THE WEIGHTED L^p-BOUNDS FOR CARLESON TYPE MAXIMAL OPERATOR WITH KERNEL SATISFYING MILD REGULARITY

CHENYAN WANG, HUOXIONG WU*

Abstract. This paper is concerned with the Carleson type maximal operator $\mathcal{T}^*$ defined by

$$\mathcal{T}^* f(x) = \sup_{\lambda} \Big| \int_{\mathbb{R}^n} e^{iP_\lambda(y)} K(y) f(x-y)dy \Big|,$$

where $P_\lambda(y) = \sum_{2\le|\alpha|\le d} \lambda_\alpha y^\alpha$ is the polynomial in $\mathbb{R}^n$ with real coefficients $\lambda := (\lambda_\alpha)_{2\le|\alpha|\le d}$. Under the assumption of that the kernel function K satisfies an L^r-Hörmander condition with $1 < r \le \infty$, the authors show that $\mathcal{T}^*$ is bounded on the weighted Lebesgue spaces $L^p(\omega)$ for $r' < p < \infty$ and $\omega \in A_{p/r'}$, which improves and generalizes the previous results obtained by Stein and Wainger in [Math. Res. Lett. 8(2001), 789-800], Ding and Liu in [Proc. Amer. Math. Soc. 140(2012), 2739-2751].

1. Introduction

Let K be an appropriate Calderón-Zygmund kernel, $P_\lambda(x) = \sum_{1\le|\alpha|\le d} \lambda_\alpha x^\alpha$ be a polynomial in $\mathbb{R}^n$ with real coefficients $\lambda := (\lambda_\alpha)_{1\le|\alpha|\le d}$, where $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}_+^n$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Define

$$T_\lambda(f)(x) = \int_{\mathbb{R}^n} e^{iP_\lambda(y)} K(y) f(x-y)dy,$$

and the corresponding Carleson type maximal operator $\mathcal{T}^*$ by

$$\mathcal{T}^* f(x) = \sup_{\lambda} |T_\lambda(f)(x)|, \tag{1.1}$$

where the supremum is taken over all the real coefficients λ of P_λ.

It is well-known that $\mathcal{T}^*$ is the generalization of the classical Carleson maximal operators, which were studied successively in [1, 4, 5, 6, 7]. For the general operator $\mathcal{T}^*$, Stein and Wainger [8] established following result:

Theorem A. *Suppose that $P_\lambda(x) = \sum_{2\le|\alpha|\le d} \lambda_\alpha x^\alpha$ and K satisfies the following conditions:*
(1) *K is a tempered distribution and agrees with a C^1-function $K(x)$ for $x \ne 0$;*
(2) *$\widehat{K} \in L^\infty$;*
(3) *$|\partial_x^\gamma K(x)| \le A|x|^{-n-|\gamma|}$ for $0 \le |\gamma| \le 1$.*
Then $\|\mathcal{T}^*(f)\|_{L^p} \le \|f\|_{L^p}$ for $1 < p < \infty$.

Subsequently, Ding and Liu [2] extended the result above to the weighted L^p spaces with homogeneous kernels as follows.

Theorem B. *Suppose that $P_\lambda(x) = \sum_{2\le|\alpha|\le d} \lambda_\alpha x^\alpha$ and $K(x) = \Omega(x/|x|)|x|^{-n}$, where Ω is an integrable function of the unit sphere S^{n-1} with mean value zero. If Ω satisfies the*

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*Corresponding author.

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L^r-Dini condition for some $1 < r \le \infty$, then for $r' := r/(r-1) < p < \infty$ and $\omega \in A_{p/r'}$, there exists a constant $C > 0$ such that

$$\|\mathcal{T}^* f\|_{L^p(w)} \le C\|f\|_{L^p(w)}, \quad \forall f \in L^p(\omega).$$

Here A_q $(1 < q < \infty)$ denote the classical Muckenhoupt classes (see [3] for the properties of A_q), and Ω is said to satisfy an L^r-Dini condition if

$$\int_0^1 \frac{\omega_r(\delta)}{\delta} d\delta < \infty,$$

where $\omega_r(\delta)\,(0 < \delta \le 1)$ is called the integral continuous modulus of Ω of degree r, which is defined by

$$\omega_r(\delta) := \sup_{|\rho\| < \delta} \Big(\int_{S^{n-1}} |\Omega(\rho x') - \Omega(x')|^r d\sigma(x') \Big)^{1/r} \quad \text{for } 1 \le r < \infty,$$

and $\omega_\infty(\delta) := \sup_{\|\rho\| < \delta} |\Omega(\rho x') - \Omega(x')$, where ρ is a rotation in $\mathbb{R}^n$ and $\|\rho\| = \sup\{|\rho x' - x'| : x' \in S^{n-1}\}$, S^{n-1} denotes the unit sphere in $\mathbb{R}^n$ equipped with normalized Lebesgue measure $d\sigma = d\sigma(\cdot)$.

On the other hand, in studying the weighted L^p-estimates of singular integrals, Waston [10] considered the following L^r-Hörmander's conditions $\mathcal{H}_r$ $(1 \le r \le \infty)$:

Definition 1.1. *Let $1 \le r \le \infty$. We say that the kernel K satisfies the L^r-Hörmander condition (we write $K \in \mathcal{H}_r$), if there exist constant $c > 1$, $C_r > 0$ such that for any $y \in \mathbb{R}^n$ and $R > c|y|$,*

$$\sum_{k=1}^{\infty} (2^k R)^{n/r'} \Big(\int_{2^k R < |x| \le 2^{k+1} R} |K(x-y) - K(x)|^r dx \Big)^{1/r} \le C_r$$

for $r < \infty$, and

$$\sum_{k=1}^{\infty} (2^k R)^n \sup_{2^k R < |x| \le 2^{k+1} R} |K(x-y) - K(x)| \le C_\infty$$

for $r = \infty$.

In [10], Waston established the weighted L^p-boundedness of the Calderón-Zygmund operator associated with the kernel $K \in \mathcal{H}_r$ for $1 < r \le \infty$.

Remark 1.2. *It is easy to check that for $1 < r < s < \infty$,*

$$\mathcal{H}^*_\infty \subsetneq \mathcal{H}_\infty \subsetneq \mathcal{H}_s \subsetneq \mathcal{H}_r \subsetneq \mathcal{H}_1, \tag{1.2}$$

where $\mathcal{H}^*_\infty$ denotes the classical Lipschitz condition, and that $\mathcal{H}_1$ is the classical Hörmander condition. Moreover, Waston [10] showed that if Ω satisfies the L^r-Dini condition, then $K(x) = \Omega(x/|x|)|x|^{-n} \in \mathcal{H}_r$ for $1 \le r \le \infty$.

Based on the above, it is natural to consider the weighted L^p-estimates of $\mathcal{T}^*$ under the assumption of that $K \in \mathcal{H}_r$ for $1 < r \le \infty$. Our main result can be formulated as follows.

Theorem 1.3. *Suppose that $P_\lambda(x) = \sum_{2 \le |\alpha| \le d} \lambda_\alpha x^\alpha$, $1 < r \le \infty$, $K \in \mathcal{H}_r$ and satisfies*

$$|K(y)| \le C|x|^{-n}, \tag{1.3}$$

$$\widehat{K} \in L^\infty. \tag{1.4}$$

Then for $r' < p < \infty$ and $\omega \in A_{p/r'}$, the Carleson type maximal operator $\mathcal{T}^*$ is bounded on the weighted space $L^p(\omega)$, that is, there is a constant $C > 0$ such that for all $f \in L^p(\omega)$

$$\|\mathcal{T}^* f\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}. \tag{1.5}$$

Remark 1.4. *It follows from Remark 1.2 that Theorem 1.3 can be regarded as an essential improvement of Theorem A, even in the un-weighted case, and an extension to Theorem B.*

The rest of this paper is organized as follows. In Section 2 we will give some auxiliary lemmas. The proof of our main result will be given in Section 3. We remark that the main ideas in our arguments are taken from [2], which were based on the ideal of linearized maximal operators and TT^* method in [8].

2. Preliminaries

In this section, we recall some auxiliary facts and lemmas, which will be used in our arguments.

Lemma 2.1. ([8]) *For any set $E \in B_2$, with $B_2 = \{x : |x| \le 2\}$, we denote by χ_E its characteristic function, and $(\chi_E)_a(x) = a^{-n}\chi_E(a^{-1}x)$ for $a > 0$. We define*

$$\mathcal{M}_\varepsilon(f)(x) = \sup_{\substack{|E|\le\varepsilon \\ a>0}} |f| * (\chi_E)_a(x)$$

with the supremum taken over all subsets E of B_2 of measure $\le \epsilon$,and all $a > 0$. Then for $1 < p < \infty$, there exists constant $C > 0$, independent of ε, such that

$$\|\mathcal{M}_\varepsilon(f)\|_{L^p} \le \varepsilon^{1-1/p}\|f\|_{L^p}.$$

Lemma 2.2. ([8]) *Let $P(x) = \sum_{1\le|\alpha|\le d} \lambda_\alpha x^\alpha$ be a polynomial in $\mathbb{R}^n$ of degree $\le d$, with real coefficients and no constant term. Denote $|\lambda| = \sum_{1\le|\alpha|\le d} |\lambda_\alpha|$. Assume that φ is a given $C^{(1)}$ function defined in the unit ball, $B = \{x : |x| \le 1\}$, and let D be any convex subset of B. Then*

$$\Big|\int_D e^{iP(x)}\varphi(x)dx\Big| \le c|\lambda|^{-1/d} \sup_{x\in B}(|\varphi(x)| + |\nabla\varphi(x)|).$$

The constant c depends on the dimension n and the degree d, but not otherwise on P, φ, or D.

Lemma 2.3. ([8]) *With the same notation as in Lemma 2.2,*

$$|\{x \in B : |P(x)| \le \varepsilon\}| \le c\varepsilon^{1/d}|\lambda|^{-1/d}, \ \forall\, \varepsilon > 0.$$

Again, c does not depend on the coefficient of P, but only on n and d.

To prove Theorem 1.3, we will use the weighted estimates of the Calderón-Zygmund operator T_K associated to K, and the corresponding maximal truncated operator T_K^*, which are defined by

$$T_K f(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x)f(x-y)dy, \tag{2.1}$$

and

$$T_K^* f(x) = \sup_{\varepsilon>0} \Big|\int_{|y|\geqslant\varepsilon} K(y)f(x-y)dy\Big|. \tag{2.2}$$

Lemma 2.4. ([10]) *Let $K \in \mathcal{H}_r$, $1 < r \le \infty$, and $\widehat{K} \in L^\infty(\mathbb{R}^n)$. Then*

(i) *for $r' < p < \infty$ and $\omega \in A_{p/r'}$, there is a constant $C > 0$ such that*

$$\|T_K f\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}, \quad \forall\, f \in L^p(\omega);$$

(ii) *for $\omega^{r'} \in A_1$,*

$$\|T_K f\|_{L^{1,\infty}(\omega)} \le C\|f\|_{L^1(\omega)}, \quad \forall\, f \in L^1(\omega).$$

Lemma 2.5. *With the same nation and assumptions as in Lemma 2.4, we have*

$$\|T_K^* f\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}, \quad \forall\, f \in L^p(\omega).$$

Proof. There is a positive sequence $\{\varepsilon_j\}$ tend to zero, such that

$$\lim_{j\to\infty} T_{K,\varepsilon_j} f(z) = T_K(z) \quad a.e.\ z \in \mathbb{R}^n, \tag{2.3}$$

where $T_{K,\varepsilon_j}(z) = \int_{|z-y|>\varepsilon_j} K(z-y)f(y)dy$. Fix $\varepsilon > 0$, let $Q = B(x, \varepsilon/4)$ and $4Q = B(x, \varepsilon)$. Define $f_1 = f\chi_{4Q}$ and $f_2 = f - f_1$. Write

$$\begin{aligned} T_K f_2(x) &= \int_{|x-y|>\varepsilon} K(x-y)f(y)dy \\ &= \int_{|x-y|>\varepsilon} [K(x-y) - K(z-y)]f(y)dy \\ &\quad + \left\{ \int_{|x-y|>\varepsilon} K(z-y)f(y)dy - \int_{|z-y|>\varepsilon_j} K(z-y)f(y)dy \right\} \\ &\quad + \int_{|z-y|>\varepsilon_j} K(z-y)f(y)dy \\ &=: I_1 + I_2 + I_3, \end{aligned}$$

where $z \in Q$ and satisfies (2.3). Obviously,

$$\lim_{j\to\infty} I_3 = T_K f(z).$$

For I_2, when ε_j is small enough, we have $B(z, \varepsilon_j) \subset B(x, \varepsilon)$, thus

$$f(y)\chi_{\mathbb{R}^n\setminus B(x,\varepsilon)}(y) = f(y)\chi_{\mathbb{R}^n\setminus B(x,\varepsilon)}(y)\,\chi_{\mathbb{R}^n\setminus B(z,\varepsilon_j)}(y).$$

Thus I_2 can be written as

$$I_2 = -\int_{|z-y|>\varepsilon_j} K(z-y)f_1(y)dy,$$

and by (2.3), we obtain

$$\lim_{j\to\infty} I_2 = -T_K f_1(z).$$

By Hölder's inequality and $K \in \mathcal{H}_r$, we get

$$\begin{aligned}|I_1| =&\Big|\int_{|x-y|>\varepsilon}[K(x-y)-K(z-y)]f(y)dy\Big|\\ \le&\sum_{k=0}^{\infty}\int_{2^k\varepsilon<|x-y|<2^{k+1}\varepsilon}|K(x-y)-K(z-y)||f(y)|dy\\ \le&\sum_{k=0}^{k}\left(2^k\varepsilon\right)^n\left(\left(2^k\varepsilon\right)^{-n}\int_{2^k\varepsilon<|x-y|<2^{k+1}\varepsilon}|K(x-y)-K(z-y)|^r dy\right)^{1/r}\\ &\times\left(\left(2^k\varepsilon\right)^{-n}\int_{2^{k+1}4Q}|f(y)|^{r'}dy\right)^{1/r'}\\ \le&C_r M_{r'}f(x),\end{aligned}$$

where $M_\delta f(x) := (M(|f|^\delta)(x))^{1/\delta}$ for $\delta > 0$, the variant of Hardy-Littlewood maximal operator M.

Denote $T_K f_2(x)$ by $T_\varepsilon f(x)$ for simplicity. Summing up the estimates for I_1, I_2 and I_3, we obtained

$$|T_\varepsilon f(x)| \le C_r M_{r'}f(x) + |T_K f_1(z)| + |T_K f(z)| \;\; a.e. \;\; z \in B(x,\varepsilon/4).$$

If $T_\varepsilon f(x) = 0$ then there is nothing to prove. If not, fix λ such that $0 < \lambda < |T_\varepsilon f(x)|$ and let

$$\begin{aligned}Q_1 &= \{z \in Q : |T_K f(z)| > \lambda/3\},\\ Q_2 &= \{z \in Q : |T_K f_1(z)| > \lambda/3\},\end{aligned}$$

and

$$Q_3 = \begin{cases}\emptyset & \text{if } C_r M_{r'}f(x) \le \lambda/3,\\ Q & \text{if } C_r M_{r'}f(x) > \lambda/3.\end{cases}$$

Then $Q = Q_1 \cup Q_2 \cup Q_3$, so $|Q| \le |Q_1| + |Q_2| + |Q_3|$. However,

$$\left(\frac{\lambda}{3}\right)^\eta |Q_1| \le \int_{Q_1}|T_K f(z)|^\eta dz, \quad 0<\eta<1.$$

By the weak $(1,1)$ inequality for T_ε (see Lemma 2.4) and Kolmogorov's inequality,

$$\begin{aligned}\left(\frac{\lambda}{3}\right)^\eta |Q_2| &\le \int_{Q_2}|T_K f_1(z)|^\eta\, dz\\ &\le C\,|Q_2|^{1-\eta}\,\|f_1\|_1^\eta,\end{aligned}$$

by a simple calculation, we have

$$\lambda\,|Q_2| \le C\int_{4Q}|f(y)|dy.$$

As for Q_3, or equal to Q, or empty set, in every case we have that

$$|Q_3| \le 3C_r|Q|M_{r'}f(x)/\lambda.$$

Based on the above results, we obtained

$$\begin{aligned}\lambda &\le C\lambda^{1-\eta}\frac{1}{|Q|}\int_Q|T_K f(z)|^\eta dz + C\frac{1}{|Q|}\int_{4Q}|f(y)|dy + 3C_r M_{r'}f(x)\\ &\lesssim M_\eta(|T_K f|)(x) + Mf(x) + M_{r'}f(x).\end{aligned}$$

Let $\lambda \to |T_\varepsilon f(x)|$ and using the weighted L^p boundedness of M (see [3]), we deduce Lemma 2.5. □

3. Proof of Theorem 1.3

Note that

$$\mathcal{T}^*(f)(x) = \sup_{\lambda} |T_\lambda(f)(x)| \le \sup_{\lambda\neq 0} |T_\lambda(f)(x)| + |T_K(f)(x)|,$$

where T_K is as in (2.1). By Lemma 2.4, we may assume that the supremum is taken over all vectors $\lambda = (\lambda_\alpha)_{2\le|\alpha|\le d}$ with $|\lambda| > 0$, where

$$|\lambda| = \sum_{2\le|\alpha|\le d} |\lambda_\alpha|.$$

Let $P_\lambda(x) = \sum_{2\le|\alpha|\le d} \lambda_\alpha x^\alpha$ be the polynomial of degree d with coefficients $\lambda = (\lambda_\alpha)$. For $f \in L^p$ and $x \in \mathbb{R}^n$, denote by $\lambda(x)$ the nonzero vector $(\lambda_\alpha(x))_{2\le|\alpha|\le d}$ such that

$$|T_{\lambda(x)}(f)(x)| \ge \frac{1}{2} \sup_{\lambda} |T_\lambda(f)(x)|.$$

Therefore, to prove (1.5), it suffices to prove that, under the conditions of Theorem 1.3, there exists a constant C, independent of the choice of $\lambda(\cdot)$, such that

$$\|T_{\lambda(\cdot)}(f)\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}. \tag{3.1}$$

Employing the arguments in [2, 8], take a nonnegative function $\psi \in C_0^\infty(\mathbb{R}^n)$ such that $\mathrm{supp}(\psi) \subseteq \{1/4 < |y| \le 1\}$ and ψ satisfies

$$\sum_{j=-\infty}^{\infty} \psi_j(y) = 1 \quad \text{for } y \neq 0,$$

where $\psi_j(y) = \psi(2^{-j}y)$. Denote $N(\lambda) = \sum_{2\le|\alpha|\le d} |\lambda_\alpha|^{1/|\alpha|}$. Then $N(\lambda(x)) > 0$ for $x \in \mathbb{R}^n$, and

$$\sum_{j=-\infty}^{\infty} \psi_j(N(\lambda(x))y) = 1 \quad \text{for } y \neq 0.$$

For $x \in \mathbb{R}^n$, let $\psi_{j,\lambda}(y) = \psi_j(N(\lambda(x))y)$ for short. Write K as

$$K(y) = \sum_{j=0}^{\infty} K_j(y),$$

where $K_0(y) = \sum_{j=-\infty}^{0} \psi_{j,\lambda}(y)K(y)$ and $K_j(y) = \psi_{j,\lambda}(y)K(y)$ for $j \ge 1$. Hence,

$$\begin{aligned}|T_{\lambda(x)}(f)(x)| &\le \Big| \int_{\mathbb{R}^n} e^{iP_{\lambda(x)}(y)} K_0(y) f(x-y)dy \Big| \\ &\quad + \sum_{j=1}^{\infty} \Big| \int_{\mathbb{R}^n} e^{iP_{\lambda(x)}(y)} K_j(y) f(x-y)dy \Big| \\ &=: T^0_{\lambda(x)}(f)(x) + \sum_{j=1}^{\infty} T^j_{\lambda(x)}(f)(x).\end{aligned} \tag{3.2}$$

For $T^0_{\lambda(\cdot)}(f)(x)$, note that $\mathrm{supp}(K_0) \subseteq \{y \in \mathbb{R}^n : |y| \le \frac{1}{N(\lambda(x))}\}$ and $K_0(y) = K(y)$ for $|y| \le \frac{1}{2N(\lambda(x))}$, we have

$$\begin{aligned}|T^0_{\lambda(x)}(f)(x)| &\le \Big| \int_{|y|\le \frac{1}{2N(\lambda(x))}} e^{iP_{\lambda(x)}(y)} K(y) f(x-y)dy \Big| \\ &\quad + C \int_{\frac{1}{2N(\lambda(x))}\le |y| \le \frac{1}{N(\lambda(x))}} |K(y)||f(x-y)|dy \\ &\le \Big| \int_{|y|\le \frac{1}{2N(\lambda(x))}} e^{iP_{\lambda(x)}(y)} K(y) f(x-y)dy \Big| + CMf(x) \\ &=: I(x) + CMf(x),\end{aligned}$$

where M denotes the usual Hardy-Littlewood maximal operator. Since $N(\lambda(x))|y| \le 1$ and $|\lambda_\alpha(x))| \le N(\lambda(x))^{|\alpha|}$, we get

$$\Big| e^{iP_{\lambda(x)}(y)} - 1 \Big| \le C \sum_{2\le|\alpha|\le d} |\lambda_\alpha(x)||y^\alpha| \le C \sum_{2\le|\alpha|\le d} N(\lambda(x))^{|\alpha|}|y|^{|\alpha|} \le CN(\lambda(x))|y|.$$

Thus,

$$\begin{aligned}I(x) \le& \Big| \int_{|y|\le \frac{1}{2N(\lambda(x))}} (e^{iP_{\lambda(x)}(y)} - 1) K(y) f(x-y)dy \Big| \\ &+ \Big| \int_{|y|\le \frac{1}{2N(\lambda(x))}} K(y) f(x-y)dy \Big| \\ \le& CN(\lambda(x)) \int_{|y|\le \frac{1}{2N(\lambda(x))}} \frac{1}{|y|^{n-1}} |f(x-y)|dy \\ &+ \Big| p.v. \int_{\mathbb{R}^n} K(y) f(x-y)dy \Big| + \sup_{\epsilon>0} \Big| \int_{|y|\ge\epsilon} K(y) f(x-y)dy \Big| \\ \le& CM(f)(x) + |T_K(f)(x)| + T^*_K(f)(x).\end{aligned}$$

Consequently,

$$|T^0_{\lambda(x)}(f)(x)| \le |T_K(f)(x)| + T^*_K(f)(x) + CM(f)(x),$$

where C is independent of the choice of $\lambda(\cdot)$. By Lemmas 2.4 and 2.5 and the weighted L^p-boundedness of M, we know that, under the conditions of Theorem 1.3,

$$\|T^0_{\lambda(\cdot)}(f)\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}, \quad r' < p < \infty, \omega \in A_{p/r'}, \tag{3.3}$$

where the constant C is independent of $\lambda(\cdot)$.

Next, we will estimate $\|T^j_{\lambda(\cdot)}(f)\|_{L^p(\omega)}$ for each $j \in \mathbb{N}$. We take $\phi \in C^\infty_c(\mathbb{R}^n)$ such that $\mathrm{supp}(\phi) \subseteq \{y \in \mathbb{R}^n : |y| \le 2^{-5}\}$ and $\|\phi\|_{L^1} = 1$. For $t > 0$, denote $\phi_t(x) = t^{-n}\phi(x/t)$. For each $j \in \mathbb{N}$ and some $\sigma > 0$ small enough, which will be chosen later, let

$$L_{j,\lambda(x)}(y) := K_j * \phi_{\frac{2^{j(1-\sigma)}}{N(\lambda(x))}}(y) \text{ and } R_{j,\lambda(x)}(y) := K_j(y) - L_{j,\lambda}(y).$$

Define operators $\mathcal{L}^j_{\lambda(\cdot)}$ and $\mathcal{R}^j_{\lambda(\cdot)}$ by

$$\mathcal{L}^j_{\lambda(x)}(f)(x) := \int_{\mathbb{R}^n} e^{iP_{\lambda(x)}(y)} L_{j,\lambda(x)}(y) f(x-y)dy,$$

and

$$\mathcal{R}^j_{\lambda(x)}(f)(x) := \int_{\mathbb{R}^n} e^{iP_{\lambda(x)}(y)} R_{j,\lambda(x)}(y) f(x-y)dy,$$

respectively. Then

$$|T^j_{\lambda(x)}(f)(x)| \le |\mathcal{L}^j_{\lambda(x)}(f)(x)| + |\mathcal{R}^j_{\lambda(x)}(f)(x)|. \tag{3.4}$$

In what follows, we will estimate $|\mathcal{L}^j_{\lambda(x)}(f)(x)|$ and $|\mathcal{R}^j_{\lambda(x)}(f)(x)|$ for each $j \in \mathbb{N}$, respectively. It is obvious that $L_{j,\lambda}$ is smooth by its definition and

$$\operatorname{supp}(L_{j,\lambda}) \subseteq \Big\{ y \in \mathbb{R}^n : \frac{2^{j-3}}{N(\lambda(x))} \le |y| \le \frac{2^{j+1}}{N(\lambda(x))} \Big\}. \tag{3.5}$$

By a simple calculation, we have

$$|L_{j,\lambda}(y)| \le C(2^{-j} N(\lambda(x)))^n 2^{jn\sigma}, \tag{3.6}$$

where C is independent of j and $\lambda(\cdot)$. It follows from (3.5) and (3.6) that

$$\mathcal{L}^j_{\lambda(x)}(f)(x) \le C 2^{jn\sigma} M(f)(x). \tag{3.7}$$

Recall $\lambda(x) = (\lambda_\alpha(x))_{2\le|\alpha|\le d}$. For $j \in \mathbb{N}$, let

$$A_{j,\lambda} \circ \lambda = \Big(\Big(\frac{2^j}{N(\lambda(x))} \Big)^{|\alpha|} \lambda_\alpha(x) \Big)_{2\le|\alpha|\le d}$$

for convenience. Note that

$$\begin{aligned} P_{\lambda(x)}(y) &= \sum_{2\le|\alpha|\le d} \lambda_\alpha(x) y^\alpha = \sum_{2\le|\alpha|\le d} \lambda_\alpha(x) \Big(\frac{2^j}{N(\lambda(x))} \Big)^{|\alpha|} (2^{-j} N(\lambda(x)) y)^\alpha \\ &= P_{A_{j,\lambda}\circ\lambda}(2^{-j} N(\lambda(x)) y). \end{aligned} \tag{3.8}$$

Thus,

$$\mathcal{L}^j_{\lambda(x)}(f)(x) = \int_{\mathbb{R}^n} e^{iP_{A_{j,\lambda}\circ\lambda}(2^{-j}N(\lambda(x))y)} L_{j,\lambda(x)}(y) f(x-y)dy.$$

From now on, we denote $\mathcal{L}^j_{\lambda(x)}(f)(x)$ by $\mathcal{L}_j(f)(x)$ for simplicity. Note that for a $(\nu_\alpha)_{2\le|\alpha|\le d}$,

$$N(A_{j,\lambda} \circ \lambda) = \sum_{2\le|\alpha|\le d} \Big(\Big(\frac{2^j}{N(\nu)} \Big)^{|\alpha|} |\nu_\alpha| \Big)^{1/|\alpha|} = \frac{2^j}{N(\nu)} \sum_{2\le|\alpha|\le d} |\nu_\alpha|^{1/|\alpha|} = 2^j.$$

Since there exists a constant $c_0 > 0$ such that $N(\nu) \le c_0|\nu|$ for any vector ν satisfying $N(\nu) \ge 1$, we get

$$\Big| A_{j,\lambda} \circ \lambda \Big| \ge 2^j / c_0 \ \text{ for all } \ \lambda(x),\, x \in \mathbb{R}^n. \tag{3.9}$$

For $r \ge 2^j/c_0$, let

$$U_{j,r} = \{x : r \le |A_{j,\lambda} \circ \lambda| \le 2r\},$$

and

$$\mathcal{L}_{j,r}(f)(x) = \mathcal{L}_j(f)(x) \chi_{U_{j,r}}(x).$$

$\mathcal{L}^*_{j,r}$ denotes the adjoint operator of $\mathcal{L}_{j,r}$. It is easy to check that

$$\mathcal{L}^j_{\lambda(x)}(g)(x) = \int_{\mathbb{R}^n} e^{-iP_{\lambda(x)}(x-y)} L_{j,\lambda(x)}(x-y) g(x) \chi_{U_{j,r}}(x) dx,$$

and

$$(\mathcal{L}_{j,r}\mathcal{L}_{j,r}^*)(f)(x) = \int_{\mathbb{R}^n} \mathcal{K}(x,z)f(z)dz,$$

where

$$\begin{aligned}\mathcal{K}(x,z) &= \int_{\mathbb{R}^n} e^{iP_{\lambda(x)}(y)}e^{-iP_{\lambda(z)}(z-x+y)}L_{j,\lambda(x)}(y)L_{j,\lambda(z)}(z-x+y)dy\chi_{U_{j,r}}(x)\chi_{U_{j,r}}(z)\\ &= (e^{iP_{\lambda(x)}(\cdot)}L_{j,\lambda(x)}(\cdot)) * (e^{-iP_{\lambda(z)}(-\cdot)}L_{j,\lambda(z)}(-\cdot))(x-z)\chi_{U_{j,r}}(x)\chi_{U_{j,r}}(z).\end{aligned}$$

We claim that for $r \geq 2^j/c_0$ and fixed $x, z \in U_{j,r}$,

$$\begin{aligned}\Big|\mathcal{K}(x,z)\Big| \leq\ & C(2^{-j}N(\lambda(z)))^n 2^{2j\sigma(n+1)}[r^{-2\delta}\chi_{B_4}(2^{-j}N(\lambda(z))(x-z))\\ &\qquad + \chi_{E^j_{\lambda(z)}}(2^{-j}N(\lambda(z))(x-z))]\\ &+C(2^{-j}N(\lambda(x)))^n 2^{2j\sigma(n+1)}[r^{-2\delta}\chi_{B_4}(2^{-j}N(\lambda(x))(x-z))\\ &\qquad + \chi_{E^j_{\lambda(x)}}(2^{-j}N(\lambda(x))(x-z))].\end{aligned} \tag{3.10}$$

where the sets $E^j_{\lambda(x)}$, $E^j_{\lambda(z)} \subset B_4 = \{|y| \leq 4\}$ satisfy $|E^j_{\lambda(x)}|$, $|E^j_{\lambda(z)}| \leq r^{-4\delta}$ with $\delta = (6d)^{-1}$.

Accepting (3.10), we first show how to deduce the weighted boundedness of $\mathcal{L}_j(f)$ from (3.10). To do this, we need a variant of the Hardy-Littlewood maximal operator M and its L^p boundedness. Let $B_4 = \{x \in \mathbb{R}^n : |x| \leq 4\}$. For a measurable set $E \subset B_4$, denote by χ_E the characteristic function of E. For $\varepsilon > 0$, the maximal operator $\mathcal{M}_\varepsilon$ is defined by

$$\mathcal{M}_\varepsilon(f)(x) = \sup_{\substack{E\subset B_4,\,|E|\leq\varepsilon,\\ a>0}} |f| * (\chi_E)_a(x),$$

where $(\chi_E)_a(x) = a^{-n}\chi_E(x/a)$. Denote $\varepsilon = r^{-4\delta}$ and by (3.10), we have

$$\begin{aligned}&|(\mathcal{L}_{j,r}\mathcal{L}_{j,r}^*(f),\, g)|\\ &\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\mathcal{K}(x,z)||f(z)||g(x)|dzdx\\ &\leq Cr^{-2\delta}2^{2j\sigma(n+1)}\int_{\mathbb{R}^n}|f(z)|(2^{-j}N(\lambda(z)))^n\int_{|x-z|\leq\frac{4\cdot 2^j}{N(\lambda(z))}}|g(x)|dxdz\\ &\quad + C2^{2j\sigma(n+1)}\int_{\mathbb{R}^n}|f(z)|(2^{-j}N(\lambda(z)))^n\int_{\mathbb{R}^n}\chi_{E^j_{\lambda(z)}}(2^{-j}N(\lambda(z))(x-z))|g(x)|dxdz\\ &\quad + Cr^{-2\delta}2^{2j\sigma(n+1)}\int_{\mathbb{R}^n}|g(x)|(2^{-j}N(\lambda(x)))^n\int_{|x-z|\leq\frac{4\cdot 2^j}{N(\lambda(x))}}|f(z)|dzdx\\ &\quad + C2^{2j\sigma(n+1)}\int_{\mathbb{R}^n}|g(x)|(2^{-j}N(\lambda(x)))^n\int_{\mathbb{R}^n}\chi_{E^j_{\lambda(x)}}(2^{-j}N(\lambda(x))(x-z))|f(z)|dzdx\\ &\leq Cr^{-2\delta}2^{2j\sigma(n+1)}\int_{\mathbb{R}^n}|f(z)|M(g)(z)dz + 2^{2j\sigma(n+1)}\int_{\mathbb{R}^n}|f(z)|\mathcal{M}_\varepsilon(g)(z)dz\\ &\leq Cr^{-2\delta}2^{2j\sigma(n+1)}\int_{\mathbb{R}^n}|g(x)|M(f)(x)dx + 2^{2j\sigma(n+1)}\int_{\mathbb{R}^n}|g(x)|\mathcal{M}_\varepsilon(f)(x)dx.\end{aligned}$$

Using Hölder's inequality, the L^2 boundedness of M and Lemma 2.1, we get

$$|(\mathcal{L}_{j,r}\mathcal{L}_{j,r}^*(f),\, g)| \leq Cr^{-2\delta}2^{2j\sigma(n+1)}\|f\|_{L^2}\|g\|_{L^2}. \tag{3.11}$$

Note that

$$\Big\{x\in\mathbb{R}^n: |A_{j,\lambda}\circ\lambda|\ge \frac{2^j}{c_0}\Big\}=\bigcup_{k=0}^{\infty}\Big\{x: \frac{2^{j+k}}{c_0}\le |A_{j,\lambda}\circ\lambda|<\frac{2^{j+k+1}}{c_0}\Big\}.$$

Now we take $r=2^{j+k}/c_0$ successively for $k=0,1,\cdots$, and denote $\mathcal{L}_j^{(k)}:=\mathcal{L}_{j,r}$. Then $\mathcal{L}_j(f)(x)=\sum_{k=0}^{\infty}\mathcal{L}_j^{(k)}(f)(x)$ and by (3.11) we have

(3.12)

$$\|\mathcal{L}_j(f)\|_{L^2\to L^2}\le\sum_{k=0}^{\infty}\|\mathcal{L}_j^{(k)}(f)\|_{L^2\to L^2}\le C2^{2j\sigma(n+1)}\sum_{k=0}^{\infty}(2^{j+k}/c_0)^{-\delta}\le C2^{2j\sigma(n+1)}2^{-j\delta}.$$

By (3.7), we also have

$$\|\mathcal{L}_j(f)\|_{L^s}\le 2^{j\sigma n}\|f\|_{L^s}\ \text{ for }\ 1<s<\infty. \tag{3.13}$$

Using the Riesz-Thörin interpolation theorem between (3.12) and (3.13), we get

$$\|\mathcal{L}_j(f)\|_{L^p}\le 2^{j\sigma(n+1)}2^{-j\gamma(p)\delta}\|f\|_{L^p},\ r'<p<\infty, \tag{3.14}$$

where $\gamma(p)=\min\{2/p,2/p'\}$.

On the other hand, using (3.7) again and the weighted L^p boundedness of M for $1<p<\infty$ and $\omega\in A_p$, we have

$$\|\mathcal{L}_j(f)\|_{L^p(\omega)}\le 2^{j\sigma n}\|f\|_{L^p(\omega)}. \tag{3.15}$$

Since $\omega\in A_{p/r'}$, there is an $\varepsilon>0$ such that $\omega^{1+\varepsilon}\in A_{p/r'}\subset A_p$ (see [3]). Thus by (3.15), we have

$$\|\mathcal{L}_j(f)\|_{L^p(\omega^{1+\varepsilon})}\le 2^{j\sigma n}\|f\|_{L^p(\omega^{1+\varepsilon})}. \tag{3.16}$$

Applying the Stein-Weiss interpolation theorem with change of measure (see [9]) between (3.14) and (3.16), we get for $r'<p<\infty$ and $\omega\in A_{p/r'}$,

$$\|\mathcal{L}_j(f)\|_{L^p(\omega)}\le 2^{j\sigma(n+1)}2^{-j\gamma(p)\delta\frac{\varepsilon}{1+\varepsilon}}\|f\|_{L^p(\omega)}. \tag{3.17}$$

Hence, taking $0<\sigma<\frac{\gamma(p)\delta\varepsilon}{(n+1)(1+\varepsilon)}$, and letting $\theta=\gamma(p)\delta\frac{\varepsilon}{1+\varepsilon}-(n+1)\sigma$, we obtain

$$\|\mathcal{L}_j(f)\|_{L^p(\omega)}\le C2^{-j\theta}\|f\|_{L^p(\omega)}. \tag{3.18}$$

Now, we give the proof of (3.10). Define $\mathcal{F}_j^{\mu,\nu}$ by

$$\mathcal{F}_j^{\mu,\nu}(u)=(e^{iP_\nu(\cdot)}L_{j,\nu}(\cdot))*(e^{-iP_\mu(-\cdot)}L_{j,\mu}(-\cdot))(u),$$

where $\nu=(\nu_\alpha)_{2\le|\alpha|\le d}$ and $\mu=(\mu_\alpha)_{2\le|\alpha|\le d}$, to satisfy

$$r\le|A_{j,\nu}\circ\nu|,\ |A_{j,\mu}\circ\mu|<2r.$$

Let $h=\frac{N(\mu)}{N(\nu)}$. We assume that $h=\frac{N(\mu)}{N(\nu)}\le 1$. Hence, by (3.8) we have

(3.19)

$$\begin{aligned}&\mathcal{F}_j^{\mu,\nu}\Big(\frac{2^ju}{N(\mu)}\Big)\\&=\int_{\mathbb{R}^n}e^{i[P_{A_{j,\nu}\circ\nu}(y)-P_{A_{j,\mu}\circ\mu}(hy-u)]}L_{j,\nu}\Big(\frac{2^j}{N(\nu)}y\Big)L_{j,\mu}\Big(\frac{2^j}{N(\mu)}(hy-u)\Big)\Big(\frac{2^j}{N(\nu)}y\Big)^n dy.\end{aligned}$$

Below we give the estimate of $\mathcal{F}_j^{\mu,\nu}\Big(\frac{2^ju}{N(\mu)}\Big)$ by dividing into two cases: h is near the origin and h is away from the origin.

Case1. $0 < h \le \eta \ll 1$, where η will be chosen later. By (3.5), we get

$$|u| \le |hy - u| + h|y| \le 4.$$

By a trivial calculation, we have

$$P_{A_{j,\nu}\circ\nu}(y) - P_{A_{j,\mu}\circ\mu}(hy-u) = \sum_{2\le|\alpha|\le d} ((A_{j,\nu}\circ\nu)_\alpha + O(h|A_{j,\mu}\circ\mu|))y^\alpha \\ - P_{A_{j,\mu}\circ\mu}(-u).$$

Note that

$$r \le |A_{j,\nu}\circ\nu| = \sum_{2\le|\alpha|\le d} |(A_{j,\nu}\circ\nu)_\alpha| < 2r,$$

and $|A_{j,\mu}\circ\mu| < 2r$. Hence if η is chosen small enough, then

$$\sum_{2\le|\alpha|\le d} |((A_{j,\nu}\circ\nu)_\alpha + O(h|A_{j,\mu}\circ\mu|))| \ge \sum_{2\le|\alpha|\le d} |(A_{j,\nu}\circ\nu)_\alpha| + C\eta|A_{j,\mu}\circ\mu| \\ \ge C\sum_{2\le|\alpha|\le d} |(A_{j,\nu}\circ\nu)_\alpha| \ge Cr.$$

Thus, by Lemma 2.2 and (3.19), we have

$$\Big|\mathcal{F}_j^{\mu,\nu}\Big(\frac{2^j u}{N(\mu)}\Big)\Big| \le C(2^{-j}N(\mu))^n 2^{j\sigma(n+1)} r^{-1/d}\chi_{B_4}(u). \tag{3.20}$$

Case 2. $\eta < h \le 1$. By the assumption on polynomials in Theorem 1.3, there was no first-order term in y in $P_{A_{j,\nu}\circ\nu}(y)$. Thus, the first-order term in y in $P_{A_{j,\nu}\circ\nu}(y) - P_{A_{j,\mu}\circ\mu}(hy-u)$ is

$$-h\sum_{k=1}^n P^{(k)}_{A_{j,\mu}\circ\mu}(u)y_k,$$

where

$$P^{(k)}_{A_{j,\mu}\circ\mu}(u) = \sum_{2\le|\alpha|\le d} \alpha_k\Big(\frac{2^j}{N(\mu)}\Big)^{|\alpha|}\mu_\alpha u^{\alpha-e_k},$$

and $e_k = (0,\cdots,1,0,\cdots)$ with 1 in the k^{th} component. Applying Lemma 2.2 and (3.19) again, we get

$$\Big|\mathcal{F}_j^{\mu,\nu}\Big(\frac{2^j u}{N(\mu)}\Big)\Big| \le C(2^{-j}N(\mu))^n 2^{j\sigma(n+1)}\Big(\sum_{k=1}^n |P^{(k)}_{A_{j,\mu}\circ\mu}(u)|\Big)^{-1/d}\chi_{B_4}(u).$$

For $\rho > 0$, denote

$$E^j_\mu = \Big\{u\in B_4 : \sum_{k=1}^n |P^{(k)}_{A_{j,\mu}\circ\mu}(u)| \le \rho\Big\}.$$

Thus, for $u \in (E^j_\mu)^c$, it is obvious that

$$\Big|\mathcal{F}_j^{\mu,\nu}\Big(\frac{2^j u}{N(\mu)}\Big)\Big| \le C(2^{-j}N(\mu))^n 2^{2j\sigma(n+1)}\rho^{-1/d}\chi_{B_4}(u). \tag{3.21}$$

By Lemma 2.3, we obtain

$$|E^j_\mu| \le C_{n,d}\Big(\sum_{k=1}^n\sum_{2\le|\alpha|\le d}\alpha_k\Big(\frac{2^j}{N(\mu)}\Big)^{|\alpha|}|\mu_\alpha|\Big)^{-1/d}\rho^{1/d}.$$

Note that

$$\sum_{k=1}^{n}\sum_{2\le|\alpha|\le d}\alpha_k\Big(\frac{2^j}{N(\mu)}\Big)^{|\alpha|}|\mu_\alpha|\ge\sum_{2\le|\alpha|\le d}\alpha_k\Big(\frac{2^j}{N(\mu)}\Big)^{|\alpha|}|\mu_\alpha|=|A_{j,\mu}\circ\mu|\ge r.$$

Thus for $u\in E^j_\mu$, we have

$$\Big|\mathcal{F}^{\mu,\nu}_j\Big(\frac{2^ju}{N(\mu)}\Big)\Big|\le C(2^{-j}N(\mu))^n2^{2j\sigma(n+1)}\chi_{E^j_\mu}(u),$$

with $|E^j_\mu|\le C_{n,d}(\rho/r)^{1/d}$. We now take $\rho=(C_{n,d})^{-d}r^{1/3}$ and denote $\delta=1/6d$. Then, for $u\in E^j_\mu$, we have

$$\Big|\mathcal{F}^{\mu,\nu}_j\Big(\frac{2^ju}{N(\mu)}\Big)\Big|\le C(2^{-j}N(\mu))^n2^{2j\sigma(n+1)}\chi_{E^j_\mu}(u),\tag{3.22}$$

and $|E^j_\mu|\le r^{-4\delta}$. Note that $r\ge 2^j/c_0$. It follows from (3.20), (3.21) and (3.22) that

$$\Big|\mathcal{F}^{\mu,\nu}_j\Big(\frac{2^ju}{N(\mu)}\Big)\Big|\le C(2^{-j}N(\mu))^n2^{2j\sigma(n+1)}[r^{-2\delta}\chi_{B_4}(u)+\chi_{E^j_\mu}(u)].$$

Then, for μ and ν with

$$r\le|A_{j,\nu}\circ\nu|\le 2r,\ \ r\le|A_{j,\mu}\circ\mu|\le 2r\ \text{ and }\ h\le 1,$$

we have

$$|\mathcal{F}^{\mu,\nu}_j(u)|\le C(2^{-j}N(\mu))^n2^{2j\sigma(n+1)}[r^{-2\delta}\chi_{B_4}(2^{-j}N(\mu)u)+\chi_{E^j_\mu}(2^{-j}N(\mu)u)].\tag{3.23}$$

For fixed $x,z\in U_{j,r}$, let $\nu=\lambda(z)$, $\mu=x-z$. By the symmetry of μ and ν and by (3.23), we obtain (3.10).

Finally, we will estimate $R^j_{\lambda(x)}(f)$. Recall that

$$\mathcal{R}^j_{\lambda(x)}(f)(x)=\int_{\mathbb{R}^n}e^{iP_{\lambda(x)}(y)}R_{j,\lambda}(y)f(x-y)dy,$$

where $R_{j,\lambda}(y)=K_j(y)-L_{j,\lambda}(y)$. Thus,

$$R_{j,\lambda}(y)=\int_{\mathbb{R}^n}[K(y)\psi_{j,\lambda}(y)-K(y-z)\psi_{j,\lambda}(y-z)]\phi_{\frac{2^{j(1-\sigma)}}{N(\lambda(x))}}(z)dz.$$

By

$$\operatorname{supp}(R_{j,\lambda})\subseteq\Big\{y\in\mathbb{R}^n:\frac{2^{j-3}}{N(\lambda(x))}\le|y|\le\frac{2^{j+1}}{N(\lambda(x))}\Big\},$$

and $|z|\le\frac{2^{j(1-\sigma)-5}}{N(\lambda(x))}$, we have

$$\begin{aligned}|R_{j,\lambda}(y)|&\le\int_{\mathbb{R}^n}|\psi_{j,\lambda}(y-z)||K(y)-K(y-z)|\phi_{\frac{2^{j(1-\sigma)}}{N(\lambda(x))}}(z)dz\\&\quad+\int_{\mathbb{R}^n}|K(y)||\psi_{j,\lambda}(y)-\psi_{j,\lambda}(y-z)|\phi_{\frac{2^{j(1-\sigma)}}{N(\lambda(x))}}(z)dz\\&\le C\int_{\mathbb{R}^n}|K(y)-K(y-z)|\phi_{\frac{2^{j(1-\sigma)}}{N(\lambda(x))}}(z)dz\\&\quad+\frac{C}{|y|^n}\int_{\mathbb{R}^n}|2^{-j}N(\lambda)z|\phi_{\frac{2^{j(1-\sigma)}}{N(\lambda(x))}}(z)dz\end{aligned}$$

$$\leq C\int_{\mathbb{R}^n}|K(y)-K(y-z)|\phi_{\frac{2^{j(1-\sigma)}}{N(\lambda(x))}}(z)dz+C2^{-j\sigma}\frac{1}{|y|^n}.$$

Consequently,

$$|\mathcal{R}^j_{\lambda(x)}(f)(x)|\leq\int_{\mathbb{R}^n}\phi_{\frac{2^{j(1-\sigma)}}{N(\lambda(x))}}(z)\int_{\frac{2^{j-3}}{N(\lambda(x))}\leq|y|\leq\frac{2^{j+1}}{N(\lambda(x))}}|K(y)-K(y-z)||f(x-y)|dydz$$
$$+C2^{-j\sigma}M(f)(x).$$

Set $r_j=\frac{2^{j(1-\sigma)}}{N(\lambda(x))}$. Then we can write $|y|\approx 2^{j\sigma}r_j$ for $\frac{2^{j-3}}{N(\lambda(x))}\leq|y|\leq\frac{2^{j+1}}{N(\lambda(x))}$. Then by Hölder's inequality and $K\in\mathcal{H}_r$, the first term on the right side above is controlled by

$$\int_{\mathbb{R}^n}\phi_{\frac{2^{j(1-\sigma)}}{N(\lambda(x))}}(z)\Big(\int_{|y|\approx 2^{j\sigma}r_j}|K(y)-K(y-z)|^r dy\Big)^{1/r}\Big(\int_{|y|\approx 2^{j\sigma}r_j}|f(x-y)|^{r'}dy\Big)^{1/r'}dz$$
$$=\int_{\mathbb{R}^n}\phi_{\frac{2^{j(1-\sigma)}}{N(\lambda(x))}}(z)(2^{j\sigma}r_j)^{n/r'}\Big(\int_{|y|\approx 2^{j\sigma}r_j}|K(y)-K(y-z)|^r dy\Big)^{1/r}$$
$$\times\Big((2^{j\sigma}r_j)^{-n}\int_{|y|\approx 2^{j\sigma}r_j}|f(x-y)|^{r'}dy\Big)^{1/r'}dz$$
$$\leq C\int_{\mathbb{R}^n}\phi_{\frac{2^{j(1-\sigma)}}{N(\lambda(x))}}(z)(2^{j\sigma}r_j)^{n/r'}\Big(\int_{|y|\approx 2^j r_j}|K(y)-K(y-z)|^r dy\Big)^{1/r}dzM_{r'}(f)(x)$$
$$=C\int_{\mathbb{R}^n}\phi_{\frac{2^{j(1-\sigma)}}{N(\lambda(x))}}(z)2^{j(\sigma-1)n/r'}(2^j r_j)^{n/r'}\Big(\int_{|y|\approx 2^j r_j}|K(y)-K(y-z)|^r dy\Big)^{1/r}dzM_{r'}(f)(x)$$
$$\leq C2^{j(\sigma-1)n/r'}M_{r'}(f)(x).$$

Hence

$$|\mathcal{R}^j_{\lambda(x)}(f)(x)|\leq C[2^{-j\sigma}M(f)(x)+2^{j(\sigma-1)n/r'}M_{r'}(f)(x)],$$

where C is independent of the choice of $\lambda(\cdot)$. Since $p>r'$ and $\omega\in A_{p/r'}$, applying the weighted L^p boundedness of M again, we have

$$\|\mathcal{R}^j_{\lambda(\cdot)}(f)\|_{L^p(\omega)}\leq C(2^{-j\sigma}+2^{j(\sigma-1)n/r'})\|f\|_{L^p(\omega)}.$$

This, combing (3.4) with (3.18), implies that

$$\Big\|\sum_{j=1}^{\infty}T^j_{\lambda(\cdot)}(f)\Big\|_{L^p(\omega)}\leq\sum_{j=1}^{\infty}\|T^j_{\lambda(\cdot)}(f)\|_{L^p(\omega)}\leq C\|f\|_{L^p(\omega)},$$

which, together with (3.2) and (3.3), leads to (3.1) and completes the proof of Theorem 1.3.

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School of Mathematical Sciences, Xiamen University, Xiamen 361005, China
E-mail address: chenyanwangxmu@163.com

School of Mathematical Sciences, Xiamen University, Xiamen 361005, China
E-mail address: huoxwu@xmu.edu.cn