SOLVABILITY OF NONLINEAR FRACTIONAL LANE-EMDEN TYPE DELAY EQUATIONS WITH TIME-SINGULAR COEFFICIENTS

NGUYEN MINH DIEN

Abstract. We focus on the nonlinear Lane-Emden type delay problems connected with g-Caputo fractional derivatives. Here, we investigate the problems under the assumption that the source functions have time-singular coefficients. Via the Bielecki type norm, we derive the existence and uniqueness of mild solutions for the problem. In a special case, by subdividing the interval of time, we obtain a unique mild solution for the problem under a weaker condition than the previous one. Besides, we propose and discuss a new Ulam-Hyers type stability for the main equation. Meanwhile, a new inequality is established to prove the main results of the paper. Some examples are provided to illustrate the theoretical results.

1. Introduction

1.1. Statement of the problem. To state the problem, let us set up some notations. For $a, b \in \mathbb{R}$ with $a < b$, we denote

$$Mc_+^1 = \{g : g \in C[a,b] \text{ and } g'(t) > 0 \text{ on } (a,b)\}.$$

Let $0 < \alpha, \beta \leq 1$, and let ℓ, η be real numbers with $\ell < a < b$. Let $\varrho \in C([a,b],[\ell,b])$ such that $\varrho(t) \leq t$, $\varphi \in C([\ell,a],\mathbb{R})$, and let $\vartheta \in C(Mc_+^1 \times (a,b],\mathbb{R})$ and $\lim_{t\to a^+} \vartheta(g,t) = \infty$ (singular at $t = a$). We examine nonlinear fractional Lane-Emden type delay equation as follows

$$ {}^C D_{a+}^{\beta,g} \left({}^C D_{a+}^{\alpha,g} + \vartheta(g,t) \right) u(t) = f(t, g, u(t), u(\rho(t))), \quad t \in (a,b] \tag{1.1}$$

subject to the initial conditions

$$u(t) = \varphi(t), \ \ell \leq t \leq a, \ \lim_{t\to a^+} \left({}^C D_{a+}^{\alpha,g} + \vartheta(g,t) \right) u(t) = \eta, \tag{1.2}$$

where ${}^C D_{a+}^{\alpha,g}(\cdot)$, ${}^C D_{a+}^{\beta,g}(\cdot)$ are g-Caputo fractional derivatives. It is worth noting that if $\vartheta(g,t)$ is a continuous function on $[a,b]$, the equation (1.1) become Langevin equation and then condition

$$\lim_{t\to a^+} \left({}^C D_{a+}^{\alpha,g} + \vartheta(g,t) \right) u(t) = \eta$$

can be replaced with condition $D_{a+}^{\alpha,g} u(a) = \eta$.

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1.2. **Relevant works and motivations.** The classical Lane-Emden equation has the following form

$$x''(t) + \frac{a}{t}x'(t) = f(x,t),\ t \in (0,1]$$

subject to the initial conditions

$$x(0) = A, \quad x'(0) = B,$$

where A, B are constants, and f is a continuous function. The problem has attracted the great attention of many scientists because it can be used to describe various phenomena in physics and astrophysics, such as the thermal history of a spherical cloud of gas, isothermal gas spheres and thermionic currents, and aspects of stellar structure and so on (see [5]). One of the most difficult things in solving Lane-Emden problems is the appearance of the singularity at $t = 0$. However, there are some suggested methods to deal with them. For instance, to find exact solutions of these types of problems, one may use the variational iteration methods as in [27]. To solve numerical solutions, one can use the methods as in [1, 13, 18] and some references therein.

Fractional Lane-Emden problems, in recent years, have been studied by many authors. Indeed, in [14], the authors obtained the existence and uniqueness results for a nonlinear singular integro-differential equation of Lane-Emden type with nonlocal multi-point integral conditions. Tablennehas et al [24, 25] studied some types of Ulam stability for a nonlinear fractional differential equation of Lane-Emden type with anti-periodic conditions. Gouari and Dahmani [15] considered a system of Lane-Emden type equations involves Caputo derivatives and Riemann-Liouville integral, obtained some existence and uniqueness results. In [3, 23], the authors investigated the existence and uniqueness as well as the Ulam-Hyers stability for some systems of Lane-Emden type equations. Sabir et al [21] considered numerical solutions for a fractional order pantograph differential model of the Lane-Emden type. Gupta and Kumar [16] investigated numerical methods for a variable-order fractional differential equation of nonlinear Lane-Emden type appearing in astrophysics. Very recently, Rufai and Ramos [19, 20] proposed one-step hybrid block techniques and a variable stepsize formulation of a pair of block methods to solve Lane-Emden-Fowler type equations.

Fractional differential equations with time-singular coefficients have recently been considered in many works. In fact, Webb [26] looked at some differential equations with Riemann-Liouville where the sources are singular, derived existence and uniqueness results. Recently, we considered differential equations with Hilfer fractional derivative [8], Langevin equations with generalized Caputo fractional derivatives [7, 9], and differential equations with sequential Hilfer fractional derivatives [12]. In these works, we discussed many aspects such as Lyapunov inequality, existence and uniqueness, and stability of solutions to problems where the source functions have singularities. These types of sources were also considered for fractional parabolic equations [10, 11]. However, fractional differential time-delay equations with time-singular coefficients are still not studied.

Motivated from the problems having time-singular coefficients in [7, 9, 26], the concept of generalized Caputo fractional derivative [2], and the above analyses, we investigate the problem (1.1)-(1.2) and make some new features as follows: (i) prove the existence and uniqueness of mild solutions of the problem (ii) relax the condition to problem possesses a mild solution uniquely (iii) propose and discuss a new type of Ulam-Hyers stability. It should be emphasized that our problem is more difficult to solve than the usual cases because the source has a singularity.

1.3. **Organization of the paper.** The structure of the remainder of the paper is as follows. In Section 2, we introduce concepts of fractional integral and fractional derivative of a function with respect to an appropriate function. Besides, some auxiliary lemmas are introduced. Section 3 presents the results of the existence and uniqueness of mild solutions to the problem. A new type of Ulam-Hyes stability is also proven. In Section 4, we provide some examples to show the applicability of the obtained results. Conclusions of the paper are given in section 5.

2. Definitions and Lemmas

We present the definitions of the Gamma and Beta functions, recalling the definitions of fractional integral and fractional derivative of a function with respect to another function. The definition of new Ulam-Hyers type stability and some preliminary lemmas are also introduced in this section.

We first recall the definitions of the Gamma and Beta functions

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}\,\mathrm{d}t, \quad B(\alpha,\beta) = \int_0^1 (1-t)^{\alpha-1}t^{\beta-1}\,\mathrm{d}t, \quad \alpha,\beta > 0.$$

We list here two identities related to the Gamma and Beta functions

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \quad B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \text{ for all } \alpha,\beta > 0.$$

Definition 2.1 (see [2, 17])**.** *Let $\alpha > 0$, $a, b \in \mathbb{R}$ with $a < b$, and $g \in Mc^1_+$.*

(i). The fractional integral of a function $f \in L^1(a,b)$ with respect to the function g is defined as follows

$$I^{\alpha,g}_{a+}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t g'(\tau)(g(t)-g(\tau))^{\alpha-1}f(\tau)\,\mathrm{d}\tau.$$

(ii). The Caputo fractional derivative of a function $f \in C^n[a,b]$ with respect to the function g is defined as follows

$${}^CD^{\alpha,g}_{a+}f(t) = I^{n-\alpha,g}_{a+}\left(\frac{1}{g'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n f(t),$$

where $n = [\alpha]+1$ for $n \neq \mathbb{N}$ and $n = \alpha$ for $\alpha \in \mathbb{N}$.

Next, we introduce some properties of the Caputo derivative that we will use in the subsequent sections of this paper.

Lemma 2.2 (see [2])**.** *Let $\alpha, \beta > 0$, $n = [\alpha]+1 \in \mathbb{N}$, and $g \in Mc^1_+$.*

(i). ${}^CD^{\alpha,g}_{a+}(g(t)-g(a))^p = \frac{p!}{\Gamma(p+1-\alpha)}(g(t)-g(a))^{p-\alpha}$ for all $p \in \mathbb{N}$ and $p \geq \alpha$.

(ii). $I^{\alpha,g}_{a+}I^{\beta,g}_{a+}f(t) = I^{\alpha+\beta,g}_{a+}f(t)$.

(iii). If $f \in C^1[a,b]$ then ${}^CD^{\alpha,g}_{a+}I^{\alpha,g}_{a+}f(t) = f(t)$.

(iv). If $f \in C^n[a,b]$ then $I^{\alpha,g}_{a+}{}^CD^{\alpha,g}_{a+}f(t) = f(t) + \sum_{k=0}^{n-1}c_k(g(t)-g(a))^k$.

Continuously, we present the concept of the new Ulam-Hyers type stable.

Definition 2.3. *Equation (1.1) is called Ulam-Hyers ω-type stable if there exist $C > 0$ and $0 < \omega < \min\{1, \alpha+\beta\}$ such that for each $\epsilon > 0$ and for each solution $v \in C([\ell, b], \mathbb{R})$ of the following inequality*

$$\left| {}^{C}D^{\beta,g}_{a+}\left({}^{C}D^{\alpha;g}_{a+} + \vartheta(g,t)\right)v(t) - f(t, g, v(t), v(\rho(t)))\right| \le \epsilon(g(t)-g(a))^{-\omega} \quad t \in (a, b], \tag{2.1}$$

and $v(t) = \varphi(t)$ for $t \in [\ell, a]$, there exists a solution $u \in C([\ell, b], \mathbb{R})$ of Equation (1.1) such that

$$|u(t) - v(t)| \le C\epsilon, \quad t \in [\ell, b].$$

Remark 2.4. *From the concept of Ulam-Hyers ω-type stable, we have*

(i). If an equation is of the Ulam-Hyers ω-type stable then it is Ulam-Hyers stable in common sense as defined in [22], but the converse is not true. Furthermore, in this work, the source f has time-singular coefficients at $t = a$ and then the left-side of (2.1) may be unbounded as $t \to a^+$, so this inequality seems suitable for this type of the problem.

(ii). A function v is a solution of inequality (2.1) if there exists a function $\chi \in C((a, b], \mathbb{R})$ such that

$$|\chi(t)| \le \epsilon(g(t) - g(a))^{-\omega}$$

for all $t \in (a, b]$ and satisfying the following equation

$${}^{C}D^{\beta;g}_{a+}\left({}^{C}D^{\alpha,g}_{a+} + \vartheta(g,t)\right)v(t) = f(t, g, v(t), v(\rho(t))) + \chi(t), \quad t \in (a, b].$$

To end this section, we present a new inequality that plays an important role in proving the main results of the paper.

Lemma 2.5. *Let $\alpha > 0$, $0 \le \beta < \min\{1, \alpha\}$, and $\theta > 0$. Let $a, b \in \mathbb{R}$ with $a < b$ and $g \in Mc^1_+$. Then there exists a positive D independent of θ such that*

$$\int_a^t g'(\tau)(g(t)-g(\tau))^{\alpha-1}(g(\tau)-g(a))^{-\beta}e^{\theta g(\tau)}\,\mathrm{d}\tau \le D\frac{e^{\theta g(t)}}{\min\{\theta^{1-\beta}, \theta^{\alpha-\beta}\}}$$

for all $a < t \le b$.

Proof. First, we recall the following equality (see [8, 9])

$$\int_a^t g'(\tau)(g(t)-g(\tau))^{\alpha-1}(g(\tau)-g(a))^{-\beta}\,\mathrm{d}\tau = B(\alpha, 1-\beta)(g(t)-g(a))^{\alpha-\beta}. \tag{2.2}$$

Based on the above inequality, we will prove the result of Lemma. The proof is divided into two cases.

The first case: $g(t) - g(a) \le 1/\theta$ for any $t \in (a, b)$. Using (2.2), we get

$$\begin{aligned}
&\int_a^t g'(\tau)(g(t)-g(\tau))^{\alpha-1}(g(\tau)-g(a))^{-\beta}e^{\theta g(\tau)}\,\mathrm{d}\tau\\
&\le e^{\theta g(t)}\int_a^t g'(\tau)(g(t)-g(\tau))^{\alpha-1}(g(\tau)-g(a))^{-\beta}\,\mathrm{d}\tau\\
&= e^{\theta g(t)}B(\alpha, 1-\beta)(g(t)-g(a))^{\alpha-\beta}\\
&\le B(\alpha, 1-\beta)\frac{e^{\theta g(t)}}{\theta^{\alpha-\beta}}.
\end{aligned}$$

The second case: there exists $t_0 \in (a, b)$ such that $g(t_0) - g(a) = 1/\theta$. In this case, we note that $0 \le g(t) - g(a) \le 1/\theta$ for any $t \in [a, t_0]$ and $g(t) - g(a) > 1/\theta$ for all

$t > t_0$.
If $t \in [a, t_0]$, applying (2.2) and directly computes, we have

$$\begin{aligned}
&\int_a^t g'(\tau)(g(t)-g(\tau))^{\alpha-1}(g(\tau)-g(a))^{-\beta}e^{\theta g(\tau)}\,\mathrm{d}\tau \\
&\le e^{\theta g(t)}\int_a^t g'(\tau)(g(t)-g(\tau))^{\alpha-1}(g(\tau)-g(a))^{-\beta}\,\mathrm{d}\tau \\
&= e^{\theta g(t)}B(\alpha,1-\beta)(g(t)-g(a))^{\alpha-\beta} \\
&\le B(\alpha,1-\beta)\frac{e^{\theta g(t)}}{\theta^{\alpha-\beta}}. \qquad (2.3)
\end{aligned}$$

If $t > t_0$, we have

$$\int_a^t g'(\tau)(g(t)-g(\tau))^{\alpha-1}(g(\tau)-g(a))^{-\beta}e^{\theta g(\tau)}\,\mathrm{d}\tau = I_1(t,t_0)+I_2(t,t_0), \qquad (2.4)$$

where

$$\begin{aligned}
I_1(t,t_0) &= \int_a^{t_0} g'(\tau)(g(t)-g(\tau))^{\alpha-1}(g(\tau)-g(a))^{-\beta}e^{\theta g(\tau)}\,\mathrm{d}\tau, \\
I_2(t,t_0) &= \int_{t_0}^{t} g'(\tau)(g(t)-g(\tau))^{\alpha-1}(g(\tau)-g(a))^{-\beta}e^{\theta g(\tau)}\,\mathrm{d}\tau.
\end{aligned}$$

Estimate for $I_1(t,t_0)$. If $0 < \alpha \le 1$, we have $(g(t)-g(\tau))^{\alpha-1} \le (g(t_0)-g(\tau))^{\alpha-1}$ for any $t_0 < \tau \le t$. So, similarly the first case, we get

$$\begin{aligned}
I_1(t,t_0) &\le e^{\theta g(t)}\int_a^{t_0} g'(\tau)(g(t_0)-g(\tau))^{\alpha-1}(g(\tau)-g(a))^{-\beta}\,\mathrm{d}\tau \\
&\le B(\alpha,1-\beta)\frac{e^{\theta g(t)}}{\theta^{\alpha-\beta}}. \qquad (2.5)
\end{aligned}$$

If $\alpha > 1$, we have $(g(t)-g(\tau))^{\alpha-1} \le (g(b)-g(a))^{\alpha-1}$ for any $t_0 < \tau \le t$. It follows

$$\begin{aligned}
I_1(t,t_0) &\le e^{\theta g(t)}(g(b)-g(a))^{\alpha-1}\int_a^{t_0} g'(\tau)(g(\tau)-g(a))^{-\beta}\,\mathrm{d}\tau \\
&= e^{\theta g(t)}(g(b)-g(a))^{\alpha-1}\frac{1}{1-\beta}(g(t_0)-g(a))^{1-\beta} \\
&= \frac{1}{1-\beta}(g(b)-g(a))^{\alpha-1}\frac{e^{\theta g(t)}}{\theta^{1-\beta}}. \qquad (2.6)
\end{aligned}$$

Combining (2.5) and (2.6), we get

$$I_1(t,t_0) \le \max\left\{B(\alpha,1-\beta), \frac{1}{1-\beta}(g(b)-g(a))^{\alpha-1}\right\}\frac{e^{\theta g(t)}}{\min\{\theta^{1-\beta},\theta^{\alpha-\beta}\}} \qquad (2.7)$$

Estimate for $I_2(t,t_0)$. Using the fact that $(g(\tau)-g(a))^{-\beta} \le (g(t_0)-g(a))^{-\beta} = \theta^\beta$, we obtain

$$\begin{aligned}
I_2(t,t_0) &\le \theta^\beta \int_{t_0}^t g'(\tau)(g(t)-g(\tau))^{\alpha-1} e^{\theta g(\tau)}\,\mathrm{d}\tau \\
&\le \theta^\beta \int_a^t g'(\tau)(g(t)-g(\tau))^{\alpha-1} e^{\theta g(\tau)}\,\mathrm{d}\tau \\
&= \theta^\beta e^{\theta g(t)} \int_0^{g(t)-g(a)} y^{\alpha-1} e^{-\theta y}\,\mathrm{d}y \\
&= \frac{e^{\theta g(t)}}{\theta^{\alpha-\beta}} \int_0^{\theta(g(t)-g(a))} z^{\alpha-1} e^{-z}\,\mathrm{d}z \\
&\le \frac{e^{\theta g(t)}}{\theta^{\alpha-\beta}} \int_0^{+\infty} z^{\alpha-1} e^{-z}\,\mathrm{d}z \\
&= \Gamma(\alpha)\frac{e^{\theta g(t)}}{\theta^{\alpha-\beta}}. \qquad (2.8)
\end{aligned}$$

Pushing (2.7) and (2.8) into (2.4), we infer that

$$\int_a^t g'(\tau)(g(t)-g(\tau))^{\alpha-1}(g(\tau)-g(a))^{-\beta} e^{\theta g(\tau)}\,\mathrm{d}\tau \le D\frac{e^{\theta g(t)}}{\min\{\theta^{1-\beta}, \theta^{\alpha-\beta}\}} \qquad (2.9)$$

for any $t > t_0$, where $D = \Gamma(\alpha) + \max\left\{B(\alpha, 1-\beta), \frac{1}{1-\beta}(g(b)-g(a))^{\alpha-1}\right\}$. Combining (2.3) and (2.9), we conclude that

$$\int_a^t g'(\tau)(g(t)-g(\tau))^{\alpha-1}(g(\tau)-g(a))^{-\beta} e^{\theta g(\tau)}\,\mathrm{d}\tau \le D\frac{e^{\theta g(t)}}{\min\{\theta^{1-\beta}, \theta^{\alpha-\beta}\}}$$

for all $a < t \le b$. This completes the proof of Lemma. $\square$ $\square$

3. Fundamental results

In this section, we establish the formula solution of the problem. We also investigate the unique mild solution of the problem and Ulam-Hyers ω-type stability for the main equation. We begin by presenting a solution formula for the problem. Precisely, we have the following Lemma.

Lemma 3.1. *Let u be a solution of the problem (1.1)-(1.2). Then u is a solution of the following integral equation*

$$\begin{aligned}
u(t) &= \varphi(0) + \eta(g(t)-g(a))^\alpha + \frac{1}{\Gamma(\alpha)}\int_a^t g'(\tau)(g(t)-g(\tau))^{\alpha-1}\vartheta(g,t)u(\tau)\,\mathrm{d}\tau \\
&\quad + \frac{1}{\Gamma(\alpha+\beta)}\int_a^t g'(\tau)(g(t)-g(\tau))^{\alpha+\beta-1} f(\tau, g, u(\tau), u(\rho(\tau)))\,\mathrm{d}\tau \qquad (3.1)
\end{aligned}$$

for $t \in (a,b]$ and $u(t) = \varphi(t)$ for $t \in [\ell, a]$.

Proof. Since $\left({}^C D_{a+}^{\alpha,g} + \vartheta(g,\cdot)\right)u(\cdot) \in C^1[a,b]$, applying Lemma 2.2, we have

$$\left({}^C D_{a+}^{\alpha,g} + \vartheta(g,t)\right)u(t) = c + I_{a+}^{\beta,g} f(t,g,u(t),u(\rho(t))). \qquad (3.2)$$

Letting $t \to a$ and using the condition (1.2), we get $c = \eta$. Again, apply Lemma 2.2 for Eq. (3.2), we have

$$u(t) = d + c(g(t)-g(a))^\alpha + I_{a+}^{\alpha,g}\vartheta(g,t)u(t) + I_{a+}^{\alpha+\beta,g} f(t,g,u(t),u(\rho(t))) \qquad (3.3)$$

due to $I^{\alpha,g}_{a+}c(t) = \frac{c}{\Gamma(\alpha)}\int_a^t g'(\tau)(g(t)-g(\tau))^{\alpha-1}\,\mathrm{d}\tau = \frac{c}{\Gamma(\alpha+1)}(g(t)-g(a))^{\alpha}$. Using the condition $u(0)=\varphi(0)$, we get $d=\varphi(0)$. Pushing the obtained coefficients c and d into (3.3), we obtain the desired result of Lemma. $\square$

Definition 3.2. *A function $u \in C([\ell,b],\mathbb{R})$ satisfies the integral equation (3.1) is called mild solution of the problem (1.1)-(1.2).*

To prepare for the presentation of the main results of the paper, we will make some hypotheses.

- **Hypothesis** $(\mathcal{H}1)$. $g \in Mc^1_+[a,b]$, $\rho \in C([a,b],[\ell,b])$ with $\rho(t) \le t$ for any $t \in [a,b]$, and $\varphi \in C([\ell,a],\mathbb{R})$.
- **Hypothesis** $(\mathcal{H}2)$. $\vartheta \in C(Mc^1_+ \times (a,b],\mathbb{R})$ and there exist $L>0$ and $0<\gamma<\alpha$ such that

$$|\vartheta(g,t)| \le L(g(t)-g(a))^{-\gamma}.$$

- **Hypothesis** $(\mathcal{H}3)$. $f \in C((a,b]\times Mc^1_+ \times \mathbb{R}\times\mathbb{R},\mathbb{R})$, and there exist $K>0$ and $0 \le \kappa < \min\{1,\alpha+\beta\}$ such that

$$|f(t,g,u_1,v_1) - f(t,g,u_2,v_2)| \le K(g(t)-g(a))^{-\kappa}\left(|u_1-u_2|+|v_1-v_2|\right)$$

  for every $u_1,u_2,v_1,v_2 \in \mathbb{R}$ and for any $t>a$.
- **Hypothesis** $(\mathcal{H}4)$. $f \in C((a,b]\times Mc^1_+ \times \mathbb{R}\times\mathbb{R},\mathbb{R})$, and there exist $K>0$ and $0 \le \kappa < \min\{1,\alpha+\beta\}$ such that

$$|f(t,g,u_1,v) - f(t,g,u_2,v)| \le K(g(t)-g(a))^{-\kappa}|u_1-u_2|$$

  for all $u_1,u_2,v \in \mathbb{R}$ and for any $t>a$.
- **Hypothesis** $(\mathcal{H}5)$. There exists a constant $0<r\le a-\ell$ such that $\rho(t)\le t-r$ for any $t\in(a,b]$.

Throughout this paper, we use the Bielecki type norm given by

$$\|u\|_{\theta,b} = \max\left\{\Theta(\theta,t)|u(t)| : \ell \le t \le b\right\}, \tag{3.4}$$

where $\theta>0$, and Θ is defined as follows

$$\Theta(\theta,t) = \begin{cases} e^{-\theta g(t)} & \text{for } t\in[a,b] \\ e^{-\theta g(a)} & \text{for } t\in[\ell,a). \end{cases}$$

Remark 3.3. *Remark that in some case the function g may not be defined for every $t\in[\ell,a)$, for instance, $g(t)=\ln t$ (Hadamard fractional derivative) and $\ell=0$. Here is the reason why we introduce the weighted norm associated to the function Θ as above.*

Theorem 3.4. *Assume that the hypotheses $(\mathcal{H}1)-(\mathcal{H}3)$ are valid. Then, the problem (1.1)-(1.2) possesses a mild solution uniquely in $C([\ell,b],\mathbb{R})$.*

Remark 3.5. *Unlike previous results in literature, in Theorem 3.4, we obtained the existence and uniqueness results for the problem where the source function may have time-singular coefficients.*

PROOF. Consider the operator $\Phi : C([\ell, b], \mathbb{R}) \to C([\ell, b], \mathbb{R})$ defined as follows

$$\Phi u(t) = \begin{cases} \varphi(0) + \eta(g(t) - g(a))^{\alpha} \\ + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t g'(\tau)(g(t) - g(\tau))^{\alpha-1} \vartheta(g, t) u(\tau) \, \mathrm{d}\tau \\ + \dfrac{1}{\Gamma(\alpha+\beta)} \displaystyle\int_a^t g'(\tau)(g(t) - g(\tau))^{\alpha+\beta-1} \\ \qquad \times f(\tau, g, u(\tau), u(\rho(\tau))) \, \mathrm{d}\tau & \text{for } t \in (a, b], \\ \varphi(t) & \text{for } t \in [\ell, a]. \end{cases}$$

Using the Bielecki type norm given by (3.4), we intend to prove that Φ is a contraction mapping with θ large enough. Observe that $|\Phi u(t) - \Phi v(t)| = 0$ for any $t \in [\ell, a]$, hence, the next step is to evaluate for $t \in (a, b]$.

Using the hypotheses $(\mathcal{H}2) - (\mathcal{H}3)$ and direct computation, we have

$$\begin{aligned} |\Phi u(t) - \Phi v(t)| &\le \frac{1}{\Gamma(\alpha)} \int_a^t g'(\tau)(g(t) - g(\tau))^{\alpha-1} |\vartheta(g, \tau)| |u(\tau) - v(\tau)| \, \mathrm{d}\tau \\ &\quad + \frac{1}{\Gamma(\alpha+\beta)} \int_a^t g'(\tau)(g(t) - g(\tau))^{\alpha+\beta-1} \\ &\qquad \times |f(\tau, g, u(\tau), u(\rho(\tau))) - f(\tau, g, v(\tau), v(\rho(\tau)))| \, \mathrm{d}\tau \\ &\le \frac{L}{\Gamma(\alpha)} \int_a^t g'(\tau)(g(t) - g(\tau))^{\alpha-1} (g(\tau) - g(a))^{-\gamma} |u(\tau) - v(\tau)| \, \mathrm{d}\tau \\ &\quad + \frac{K}{\Gamma(\alpha+\beta)} \int_a^t g'(\tau)(g(t) - g(\tau))^{\alpha+\beta-1} (g(\tau) - g(a))^{-\kappa} \\ &\qquad \times \left(|u(\tau) - v(\tau)| + |u(\rho(\tau)) - v(\rho(\tau))|\right) \, \mathrm{d}\tau. \end{aligned} \tag{3.5}$$

On the other hand, it is clear that Θ is a non-increasing function (with respect to the second variable). Therefore, for any $\tau \in [a, b]$, we have

$$\begin{aligned} e^{-\omega g(\tau)} |u(\rho(\tau)) - v(\rho(\tau))| &= \Theta(g, \tau) |u(\rho(\tau)) - v(\rho(\tau))| \\ &\le \max_{a \le \tau \le b} \Theta(g, \rho(\tau)) |u(\rho(\tau)) - v(\rho(\tau))| \\ &\le \max_{\ell \le z \le b} \Theta(g, z) |u(z) - v(z)| = \|u - v\|_{\theta, b}, \end{aligned}$$

due to $\ell \le \rho(\tau) \le \tau \le b$. Thus, pushing the results just obtained into (3.5) and using Lemma 2.5, we obtain

$$\begin{aligned} &|\Phi u(t) - \Phi v(t)| \\ &\le \frac{L}{\Gamma(\alpha)} \|u - v\|_{\theta, b} \int_a^t g'(\tau)(g(t) - g(\tau))^{\alpha-1} (g(\tau) - g(a))^{-\gamma} e^{\theta g(\tau)} \, \mathrm{d}\tau \\ &+ \frac{2K}{\Gamma(\alpha+\beta)} \|u - v\|_{\theta, b} \int_a^t g'(\tau)(g(t) - g(\tau))^{\alpha+\beta-1} (g(\tau) - g(a))^{-\kappa} e^{\theta g(\tau)} \, \mathrm{d}\tau \\ &\le \left(\frac{L D_1}{\Gamma(\alpha) \min\{\theta^{1-\gamma},\ \theta^{\alpha-\gamma}\}} + \frac{2K D_2}{\Gamma(\alpha+\beta) \min\{\theta^{1-\kappa},\ \theta^{\alpha+\beta-\kappa}\}} \right) e^{\theta g(t)} \|u - v\|_{\theta, b}, \end{aligned}$$

where D_1, D_2 is independent of θ. This leads to

$$\begin{aligned} & e^{-\theta g(t)}|\Phi u(t) - \Phi v(t)| \\ & \leq \left(\frac{LD_1}{\Gamma(\alpha)\min\{\theta^{1-\gamma},\ \theta^{\alpha-\gamma}\}} + \frac{2KD_2}{\Gamma(\alpha+\beta)\min\{\theta^{1-\kappa},\ \theta^{\alpha+\beta-\kappa}\}} \right) \|u-v\|_{\theta,b}. \end{aligned}$$

It follows

$$\begin{aligned} & \|\Phi u - \Phi v\|_{\theta,b} \\ & \leq \left(\frac{LD_1}{\Gamma(\alpha)\min\{\theta^{1-\gamma},\ \theta^{\alpha-\gamma}\}} + \frac{2KD_2}{\Gamma(\alpha+\beta)\min\{\theta^{1-\kappa},\ \theta^{\alpha+\beta-\kappa}\}} \right) \|u-v\|_{\theta,b}. \end{aligned}$$

Since

$$\left(\frac{LD_1}{\Gamma(\alpha)\min\{\theta^{1-\gamma},\ \theta^{\alpha-\gamma}\}} + \frac{2KD_2}{\Gamma(\alpha+\beta)\min\{\theta^{1-\kappa},\ \theta^{\alpha+\beta-\kappa}\}} \right) = 0,$$

we conclude from the latter inequality that Φ is contraction with θ large enough. As a consequence, Φ possesses a solution uniquely in $C([\ell,b],\mathbb{R})$. This finishes the proof of Theorem. □

In the next theorem, we show that if hypothesis ($\mathcal{H}5$) is valid, then we can replace hypothesis ($\mathcal{H}3$) in Theorem 3.4 with hypothesis ($\mathcal{H}4$). It is clear that the hypothesis ($\mathcal{H}4$) is weaker than the hypothesis ($\mathcal{H}3$). More precisely, we have the following theorem.

Theorem 3.6. *Suppose that the hypotheses ($\mathcal{H}1$), ($\mathcal{H}2$), ($\mathcal{H}4$) and ($\mathcal{H}5$) are satisfied. Then the problem (1.1)-(1.2) possesses a mild solution uniquely in $C([\ell,b],\mathbb{R})$.*

Remark 3.7. *The result of Theorem 3.6 seems to be new and still not consider for Lane-Emden in literature. Besides, considering the problem with the source function having time-singular coefficients is also a new aspect of our result.*

Proof. We divide the interval $[a,b]$ into parts by $a = A_0 < A_1 < .. < A_n = b$ with $A_i - A_{i-1} = \delta$ $(i = \overline{1,n})$, where $\delta \leq r$ (related to this idea, we can refer to [4, 6]). The proof of Theorem is separated into 3 steps.

Step 1. We denote by V_1 the Banach space of all continuous functions on $[\ell, A_1]$ such that $u(t) = \varphi(t)$ for all $u \in V_1$ and $\ell \leq t \leq a$ with the Bielecki type norm $\|\cdot\|_{\theta,A_1}$. Define the operator $\Phi_1 : V_1 \to V_1$ by

$$\Phi_1 u(t) = \begin{cases} \varphi(0) + \eta(g(t)-g(a))^{\alpha} \\ +\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_a^t g'(\tau)(g(t)-g(\tau))^{\alpha-1}\vartheta(g,t)u(\tau)\,\mathrm{d}\tau \\ +\dfrac{1}{\Gamma(\alpha+\beta)}\displaystyle\int_a^t g'(\tau)(g(t)-g(\tau))^{\alpha+\beta-1} \\ \qquad\qquad \times f(\tau, g, u(\tau), u(\rho(\tau)))\,\mathrm{d}\tau & \text{for } t \in (a, A_1] \\ \varphi(t) & \text{for } t \in [\ell, a]. \end{cases}$$

We have $\Phi_1 u(t) = \Phi_1 v(t) = \varphi(t)$ for any $t \in [\ell, a]$. We next consider $t \in [a, A_1]$. Since $\rho(t) \leq t - r \leq t - \delta \leq a$ for any $t \leq A_1$, it follows $u(\rho(t)) = v(\rho(t)) = \varphi(t)$ for any $t \leq T_1$. Using the hypothesis ($\mathcal{H}4$), we get

$$
\begin{aligned}
&|\Phi_1 u(t) - \Phi_1 v(t)| \\
&\leq \frac{L}{\Gamma(\alpha)} \int_a^t g'(\tau)(g(t)-g(\tau))^{\alpha-1}(g(\tau)-g(a))^{-\gamma}|u(\tau)-v(\tau)| \, \mathrm{d}\tau \\
&+ \frac{K}{\Gamma(\alpha+\beta)} \int_a^t g'(\tau)(g(t)-g(\tau))^{\alpha+\beta-1}(g(\tau)-g(a))^{-\kappa}|u(\tau)-v(\tau)| \, \mathrm{d}\tau.
\end{aligned}
$$

Similarly the process of the proof of Step 1, we obtain

$$
\begin{aligned}
&|\Phi_1 u(t) - \Phi_1 v(t)| \\
&\leq \frac{L}{\Gamma(\alpha)} \|u-v\|_{\theta,A_1} \int_a^t g'(\tau)(g(t)-g(\tau))^{\alpha-1}(g(\tau)-g(a))^{-\gamma} e^{\theta g(\tau)} \, \mathrm{d}\tau \\
&+ \frac{K}{\Gamma(\alpha+\beta)} \|u-v\|_{\theta,A_1} \int_a^t g'(\tau)(g(t)-g(\tau))^{\alpha+\beta-1}(g(\tau)-g(a))^{-\kappa} e^{\theta g(\tau)} \, \mathrm{d}\tau \\
&\leq \left(\frac{LD_1}{\Gamma(\alpha)\min\{\theta^{1-\gamma},\ \theta^{\alpha-\gamma}\}} + \frac{KD_2}{\Gamma(\alpha+\beta)\min\{\theta^{1-\kappa},\ \theta^{\alpha+\beta-\kappa}\}} \right) e^{\theta g(t)} \|u-v\|_{\theta,A_1},
\end{aligned}
$$

where D_1, D_2 is independent of θ. From the latter inequality, we deduce that

$$
\begin{aligned}
&\|\Phi_1 u - \Phi_1 v\|_{\theta,A_1} \\
&\leq \left(\frac{LD_1}{\Gamma(\alpha)\min\{\theta^{1-\gamma},\ \theta^{\alpha-\gamma}\}} + \frac{KD_2}{\Gamma(\alpha+\beta)\min\{\theta^{1-\kappa},\ \theta^{\alpha+\beta-\kappa}\}} \right) \|u-v\|_{\theta,A_1}.
\end{aligned}
$$

For θ large enough, we find that Φ_1 is a contraction mapping on V_1 and Φ_1 admits a unique solution in V_1. As a result, the problem (1.1)-(1.2) possesses a mild solution uniquely on $[\ell, A_1]$.

Step 2. Next, we extend the existence and uniqueness of solutions of the problem on $[\ell, A_2]$. We denote by V_2 the Banach space of all continuous functions on $[a, A_2]$ such that $u(t) = u_1(t)$ for any $u \in V_2$ and $\ell \leq t \leq A_1$ with the Bielecki type norm $\|\cdot\|_{\theta,A_2}$. Define the operator $\Phi_2 : V_2 \to V_2$ by

$$
\Phi_2 u(t) = \begin{cases}
\varphi(0) + \eta(g(t)-g(a))^{\alpha} \\
+\dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t g'(\tau)(g(t)-g(\tau))^{\alpha-1}\vartheta(g,t)u(\tau) \, \mathrm{d}\tau \\
+\dfrac{1}{\Gamma(\alpha+\beta)} \displaystyle\int_a^t g'(\tau)(g(t)-g(\tau))^{\alpha+\beta-1} \\
\qquad \times f(\tau, g, u(\tau), u(\rho(\tau))) \, \mathrm{d}\tau & \text{for } t \in (A_1, A_2] \\
u_1(t) & \text{for } t \in [\ell, A_1].
\end{cases}
$$

It is clear that $\Phi_2 u(t) = \Phi_2 v(t) = u_1(t)$ for $t \in [\ell, A_1]$. For $t \in (A_1, A_2]$, we find that $\rho(t) \leq t - r \leq t - \delta \leq A_1$ for any $t \leq A_2$. This implies $u(\rho(t)) = v(\rho(t)) = u_1(t)$ for $t \leq A_2$. By using the same method used to prove Step 1, we conclude that

$$
\begin{aligned}
&\|\Phi_2 u - \Phi_2 v\|_{\theta,A_2} \\
&\leq \left(\frac{LD_1}{\Gamma(\alpha)\min\{\theta^{1-\gamma},\ \theta^{\alpha-\gamma}\}} + \frac{KD_2}{\Gamma(\alpha+\beta)\min\{\theta^{1-\kappa},\ \theta^{\alpha+\beta-\kappa}\}} \right) \|u-v\|_{\theta,A_2}.
\end{aligned}
$$

This leads to the existence and unique of mild solutions of the problem (1.1)-(1.2) on $[\ell, A_2]$.

Step 3. By continuing this process to n^{th} Step, we obtain a unique continuous function $u_n := u$ on $[\ell, A_n] = [\ell, b]$, which is a mild solution of the problem (1.1)-(1.2). The proof of the theorem is complete. □

Finally, we give a results on the Ulam-Hyers ω-type stability for the main equation.

Theorem 3.8. *Assume that the hypotheses $(\mathcal{H}1) - (\mathcal{H}3)$ hold. Then, Equation (1.1) is Ulam-Hyers ω-type stable for some $0 < \omega < \min\{1, \alpha + \beta\}$.*

Remark 3.9. *It should be noted that the functions ϑ and f are singular at $t = a$, so, for fixed $v \in C([\ell, b], \mathbb{R})$, we may have*

$$\left| {}^C D_{a+}^{\beta,g} \left({}^C D_{a+}^{\alpha,g} + \vartheta(g,t) \right) v(t) - f(t, g, v(t), v(\rho(t))) \right| \to \infty \text{ as } t \to a^+.$$

This shows that it seems justified to consider the condition

$$\left| {}^C D_{a+}^{\beta,g} \left({}^C D_{a+}^{\alpha,g} + \vartheta(g,t) \right) v(t) - f(t, g, v(t), v(\rho(t))) \right| \le \epsilon (g(t) - g(a))^{-\omega}, \quad t \in (a, b]$$

for problems with time-singular coefficients. Here is the main reason why the Ulam-Hyers ω-type stable is proposed and studied in this paper. To the best of our knowledge, it is the first time in the literature the concept Ulam-Hyers ω-type stability has been proposed and investigated. Furthermore, as we mentioned in Remark 2.4, the obtained result is stronger than the previous ones.

Proof. Note that the hypotheses $(\mathcal{H}1) - (\mathcal{H}3)$ ensures that the problem (1.1)-(1.2) possesses a mild solution uniquely u on $[\ell, b]$, i.e., it satisfies the following integral equation

$$\begin{aligned} u(t) &= \varphi(0) + \eta (g(t) - g(a))^{\alpha} + \frac{1}{\Gamma(\alpha)} \int_a^t g'(\tau)(g(t) - g(\tau))^{\alpha-1} \vartheta(g,t) u(\tau) \, \mathrm{d}\tau \\ &\quad + \frac{1}{\Gamma(\alpha+\beta)} \int_a^t g'(\tau)(g(t) - g(\tau))^{\alpha+\beta-1} f(\tau, g, u(\tau), u(\rho(\tau))) \, \mathrm{d}\tau \end{aligned}$$

for $t \in (a, b]$.
We now consider a solution v of the following inequality

$$\left| {}^C D_{a+}^{\beta,g} \left({}^C D_{a+}^{\alpha,g} + \vartheta(g,t) \right) v(t) - f(t, g, v(t), v(\rho(t))) \right| \le \epsilon (g(t) - g(a))^{-\omega}, \quad t \in (a, b]$$

and $v(t) = \varphi(t)$ for $t \in [\ell, a]$. This implies from Remark 2.4 that for each $\epsilon > 0$ there exists a function $\chi \in C((a, b], \mathbb{R})$ with $|\chi(t)| \le \epsilon (g(t) - g(a))^{-\omega}$ for some $0 < \omega < \min\{1, \alpha + \beta\}$ such that

$$\begin{aligned} v(t) &= \varphi(0) + \eta (g(t) - g(a))^{\alpha} + \frac{1}{\Gamma(\alpha)} \int_a^t g'(\tau)(g(t) - g(\tau))^{\alpha-1} \vartheta(g,t) v(\tau) \, \mathrm{d}\tau \\ &\quad + \frac{1}{\Gamma(\alpha+\beta)} \int_a^t g'(\tau)(g(t) - g(\tau))^{\alpha+\beta-1} (f(\tau, g, v(\tau), v(\rho(\tau))) + \chi(\tau)) \, \mathrm{d}\tau. \end{aligned}$$

This gives

$$
\begin{aligned}
|v(t)-u(t)| \le \Big| & v(t)-\varphi(0)-\eta(g(t)-g(a))^{\alpha} \\
& - \frac{1}{\Gamma(\alpha)}\int_a^t g'(\tau)(g(t)-g(\tau))^{\alpha-1}\vartheta(g,t)u(\tau)\,\mathrm{d}\tau \\
& - \frac{1}{\Gamma(\alpha+\beta)}\int_a^t g'(\tau)(g(t)-g(\tau))^{\alpha+\beta-1}f(\tau,g,u(\tau),u(\rho(\tau)))\,\mathrm{d}\tau\Big| \\
\le\; & I_1(t)+I_2(t)+I_3(t), \qquad (3.6)
\end{aligned}
$$

where

$$
\begin{aligned}
I_1(t) = \Big| & v(t)-\varphi(0)-\eta(g(t)-g(a))^{\alpha} \\
& - \frac{1}{\Gamma(\alpha)}\int_a^t g'(\tau)(g(t)-g(\tau))^{\alpha-1}\vartheta(g,t)v(\tau)\,\mathrm{d}\tau \\
& - \frac{1}{\Gamma(\alpha+\beta)}\int_a^t g'(\tau)(g(t)-g(\tau))^{\alpha+\beta-1}f(\tau,g,v(\tau),v(\rho(\tau)))\,\mathrm{d}\tau\Big| \\
I_2(t) = & \int_a^t g'(\tau)(g(t)-g(\tau))^{\alpha-1}|\vartheta(g,\tau)||v(\tau)-u(\tau)|\,\mathrm{d}\tau \\
I_3(t) = & \int_a^t g'(\tau)(g(t)-g(\tau))^{\alpha+\beta-1}|f(\tau,g,v(\tau),v(\rho(\tau)))-f(\tau,g,u(\tau),u(\rho(\tau)))|\,\mathrm{d}\tau.
\end{aligned}
$$

To obtain the desired result, we estimate for three terms in the right-hand side of (3.6). For the first term, thank to equality (2.2), we have

$$
\begin{aligned}
I_1(t) &= \Big|\varphi(0)-\eta(g(t)-g(a))^{\alpha} - \frac{1}{\Gamma(\alpha)}\int_a^t g'(\tau)(g(t)-g(\tau))^{\alpha-1}\vartheta(g,t)v(\tau)\,\mathrm{d}\tau \\
&\qquad - \frac{1}{\Gamma(\alpha+\beta)}\int_a^t g'(\tau)(g(t)-g(\tau))^{\alpha+\beta-1}f(\tau,g,v(\tau),v(\rho(\tau)))\,\mathrm{d}\tau\Big| \\
&\le \frac{1}{\Gamma(\alpha+\beta)}\int_a^t g'(\tau)(g(t)-g(\tau))^{\alpha+\beta-1}|\chi(\tau)|\,\mathrm{d}\tau \\
&\le \frac{\epsilon}{\Gamma(\alpha+\beta)}\int_a^t g'(\tau)(g(t)-g(\tau))^{\alpha+\beta-1}(g(\tau)-g(a))^{-\omega}\,\mathrm{d}\tau \\
&= \frac{B(\alpha+\beta,1-\omega)}{\Gamma(\alpha+\beta)}(g(t)-g(a))^{\alpha+\beta-\omega}\epsilon \\
&\le \frac{\Gamma(1-\omega)}{\Gamma(\alpha+\beta+1-\omega)}(g(b)-g(a))^{\alpha+\beta-\omega}\epsilon. \qquad (3.7)
\end{aligned}
$$

For the second term, using the hypothesis $(\mathcal{H}2)$, equality (2.2) and Lemma 2.5, we get

$$
\begin{aligned}
I_2(t) &\le \frac{L}{\Gamma(\alpha)}\int_a^t g'(\tau)(g(t)-g(\tau))^{\alpha-1}(g(\tau)-g(a))^{-\gamma}|v(\tau)-u(\tau)|\,\mathrm{d}\tau \\
&\le \frac{L}{\Gamma(\alpha)}\|u-v\|_{\theta,b}\int_a^t g'(\tau)(g(t)-g(\tau))^{\alpha-1}(g(\tau)-g(a))^{-\gamma}e^{\theta g(\tau)}\,\mathrm{d}\tau \\
&\le \frac{LD_1}{\Gamma(\alpha)\min\{\theta^{1-\gamma},\ \theta^{\alpha-\gamma}\}}e^{\theta g(t)}\|u-v\|_{\theta,b}. \qquad (3.8)
\end{aligned}
$$

For the third term, using the hypothesis ($\mathcal{H}3$) and equality (2.2) together with Lemma 2.5, we also have

$$\begin{aligned}
I_3(t) &\le \frac{K}{\Gamma(\alpha+\beta)} \int_a^t g'(\tau)(g(t)-g(\tau))^{\alpha+\beta-1}(g(\tau)-g(a))^{-\kappa} \\
&\qquad\qquad \times \left(|v(\tau)-u(\tau)| + |v(\rho(\tau))-u(\rho(\tau))|\right)\, \mathrm{d}\tau \\
&\le \frac{2K}{\Gamma(\alpha+\beta)} \|u-v\|_{\theta,b} \int_a^t g'(\tau)(g(t)-g(\tau))^{\alpha-1}(g(\tau)-g(a))^{-\kappa} e^{\theta g(\tau)}\, \mathrm{d}\tau \\
&\le \frac{2KD_2}{\Gamma(\alpha+\beta)\min\{\theta^{1-\kappa},\ \theta^{\alpha+\beta-\kappa}\}} e^{\theta g(t)} \|u-v\|_{\theta,b}. \qquad (3.9)
\end{aligned}$$

Substituting (3.7), (3.8) and (3.9) into (3.6), we obtain

$$\begin{aligned}
|v(t)-u(t)| &\le \frac{\Gamma(1-\omega)}{\Gamma(\alpha+\beta+1-\omega)}(g(b)-g(a))^{\alpha+\beta-\omega}\epsilon \\
&+ \left(\frac{LD_1}{\Gamma(\alpha)\min\{\theta^{1-\gamma},\ \theta^{\alpha-\gamma}\}} + \frac{2KD_2}{\Gamma(\alpha+\beta)\min\{\theta^{1-\kappa},\ \theta^{\alpha+\beta-\kappa}\}} \right) e^{\theta g(t)} \|u-v\|_{\theta,b}.
\end{aligned}$$

This deduces that

$$\begin{aligned}
\|u-v\|_{\theta,b} &\le \frac{\Gamma(1-\omega)}{\Gamma(\alpha+\beta+1-\omega)}(g(b)-g(a))^{\alpha+\beta-\omega}\epsilon \\
&+ \left(\frac{LD_1}{\Gamma(\alpha)\min\{\theta^{1-\gamma},\ \theta^{\alpha-\gamma}\}} + \frac{2KD_2}{\Gamma(\alpha+\beta)\min\{\theta^{1-\kappa},\ \theta^{\alpha+\beta-\kappa}\}} \right) \|u-v\|_{\theta,b}.
\end{aligned}$$

We choose θ large enough such that

$$E = \left(\frac{LD_1}{\Gamma(\alpha)\min\{\theta^{1-\gamma},\ \theta^{\alpha-\gamma}\}} + \frac{2KD_2}{\Gamma(\alpha+\beta)\min\{\theta^{1-\kappa},\ \theta^{\alpha+\beta-\kappa}\}} \right) < 1$$

and we obtain

$$\begin{aligned}
&e^{-\theta g(t)}|v(t)-u(t)| \\
&\le \|u-v\|_{\theta,b} \le \frac{\Gamma(1-\omega)}{(1-E)\Gamma(\alpha+\beta+1-\omega)}(g(b)-g(a))^{\alpha+\beta-\omega}\epsilon e^{-\theta g(a)}\epsilon
\end{aligned}$$

or

$$\begin{aligned}
|v(t)-u(t)| &\le \frac{\Gamma(1-\omega)}{(1-E)\Gamma(\alpha+\beta+1-\omega)}(g(b)-g(a))^{\alpha+\beta-\omega} e^{\theta(g(t)-g(a))}\epsilon \\
&\le \frac{\Gamma(1-\omega)}{(1-E)\Gamma(\alpha+\beta+1-\omega)}(g(b)-g(a))^{\alpha+\beta-\omega} e^{\theta(g(b)-g(a))}\epsilon.
\end{aligned}$$

So, the main equation is Ulam-Hyers ω-type stable. The theorem is completely proven. □

4. Applications

We present two examples to show the applicability of the obtained results of the paper.

Example 4.1. *We consider the common Caputo fractional derivative, i.e., $g(t) = t$. We examine the Lane-Emden problem in the following form*

$$
\begin{cases} {}^C D_{0+}^{t,2/3} \left({}^C D_{0+}^{t,9/10} + t^{-1/2} \right) u(t) \\ = t^{-2/5}(u(t) + u^3(t - (t+1)/(t+2))), \ t \in (0,1] \\ u(t) = 1 - \cos t, \ t \in [-1, 0], \ \lim_{t \to 0+} \left(t^{-1/2} u(t) + {}^C D_{0+}^{t,9/10} u(t) \right) = 0, \end{cases} \tag{4.1}
$$

where $a = 0, b = 1$, $\ell = -1$, $\alpha = 9/10$, $\beta = 2/3$, $\vartheta(g,t) = t^{-1/2}$, and $f(t,g,u,v) = t^{-2/5}(u(t) + v^3(t))$ with $\rho(t) = t - (t+1)/(t+2)$. It is clear that $f \in C((0,1] \times Mc_+^1 \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, and

$$
|f(t,u_1,v) - f(t,u_2,v)| \leq t^{-2/5} |u_1 - u_2|
$$

for any $u_1, u_2, v \in \mathbb{R}$ and $t \in (0,1]$. Since $\gamma = 1/2 < \alpha = 9/10$, $\kappa = 2/5 < \alpha + \beta = 47/30$ and $\rho(t) = t - (t+1)/(t+2) \leq t - 1/2$ for any $t \in (0,1]$, so, we can use Theorem 3.6 to conclude that the problem (4.1) possesses a mild solution uniquely on $[-1,1]$.

Example 4.2. *In the second example, we consider the Hadamard fractional derivative, i.e., $g(t) = \ln t$ and examine the Lane-Emden problem as follows*

$$
\begin{cases} {}^C D_{1+}^{\ln,1/2} \left({}^C D_{1+}^{\ln,5/6} + (\ln t)^{-1/3} \right) u(t) \\ = t^{-2/3} \left[2u(t) + 1/(|u(t - 0.1(t-1))| + 1) \right], \quad t \in (1, e] \\ u(t) = t + t^2 - 2, \ t \in [0,1], \\ \lim_{t \to 1+} \left((\ln t)^{-1/3} u(t) + {}^C D_{1+}^{\ln,5/6} u(t) \right) = 0, \end{cases} \tag{4.2}
$$

where $a = 1, b = e$, $\ell = 0$, $\alpha = 1/2$, $\beta = 4/6$, $\vartheta(g,t) = (\ln t)^{-1/3}$, and $f(t,g,u,v) = t^{-2/3} \left[2u(t) + 1/(|v| + 1) \right]$ with $\rho(t) = t - 0.1(t-1)$. It is obvious that $f \in C((1,e] \times Mc_+^1 \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, and

$$
|f(t,u_1,v_1) - f(t,u_2,v_2)| \leq 2t^{-2/3}(|u_1 - u_2| + |v_1 - v_2|)
$$

for all $u_1, u_2, v_1, v_2 \in \mathbb{R}$ and $t \in (1,e]$. We can find that $\gamma = 1/3 < \alpha = 5/6$ and $\kappa = 2/3 < \alpha + \beta = 4/3$. So, using Theorem 3.4, we conclude that the problem (4.2) possesses a mild solution uniquely. Moreover, using Theorem 3.8, we also conclude that the main equation is Ulam-Hyers ω-type stable for any $\omega \in (0,1)$.

5. Conclusions

We studied the existence and uniqueness of mild solutions of nonlinear Lane-Emden type delay problem with generalized Caputo fractional derivatives. In some cases, we obtained a unique result with a condition that is weaker than the previous one. We also proved that the main equation is Ulam-Hyers ω-type stable. In the future work, we would like to study nonlinear Lane-Emden type delay equations with Hilfer fractional derivatives and investigate the continuous dependence of solutions of problems with respect to fractional orders and associated parameters.

Conflict of interest

The author has no conflicts of interest to declare.

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Faculty of Education, Thu Dau Mot University, Binh Duong Province, Vietnam
Email address: diennm@tdmu.edu.vn