpoint property or decomposably saturated for an allowable collection of arcs will give indecomposability results for inverse limits on graphs where the bonding mappings are n-pass maps for appropriate allowable collections of arcs in a subsequence of factor spaces. Particularly, for a given inverse sequence on a single graph G and a single bonding mapping g, it is easy to spot if, for some $k \geq 1$, g^k is an n-pass map on an allowable collection of n arcs. We simply find arcs in G that iterate under g to all of G, and check if a sufficient number of them, according to our theorems, have pairwise disjoint interiors, and pairwise have no arc connecting their interiors. This process is illustrated in the examples in Section 5. Furthermore, how to easily construct examples of indecomposable inverse limits on trees and graphs is made clear, respectively, from Corollaries 1 and 2 and from results in this section. The author suspects that using the multicoherence of a graph may lead to a positive answer to Question 2(b), although such an investigation was not pursued. See Section 12 of Chapter IV in [**44**] for a definition of multicoherence.

Theorem 7 gives a partial answer to Question 1(a).

Theorem 7. *Let G be a graph with cyclomatic number $n \geq 0$, and let $\mathcal{J} = \{J_1, \ldots, J_{n+2}\}$ be an allowable collection of arcs such that, for $1 \leq i \leq n+2$, either J_i is a unique arc in G or J_i is contained in an edge of G. Then $\mathcal{J}$ has the 3-endpoint property.*

Proof. We use induction on n. If G has cyclomatic number zero, then G has no cycles and the result follows from Theorem 5. Let G and $\mathcal{J}$ satisfy the hypothesis for some $n \geq 1$, and assume the result holds for graphs with cyclomatic number less than n.

If two members of $\mathcal{J}$ are unique arcs, then, by Theorem 5, the result follows. So, some member of $\mathcal{J}$ lies in an edge of a cycle S in G. Assume, without loss of generality, that J_1 is contained in an edge of S. Let $G' = G \setminus J_1^\circ$. Now, G' is a connected graph that has one less edge than G, and has the same number of vertices as G. Hence, the cyclomatic number of G' is $n-1$. Since $\mathcal{J}' = \{J_2, \ldots, J_{n+2}\}$ is an allowable collection of arcs in G', where each member of $\mathcal{J}'$ is either a unique arc or is contained in an edge of G', we have by inductive assumption, that $\mathcal{J}'$ has the 3-endpoint property for G'. Let a, b, and c be the endpoints satisfying the 3-endpoint property for $\mathcal{J}'$ in G'. We show that a, b, and c also satisfy the 3-endpoint property for $\mathcal{J}$ in G.

Let L be an arc in G with endpoints in $\{a, b, c\}$. Assume, without loss of generality, that a and b are the endpoints of L. If $L \subset G'$, then L contains J_i for some $2 \leq i \leq n+2$. Otherwise, since J_1 lies in an edge of S, no vertex of G is an interior point of J_1, and it follows that $J_1 \subset L$. We have that $\mathcal{J}$ has the 3-endpooint property for G. $\square$

The proofs of Lemmas 4 and 5 are straightforward by simply picking the three appropriate endpoints.

Lemma 4. Let G be a graph, and let $\mathcal{J} = \{J_1, J_2, J_3\}$ be an allowable collection of arcs in G. If there exists an edge of G that contains all three members of $\mathcal{J}$, then $\mathcal{J}$ has the 3-endpoint property in G.

Lemma 5. Let G be a graph, and let $\mathcal{J} = \{J_1, J_2, J_3, J_4\}$ be an allowable collection of arcs in G. If there exist two edges E_1 and E_2 of G where $J_1 \cup J_2 \subset E_1$ and $J_3 \cup J_4 \subset E_2$, then $\mathcal{J}$ has the 3-endpoint property in G.

Theorem 8. *Let S be a unique cycle in a graph G. Suppose $\mathcal{J} = \{J_1, J_2, J_3\}$ is an allowable collection of three arcs in G, where for $i \in \{1, 2, 3\}$, if K is a subarc of J_i that does not overlap S, then K is a unique arc in G. Then $\mathcal{J}$ has the 3-endpoint property in G.*

Proof. Suppose, for some $i \in \{1, 2, 3\}$, $J_i \subset S$, and if p is a branchpoint of G in the interior of J_i and C is the closure of a component of $G \setminus S$ that meets S at p, then for $k \in \{1, 2, 3\}$ with $k \neq i$, we have that $J_k \not\subset C$. We begin by showing that if this supposition is satisfied, then $\mathcal{J}$ has the 3-endpoint property in G. The proof of this implication is similar to the proof of the inductive step in Theorem 7.

To see this, we assume, without loss of generality, that J_1 has the property in the previous paragraph. Let u and w be the endpoints of J_1. By Lemma 2(b), there are exactly two arcs in G with endpoints u and w; one is J_1, and the other is the closure of the complement of J_1 in S, which we denote by J_1'. Let G' be the component of $G \setminus J_1^\circ$ that contains J_1'. We note that the closure of $G \setminus G'$ is connected, contains J_1, and meets G' at the two point set $\{u, w\}$. By our supposition concerning J_1, it follows that $J_2 \cup J_3 \subset G'$. Furthermore, we note that J_1' is a unique arc in G'. So, by hypothesis and Lemma 1(c), it follows that J_2 and J_3 are unique arcs in G'. Hence, G' and the collection $\{J_2, J_3\}$ satisfy the conditions of Theorem 5. So, choose points a, b, and c among the endpoints of J_2 and J_3 that satisfy the 3-endpoint property in G'. We claim that a, b, and c also satisfy the 3-endpoint property in G. Let L be an arc in G with endpoints in $\{a, b, c\}$. Assume, without loss of generality, that L has endpoints a and b. If $L \subset G'$, then L contains one of J_2 or J_3. Otherwise, $J_1 \subset L$. So, $\mathcal{J}$ has the 3-endpoint property in G.

Hence, we will assume hereafter that the statement $(*)$ below holds.

$(*)$ If, for some $i \in \{1, 2, 3\}$, $J_i \subset S$, then there exist $k \in \{1, 2, 3\}$ with $k \neq i$ and an interior point p of J_i such that either p is an endpoint of J_k or each arc in G joining J_k to S meets S at p. We will say that J_k *is linked to* S *through* $p \in J_i^\circ$.

If J_i doesn't overlap S for some $i \in \{1, 2, 3\}$, then, by hypothesis, J_i is a unique arc in G. So, either $J_i \cap S = \emptyset$, or $J_i \cap S$ is degenerate since S is a unique cycle. If two members of $\mathcal{J}$, say J_i and J_k, don't overlap S, then, by Theorem 5, $\{J_i, J_k\}$ has the 3-endpoint property. So, $\mathcal{J}$ has the 3-endpoint property. Hence, we assume throughout that two members of $\mathcal{J}$, say J_2 and J_3, overlap S. By Lemma 2(c), $J_2 \cap S$ and $J_3 \cap S$ are arcs.

We consider two cases with various subcases.

Case 1. Suppose J_1 doesn't overlap S. So, J_1 is a unique arc in G. We will assume that J_1 is disjoint from S throughout this case, and let $[u, v]$ be an arc in G joining J_1 to S. If J_1 meets S at a point, the proof of this case is similar, but easier.

(a) Suppose u is an interior point of J_1. By Lemma 3, for $i \in \{2, 3\}$, either J_i has an endpoint in $[u, v]$ or $J_i \subset S \setminus \{v\}$. Furthermore, if J_i has an endpoint in $[u, v]$, then $J_i \cap [u, v]$ is either $\{v\}$ or a subarc of $[u, v]$ containing $\{v\}$. This follows from Lemma 2, and from the assumption about subarcs of members of $\mathcal{J}$ in the hypothesis.

(i) Suppose both J_2 and J_3 have an endpoint in $[u, v]$. Since J_2 and J_3 do not overlap, v must be an endpoint of one of J_2 or J_3. We assume, without loss of generality, that v is an endpoint of J_3, and that v is the largest point of the arc $J_3 \cap S$ in the orientation on S. It follows that v must be the least point of the arc $J_2 \cap S$. It also follows that either J_2 is a subset of $S \cup [u, v]$ or J_3 is a subset of S, for otherwise either the interiors of J_2 and J_3 meet or an arc in S joins the interiors of J_2 and

J_3. If $J_3 \subset S$, by $(*)$, one of J_1 or J_2 is linked to J_3 at an interior point of J_3, which is a contradiction. So, we have that $J_2 \subset S \cup [u,v]$, and $J_3 \not\subset S$. Let u' be the endpoint of J_2 in $[u,v]$. We assume that $u' \neq v$, for otherwise, $J_2 \subset S$, and again we violate $(*)$. The subarc $[u',v]$ of J_2 is unique, and $J_2 \cap [u,v] = [u',v]$.

Let c be the endpoint of J_3 that is not in S, and let a and b be the endpoints of J_1. We show that a, b, and c satisfy the 3-endpoint property for $\mathcal{J}$. Since J_1 is a unique arc, the only arc in G with endpoints a and b is J_1. Suppose L is an arc in G with endpoints a and c. Let $[c,w]$ be the subarc of J_3 joining c to S. By hypothesis, $[c,w]$ is a unique arc. Suppose $L \cap S$ is empty. Let $[c,c']$ be a subarc of L joining c to $[a,u] \cup [u,v]$. Then $[c,c'] \cup [c',v]$ is an arc, distinct from $[c,w]$ joining c to the unique cycle S at the point $v \neq w$. This contradicts Lemma 2(a). So, L must meet S. Let L_1 be the subarc of L joining c to S. By Lemma 2(a), L_1 meets S at w. Since $[c,w]$ is a unique arc, $L_1 = [c,w]$. Let L_2 be the subarc of L joining a to S. Now, $[a,u] \cup [u,v]$ is an arc joining a to S that contains the unique subarcs $[a,u]$ and $[u',v]$. By Lemma 1(b), $[a,u] \cup [u',v] \subset L_2$. By Lemma 2(b), there are two arcs in G with endpoints v and w. So, L contains one of them. That is, L contains either $J_2 \cap S$ or $J_3 \cap S$. Hence, either $J_3 = [c,w] \cup (J_3 \cap S) \subset L$ or $J_2 = [u',v] \cup (J_2 \cap S) \subset L$. An analogous argument shows that each arc in G from c to b contains one of J_2 or J_3.

(ii) Suppose one of J_2 or J_3 does not have an endpoint in $[u,v]$. We assume, without loss of generality, that J_3 does not have an endpoint in $[u,v]$. By Lemma 3, $J_3 \subset S \setminus \{v\}$, and by $(*)$, either J_1 or J_2 is linked to S through some interior point of J_3. However, we show that this is not the case. We see that J_1 is not linked to S through an interior point of J_3 since J_1 is disjoint from S, and the arc $[u,v]$ from J_1 to S meets S only at v, which is not an interior point of J_3. To see that J_2 is not linked to S through an interior point of J_3, we recall that J_2 overlaps S and S is a unique cycle. Hence, neither endpoint of J_2 is in the interior of J_3. Similarly, there is no arc joining J_2 to S. So, we have that neither J_1 nor J_2 is linked to S through an interior point of J_3, a contradiction.

(b) Suppose u is an endpoint of J_1. Let a be the other endpoint of J_1. Let p_1 and p_2, and q_1 and q_2 be, respectively, the least and largest points of the arcs $J_2 \cap S$ and $J_3 \cap S$. We assume that $p_1 < p_2 \leq q_1 < q_2 \leq p_1$

in the orientation on S.

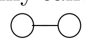
(i) Suppose one of J_2 or J_3 is a subset of S, say $J_2 \subset S$. Then, v must be an interior point of J_2, for otherwise, by $(*)$, one of J_1 or J_3 is linked to J_2 through an interior point of J_2 other than v, which is a contradiction. Similarly, if $J_3 \subset S$, v must be an interior point of J_3, which would be a contradiction. So, we have that J_3 is not a subset of S. So, one of q_1 or q_2 is an interior point of J_3. If both q_1 and q_2 are interior points of J_3, let b and c be the endpoints of J_3. We choose the points a, b, and c for the 3-endpoint property. If L is an arc in G with endpoints a and b, then since J_1 is unique in G and lies in the arc $J_1 \cup [u,v]$, by Lemma 1(c), J_1 is a subset of L. The situation is analogous for an arc in G with endpoints a and c. Assume $b < q_1 < q_2 < c$ in the order on J_3 from b to c. The arcs $[b,q_1] \subset J_3$ and $[q_2,c] \subset J_3$ are unique arcs in G by hypothesis. Since S is a unique cycle, by Lemma 2(b), there are two arcs in G with endpoints q_1 and q_2 and their union is S. It follows that there are exactly two arcs in G with endpoints b and c. One is J_3, and the other contains J_2. So, the proof is complete when both q_1 and q_2 are interior points of J_3. We suppose, without loss of generality, that q_1 is an interior point of J_3 and q_2 is an endpoint. Let d be the other endpoint of J_3 that is not in S. We pick the points a, q_2, and d. An analysis similar to the one in the previous paragraph gives that these three points satisfy the 3-endpoint property.

(ii) Suppose neither J_2 nor J_3 is a subset of S. So, one of p_1 or p_2 is an interior point of J_2. We assume, without loss of generality, that p_1 is an interior point of J_2. It follows that q_2 must be an endpoint of J_3, for otherwise, either J_2 and J_3 have an interior point, namely $p_1 = q_2$, in common, or an arc in S joining J_2 to J_3 meets each arc in its interior, a contradiction. Hence, q_1 must be an interior point of J_3, and p_2 must be an endpoint of J_2. Let b and c be, respectively, the endpoints of J_2 and J_3 that are not in S. The points a, b, and c are the desired endpoints for the 3-endpoint property. Checking arcs between each pair is similar to previous cases.

Case 2. Suppose J_1 also overlaps S. By Lemma 2(c), for each $i = 1,2,3$, $J_i \cap S$ is an arc. We let $J_1 \cap S = [p_1,p_2]$, $J_2 \cap S = [q_1,q_2]$, and $J_3 \cap S = [t_1,t_2]$. We also assume, without loss of generality, that $p_1 < p_2 \leq q_1 < q_2 \leq t_1 < t_2 \leq p_1$ in the orientation on S. By $(*)$, no J_i is a subset of S. Assume, without loss of generality, that p_2 is an interior point of J_1. As we saw in the previous case, it follows that q_1 is an endpoint of J_2, making q_2 an interior point of J_2, t_1 an endpoint of J_3, t_2 an interior point of J_3, and p_1 an endpoint of J_1. Let a, b, and c be, respectively, the endpoints of J_1, J_2, and J_3 that are in $G \setminus S$. The points a, b, and c are the desired endpoints, and checking arcs in G with endpoints in $\{a, b, c\}$ is similar to previous cases. $\square$

Corollary 3. Suppose G is a graph with one cycle, and $\mathcal{J} = \{J_1, J_2, J_3\}$ is an allowable collection of three arcs in G. Then $\mathcal{J}$ has the 3-endpoint property.

Proof. Since G has exactly one cycle S, by definition, S is a unique cycle in G. So, the arcs J_i, for $i = 1, 2, 3$, satisfy the hypothesis of Theorem 8. Hence, $\mathcal{J}$ has the 3-endpoint property. $\square$

A graph homeomorphic to the figure  is commonly called a *figure-eight*. A graph homeomorphic to the figure  is commonly called a *theta-curve*. We call a graph homeomorphic to the figure  a *dumbbell*.

Corollary 4. If G is either a figure-eight, a theta-curve, or a dumbbell, and $\mathcal{J} = \{J_1, J_2, J_3, J_4\}$ is an allowable collection of arcs in G, then $\mathcal{J}$ has the 3-endpoint property in G.

Proof. We prove the result for G a dumbbell. Proofs for a figure-eight and a theta-curve are similar.

Let $G = S_1 \cup [v_1, v_2] \cup S_2$ be a dumbbell, where S_1 and S_2 are unique cycles in G, and $[v_1, v_2]$ is the unique arc in G joining S_1 to S_2. We assume no two members of $\mathcal{J}$ are subsets of $[v_1, v_2]$, for otherwise the result follows from Theorem 5. For $i \in \{1, 2\}$, v_i is an interior point of at most one member of $\mathcal{J}$. It follows that some member of $\mathcal{J}$ is a subset of one of S_1 or S_2, and does not contain either v_1 or v_2 in its interior. Assume, without loss of generality, that $J_1 \subset S_1$, and $v_1 \notin J_1^\circ$. Removing the interior of J_1 from G, we obtain a subgraph G' of G for which $\mathcal{J}' = \{J_2, J_3, J_4\}$ and G' satisfy the conditions of Corollary 3. Hence, $\mathcal{J}'$ has the 3-endpoint property in G'. Analogously as in the proof of Theorem 7, and as in the proof of $(*)$ in Theorem 8, the three endpoints that satisfy the 3-endpoint property for $\mathcal{J}'$ in G' also satisfy the 3-endpoint property for $\mathcal{J}$ in G.

□

Theorem 9. *Let G be a graph containing unique cycles S_1 and S_2. Suppose $\mathcal{J} = \{J_1, J_2, J_3, J_4\}$ is an allowable collection of four arcs in G where J_1 and J_2 overlap S_1, and J_3 and J_4 overlap S_2. Suppose also that, for $i = 1, 2$ $(i = 3, 4)$, if K is a subarc of J_i that does not overlap S_1 (S_2), then K is a unique arc in G. Then $\mathcal{J}$ has the 3-endpoint property.*

Proof. It is clear from the hypothesis that no J_i overlaps both S_1 and S_2. Either $S_1 \cap S_2$ is degenerate, or $S_1 \cap S_2$ is empty. We assume the latter. The proof for the former case is similar.

By Lemma 2(a), there exist vertices $v_1 \in S_1$ and $v_2 \in S_2$ such that each arc in G joining S_1 and S_2 has endpoints v_1 and v_2. Let $[v_1, v_2]$ denote an arc in G joining S_1 to S_2.

Suppose, for some $1 \leq i \leq 4$, J_i is a subset of one of S_1 or S_2, and neither v_1 nor v_2 is an interior point of J_i. We assume, without loss of generality, that $J_1 \subset S_1$, and v_1 is not an interior point of J_1. Let u and w be the endpoints of J_1. Let C be the component of $G \setminus \{u, w\}$ that contains J_1°, and note that, for $i \neq 1$, $J_i \cap C = \emptyset$, for otherwise, S_1 is not a unique cycle. As in the proof of $(*)$ in Theorem 8, we have that $\mathcal{J}' = \{J_2, J_3, J_4\}$ is an allowable collection of arcs in the graph $G' = G \setminus C$, and the hypothesis of Theorem 8 is satisfied for $\mathcal{J}'$ in G'. Hence, $\mathcal{J}'$ has the 3-endpoint property in G'. Analogously, as in the proof of $(*)$ in Theorem 8, it follows that $\mathcal{J}$ has the 3-endpoint property in G, and the proof is complete. So, hereafter, we assume that

$(**)$ if J_i is a subset of S_1 or S_2 for some $1 \leq i \leq 4$, then J_i contains, respectively, v_1 or v_2 in its interior.

Case 1. Suppose, for some $1 \leq i \leq 4$, J_i is a subset of one of S_1 or S_2. We assume that $J_1 \subset S_1$. It follows from $(**)$ that $v_1 \in J_1^\circ$, and that $J_2 \not\subset S_1$. By Lemma 3, J_3 and J_4 must have an endpoint in $[v_1, v_2]$. So, v_2 must be an endpoint of one of J_3 or J_4. Assume v_2 is an endpoint of J_3. By Lemma 2(c), we let $J_3 \cap S_2 = [v_2, p]$, and $J_4 \cap S_2 = [v_2, q]$. We assume, without loss of generality, that $v_2 < p \leq q < v_2$ in the orientation of S_2. By $(**)$, p must be an interior point of J_3. Let c be the endpoint of J_3 that is not in S_2. It follows that q must be an endpoint of J_4, for otherwise, either J_3 and J_4 have an interior point in common or an arc in S_2 joins interior points of J_3 and J_4. By $(**)$, v_2 is an interior point of J_4. Since J_4 has an endpoint in $[v_1.v_2]$, as we saw in the first paragraph of Case 1(a) in Theorem 8, it follows that $J_4 \subset [v_1, v_2] \cup S_2$. Letting v be the endpoint of J_4 in $[v_1.v_2]$, we have that $J_4 = [v, v_2] \cup [v_2, q]$ with $[v, v_2] \subset [v_1, v_2]$. Let b_1 and b_2 be the endpoints of J_2. Recalling that $J_2 \not\subset S_1$, at least one of b_1 and b_2 is not in S_1; possibly both endpoints are not in S_1. In either case, the points b_1, b_2, and c satisfy the 3-endpoint property for $\mathcal{J}$ in G. Verifying this is similar to proofs given in previous results.

Case 2. Suppose no J_i is a subset of either S_1 or S_2. Let $J_1 \cap S_1 = [p_1, p_2]$, $J_2 \cap S_1 = [q_1, q_2]$, $J_3 \cap S_2 = [r_1, r_2]$, and $J_4 \cap S_2 = [t_1, t_2]$. We assume, without loss of generality, that $p_1 < p_2 \leq v_1 \leq q_1 < q_2 \leq p_1$ in the orientation on S_1, and that $v_2 \leq r_1 < r_2 \leq t_1 < t_2 \leq v_2$ in the orientation on S_2. By supposition in this case, one of p_1 and p_2 is an interior point of J_1. Assume p_2 is an interior point of J_1. As we have previously seen, it follows that q_1 is an endpoint of J_2, q_2 is an interior point of J_2, and p_1 is an endpoint of J_1. Analogously, for the arcs J_3 and J_4 overlapping S_2, we assume, without loss of generality, that r_1 is an interior point of J_3, r_2 is an endpoint of J_3, t_1 is an interior point of J_4, and t_2 is an endpoint of J_4.

(a) Suppose one of p_2 or q_1 is not v_1. We assume that $q_1 \neq v_1$. Then the union of $[v_1, v_2]$ and one of the two arcs in S_1 from v_1 to q_2 is an arc joining S_2 to an interior point of J_2. Thus, by Lemma 3, both J_3 and J_4 have an endpoint in $[v_1, v_2]$. We have that $v_2 = t_2$ is an endpoint of J_4, $r_1 = v_2$ is an interior point of J_3, and J_3 has an endpoint in $[v_1, v_2]$. This arrangement of J_3 and J_4 relative to S_2 is analogous to that in Case 1. Let a and b be, respectively, the endpoints of J_1 and J_2 that are not in S_1, and let c be the endpoint of J_4 that is not in S_2. We see that a, b, and c satisfy the 3-endpoint property for $\mathcal{J}$ in G.

(b) Suppose $p_2 = v_1$ and $q_1 = v_1$. An analogous argument as in (a) for one of r_1 and t_2 not being equal to v_2 would complete the proof. So, we also suppose $r_1 = v_2$ and $t_2 = v_2$. Recall that $p_2 = v_1$ and $r_1 = v_2$ are, respectively, interior points of J_1 and J_3. Let b and d be, respectively, the endpoints of J_1 and J_3 that are not in S_1 and S_2. By hypothesis, the subarcs $[v_2, b]$ and $[v_2, d]$ of, respectively, J_1 and J_3 are unique arcs in G. So, one must be a subset of $[v_1, v_2]$, for otherwise, either J_1 and J_3 have an interior point in common, or an arc in $[v_1, v_2]$ joins the interiors of J_1 and J_3. So, we assume $[v_2, d] \subset [v_1, v_2]$. Again,

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the arrangement of J_3 and J_4 relative to S_2 is analogous to that in Case 1 and Case 2(a). Letting a be the endpoint of J_2 not in S_1, and c be the endpoint of J_4 not in S_2, the points a, b, and c satisfy the 3-endpoint property for $\mathcal{J}$ in G. □

Theorem 10. *Let G be a graph containing unique cycles $S_1, \ldots, S_n$. Suppose $\mathcal{J} = \{J_1, \ldots, J_{n+2}\}$ is an allowable collection of arcs in G, where for each $1 \leq i \leq n+2$, either J_i is a unique arc in G, or there exist $1 \leq k_i \leq n$ such that J_i overlaps S_{k_i}, and if K is a subarc of J_i that does not overlap S_{k_i}, then K is a unique arc in G. Then $\mathcal{J}$ has the* 3-*endpoint property.*

Proof. It is clear from the hypothesis that no J_i overlaps two members of $S_1, \ldots, S_n$. If two members of $\mathcal{J}$ are unique arcs, the result follows from Theorem 5. We consider two cases.

Case 1. Suppose exactly one member of $\mathcal{J}$, say J_1, is a unique arc in G. So, each member of $\mathcal{J}' = \{J_2, \ldots, J_{n+2}\}$ overlaps exactly one of $S_1, \ldots, S_n$. Since $\mathcal{J}'$ has $n+1$ members, it follows that there exist two members of $\mathcal{J}'$ that overlap the same S_i. We assume, without loss of generality, that J_2 and J_3 overlap S_1. It follows from Theorem 8 that $\{J_1, J_2, J_3\}$ has the 3-endpoint property in G. Hence, $\mathcal{J}$ has the 3-endpoint property in G.

Case 2. Suppose no member of $\mathcal{J}$ is a unique arc in G. If three members of $\mathcal{J}$ overlap the same S_i, then the result follows from Theorem 8. So, assuming otherwise, it follows that there exist four members of $\mathcal{J}$, say J_1, J_2, J_3, and J_4, where J_1 and J_2 overlap S_i for some $1 \leq i \leq n$, and J_3 and J_4 overlap S_k for some $k \neq i$. By Theorem 9, $\{J_1, J_2, J_3, J_4\}$ has the 3-endpoint property in G. Hence, $\mathcal{J}$ has the 3-endpoint property in G. □

5. Examples. The examples in this section have previously appeared in the literature. It may be helpful to the reader to have copies of the references for these examples. Most of the examples have been shown to be indecomposable by proving that they are arc continua that are not tree-like and not a simple closed curve, or not circle-like and not an arc, which gives additional useful information about the continua, but can be nontrivial to prove. As mentioned in the introduction, an arc continuum that is not an arc or a simple closed curve is indecomposable. We take another look at these examples to illustrate how easily indecomposability can be determined by looking at properties of the bonding mappings.

Since a weaker version of Theorem 5, for inverse limits on trees, appears in [**30**], examples of indecomposable tree-like continua that illustrate the use of Theorem 5 can be found in [**30**], see Remark 28, and Examples 40 and 41 in Section 6 of [**30**].

Example 2 (J.H. Case and R.E. Chamberlin [**10**]), (P.D. Roberson [**38**]). The Case-Chamberlin example is quite well-known. It is an inverse limit on the figure-eight graph with a single bonding mapping f. A definition of the bonding mapping is given in [**10,** Section 5, page 78].

Roberson defines an uncountable collection of Case-Chamberlin type continua with no model in [**38**]. Each of her examples is also an inverse limit on the figure-eight graph X, where each bonding mapping either is the Case-Chamberlin mapping f, or is a homeomorphism $r\colon X \to X$ composed with f. Roberson labels the two cycles in X, which are unit circles, by a and b, and provides a definition of f that nicely reveals the action of f on the two cycles, see [**38,** page 171]. We refer to this definition in the discussion that follows.

For $t \in [0,1]$, ta (tb) represents the point of cycle a (cycle b) that is $2\pi t$ along cycle a (cycle b) moving away from the branchpoint of the figure-eight. So, we easily see from the first line of the definition of f that the first quarter of cycle a is stretched by f onto cycle a, with the branchpoint being fixed.

We let $J_1 = \{ta \mid 0 \le t \le \frac{1}{2}\}$, $J_2 = \{ta \mid \frac{1}{2} \le t \le 1\}$, $J_3 = \{tb \mid \frac{3}{8} \le t \le \frac{5}{8}\}$, and $J_4 = \{tb \mid \frac{5}{8} \le t \le \frac{7}{8}\}$. Clearly, $\mathcal{J} = \{J_1, J_2, J_3, J_4\}$ is an allowable collection of arcs in X. Lines 1, 2, and 3 of the definition of f give that $f(J_1) = X$, lines 4 and 5 give that $f(J_2) = X$, lines 10, 11, and 12 give that $f(J_3) = X$, and lines 13 and 14 give that $f(J_4) = X$. So, f is a four-pass map for the collection $\mathcal{J}$. It follows, from Corollary 4 and Theorem 4, that the inverse limit space is indecomposable. Theorem 7 or Theorem 9 could also be applied in lieu of Corollary 4.

For the other examples in the Roberson collection, the bonding mappings are also four-pass maps on $\mathcal{J}$ since r is a homeomorphism on

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X. So, analogously, each example in the collection is indecomposable.

Example 3. (D. Sherling [**40**]) The example in [**40**] is an inverse limit on a graph with four unique cycles, and a single bonding mapping. The example X has the cone=hyperspace property, is not circle-like, is not weakly chainable, and admits a natural mapping onto Ingram's simple triod-like arc continuum with positive span. The factor space S is shown in Figure 2 with labeled edges and vertices. In Figure 2 on page 1035 of [**40**], Sherling gives a schematic indication of the bonding mapping g, which is sufficient for our purposes.

It is easy to check, using Figure 2 in [**40**], that a composition of g with itself, four times or less, maps each edge Z_i onto S. For example, one can check that $g^2(Z_1) = S = g^2(Z_2)$. Hence, g^4 is a four-pass map on the allowable collection of arcs $\mathcal{J} = \{Z_1, \ldots, Z_8\}$. Choosing any subcollection of $\mathcal{J}$ with six arcs, and applying Theorems 7 and 4, gives us that X is indecomposable.

In this example, however, the subcollection $\mathcal{J}' = \{Z_1, Z_2, Z_7, Z_8\}$ of $\mathcal{J}$ will suffice for the 3-endpoint property. Either observe that $\mathcal{J}'$ satisfies Theorem 9, or observe directly that the vertices B, J, and C satisfy the 3-endpoint property for $\mathcal{J}'$.

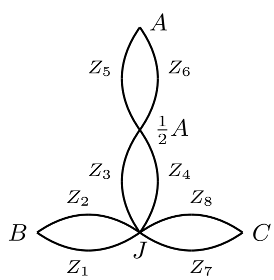

Figure 2. The graph S with four unique cycles and eight edges.

Example 4 (Ingram [**19**]). In Lemma 4.1 of [**19**], Ingram provides four methods of constructing an inverse limit on a graph with a single bonding mapping f, where f is a two-pass map on two non-overlapping arcs, but the inverse limit space is not indecomposable. It should be noted that, in each case where the factor space is a tree, the two arcs do not form an allowable collection of arcs. In the cases that involve a graph with a cycle, the two arcs may form an allowable collection of arcs, but there is no third arc to add to the collection on which f will be a three-pass map.

We wish to slightly modify an example of the type in Lemma 4.1(2) of **[19]** to get an indecomposable example that illustrates our results. Let G be the graph shown in Figure 3, with vertices p, q, and u. We define a new mapping $g\colon G \to G$ that only differs from Ingram's mapping f on the unique arc in G, which we label by γ. Ingram's mapping f is the identity mapping on γ. Subarcs α_1, α_2, β_1, β_2, and γ are labeled in Figure 1(a), and Figures 1(a) and 1(b) give schematic indications of how g maps the five subarcs into G. Our labeling is consistent with that of Ingram. We note that the endpoints of the four subarcs of the cycle in G are fixed by g. Let $\alpha = \alpha_1 \cup \alpha_2$, and $\beta = \beta_1 \cup \beta_2$.

Ingram notes that $\varprojlim\{G, f\}$ is decomposable, even though f is a two-pass map on α and β. Since f is the identity mapping on γ there is no third arc, not overlapping α or β, that is mapped onto G by a power of f.

We note, however, that g is also a two-pass map on α and β since $g = f$ on these two arcs, and $g^2(\gamma) = G$. Also, $\mathcal{J} = \{\alpha, \beta, \gamma\}$ is an allowable collection of three arcs in G. So, g^2 is a three-pass map on $\mathcal{J}$. It follows from Corollary 3 and Theorem 4 that $\varprojlim\{G, g\}$ is indecomposable.

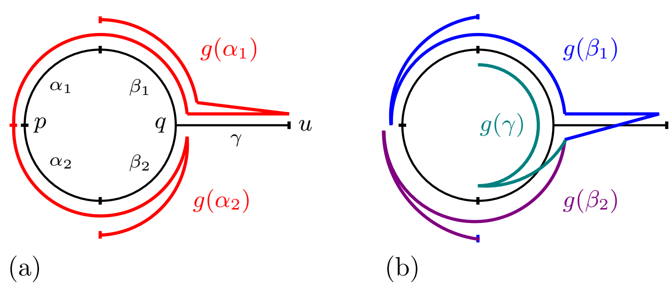

Figure 3. (a) A schematic indication of $g(\alpha)$.
(b) A schematic indication of $g(\beta)$ and $g(\gamma)$.

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