ON SOLUTION TO THE TIME-FRACTIONAL NAVIER-STOKES EQUATIONS WITH DAMPING

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Abstract: In this paper, we deal with the time-fractional Navier-Stokes equations with damping in a bounded domain Ω in $\mathbb{R}^3$. First, we establish the existence of weak solutions by Galerkin approximation for $\beta \geq 1$. We have also shown the uniqueness of weak solution for $\beta \geq 4$. Further, we prove the regularity of the solution for $\beta \geq 3$ and $4\beta\mu > 1$.

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1. Introduction

Let us consider the following system of time-fractional Navier-Stokes equations with sufficiently smooth boundary in a simply connected bounded domain Ω of $\mathbb{R}^3$

$$\partial_t^\alpha \mathbf{u} + \vartheta|\mathbf{u}|^{\beta-1}\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \mu\Delta\mathbf{u} = \mathbf{g} \qquad \text{in } \Omega\times(0,T), \tag{1.1}$$

$$\text{div } \mathbf{u} = 0 \qquad \text{in } \Omega\times(0,T), \tag{1.2}$$

$$\mathbf{u} = \mathbf{u}_0 \qquad \text{in } \Omega\times\{0\}, \tag{1.3}$$

$$\mathbf{u} = \mathbf{0} \qquad \text{on } \partial\Omega\times(0,T), \tag{1.4}$$

where ∂_t^α is the Caputo fractional derivative of order $\alpha \in (0,1)$. Here the unknown function u corresponds to the velocity of the flow and p is used to denote the pressure. $\vartheta|\mathbf{u}|^{\beta-1}\mathbf{u}$ is the damping term and $\vartheta > 0$ and $\beta \geq 1$ are the scalars appeared in the expression. Here $\mathbf{u}_0$ is the initial velocity, g represents the external force.

The Navier-Stokes equations describe the motion of the fluid flows ranging from lubrication of ball bearings to large-scale atmospheric motions and reflect the conservation of mass as well as momentum. The integer-order Navier-Stokes equations have been addressed in detailed by many authors [3, 4, 6, 10, 14, 15, 17, 16, 24] due to their essential role in turbulence problems and fluid mechanics over the last decades. On the flip side, Fractional calculus has gained much attention due to its demonstrated applications to model many vital phenomena in different fields of science and engineering, including mechanics, mathematical biology, control theory of dynamical systems, and many others. Fractional calculus [11, 12] has been known as one of the most useful tools to model anomalous diffusion and to describe the long memory processes. In comparison to the literature of integer-order Navier-Stokes equations, research on fractional Navier-Stokes equations is still in the early stages of development. Recently, some significant development has been done by Salem *et al.* [8], Ganji *et al.* [9], and Zaid [19] in direction of time-fractional Navier-Stokes equations. Zhou and He in [32] studied the well-posedness and regularity

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of mild solutions for a class of time fractional damped wave equations. In [31], Zhou *et al.* studied a backward problem for an inhomogeneous fractional diffusion equation in a bounded domain. The existence of solutions for nonlinear Rayleigh-Stokes problem for a generalized second grade fluid with Riemann-Liouville fractional derivative has been proved by Zhou *et al.* [33]. In [30] Zhou *et al.* established the existence and regularity of weak solutions to the time-fractional Navier-Stokes equations. The existence and uniqueness of mild solutions to the time-fractional Navier-Stokes equation has been proved by de Carvalho-Neto *et al.* [7]. The system (1.1) represents the flow with the obstruction to the motion. (1.1) is the adaptation of the classical time-fractional Navier-Stokes equation with the regularizing term $\vartheta|u|^{\beta-1}u$. Many authors [5, 20, 21, 22, 25, 27] studied the system (1.1)–(1.4) with $\alpha = 1$. Cai in [5] proves that, if $\beta \geq 1$, then weak solutions of the damped Navier-Stokes equations exists; if $\beta \geq 7/2$, then a strong solution ensured. Further, they restrict $5 \geq \beta \geq 7/2$ to show the uniqueness of the strong solution. Later, Zhang *et al.* [25] improved the result and showed that a global strong solution exist when $\beta > 3$ and this strong solution becomes unique when $5 \geq \beta > 3$.

To show the existence of the solutions to the time-fractional Navier-Stokes equations with damping, we will use the Galerkin approximation method. It is worth mentioning that, in comparison to the integer case, the solutions of the time-fractional Navier-Stokes equation with damping are not yet fully explored. So, it is vital to investigate the solutions to time-fractional Navier-Stokes equations. To best of our knowledge, the existence, uniqueness and regularity of solutions to the the time-fractional Navier-Stokes equation with damping has not been researched. We organize the article in the following way. The assumptions and preliminaries are discussed in Section 2. In Section 3, we discuss the existence of weak solutions of (1.1). In Section 4, we discuss the uniqueness of solutions. Also, We have addressed the regularity of the solutions in Section 5.

2. Preliminaries and Assumptions

In this section we accumulate the solution spaces, Lemmas and notations. We refer to Zhou [28] and Temam [23] for further details. Let Ω be a simply connected bounded domain in $\mathbb{R}^3$ having smooth boundary. Let X be a Banach space. Let $\alpha \in (0, 1]$. Let $w : \mathbb{R} \to X$. Let us define the Caputo fractional derivative and the Liouville-Weyl integral on real axis by

$$_{-\infty}^{\;\;C}D_t{}^{\alpha} w(t) = \int_{-\infty}^{t} h_{1-\alpha}(t-s)\frac{d}{ds}w(s)ds, \qquad {}_{-\infty}I_t{}^{\alpha} w(t) = \int_{-\infty}^{t} h_{\alpha}(t-s)w(s)ds$$

respectively, where h_α represents the Riemann-Liouville kernel and $h_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$. Let us define left and right Riemann-Liouville integrals of w as follows

$${}_0I_t{}^{\alpha} w(t) = \int_0^t h_\alpha(t-s)w(s)ds, \qquad {}_tI_T{}^{\alpha} w(t) = \int_t^T h_\alpha(s-t)w(s)ds,$$

where $w : [0, T] \to X$. Further, we define the left Caputo and right Riemann-Liouville fractional derivatives of order α by

$${}_0^CD_t{}^{\alpha} w(t) = \int_0^t h_{1-\alpha}(t-s)\frac{d}{ds}w(s)ds \qquad {}_tD_T{}^{\alpha} w(t) = -\frac{d}{dt}\int_t^T h_{1-\alpha}(s-t)w(s)ds.$$

Let $v : \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}^3$, the left Caputo fractional derivative with respect to time of the function v is denoted by

$$\partial_t^\alpha v(x,t) = \int_0^t h_{1-\alpha}(t-s)\frac{\delta}{\delta_s}v(x,s)ds,\ t>0.$$

We will use the following fractional integral by parts formula [1]

$$\int_0^T (\partial_t^\alpha v(t),\psi(t))dt = (v(t), {}_tI_T{}^{1-\alpha}\psi(t))|_0^T + \int_0^T (v(t), {}_tD_T{}^\alpha \psi(t))dt.$$

If $\psi \in C_0^\infty([0,T],X)$, then $\lim_{t\to T} {}_tI_T{}^{1-\alpha}\psi(t) = 0$ and

$$\int_0^T (\partial_t^\alpha v(t),\psi(t))dt = -(v(0), {}_0I_T{}^{1-\alpha}\psi(t)) + \int_0^T (v(t), {}_tD_T{}^\alpha \psi(t))dt.$$

We recall the following Lemmas from [30, Lemma 2.3, Lemma 2.4].

Lemma 2.1. *Let X is a real Hilbert space and $w : [0,T] \to X$ have derivative, then $(w(t), \partial_t^\alpha w(t)) \geq \frac{1}{2}\partial_t^\alpha |w(t)|^2$.*

Lemma 2.2. *Suppose the function $w(t) \geq 0$ satisfies $\partial_t^\alpha w(t) + k_1 w(t) \leq k_2(t)$ for almost all $t \in [0,T]$, where $k_1 > 0$ and $k_2(t)$ is nonnegative and integrable function in $[0,T]$. Then*

$$w(t) \leq w(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}k_2(s)ds.$$

Let $L^p(\Omega)$ and $H^r(\Omega)$ be the Lebesgue space and the Sobolev space respectively. Let $Z = \{u \in (C_0^\infty(\Omega))^3, \text{div } u = 0\}$. H is the closure of Z in $L^2(\Omega)^3$ and V is the closure of Z in $H^1(\Omega)^3$. $(\cdot,\cdot)$ represents inner product in L^2 and

$$((u,v)) = \sum_{j=1}^3 \int_\Omega \frac{\partial u_j}{\partial x_i}\frac{\partial v_j}{\partial x_i}.$$

Let P be the orthogonal projection from $(L^2(\Omega))^3$ to H. We define the Stokes operator A by $A = -P\Delta$, with $D(A) = H^2(\Omega) \cap V$. Then

$$(Au_1,u_2) = ((u_1,u_2)) \qquad \forall u_1,u_2 \in D(A).$$

We define the trilinear form b as

$$b(u_1,u_2,u_3) = \int_\Omega (u_1\cdot\nabla u_2)\cdot u_3,\ \forall u_1,u_2,u_3 \in V. \tag{2.5}$$

For $u_1,u_2 \in V$, we define $B(u_1,u_2)$ by

$$(B(u_1,u_2),u_3) = b(u_1,u_2,u_3),\ \forall u_3 \in V,$$

and we set $B(u_1) = B(u_1,u_1) \in V'$, $\forall u_1 \in V$. So,

$$(Bu_1,u_2) = \int_\Omega (u_1\cdot\nabla u_1)\cdot u_2. \tag{2.6}$$

We recall the following Lemma from Temam [23].

Lemma 2.3. *If $u \in L^2(0,T;V)$, then $Bu \in L^1(0,T;V')$. Further, we have $\|Bu\|_{V'} \leq C\|u\|_{H^1}^2$.*

We recall the following Lemma from Cai[5].

Lemma 2.4. *Let $0 < \gamma \leq 1$ and Y_0, Y be Hilbert space having compact imbedding $Y_0 \hookrightarrow Y$. If $(z_j)_{j=1}^{\infty}$ be a sequence in $L^2(R, Y_0)$ and fulfilling the following condition*

$$\sup_j \left(\int_{-\infty}^{+\infty} \|z_j\|_{Y_0}^2 dt \right) < \infty, \ \sup_j \left(\int_{-\infty}^{+\infty} |\eta|^{2\gamma} \|\hat{v}_j\|_Y^2 d\eta \right) < \infty,$$

where

$$\hat{z_j}(\eta) = \int_{-\infty}^{\infty} z_j(t) \exp(-2\pi i \eta t) dt$$

represent the Fourier transformation on the time variable of $z_j(t)$. Then there is a subsequence of $(z_j)_{j=1}$ which strongly converge to some $z \in L^2(R; Y)$.

3. Existence Theorem

In the following section, we will established the existence of weak solutions to the time-fractional damped Navier-Stokes system. The idea of the proof is inspired from [23]. First, we give the definition of weak solutions for (1.1)–(1.4).

Definition 3.1. *If $\boldsymbol{u} \in L^\infty(0, T; H) \cap L^2(0, T; V) \cap L^{\beta+1}(0, T; L^{\beta+1}(\Omega))$ and for any $\boldsymbol{z} \in V$, $\boldsymbol{u}$ satisfies*

$$(\partial_t^\alpha \boldsymbol{u}, \boldsymbol{z}) + (\vartheta |\boldsymbol{u}|^{\beta-1} \boldsymbol{u}, \boldsymbol{z}) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{z}) + \mu((\boldsymbol{u}, \boldsymbol{z})) = (\boldsymbol{g}, \boldsymbol{z}). \tag{3.7}$$

Then, we called $\boldsymbol{u}$ is a weak solutions of the time fractional Navier-Stokes equations.

Theorem 3.2. *Let $\boldsymbol{u}_0 \in H$, $\boldsymbol{g} \in L^{\frac{2}{\alpha_1}}(0, T; H)$ where $\alpha_1 \in (0, \alpha)$ and, $\beta \geq 1$. Then for positive T, there exists a weak solutions **u(x,t)** of (1.1)–(1.4) where $\boldsymbol{u} \in L^\infty(0, T; H) \cap L^2(0, T; V) \cap L^{\beta+1}(0, T; L^{\beta+1}(\Omega))$ and **u(x,t)** will fulfill (3.7).*

Proof. The Galerkin approximations will be used to prove the theorem. We choose a sequence of elements $\mathbf{z}_1, \mathbf{z}_2, ..., \mathbf{z}_m$ from Z, where $Z = \{\mathbf{u} \in (C_0^\infty(\Omega))^3, \text{div } \mathbf{u} = 0\}$. Since Z is separable and dense in V, elements $\mathbf{z}_i$ are free and total in V. We consider the approximate of the solution $\mathbf{u}_m$ by $\mathbf{u}_m(t) = \sum_{i=1}^m f_{im}(t) \mathbf{z}_i, m > 0$. We put $\mathbf{u}_m$ in (3.7), we obtain

$$\begin{aligned}(\partial_t^\alpha \mathbf{u}_m(t), z_j) + (\vartheta |\mathbf{u}_m|^{\beta-1} \mathbf{u}_m(t), \mathbf{z}_j) + b(\mathbf{u}_m(t), \mathbf{u}_m(t), \mathbf{z}_j) + \mu((\mathbf{u}_m(t), \mathbf{z}_j)) \\ = (\mathbf{g}(t), \mathbf{z}_j),\end{aligned} \tag{3.8}$$

$$\mathbf{u}_m(0) = \sum_{j=1}^m (\mathbf{u}_0, \mathbf{z}_j) \mathbf{z}_j = \mathbf{u}_{0m}, \quad j = 1, ..., m. \tag{3.9}$$

We can write (3.8) –(3.9) as follows

$$\begin{aligned}\sum_{i=1}^m (\mathbf{z}_i, \mathbf{z}_j) \partial_t^\alpha f_{im}(t) + \sum_{i=1}^m \vartheta |\mathbf{u}_m|^{\beta-1} (\mathbf{z}_i, \mathbf{z}_j) f_{im}(t) + \sum_{i,l=1}^m b(\mathbf{z}_i, \mathbf{z}_l, \mathbf{z}_j) f_{im}(t) f_{lm}(t) \\ + \mu \sum_{i=1}^m ((\mathbf{z}_i, \mathbf{z}_j)) f_{im}(t) = (\mathbf{g}(t), \mathbf{z}_j),\end{aligned} \tag{3.10}$$

$$\mathbf{u}_m(0) = \sum_{j=1}^m (\mathbf{u}_0, \mathbf{z}_j) \mathbf{z}_j = \mathbf{u}_{0m} \qquad j = 1, ..., m. \tag{3.11}$$

By using Picard's Theorem we can conclude that nonlinear differential equations (3.10)–(3.11) has a maximal solutions in $[0, T_m]$ contained in $[0, T]$. The priori estimate for the approximate solutions u_m has been deduced in the following way. Multiplying (3.8) by $f_{jm}(t)$ and adding all for $j = 1, ..., m$. We get

$$\frac{1}{2}\partial_t^\alpha \|\mathbf{u}_m(t)\|^2_{L^2} + \vartheta\|\mathbf{u}_m\|^{\beta+1}_{L^{\beta+1}} + \mu\|\mathbf{u}_m(t)\|^2_{H^1} \le |(\mathbf{g}(t), \mathbf{u}_m(t))| \le \frac{\mu}{2}\|\mathbf{u}_m(t)\|^2_{H^1} + \frac{1}{2\mu}\|\mathbf{g}(t)\|^2_{V'}. \quad (3.12)$$

After integrating (3.12) and using Young inequality, we get

$$\begin{aligned}\|\mathbf{u}_m(t)\|^2_{L^2} &+ 2\vartheta\int_0^t (t-s)^{\alpha-1}\|\mathbf{u}_m\|^{\beta+1}_{L^{\beta+1}}ds + \mu\int_0^t (t-s)^{\alpha-1}\|\mathbf{u}_m\|^2_{H^1}ds \\ &\le \|\mathbf{u}_0\|^2_{L^2} + \frac{1}{\mu}\int_0^t (t-s)^{\alpha-1}\|\mathbf{g}\|^2_H \\ &\le \|\mathbf{u}_0\|^2_{L^2} + \frac{1}{\mu}\int_0^t \|\mathbf{g}\|^{\frac{2}{\alpha_1}}_H ds + \frac{1}{\mu}\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_1}}ds \\ &\le \|\mathbf{u}_0\|^2_{L^2} + \frac{1}{\mu}\int_0^T \|\mathbf{g}\|^{\frac{2}{\alpha_1}}_H ds + \frac{T^{1+b}}{\mu(1+b)} \qquad (3.13)\end{aligned}$$

where $\alpha_1 \in (0, \alpha)$, $b = \frac{\alpha-\alpha_1}{1-\alpha_1}$. Also,

$$\begin{aligned}T^{\alpha-1}\int_0^t \|\mathbf{u}_m\|^2_{H^1}ds &\le \int_0^t (t-s)^{\alpha-1}\|\mathbf{u}_m\|^2_{H^1}ds \\ &\le \|\mathbf{u}_0\|^2_{L^2} + \frac{1}{\mu}\int_0^T \|\mathbf{g}\|^{\frac{2}{\alpha_1}}_H ds + \frac{T^{1+b}}{\mu(1+b)}. \qquad (3.14)\end{aligned}$$

From (3.13) and (3.14), we achieve that the sequence $\{\mathbf{u}_m\}$ in a bounded set $L^\infty(0, T; H)$ $\cap\, L^2(0, T; V) \cap L^{\beta+1}(0, T; L^{\beta+1}(\Omega))$. Now, we will show the strong convergence of $\mathbf{u}_m$ in $L^2 \cap L^\beta([0, T] \times \Omega)$ by using Lemma 2.4. Let $\tilde{\mathbf{u}}_m$ the function from $\mathbb{R}$ to V, which is equal to $\mathbf{u}_m$ on $[0, T]$ and to 0 otherwise. In the same way, we construct $\tilde{f}_{im}(t)$, $f_{im}(t)$ to $\mathbb{R}$ by assigning $\tilde{f}_{im}(t) = f_{im}$ on $[0, T]$ and 0 otherwise. We use $\hat{\tilde{\mathbf{u}}}_m$ and $\hat{\tilde{\mathbf{g}}}_{im}$ to represent the Fourier transformations on time variable of $\tilde{\mathbf{u}}_m$ and $\tilde{\mathbf{g}}_{im}$. We will prove for any positive constant C

$$\int_{-\infty}^{+\infty} |\eta|^{2\gamma}\|\hat{\tilde{\mathbf{u}}}_m(\eta)\|^2_{L^2}d\eta \le C, \text{ for some } \gamma > 0 \qquad (3.15)$$

We have for $j = 1, ..., m$,

$$\begin{aligned}(\partial_t^\alpha \tilde{\mathbf{u}}_m(t), \mathbf{z}_j) &= (\mathbf{u}_{0m}, \mathbf{z}_j)\, {}_{-\infty}{I_t}^{1-\alpha}\delta_0 - (\mathbf{u}_m(T), \mathbf{z}_j)\, {}_{-\infty}{I_t}^{1-\alpha}\delta_T + (\tilde{\mathbf{g}}_m(t), \mathbf{z}_j) \\ &\quad + (\vartheta|\tilde{\mathbf{u}}_m|^{\beta-1}\tilde{\mathbf{u}}_m(t), \mathbf{z}_j) \qquad (3.16)\end{aligned}$$

where δ_0 is the Dirac distributions at 0 and δ_T is the Dirac distributions at T. Let $\mathbf{g}_m = \mathbf{g} - \mu A\mathbf{u}_m - B\mathbf{u}_m$, $\hat{\mathbf{g}}_m = \mathbf{g}_m$ on $[0, T]$, 0 otherwise. Note that $\tilde{\mathbf{u}}_m$ has two discontinues at T and 0.

$$\begin{aligned}{}_{-\infty}^{\ \ c}{D_t}^\alpha \tilde{\mathbf{u}}_m &= {}_{-\infty}{I_t}^{1-\alpha}\left(\frac{d}{dt}\tilde{\mathbf{u}}_m\right) \\ &= {}_{-\infty}{I_t}^{1-\alpha}\left(\frac{d}{dt}\mathbf{u}_m + \mathbf{u}_m(0)\delta_0 - \mathbf{u}_m(T)\delta_T\right) \\ &= {}_0^c{D_t}^\alpha \mathbf{u}_m + {}_{-\infty}{I_t}^{1-\alpha}(\mathbf{u}_m(0)\delta_0 - \mathbf{u}_m(T)\delta_T).\end{aligned}$$

From (3.16), we get

$$(2\pi i\eta)^{\alpha}(\hat{\tilde{\mathbf{u}}}_m(\eta),\mathbf{z}_j) = (\hat{\tilde{\mathbf{g}}}_m(\eta),\mathbf{z}_j) + \vartheta(|\tilde{\mathbf{u}}_m|^{\hat{\beta-1}}\tilde{\mathbf{u}}_m(\eta),\mathbf{z}_j) + (\mathbf{u}_{0m},\mathbf{z}_j)(2\pi i\eta)^{\alpha-1} - (\mathbf{u}_m(T),\mathbf{z}_j)(2\pi i\eta)^{\alpha-1}\exp(-2\pi i T\eta), \quad (3.17)$$

where Fourier transformation have been applied on the time variable. We multiply (3.17) by $\hat{\tilde{\mathbf{g}}}_{jm}(\eta)$ and adding all the equations, we get

$$(2\pi i\eta)^{\alpha}\|\hat{\tilde{\mathbf{u}}}_m(\eta)\|_{L^2}^2 = (\hat{\tilde{\mathbf{g}}}_m(\eta),\hat{\tilde{\mathbf{u}}}_m(\eta)) + \vartheta(|\tilde{\mathbf{u}}_m|^{\hat{\beta-1}}\tilde{\mathbf{u}}_m(\eta),\hat{\tilde{\mathbf{u}}}_m(\eta)) + (\mathbf{u}_{0m},\hat{\tilde{\mathbf{u}}}_m(\eta))(2\pi i\eta)^{\alpha-1} + (\mathbf{u}_m(T),\hat{\tilde{\mathbf{u}}}_m(\eta))(2\pi i\eta)^{\alpha-1}\exp(-2\pi i T\eta). \quad (3.18)$$

When $\mathbf{z}\in L^{\beta+1}(0,T;L^{\beta+1})\cap L^2(0,T;V)$ we have

$$(\mathbf{g}_m(t),\mathbf{z}) \le C\Big(c_1\|\mathbf{u}_m(t)\|_{H^1}^2 + \mu\|\mathbf{u}_m(t)\|_{H^1} + \|\mathbf{g}(t)\|_{V'}\Big)\|\mathbf{z}\|_{H^1}.$$

So for any positive T,

$$\begin{aligned}\sup_{\eta\in\mathbb{R}}\|\hat{\tilde{\mathbf{g}}}_m(\eta)\|_{V'} &\le \int_0^T \|\mathbf{g}_m(t)\|_{V'}dt\\ &\le \int_0^T C\Big(d_1\|\mathbf{u}_m(t)\|_{H^1}^2 + \mu\|\mathbf{u}_m(t)\|_{H^1} + \|\mathbf{g}(t)\|_{V'}\Big)\\ &\le C.(\text{ Where } d_1 \text{ is arbitrary constant}) \qquad (3.19)\end{aligned}$$

From (3.13), we get

$$\begin{aligned}&\int_0^T \||\mathbf{u}_m|^{\beta-1}\mathbf{u}_m(t)\|_{\frac{\beta+1}{\beta}}dt \le \int_0^T \|\mathbf{u}_m(t)\|_{\beta+1}^{\beta}dt \le C.\\ &\Rightarrow \sup_{\eta\in\mathbb{R}}\||\mathbf{u}_m|^{\hat{\beta-1}}\mathbf{u}(\eta)\|_{\frac{\beta+1}{\beta}} \le C. \qquad (3.20)\end{aligned}$$

Further from (3.13), we get

$$\|\mathbf{u}_m(T)\|_{L^2} \le C_1,\ \|\mathbf{u}_m(0)\|_{L^2} \le C_1. \qquad (3.21)$$

From (3.18)–(3.21), we have

$$|\eta|^{\alpha}\|\hat{\tilde{\mathbf{u}}}_m(\eta)\|_{L^2}^2 \le C_2\|\hat{\tilde{\mathbf{u}}}_m(\eta)\|_{H^1} + C_3|\eta|^{\alpha-1}\|\hat{\tilde{\mathbf{u}}}_m(\eta)\|_{\beta+1}.$$

Note that when $0<\gamma<\frac{\alpha}{4}$, we have

$$|\eta|^{2\gamma} \le C_4(\gamma)\frac{1+|\eta|^{\alpha}}{1+|\eta|^{\alpha-2\gamma}}, \forall\eta\in\mathbb{R}.$$

Thus

$$\begin{aligned}\int_{-\infty}^{+\infty}|\eta|^{2\gamma}\|\hat{\tilde{\mathbf{u}}}_m(\eta)\|_{L^2}^2d\eta &\le C_4(\gamma)\int_{-\infty}^{+\infty}\frac{1+|\eta|^{\alpha}}{1+|\eta|^{\alpha-2\gamma}}\|\hat{\tilde{\mathbf{u}}}_m(\eta)\|_{L^2}^2d\eta\\ &\le C_5(\gamma)\int_{-\infty}^{+\infty}\|\hat{\tilde{\mathbf{u}}}_m(\eta)\|_{L^2}^2d\eta + C_6(\gamma)\int_{-\infty}^{+\infty}\frac{\|\hat{\tilde{\mathbf{u}}}_m(\eta)\|_{H^1}}{1+|\eta|^{\alpha-2\gamma}}d\eta\\ &+ C_7(\gamma)\int_{-\infty}^{+\infty}\frac{|\eta|^{\alpha-1}\|\hat{\tilde{\mathbf{u}}}_m(\eta)\|_{\beta+1}}{1+|\eta|^{\alpha-2\gamma}}d\eta. \qquad (3.22)\end{aligned}$$

Using the Parseval equality and (3.13), the first integral on the right hand side of (3.22) is bounded whenever $m \to \infty$. By the Parseval equality, Schwartz inequality, and (3.13), we get

$$\int_{-\infty}^{+\infty} \frac{\|\hat{\tilde{\mathbf{u}}}_m(\eta)\|_{H^1}}{1+|\eta|^{\alpha-2\gamma}} d\eta \le \left(\int_{-\infty}^{+\infty} \frac{d\eta}{(1+|\eta|^{\alpha-2\gamma})^2} \right)^{\frac{1}{2}} \left(\int_{-\infty}^{+\infty} \|\mathbf{u}_m(\eta)\|_{H^1}^2 d\eta \right)^{\frac{1}{2}} \le C_8 \tag{3.23}$$

Similarly, for $0 < \gamma < \frac{\alpha}{2(\beta+1)}$, we have

$$\begin{aligned} &\int_{-\infty}^{+\infty} \frac{|\eta|^{\alpha-1}\|\hat{\tilde{\mathbf{u}}}_m(\eta)\|_{\beta+1}}{1+|\eta|^{\alpha-2\gamma}} d\eta \\ &\le \left(\int_{-\infty}^{+\infty} \frac{d\eta}{(1+|\eta|^{\alpha-2\gamma})^{\frac{\beta+1}{\beta}}} \right)^{\frac{\beta}{\beta+1}} \left(\int_{-\infty}^{+\infty} |\eta|^{(\alpha-1)(\beta+1)} \|\hat{\tilde{\mathbf{u}}}_m(\eta)\|_{\beta+1}^{\beta+1} d\eta \right)^{\frac{1}{\beta+1}} \\ &\le C \left(\frac{T^{1-\alpha}}{\Gamma(2-\alpha)} \right)^{\beta+1} \int_0^T \|\hat{\tilde{\mathbf{u}}}_m(t)\|_{\beta+1}^{\beta+1}, \end{aligned} \tag{3.24}$$

since

$$\begin{aligned} \int_{-\infty}^{+\infty} |\eta|^{(\alpha-1)(\beta+1)} \|\hat{\tilde{\mathbf{u}}}_m(\eta)\|_{\beta+1}^{\beta+1} d\eta &= \int_{-\infty}^{+\infty} \left(\| {}_{-\infty}I_t{}^{1-\alpha} \hat{\tilde{\mathbf{u}}}_m(t)\|_{\beta+1} \right)^{\beta+1} dt \\ &= \int_0^T \| {}_0I_t{}^{1-\alpha} \hat{\tilde{\mathbf{u}}}_m(t)\|_{\beta+1}^{\beta+1} dt \\ &\le \left(\frac{T^{1-\alpha}}{\Gamma(2-\alpha)} \right)^{\beta+1} \int_0^T \|\hat{\tilde{\mathbf{u}}}_m(t)\|_{\beta+1}^{\beta+1} dt. \end{aligned}$$

From (3.22), we can conclude that

$$\int_{-\infty}^{+\infty} |\eta|^{2\gamma} \|\hat{\tilde{\mathbf{u}}}_m(\eta)\|_{L^2}^2 d\eta \le C.$$

Using (3.13), we can deduce that $\mathbf{u} \in L^\infty(0,T;H) \cap L^2(0,T;V) \cap L^{\beta+1}(0,T;L^{\beta+1}(\Omega))$ and there is a subsequence $\{\mathbf{u}_m\}_{m=1}^\infty$, such that $\mathbf{u}_m \to \mathbf{u}$ in weak-$*$ topology in $L^\infty(0,T;H)$, $\mathbf{u}_m \to \mathbf{u}$ weakly in $L^2(0,T;V)$, and $\mathbf{u}_m \to \mathbf{u}$ weakly in $L^{\beta+1}(0,T;L^{\beta+1}(\Omega))$. Further, we take $\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \cdots$ having smooth boundary and fulfilling $\cup_{i=1}^\infty \Omega_i = \Omega$. We consider $Y_0 = V$, $Y = L^2(\Omega_i)$ in Lemma 2.4 for fixed $i = 1, 2, ...,$. From (3.13), (3.15), and Lemma 2.4 we get that $\mathbf{u}_m$ converges to u strongly in $L^2(0,T;L^2(\Omega_i))$. For $2 \le p < \beta+1$ we have $\mathbf{u}_{m_j}$ converges to $\mathbf{u}$ strongly in $L^p(0,T;L^p_{\text{loc}}(\Omega))$, Since $\int_0^T \int_\Omega |\mathbf{u}_m|^{\beta+1} dx\, dt \le C$. We integrating (3.8) (with order α) and we get,

$$\begin{aligned} (\mathbf{u}_m(t), \mathbf{z}_j) &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big((\vartheta|\mathbf{u}_m|^{\beta-1}\mathbf{u}_m(t), \mathbf{z}_j) + b(\mathbf{u}_m(t), \mathbf{u}_m(t), \mathbf{z}_j) \\ &+ \mu((\mathbf{u}_m(t), \mathbf{z}_j)) - (\mathbf{g}(t), \mathbf{z}_j) \Big) ds = (\mathbf{u}_{0m}, \mathbf{z}_j) \qquad j = 1, ...m. \end{aligned} \tag{3.25}$$

Due to Lebesgue's dominated convergence theorem and using (3.13) , we get

$$(\mathbf{u}(t),\mathbf{z})+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\Big((\vartheta|\mathbf{u}|^{\beta-1}\mathbf{u}(t),\mathbf{z})+b(\mathbf{u}(t),\mathbf{u}(t),\mathbf{z})$$
$$+\mu((\mathbf{u}(t),\mathbf{z}))-(\mathbf{g}(t),\mathbf{z})\Big)ds=(\mathbf{u}_0,\mathbf{z})$$

holds for $\mathbf{z}=\mathbf{z}_1,\mathbf{z}_2,\cdots$. So, we get the weak solution **u(x,t)** for the time fractional damped Navier-Stokes system. ☐

4. Uniqueness Theorem

Theorem 4.1. *Let $\boldsymbol{u}_0\in H$, $\boldsymbol{g}\in L^{\frac{2}{\alpha_1}}(0,T;H)$ and, $\beta\geq 4$. Then for any positive T, there exists a unique weak solutions **u(x,t)** of (1.1)–(1.4) satisfying*

$$\boldsymbol{u}\in L^{\infty}(0,T;H)\cap L^2(0,T;V)\cap L^{\beta+1}(0,T;L^{\beta+1}(\Omega)).$$

Proof. Let us choose $\mathbf{u}$ and $\mathbf{v}$ are two solutions, and consider $\mathbf{y}(t)=\mathbf{u}(t)-\mathbf{v}(t)$ and it satisfies

$$\partial_t^{\alpha}\mathbf{y}+\mu A\mathbf{y}+\vartheta|\mathbf{u}|^{\beta-1}\mathbf{u}-\vartheta|\mathbf{v}|^{\beta-1}\mathbf{v}=-B\mathbf{u}+B\mathbf{v}, \tag{4.26}$$
$$\mathbf{y}(0)=\mathbf{0}.$$

Multiplying (4.26) with $\mathbf{y}(t)$, we get

$$\frac{1}{2}\partial_t^{\alpha}\|\mathbf{y}(t)\|_{L^2}^2+\mu\|\mathbf{y}(t)\|_{H^1}^2+\int_{\Omega}\Big(|\mathbf{u}|^{\beta-1}\mathbf{u}(t)-|\mathbf{v}|^{\beta-1}\mathbf{v}(t)\Big)(\mathbf{u}(t)-\mathbf{v}(t))dx$$
$$\leq b(\mathbf{v}(t),\mathbf{v}(t),\mathbf{y}(t))-b(\mathbf{u}(t),\mathbf{u}(t),\mathbf{y}(t)). \tag{4.27}$$

Note that

$$\int_{\Omega}\Big(|\mathbf{u}|^{\beta-1}\mathbf{u}(t)-|\mathbf{v}|^{\beta-1}\mathbf{v}(t)\Big)(\mathbf{u}(t)-\mathbf{v}(t))dx$$
$$\geq\int_{\Omega}\Big(|\mathbf{u}|^{\beta+1}-|\mathbf{v}|^{\beta}|\mathbf{u}|-|\mathbf{u}|^{\beta}|\mathbf{v}|+|\mathbf{v}|^{\beta+1}\Big)dx$$
$$=\int_{\Omega}(|\mathbf{u}|^{\beta}-|\mathbf{v}|^{\beta})(|\mathbf{u}|-|\mathbf{v}|)dx\geq 0\quad\forall\mathbf{u},\mathbf{v}\in L^{\beta+1}(\Omega).$$

We have

$$\begin{aligned}\frac{1}{2}\partial_t^{\alpha}\|\mathbf{y}(t)\|_{L^2}^2+\mu\|\mathbf{y}(t)\|_{H^1}^2&\leq b(\mathbf{v}(t),\mathbf{v}(t),\mathbf{y}(t))-b(\mathbf{u}(t),\mathbf{u}(t),\mathbf{y}(t))\\&=b(\mathbf{y},\mathbf{y},\mathbf{v})\\&\leq\|\mathbf{v}\|_{L^5}\|\mathbf{y}\|_{H^1}\|\mathbf{y}\|_{L^2}^{\frac{2}{5}}\|\mathbf{y}\|_{L^6}^{\frac{3}{5}}\\&\leq C\|\mathbf{v}\|_{L^5}\|\mathbf{y}\|_{H^1}^{\frac{8}{5}}\|\mathbf{y}\|_{L^2}^{\frac{2}{5}}\\&\leq\mu\|\mathbf{y}(t)\|_{H^1}^2+C\|\mathbf{v}\|_{L^5}^5\|\mathbf{y}\|_{L^2}^2\end{aligned}$$

which implies

$$\frac{1}{2}\partial_t^{\alpha}\|\mathbf{y}(t)\|_{L^2}^2\leq C\|\mathbf{v}\|_{L^5}^5\|\mathbf{y}\|_{L^2}^2. \tag{4.28}$$

Integrating (4.28) between the limit 0 to t, we get

$$\|\mathbf{y}(t)\|_{L^2}^2\leq\|\mathbf{y}(0)\|_{L^2}^2+\frac{C}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\|\mathbf{v}\|_{L^5}^5\|\mathbf{y}\|_{L^2}^2. \tag{4.29}$$

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Since $\mathbf{y} \in L^5(0,T;L^5(\Omega))$, and applying Gronwall's inequality we can conclude that $\|\mathbf{y}(t)\| = 0$. Thus $\mathbf{u}(t) = \mathbf{v}(t)$.

□

5. Regularity Theorem

Theorem 5.1. *Suppose $\boldsymbol{u}_0 \in V$, $\boldsymbol{g} \in L^{\frac{2}{\alpha_1}}(0,T;L^2)$, $\beta \geq 3$, and $4\beta\mu > 1$, then*

$$\boldsymbol{u} \in L^\infty(0,T;V) \cap L^2(0,T;D(A)).$$

Proof. First, we take the inner product of (1.1) with $A\mathbf{u}$ and integrating by parts we get,

$$\partial_t^\alpha \|\mathbf{u}\|_{H^1}^2 + 2\mu\|A\mathbf{u}\|_{L^2}^2 + 2(\mathbf{u}\cdot\nabla\mathbf{u}, A\mathbf{u}) + 2(\nabla(|\mathbf{u}|^{\beta-1}\mathbf{u}), \nabla\mathbf{u}) \leq 2(\mathbf{g}, A\mathbf{u}). \tag{5.30}$$

For any $\kappa_1, \kappa_2 > 0$, we can deduce that

$$\begin{aligned}\partial_t^\alpha \|\mathbf{u}\|_{H^1}^2 + 2\mu\|A\mathbf{u}\|_{L^2}^2 + 2\beta\||\mathbf{u}|^{\frac{\beta-1}{2}}\nabla\mathbf{u}\|_{L^2}^2 &\leq \kappa_1\|A\mathbf{u}\|_{L^2}^2 + \frac{1}{\kappa_1}\|\mathbf{g}\|_{L^2}^2 + 2|(\mathbf{u}\cdot\nabla\mathbf{u}, A\mathbf{u})| \\ &\leq (\kappa_1+\kappa_2)\|A\mathbf{u}\|_{L^2}^2 + \frac{1}{\kappa_1}\|\mathbf{g}\|_{L^2}^2 + \frac{1}{\kappa_2}\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2}^2. \end{aligned} \tag{5.31}$$

We have the following estimates from [27]

$$\begin{aligned}\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2}^2 &= \int |\nabla\mathbf{u}|^2(|\mathbf{u}|^{\beta-1}+1)\frac{|\mathbf{u}|^2}{|\mathbf{u}|^{\beta-1}+1}dx \\ &\leq \|\frac{|\mathbf{u}|^2}{|\mathbf{u}|^{\beta-1}+1}\|_{L^\infty}\left(\|\mathbf{u}\|_{H^1}^2 + \||\mathbf{u}|^{\frac{\beta-1}{2}}\nabla\mathbf{u}\|_{L^2}^2\right) \\ &\leq \|\mathbf{u}\|_{H^1}^2 + \||\mathbf{u}|^{\frac{\beta-1}{2}}\nabla\mathbf{u}\|_{L^2}^2 \text{ for } \beta \geq 3. \end{aligned} \tag{5.32}$$

Now we put (5.32) in (5.31) and choosing $\kappa_1, \kappa_2 > 0$ such that $2\beta > \frac{1}{\kappa_2}$, $\kappa_1 + \kappa_2 < 2\mu$ hold provided $4\beta\mu > 1$, then we get

$$\begin{aligned}\partial_t^\alpha \|\mathbf{u}\|_{H^1}^2 + (2\mu - \kappa_1 - \kappa_2)\|A\mathbf{u}\|_{L^2}^2 + (2\beta - \frac{1}{\kappa_2})\||\mathbf{u}|^{\frac{\beta-1}{2}}\nabla\mathbf{u}\|_{L^2}^2 \\ \leq C(\|\mathbf{g}\|_{L^2}^2 + \|\mathbf{u}\|_{H^1}^2). \end{aligned} \tag{5.33}$$

Integrating (5.33) (with order α) and applying Gronwall's inequality, we complete the proof.

□

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