MONOTONICITY RESULTS FOR FUNCTIONS INVOLVING THE q-POLYGAMMA FUNCTIONS

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Abstract. Let $\psi_{q,n} = (-1)^{n-1}\psi_q^{(n)}$ for $n \in \mathbb{N}$, where $\psi_q^{(n)}$ are the q-polygamma functions. In this paper, by the monotonicity rules for the ratio of two power series, it is proved that, for $q \in (0,1)$ and $n \in \mathbb{N}$, the function

$$x \mapsto F_{q,n}(x;\alpha) = \frac{q^{x+\alpha}-1}{\ln q}\frac{\psi_{q,n+1}(x)}{\psi_{q,n}(x)},$$

is decreasing (increasing) on $(0,\infty)$ if and only if $\alpha \leq \log_q(2^n/(q+1))$ ($\alpha \geq 0$). The conditions for which several relevant functions are monotonic or completely monotonic on $(0,\infty)$ are obtained. Moreover, several relations involving the q-polygamma functions are established.

1. Introduction

The classical Euler's gamma function Γ is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \tag{1.1}$$

for $x > 0$, and its logarithmic derivative $\psi(x) = \Gamma'(x)/\Gamma(x)$ is known as the psi or digamma function, while ψ', ψ'', ..., $\psi^{(n)}$ are called polygamma functions. As usual, we denote by $\psi_n = (-1)^{n-1}\psi^{(n)}$ for $n \in \mathbb{N}$.

The q-gamma function [1, 2] is defined for $x > 0$ and $q \neq 1$ by

$$\Gamma_q(x) = (1-q)^{1-x}\prod_{n=0}^{\infty}\frac{1-q^{n+1}}{1-q^{n+x}}, \quad \text{if } 0 < q < 1, \tag{1.2}$$

$$\Gamma_q(x) = (q-1)^{1-x}q^{x(x-1)/2}\prod_{n=0}^{\infty}\frac{1-q^{-(n+1)}}{1-q^{-(n+x)}}, \quad \text{if } q > 1. \tag{1.3}$$

It is easy to see that $\lim_{x\to 0}\Gamma_q(x) = \infty$ and $\lim_{x\to\infty}\Gamma_q(x) = \infty$. From (1.2) and (1.3) we have that, for all $q > 0$,

$$\Gamma_q(x) = q^{(x-1)(x-2)/2}\Gamma_{1/q}(x), \quad x > 0. \tag{1.4}$$

Analogously, the logarithmic derivative of the q-gamma function $\psi_q(x) = \Gamma_q'(x)/\Gamma_q(x)$ is known as q-psi or q-digamma function, and ψ_q', ψ_q'', ..., $\psi_q^{(n)}$ are called q-polygamma

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functions. The q-digamma function $\psi_q(x)$ has a series representation:

$$\psi_q(x) = -\ln(1-q) + \sum_{k=0}^{\infty} \frac{q^{k+x}\ln q}{1-q^{k+x}} \tag{1.5}$$

$$= -\ln(1-q) + (\ln q) \sum_{k=1}^{\infty} \frac{q^{kx}}{1-q^k} \quad \text{for } 0<q<1. \tag{1.6}$$

Then

$$(-1)^{n-1}\psi_q^{(n)}(x) = (-\ln q)^{n+1} \sum_{k=1}^{\infty} \frac{k^n q^{kx}}{1-q^k} \quad \text{if } 0<q<1 \tag{1.7}$$

for $x>0$ and $n\in\mathbb{N}$. It is worth mentioning that Ismail and Muldoon [3] found that the q-psi function has the following Stieltjes integral representation:

$$\psi_q(x) = -\ln(1-q) - \int_0^{\infty} \frac{e^{-xt}}{1-e^{-t}} d\gamma_q(t), \tag{1.8}$$

where

$$\gamma_q(t) = -\ln q \sum_{k=1}^{\infty} \delta(t+k\ln q), \quad 0<q<1,$$

is a discrete measure with positive masses $-\ln q$ at the positive points $-k\ln q$, $k=1,2,\ldots$. This offered a new and simple way to investigate the q-gamma and q-polygamma functions (see [4]).

For convenience, we denote by $\psi_{q,n} = (-1)^{n-1}\psi_q^{(n)}$ for $n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, where $\psi_{q,0}=-\psi_q$. It is readily seen from (1.6) and (1.7), for $n\in\mathbb{N}$ and $q\in(0,1)$,

$$\begin{aligned} &\lim_{x\to 0+}\psi_q(x) = -\infty, \quad \lim_{x\to\infty}\psi_q(x) = -\ln(1-q), \\ &\lim_{x\to 0+}\psi_{q,n}(x) = \infty, \quad \lim_{x\to\infty}\psi_{q,n}(x) = 0. \end{aligned} \tag{1.9}$$

The close relation between the ordinary gamma function Γ and the q-gamma function Γ_q is given by $\lim_{q\to 1}\Gamma_q(x) = \Gamma(x), x>0$ (see [2], [5]). Likewise, the ordinary digamma function ψ and q-digamma function ψ_q satisfy the following limit relation: $\lim_{q\to 1}\psi_q(x) = \psi(x), x>0$ (see [6]). We claim that the ordinary polygamma function $\psi^{(n)}$ and q-polygamma function $\psi_q^{(n)}$ also satisfy a similar limit relation.

Claim 1. *Let* $n\in\mathbb{N}$. *We have*

$$\lim_{q\to 1^-}\psi_q^{(n)}(x) = \lim_{q\to 1^+}\psi_q^{(n)}(x) = \psi^{(n)}(x), \quad x>0. \tag{1.10}$$

Sketch of proof. The first equality of (1.10) follows from the relation (1.4). It was proved in [7, Eq. (2.5)] that

$$\frac{d^n}{dt^n}\left(\frac{q^t\ln q}{1-q^t}\right) = \left(\frac{\ln q}{1-q^t}\right)^{n+1} q^t P_{n-1}(q^t), \quad n\in\mathbb{N},$$

where $P_n(z)$ is a polynomial of degree n satisfying

$$P_n(z) = (z-z^2)P'_{n-1}(z) + (nz+1)P_{n-1}(z), \quad P_0(z)=1,\ n\geq 1.$$

The above relation implies that $P_n(1) = (n+1) P_{n-1}(1)$ with $P_0(1) = 1$, and therefore, $P_n(1) = (n+1)!$. From these it follows that

$$\lim_{q\to 1} \frac{d^n}{dt^n} \left(\frac{q^t \ln q}{1-q^t} \right) = \frac{(-1)^{n+1} n!}{t^{n+1}}, \quad n \in \mathbb{N}.$$

Now, using (1.5) and differentiating yield

$$\psi_q^{(n)}(x) = \sum_{k=0}^{\infty} \frac{d^n}{dx^n} \left(\frac{q^{k+x} \ln q}{1-q^{k+x}} \right), \quad q \in (0,1).$$

Then

$$\lim_{q\to 1^-} \psi_q^{(n)}(x) = \sum_{k=0}^{\infty} \lim_{q\to 1^-} \frac{d^n}{dx^n} \left(\frac{q^{k+x} \ln q}{1-q^{k+x}} \right) = \sum_{k=0}^{\infty} \frac{(-1)^{n+1} n!}{(k+x)^{n+1}} = \psi^{(n)}(x).$$

□

In 2001, Alzer [8, Lemma 2] (see also [9, Lemma 2.1]) proved that the function $x \mapsto x\psi_{n+1}(x)/\psi_n(x)$ is strictly decreasing from $(0,\infty)$ onto $(n, n+1)$. Yang [10, Corollary 2] proved that the function $x \mapsto (x+r)\psi_{n+1}(x)/\psi_n(x)$ is strictly decreasing (increasing) on $(0,\infty)$ if and only if $r \geq 0$ ($r \leq -1/2$). For the q-polygamma functions, it is natural to ask the following problem.

Problem 1. *What are the conditions for which the function*

$$x \mapsto F_{q,n}(x;\alpha) = \frac{q^{x+\alpha}-1}{\ln q} \frac{\psi_{q,n+1}(x)}{\psi_{q,n}(x)}, \tag{1.11}$$

is increasing or decreasing on $(0,\infty)$ for $n \in \mathbb{N}$ and $q > 0$ with $q \neq 1$?

The aim of this paper is to give an answer to the problem for $q \in (0,1)$. Our main result is contained in the following theorem.

Theorem 1. *Let $q \in (0,1)$ and $n \in \mathbb{N}$. The following statements are valid:*

(i) If $\alpha \leq \alpha_0 = \log_q(2^n/(q+1))$, then the function $x \mapsto F_{q,n}(x;\alpha)$ is increasing on $(0,\infty)$. In particular, for $\alpha = \alpha_0$, the inequality

$$\frac{q^{x+\alpha_0}-1}{\ln q} < \frac{\psi_{q,n}(x)}{\psi_{q,n+1}(x)}$$

holds for $x > 0$.

(ii) If $\alpha \geq 0$ then the function $x \mapsto F_{q,n}(x;\alpha)$ is decreasing on $(0,\infty)$. In particular, for $\alpha = 0$, the double inequality

$$\frac{\ln q}{q^x-1} < \frac{\psi_{q,n+1}(x)}{\psi_{q,n}(x)} < \frac{(n+1)\ln q}{q^x-1} \tag{1.12}$$

holds for $x > 0$. The lower and upper bounds are sharp.

(iii) If $\log_q(2^n/(q+1)) < \alpha < 0$, then there is an $x_0 > 0$ such that the function is increasing on $(0, x_0)$ and decreasing on (x_0, ∞).

2. Tools

To prove our results, we need several tools: the monotonicity rules for the ratio of two power series, the signs rule for the NP (PN)-type power series, and an important limit formula.

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2.1. **Monotonicity rules for the ratio of two power series.** The following lemma is due to Biernacki and Krzyz [11], which play an important role in dealing with the monotonicity of the ratio of power series.

Lemma 1. *Let* $A(t) = \sum_{n=0}^{\infty} a_n t^n$ *and* $B(t) = \sum_{n=0}^{\infty} b_n t^n$ *be two real power series converging on* $(-r, r)$ $(r > 0)$ *with* $b_n > 0$ *for all* n*. If the sequence* $\{a_n/b_n\}_{n\geq 0}$ *is increasing (decreasing), then so is the ratio* $A(t)/B(t)$ *on* $(0, r)$*.*

Another monotonicity rule in the case when the sequence $\{a_n/b_n\}_{n\geq 0}$ is piecewise monotonic was established by Yang, Chu and Wang in [12, Theorem 2.1], which is efficient to study for certain special functions, see [13], [14], [15], [16], [17].

Before stating this monotonicity rule, we introduce an auxiliary function $H_{f,g}$ given first in [18], which was called Yang's H-function in [19] by Tian et. al. For $-\infty \leq a < b \leq \infty$, let f and g be differentiable on (a, b) and $g' \neq 0$ on (a, b). Then the function $H_{f,g}$ is defined by

$$H_{f,g} := \frac{f'}{g'} g - f. \tag{2.1}$$

The following lemma is a modified version of [12, Theorem 2.1] and appeared in [20].

Lemma 2. [20] *Let* $A(t) = \sum_{k=0}^{\infty} a_k t^k$ *and* $B(t) = \sum_{k=0}^{\infty} b_k t^k$ *be two real power series converging on* $(-r, r)$ *and* $b_k > 0$ *for all* k*. Suppose that for certain* $m \in \mathbb{N}$*, the sequences* $\{a_k/b_k\}_{0\leq k\leq m}$ *and* $\{a_k/b_k\}_{k\geq m}$ *are both non-constant, and they are increasing (decreasing) and decreasing (increasing), respectively. Then the function* A/B *is strictly increasing (decreasing) on* $(0, r)$ *if and only if* $H_{A,B}(r^-) \geq (\leq) 0$*. If* $H_{A,B}(r^-) < (>) 0$*, then there exists* $t_0 \in (0, r)$ *such that the function* A/B *is strictly increasing (decreasing) on* $(0, t_0)$ *and strictly decreasing (increasing) on* (t_0, r)*.*

2.2. **Signs rule for the NP (PN)-type power series.** We begin with introducing certain special sequences containing positive (negative) sequence, NP and PN-type sequences. If every term of a real sequence is nonnegative (nonpositive) and at least one is non-zero, then this sequence is called a positive (negative) sequence. Let $m \in \mathbb{N}$. A real sequence $\{a_n\}_{n\geq 0}$ is called an negative-positive-type sequence, NP-type sequence for short, if the subsequences $\{a_n\}_{0\leq n\leq m}$ and $\{a_n\}_{n>m}$ are negative and positive sequences, respectively. $\{-a_n\}_{n\geq 0}$ is called a positive-negative-type sequence, PN-type sequence for short. The NP or PN-type power series is defined as follows.

Definition 1 ([21])**.** *The power series* $S(t) = \sum_{k=0}^{\infty} a_k t^k$ *is called an NP-type power series if the sequence* $\{a_n\}_{n\geq 0}$ *is an NP-type sequence.* $-S(t)$ *is called a PN-type power series.*

For the NP or PN-type power series, a simple but efficient criterion to determine their signs has been proven in [22], which is a revised version of the electronic preprint [23], and proven differently in [24].

Lemma 3. *Let* $S(t)$ *be an NP-type power series converging on the interval* $(0, r)$ $(r > 0)$*. (i) If* $S(r^-) \leq 0$*, then* $S(t) < 0$ *for all* $t \in (0, r)$*. (ii) If* $S(r^-) > 0$*, then there is a unique* $t_0 \in (0, r)$ *such that* $S(t) < 0$ *for* $t \in (0, t_0)$ *and* $S(t) > 0$ *for* $t \in (t_0, r)$*.*

Remark 1. *If* $r = \infty$*, then Lemma 3 is changed to* [25, Lemma 6.3].

2.3. **An important limit formula.** The following lemma was listed in [26, Problems 85].

Lemma 4. *If two given infinite sequences* $\{a_n\}_{n\geq 0}$ *and* $\{b_n\}_{n\geq 0}$ *satisfy the conditions: (i)* $b_n > 0$ *for all* $n \geq 0$*; (ii)* $\sum\limits_{n=0}^{\infty} b_n t^n$ *is convergent for* $|t| < 1$ *and divergent for* $t = 1$*; (iii)* $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = s$*. Then* $\sum_{n=0}^{\infty} a_n t^n$ *converges for* $|t| < 1$ *and*

$$\lim_{t\to 1^-} \frac{\sum_{n=0}^{\infty} a_n t^n}{\sum_{n=0}^{\infty} b_n t^n} = s.$$

3. Proof of Theorem 1

The one parameter mean of two distinct positive numbers a and b is defined by

$$J_p(a,b) = \frac{p}{p+1} \frac{a^{p+1} - b^{p+1}}{a^p - b^p} \text{ if } p \neq -1, 0$$

and

$$\begin{aligned} J_{-1}(a,b) &= \lim_{p\to -1} J_p(a,b) = ab \frac{\ln a - \ln b}{a - b} = \frac{G^2(a,b)}{L(a,b)}, \\ J_0(a,b) &= \lim_{p\to 0} J_p(a,b) = \frac{a-b}{\ln a - \ln b} = L(a,b). \end{aligned}$$

It was proved in [27, Theorem 1] that the function $p \mapsto J_p(a,b)$ is increasing on $(-\infty, \infty)$, and is log-convex on $(-\infty, -1/2)$ and log-concave on $(-1/2, \infty)$. The following lemma provides a new property of the function $p \mapsto J_p(a,b)$, which will be directly used to prove our main result.

Lemma 5. *Let* $a > b > 0$*. The function*

$$p \mapsto W_\theta(a,b;p) = \frac{p^\theta}{(p+1)^{\theta-1}} J_p(a,b)$$

is convex on $(0,\infty)$ *if and only if* $\theta \geq 1$*.*

Proof. Making a change of variable $t = \ln\sqrt{a/b}$, $J_p(a,b)$ can be expressed as

$$\begin{aligned} \frac{J_p(a,b)}{\sqrt{ab}} &= \frac{p}{p+1} \frac{(ab)^{(p+1)/2}}{\sqrt{ab}\,(ab)^{p/2}} \frac{a^{(p+1)/2}/b^{(p+1)/2} - b^{(p+1)/2}/a^{(p+1)/2}}{a^{p/2}/b^{p/2} - b^{p/2}/a^{p/2}} \\ &= \frac{p}{p+1} \frac{\left(\sqrt{a/b}\right)^{p+1} - \left(\sqrt{a/b}\right)^{-(p+1)}}{\left(\sqrt{a/b}\right)^{p} - \left(\sqrt{a/b}\right)^{-p}} = \frac{p}{p+1} \frac{\sinh(pt+t)}{\sinh(pt)}, \end{aligned}$$

and then, $W_\theta(a,b;p)$ can be represented as

$$\frac{W_\theta(a,b;p)}{\sqrt{ab}} = \frac{p^{\theta+1}}{(p+1)^\theta} \frac{\sinh(pt+t)}{\sinh(pt)} := \mathsf{w}_\theta(t,p).$$

Differentiation yields

$$\frac{\partial \mathsf{w}_\theta}{\partial p} = \frac{(\theta+p+1)\,p^\theta}{(p+1)^{\theta+1}} \frac{\sinh(pt+t)}{\sinh(pt)} - \frac{p^{\theta+1}}{(p+1)^\theta} \frac{t\sinh t}{\sinh^2(pt)},$$

$$\begin{aligned}\frac{\partial^2 \mathsf{w}_\theta}{\partial p^2} &= \frac{\theta(\theta+1)p^{\theta-1}}{(p+1)^{\theta+2}}\frac{\sinh(pt+t)}{\sinh(pt)} - 2\frac{(\theta+p+1)p^\theta}{(p+1)^{\theta+1}}\frac{t\sinh t}{\sinh^2(pt)} \\ &\quad + \frac{p^{\theta+1}}{(p+1)^\theta}\frac{2t^2\sinh t\cosh(pt)}{\sinh^3(pt)} := \frac{p^{\theta-1}}{(p+1)^{\theta+2}\sinh^3(pt)}V_\theta(t,p),\end{aligned}$$

where

$$\begin{aligned}V_\theta(t,p) = {} & \theta(\theta+1)\sinh^2(pt)\sinh(pt+t) \\ & -2p(p+1)(\theta+p+1)t\sinh t\sinh(pt) \\ & +2p^2(p+1)^2t^2\sinh t\cosh(pt).\end{aligned} \tag{3.1}$$

If $p \mapsto W_\theta(a,b;p)$ is convex on $(0,\infty)$ for $a > b > 0$, then for all $p,t > 0$,

$$\lim_{t\to 0}\frac{V_\theta(t,p)}{t^3} \geq 0.$$

Expanding in power series of t yields

$$V_\theta(t,p) = p^2(p+1)\theta(\theta-1)t^3 + O(t^5),$$

which implies that

$$\lim_{t\to 0}\frac{V_\theta(t,p)}{t^3} = p^2(p+1)\theta(\theta-1).$$

Therefore, the necessary condition for $V_\theta(t,p) \geq 0$ for all $t,p > 0$ is that: $\theta \geq 1$.

It remains to prove that $V_\theta(t,p) > 0$ for all $t,p > 0$ if $\theta \geq 1$. Applying the known inequality $x\cosh x > \sinh x$ for $x > 0$, the sum of the second and third of the expression of $V_\theta(t,p)$ is greater than

$$\begin{aligned}& -2p(p+1)(\theta+p+1)t\sinh t\sinh(pt) + 2p(p+1)^2t\sinh t\sinh(pt) \\ = {} & -2\theta p(p+1)t\sinh t\sinh(pt),\end{aligned}$$

then

$$\begin{aligned}V_\theta(t,p) &> \theta(\theta+1)\sinh^2(pt)\sinh(pt+t) - 2\theta p(p+1)t\sinh t\sinh(pt) \\ &= 2\theta p(p+1)t\sinh t\sinh(pt)\left[\frac{\theta+1}{2}\frac{\sinh(pt)}{pt}\frac{\sinh(pt+t)}{(p+1)\sinh t} - 1\right] > 0,\end{aligned}$$

where the last inequality holds due to $\theta \geq 1$, $\sinh(pt) > pt$ and $\sinh(pt+t) > (p+1)\sinh t$ for $p,t > 0$. This completes the proof. □

Lemma 6. *Let $q \in (0,1)$ and $n \in \mathbb{N}$. Then the function $\psi_{q,n}/\psi_{q,n+1}$ is increasing from $(0,\infty)$ onto $(0,-1/\ln q)$. Consequently, for $x > 0$ we have the inequality*

$$\psi_{q,n}(x)\psi_{q,n+2}(x) - \psi_{q,n+1}^2(x) > 0. \tag{3.2}$$

Proof. Using the representation (1.7) yields

$$\frac{\psi_{q,n}(x)}{\psi_{q,n+1}(x)} = \frac{(-\ln q)^{n+1}\sum_{k=1}^\infty b_k t^k}{(-\ln q)^{n+2}\sum_{k=1}^\infty k b_k t^k} = \frac{1}{-\ln q}\frac{\sum_{k=1}^\infty b_{k+1}t^k}{\sum_{k=1}^\infty (k+1)b_{k+1}t^k}, \tag{3.3}$$

where $t = q^x$ and

$$b_k = \frac{k^n}{1-q^k}. \tag{3.4}$$

Since the ratio of those coefficients of power series in (3.3) is clearly decreasing, by Lemma 1 the ratio of power series in (3.3) is so with respect to t, which implies that the function $\psi_{q,n}/\psi_{q,n+1}$ is increasing on $(0,\infty)$ with

$$\lim_{x\to\infty}\frac{\psi_{q,n}(x)}{\psi_{q,n+1}(x)}=\frac{1}{-\ln q}\lim_{t\to 0}\frac{\sum_{k=0}^{\infty}b_{k+1}t^k}{\sum_{k=0}^{\infty}(k+1)b_{k+1}t^k}=\frac{1}{-\ln q},$$

and by Lemma 4,

$$\lim_{x\to 0}\frac{\psi_{q,n}(x)}{\psi_{q,n+1}(x)}=\frac{1}{-\ln q}\lim_{t\to 1}\frac{\sum_{k=0}^{\infty}b_{k+1}t^k}{\sum_{k=0}^{\infty}(k+1)b_{k+1}t^k}=\frac{1}{-\ln q}\lim_{t\to 1}\frac{b_{k+1}}{(k+1)b_{k+1}}=0.$$

Using the increasing property of $\psi_{q,n}/\psi_{q,n+1}$ on $(0,\infty)$, we have

$$\left(\frac{\psi_{q,n}}{\psi_{q,n+1}}\right)'=\frac{\psi_{q,n}'}{\psi_{q,n+1}}+\psi_{q,n}\left(-\frac{\psi_{q,n+1}'}{\psi_{q,n+1}^2}\right)=\frac{\psi_{q,n}\psi_{q,n+2}}{\psi_{q,n+1}^2}-1>0,$$

which implies (3.2), and the proof is completed. □

We are now in a position to prove our main result.

Proof of Theorem 1. Using the representation (1.7) yields

$$(-\ln q)\,\psi_{q,n}(x)=(-\ln q)^{n+2}\sum_{k=1}^{\infty}\frac{k^n q^{kx}}{1-q^k}=(-\ln q)^{n+2}\sum_{k=1}^{\infty}b_k t^k:=g(t), \tag{3.5}$$

where $t=q^x$ and b_k is given by (3.4);

$$\begin{aligned}\left(1-q^{x+\alpha}\right)\psi_{q,n+1}(x) &= (1-tq^{\alpha})(-\ln q)^{n+2}\sum_{k=1}^{\infty}kb_k t^k\\ &= (-\ln q)^{n+2}\left[b_1 t+\sum_{k=2}^{\infty}\left(kb_k-q^{\alpha}(k-1)b_{k-1}\right)t^k\right]\\ &= (-\ln q)^{n+2}\sum_{k=1}^{\infty}a_k t^k:=f(t),\end{aligned} \tag{3.6}$$

where

$$a_1=b_1 \quad\text{and}\quad a_k=kb_k-q^{\alpha}(k-1)b_{k-1} \quad\text{for } k\ge 2. \tag{3.7}$$

Then $F_{q,n}(x;\alpha)$ can be expressed as

$$F_{q,n}(x;\alpha)=\frac{f(t)}{g(t)}=\frac{1}{-\ln q}\frac{(-\ln q)^{n+2}\sum_{k=1}^{\infty}a_k t^k}{(-\ln q)^{n+1}\sum_{k=1}^{\infty}b_k t^k}=\frac{\sum_{k=1}^{\infty}a_k t^k}{\sum_{k=1}^{\infty}b_k t^k}.$$

To prove the monotonicity of the function $F_{q,n}(x)$, we have to observe the monotonicity of the sequence $\{a_k/b_k\}_{k\ge 1}$. A simple computation leads to $a_1/b_1=1$ and for $k\ge 2$,

$$\frac{a_k}{b_k}=k-q^{\alpha}(k-1)\frac{b_{k-1}}{b_k}=k-q^{\alpha}\frac{(k-1)^{n+1}}{k^n}\frac{1-q^k}{1-q^{k-1}}.$$

Then

$$d_1:=\frac{a_2}{b_2}-\frac{a_1}{b_1}=1-q^{\alpha}\frac{1+q}{2^n}:=1-q^{\alpha}u_1,$$

and for $k \geq 2$,

$$d_k := \frac{a_{k+1}}{b_{k+1}} - \frac{a_k}{b_k} = 1 - q^{\alpha} k \frac{b_k}{b_{k+1}} + q^{\alpha} (k-1) \frac{b_{k-1}}{b_k} := 1 - q^{\alpha} u_k, \tag{3.8}$$

where

$$u_k = \frac{k^{n+1}}{(k+1)^n} \frac{1-q^{k+1}}{1-q^k} - \frac{(k-1)^{n+1}}{k^n} \frac{1-q^k}{1-q^{k-1}}. \tag{3.9}$$

Since

$$\lim_{k\to 1} d_k = 1 - q^{\alpha} \frac{1+q}{2^n} = d_1,$$

the formula (3.8) is valid for all $k \geq 1$.

Using the notation of the one parameter mean $J_p(a,b)$, u_k can be written as

$$u_k = \frac{k^n}{(k+1)^{n-1}} J_k(1,q) - \frac{(k-1)^n}{k^{n-1}} J_{k-1}(1,q)$$

for $k \geq 1$. By Lemma 5 we see that the sequence $\{(k+1)^{1-n} k^n J_k(1,q)\}_{k\geq 1}$ is convex for $k \geq 1$, and then, the sequence $\{u_k\}_{k\geq 1}$ is increasing. Moreover, we have $u_\infty = \lim_{k\to\infty} u_k = 1$. In fact, u_k can be written as

$$\begin{aligned} u_k &= \left(\frac{k-1}{k}\right)^n \frac{1-q^k}{1-q^{k-1}} - \frac{1}{k} \frac{\left(1-k^{-2}\right)^n - 1}{k^{-2}} \left(\frac{k}{k+1}\right)^n \frac{1-q^k}{1-q^{k-1}} \\ &\quad - \left(\frac{k}{k+1}\right)^n \frac{k(q-1)^2 q^{k-1}}{(1-q^k)(1-q^{k-1})}, \end{aligned}$$

which clearly tends to 1 as $k \to \infty$ for fixed $q \in (0,1)$ and $n \geq 1$.

(i) If $q^{-\alpha} \leq \min_{k\geq 1}\{u_k\} = u_1$, that is, $\alpha \leq -\log_q u_1 = \log_q(2^n/(q+1))$, then $d_k = 1 - q^{\alpha} u_k \leq 0$ for all $k \geq 1$, which indicates that the sequence $\{a_k/b_k\}_{k\geq 1}$ is decreasing. It follows from Lemma 1 that the ratio $f(t)/g(t)$ is decreasing with respect to t on $(0,1)$, and so the function $x \mapsto F_{q,n}(x;\alpha)$ is increasing on $(0,\infty)$.

(ii) If $q^{-\alpha} \geq \lim_{k\to\infty} u_k = 1$, that is, $\alpha \geq 0$, then $d_k = 1 - q^{\alpha} u_k \geq 0$ for all $k \geq 1$, which implies that the sequence $\{a_k/b_k\}_{k\geq 1}$ is increasing. It follows from Lemma 1 that the ratio $f(t)/g(t)$ is increasing with respect to t on $(0,1)$, and so the function $x \mapsto F_{q,n}(x;\alpha)$ is decreasing on $(0,\infty)$.

(iii) When $(q+1)/2^n = u_1 < q^{-\alpha} < u_\infty = 1$, that is, $\log_q(2^n/(q+1)) < \alpha < 0$, since the sequence $d_k = 1 - q^{\alpha} u_k$ is decreasing for $k \geq 1$ with

$$d_1 = 1 - q^{\alpha} u_1 > 0 \quad \text{and} \quad d_\infty = 1 - q^{\alpha} u_\infty < 0,$$

there is a positive integer $k_0 > 1$ such that $d_k > 0$ for $1 \leq k < k_0$ and $d_k < 0$ for $k > k_0$, namely, the sequence $\{a_k/b_k\}_{k\geq 1}$ is increasing for $1 \leq k \leq k_0$ and decreasing for $k > k_0$. If we prove that

$$\lim_{t\to 1^-} H_{f,g}(t) = \lim_{t\to 1^-} \left(\frac{f'(t)}{g'(t)} g(t) - f(t)\right) < 0,$$

then by Lemma 2 we deduce that there is a $t_0 \in (0,1)$ such that $f(t)/g(t)$ is increasing on $(0,t_0)$ and decreasing on $(t_0,1)$, which, due to $t = q^x$, shows that $x \mapsto F_{q,n}(x;\alpha)$ is decreasing on (x_0,∞) and increasing on $(0,x_0)$, where $x_0 =$

$(\ln t_0)/\ln q$, the third assertion then follows. Now, since

$$f'(t) = \left[\left(1-q^{x+\alpha}\right)\psi_{q,n+1}(x)\right]' \frac{d}{dt}\frac{\ln t}{\ln q} = \frac{\left(-q^{x+\alpha}\ln q\right)\psi_{q,n+1}(x) - \left(1-q^{x+\alpha}\right)\psi_{q,n+2}(x)}{t\ln q},$$

$$g'(t) = (-\ln q)\,\psi'_{q,n}(x)\frac{d}{dt}\frac{\ln t}{\ln q} = \frac{1}{t}\psi_{q,n+1}(x),$$

we derive that

$$H_{f,g}(t) = \frac{\left[\left(-q^{x+\alpha}\ln q\right)\psi_{q,n+1}(x) - \left(1-q^{x+\alpha}\right)\psi_{q,n+2}(x)\right]/(t\ln q)}{\psi_{q,n+1}(x)/t} \times (-\ln q)\,\psi_{q,n}(x) - \left(1-q^{x+\alpha}\right)\psi_{q,n+1}(x)$$
$$= \left(q^{x+\alpha}\ln q\right)\psi_{q,n}(x) + \left(1-q^{x+\alpha}\right)\frac{\psi_{q,n+2}(x)\,\psi_{q,n}(x) - \psi_{q,n+1}(x)^2}{\psi_{q,n+1}(x)}.$$

Due to $q^x(\ln q)\,\psi_{q,n}(x) < 0$, $\psi_{q,n+2}(x)\,\psi_{q,n}(x) - \psi_{q,n+1}(x)^2 > 0$ (due to (3.2)) and $\lim_{x\to 0^+}\left(1-q^{x+\alpha}\right) = 1-q^\alpha < 0$, we arrive at $\lim_{t\to 1^-} H_{f,g}(t) < 0$.

Finally, we find the limit values of $f(t)/g(t)$ as $t \to 0, 1$. Clearly, $\lim_{t\to 0^+}[f(t)/g(t)] = a_1/b_1 = 1$. To compute $\lim_{t\to 1^-}[f(t)/g(t)]$, we note that $b_k > 0$ for all $k \geq 1$, $g(t) = (-\ln q)^{n+2}\sum_{k=1}^{\infty} b_k t^k$ is convergent for all $t \in (0,1)$ and $g(t)$ is divergent for $t = 1$; moreover, since

$$k - \frac{(k-1)^{n+1}}{k^n}\frac{1-q^k}{1-q^{k-1}} = \frac{1-(1-1/k)^{n+1}}{1/k}\frac{1-q^k}{1-q^{k-1}} - \frac{kq^{k-1}(1-q)}{1-q^{k-1}} \to n+1$$

as $k \to \infty$, we have

$$\lim_{k\to\infty}\frac{a_k}{b_k} = \lim_{k\to\infty}\left[k - q^\alpha\frac{(k-1)^{n+1}}{k^n}\frac{1-q^k}{1-q^{k-1}}\right] = \begin{cases} n+1 & \text{if } \alpha = 0, \\ \operatorname{sgn}(\alpha)\,\infty & \text{if } \alpha \neq 0. \end{cases}$$

From Lemma 4 it follows that

$$\lim_{x\to 0^+} F_{q,n}(x;\alpha) = \lim_{t\to 1^-}\frac{f(t)}{g(t)} = \lim_{t\to 1^-}\frac{\sum_{k=1}^{\infty} a_k t^k}{\sum_{k=1}^{\infty} b_k t^k} = \begin{cases} n+1 & \text{if } \alpha = 0, \\ \operatorname{sgn}(\alpha)\,\infty & \text{if } \alpha \neq 0. \end{cases}$$

Using the monotonicity of the function $f(t)/g(t)$ on $(0,1)$, the required inequalities follow. This completes the proof. □

Remark 2. *From the end of the proof of Theorem 1 we see that, for $q \in (0,1)$,*

$$\lim_{x\to 0^+} F_{q,n}(x;0) = \lim_{x\to 0^+}\left(\frac{q^x-1}{\ln q}\frac{\psi_{q,n+1}(x)}{\psi_{q,n}(x)}\right) = n+1. \tag{3.10}$$

Remark 3. *For $m, n \in \mathbb{N}$ with $n > m$, since*

$$\prod_{j=m}^{n-1}\left(\frac{q^x-1}{\ln q}\frac{\psi_{q,j+1}(x)}{\psi_{q,j}(x)}\right) = \left(\frac{q^x-1}{\ln q}\right)^{n-m}\frac{\psi_{q,n}(x)}{\psi_{q,m}(x)},$$

we find that the function

$$x \mapsto \frac{q^x-1}{\ln q}\left(\frac{\psi_{q,n}(x)}{\psi_{q,m}(x)}\right)^{1/(n-m)}$$

is also decreasing from $(0,\infty)$ onto $\left(1,(n!/m!)^{1/(n-m)}\right)$.

4. Several relevant results

Letting $p \to q = 0$ in [28, Theorem 2] yields that the function $x \mapsto n\psi_n(x)/\psi_{n+1}(x) - x$ is decreasing from $(0,\infty)$ onto $(-1/2, 0)$. Further, using the monotonicity rules for the ratio of two Laplace transforms given in [29], [30], we can prove that the function $x \mapsto \lambda\psi_n(x)/\psi_{n+1}(x) - x$ is decreasing (increasing) on $(0,\infty)$ if and only if $\lambda \le n$ ($\lambda \ge n+1$). This reminds us to guess that the function

$$x \mapsto f_{q,n}(x;\beta) = \frac{\beta\psi_{q,n}(x)}{\psi_{q,n+1}(x)} - \frac{q^x - 1}{\ln q} \tag{4.1}$$

has a similar monotonicity result on $(0,\infty)$. But we find that it is difficult to deal with this problem. Fortunately, we can prove the increasing property of $x \mapsto f_{q,n}(x;\beta)$ for $\beta = n+1$ using Theorem 1 and Lemma 6.

Proposition 1. *Let $q \in (0,1)$ and $n \in \mathbb{N}$. The function $x \mapsto f_{q,n}(x;n+1)$ is increasing from $(0,\infty)$ onto $(0, -n/\ln q)$. Consequently, the double inequality*

$$\frac{(n+1)\ln q}{q^x - 1 - n} < \frac{\psi_{q,n+1}(x)}{\psi_{q,n}(x)} < \frac{(n+1)\ln q}{q^x - 1} \tag{4.2}$$

holds for $x > 0$. The lower and upper bounds are sharp.

Proof. By Theorem 1 (ii) we see that the function

$$x \mapsto n + 1 - \frac{q^x - 1}{\ln q}\frac{\psi_{q,n+1}(x)}{\psi_{q,n}(x)}$$

is positive and increasing on $(0,\infty)$; while the function $x \mapsto \psi_{q,n}(x)/\psi_{q,n+1}(x)$ is also positive and increasing on $(0,\infty)$ due to Lemma 6. Then so is the function

$$x \mapsto \left(n + 1 - \frac{q^x - 1}{\ln q}\frac{\psi_{q,n+1}(x)}{\psi_{q,n}(x)}\right)\frac{\psi_{q,n}(x)}{\psi_{q,n+1}(x)} = \frac{(n+1)\psi_{q,n}(x)}{\psi_{q,n+1}(x)} - \frac{q^x - 1}{\ln q}$$

on $(0,\infty)$. Employing those computed results shown in Lemma 6, we obtain

$$\lim_{x\to 0} f_{q,n}(x;n+1) = 0 \quad \text{and} \quad \lim_{x\to\infty} f_{q,n}(x;n+1) = \frac{n+1}{-\ln q} - \frac{1}{-\ln q} = \frac{n}{-\ln q}.$$

Then the required double inequality follows from the increasing property of $f_{q,n}(x;n+1)$ on $(0,\infty)$, which completes the proof. □

Remark 4. *Clearly, the lower bound in (4.2) is weaker than the one in (1.12) due to*

$$\frac{\ln q}{q^x - 1} - \frac{(n+1)\ln q}{q^x - 1 - n} = \frac{nq^x \ln q}{(q^x - 1)(n + 1 - q^x)} > 0$$

for $q \in (0,1)$.

Since $(\ln q)/(q^x - 1) > 1/x$ for $q \in (0,1)$ and $x > 0$, by the left hand side inequality of (1.12) we have

$$\frac{1}{x} < \frac{\ln q}{q^x - 1} < \frac{\psi_{q,n+1}(x)}{\psi_{q,n}(x)}$$

for $x > 0$. This yields the following corollary.

Corollary 1. *Let $q\in(0,1)$ and $n\in\mathbb{N}$. The inequality*

$$\psi_{q,n}(x)-x\psi_{q,n+1}(x)<0,$$

or equivalently,

$$(-1)^{n-1}\left[\psi_q^{(n)}(x)+x\psi_q^{(n+1)}(x)\right]<0$$

holds for $x>0$, In particular, when $n=1$ we have

$$\psi_q'(x)+x\psi_q''(x)<0 \tag{4.3}$$

for $x>0$.

Remark 5. *The differential inequality (4.3) was recently proved by Alzer and Salem in* [31, Theorem 3.1]*, which plays a central role in the proofs of those main results in* [31].

Let us return to Proposition 1. Since the function $x\mapsto f_{q,n}(x;n+1)$ is increasing on $(0,\infty)$, we have

$$\frac{\partial}{\partial x}f_{q,n}(x;n+1)=(n+1)\frac{\psi_{q,n}(x)\psi_{q,n+2}(x)}{\psi_{q,n+1}(x)^2}-(n+1)-q^x>0$$

for $x>0$. We thus obtain the following corollary.

Corollary 2. *Let $q\in(0,1)$ and $n\in\mathbb{N}$. Then for $x>0$, we have*

$$\frac{\psi_{q,n}(x)\psi_{q,n+2}(x)}{\psi_{q,n+1}(x)^2}>1+\frac{q^x}{n+1}. \tag{4.4}$$

Remark 6. *Clearly, the inequality (4.4) is better than (3.2).*

Alzer [8, Lemmas 1 and 2] (see also [32]) proved that the function $x\mapsto x^c\psi_n(x)$ for $n\in\mathbb{N}$ is strictly decreasing (increasing) on $(0,\infty)$ if and only if $c\le n$ ($c\ge n+1$). Similarly, we can determine the best $r\in\mathbb{R}$ such that the function

$$x\mapsto g_{q,n}(x;r)=\left(\frac{1-q^{-x}}{\ln q}\right)^r\psi_{q,n}(x) \tag{4.5}$$

is increasing or decreasing on $(0,\infty)$, which reads as follows.

Proposition 2. *Let $q\in(0,1)$ and $n\in\mathbb{N}$. The function $x\mapsto g_{q,n}(x;r)$ is increasing (decreasing) on $(0,\infty)$ if and only if $r\ge n+1$ ($r\le 1$). While if $1<r<n+1$, there is an $x_0>0$ such that $x\mapsto g_{q,n}(x;r)$ is decreasing on $(0,x_0)$ and increasing on (x_0,∞).*

Proof. Differentiation yields

$$\begin{aligned}\frac{\partial g_{q,n}}{\partial x} &= r\left(\frac{1-q^{-x}}{\ln q}\right)^{r-1}q^{-x}\psi_{q,n}(x)-\left(\frac{1-q^{-x}}{\ln q}\right)^r\psi_{q,n+1}(x)\\ &= \left(\frac{1-q^{-x}}{\ln q}\right)^{r-1}q^{-x}\psi_{q,n}(x)\left[r-F_{q,n}(x;0)\right],\end{aligned}$$

where

$$F_{q,n}(x;0)=\frac{q^x-1}{\ln q}\frac{\psi_{q,n+1}(x)}{\psi_{q,n}(x)}$$

is as in (1.11). Using Theorem 1 we deduce that $\partial g_{q,n}/\partial x\ge(\le)\,0$ if and only if

$$r\ge\sup_{x>0}F_{q,n}(x;0)=n+1 \text{ or } r\le\inf_{x>0}F_{q,n}(x;0)=1.$$

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While $1 < r < n+1$, since $x \mapsto r - F_{q,n}(x;0)$ is increasing on $(0,\infty)$ with

$$\lim_{x\to 0}(r - F_{q,n}(x;0)) = r-(n+1) < 0,$$
$$\lim_{x\to\infty}(r - F_{q,n}(x;0)) = r-1 > 0,$$

there is an $x_0 > 0$ such that $r - F_{q,n}(x;0) < 0$ for $x \in (0,x_0)$ and $r - F_{q,n}(x;0) > 0$ for $x \in (x_0,\infty)$. That is, $\partial g_{q,n}/\partial x < 0$ for $x \in (0,x_0)$ and $\partial g_{q,n}/\partial x > 0$ for $x \in (x_0,\infty)$, which completes the proof. □

Note that

$$\frac{d}{dx}\frac{x\ln q}{1-q^{-x}} = -\frac{q^{-x}\ln q}{(1-q^{-x})^2}(1+\ln q^x - q^x) < 0$$

for $x > 0$ and $q \in (0,1)$. By Proposition 2 we find that the function

$$\frac{x\ln q}{1-q^{-x}} g_{q,n}(x;1) = \frac{x\ln q}{1-q^{-x}}\frac{1-q^{-x}}{\ln q}\psi_{q,n}(x) = x\psi_{q,n}(x)$$

is also decreasing with respect to x on $(0,\infty)$.

Corollary 3. *Let $q \in (0,1)$ and $n \in \mathbb{N}$. The function $x \mapsto \xi_{q,n}(x) = x\psi_{q,n}(x)$ is decreasing on $(0,\infty)$.*

Remark 7. *Recently, several mean inequalities for the q-gamma and q-digamma functions were obtained in [33], [34]. Using the decreasing property of the function $x \mapsto x\psi_{q,n}(x)$ on $(0,\infty)$, we can prove the following mean inequality*

$$\frac{\psi_{q,n}(x) + \psi_{q,n}(1/x)}{2} \geq \psi_{q,n}(1)$$

for $x > 0$, $q \in (0,1)$ and $n \in \mathbb{N}_0$. In fact, by a differentiation we have

$$\left[\psi_{q,n}(x) + \psi_{q,n}\left(\frac{1}{x}\right)\right]' = \psi'_{q,n}(x) - \frac{1}{x^2}\psi'_{q,n}\left(\frac{1}{x}\right)$$
$$= -\frac{1}{x}\left[x\psi_{q,n+1}(x) - \frac{1}{x}\psi_{q,n+1}\left(\frac{1}{x}\right)\right] = -\frac{1}{x}\left[\xi_{q,n+1}(x) - \xi_{q,n+1}\left(\frac{1}{x}\right)\right],$$

which, by Corollary 3, is positive if $x > 1$ and negative if $0 < x < 1$. It then follows that

$$\psi_{q,n}(x) + \psi_{q,n}\left(\frac{1}{x}\right) \geq \psi_{q,n}(1) + \psi_{q,n}(1) = 2\psi_{q,n}(1)$$

for $x > 0$.

Recall that a function f is called completely monotonic on an interval I, if f has the derivative of any order on I and satisfies

$$(-1)^k f^{(k)}(x) \geq 0$$

for all $k \in \mathbb{N}_0$ on I, see [35, 36]. As early as in 1986, Ismail [37] began to investigate the complete monotonicity of the q-gamma function. Using the Stieltjes integral representation (1.8) he and coauthors in [3], [4] effectively dealt with some problems on the complete monotonicity of q-gamma and q-polygamma functions. In 2013, Salem [38, Theorem 3.1] proved a nice result, which states that the remainder of the asymptotic expansion of $\ln\Gamma_q(x)$ is completely monotonic on $(0,\infty)$, and generalized Alzer's result in [39, Theorem 8]. More completely monotonic functions involving the q-gamma and q-polygamma functions can be found in [40], [41], [42], [43], [44], [45], and references therein.

Now, by Lemma 5 and Lemma 3, we shall prove that the function

$$x \mapsto h_{q,n}(x;\eta) = q^{-x}\left[\eta\psi_{q,n}(x) - \frac{q^x-1}{\ln q}\psi_{q,n+1}(x)\right] \tag{4.6}$$

is completely monotonic on $(0,\infty)$.

Proposition 3. *Let $q \in (0,1)$ and $n \in \mathbb{N}$. The following statements are valid:*

(i) The function $x \mapsto h_{q,n}(x;\eta)$ is completely monotonic on $(0,\infty)$ if and only if $\eta \geq n+1$.

(ii) The function $x \mapsto -h_{q,n}(x;\eta)$ is completely monotonic on $(0,\infty)$ if and only if $\eta \leq 1$.

(iii) If $1 < \eta < n+1$, then for every $m \in \mathbb{N}_0$, there is an $x_m > 0$ such that $(-1)^m h_{q,n}^{(m)}(x;\eta) > 0$ for (x_m,∞) and $(-1)^m h_{q,n}^{(m)}(x;\eta) < 0$ for $(0,x_m)$.

Proof. Let $q^x = t$. Using the representation (1.7) we obtain

$$\begin{aligned} h_{q,n}(x;\eta) &= (-\ln q)^{n+1}\left(\eta\sum_{k=1}^{\infty}\frac{k^n q^{(k-1)x}}{1-q^k} - \left(1-\frac{1}{q^x}\right)\sum_{k=1}^{\infty}\frac{k^{n+1}q^{kx}}{1-q^k}\right) \\ &= (-\ln q)^{n+1}\sum_{k=0}^{\infty}(\eta - v_k)\frac{(k+1)^n}{1-q^{k+1}}q^{kx}, \end{aligned}$$

where $v_0 = 1$ and for $k \geq 1$,

$$v_k = k+1 - \frac{k^{n+1}}{(k+1)^n}\frac{1-q^{k+1}}{1-q^k} = k+1 - \frac{k^n}{(k+1)^{n-1}}J_k(1,q).$$

Then, for $m \in \mathbb{N}_0$,

$$(-1)^m h_{q,n}^{(m)}(x;\eta) = (-\ln q)^{m+n+1}\sum_{k=0}^{\infty}(\eta - v_k)\frac{k^m(k+1)^n}{1-q^{k+1}}q^{kx} := \mathcal{H}(q^x).$$

By Lemma 5 it is seen that the sequence

$$v_k - v_{k-1} = \frac{(k-1)^n}{k^{n-1}}J_{k-1}(1,q) - \frac{k^n}{(k+1)^{n-1}}J_k(1,q) + 1$$

is decreasing for $k \geq 2$, and we have

$$v_k - v_{k-1} > \lim_{k\to\infty}(v_k - v_{k-1}) = 0 \text{ for } k \geq 2,$$

and

$$v_1 - v_0 = 2 - \frac{q+1}{2^n} - 1 = \frac{2^n-(q+1)}{2^n} \geq 0,$$

which indicates that the sequence $\{v_k\}_{k\geq 0}$ is increasing.

Case 1: $\eta \geq \lim_{k\to\infty} v_k = n+1$. Then $\eta - v_k > 0$ for all $k \geq 0$, and then $(-1)^m h_{q,n}^{(m)}(x;\eta) > 0$ for $x > 0$. That is, the function $x \mapsto h_{q,n}(x;\eta)$ is completely monotonic on $(0,\infty)$.

Case 2: $\eta \leq 1$. Then $\eta - v_k \leq 1 - v_0 = 0$ for $k \geq 0$, and then $(-1)^m h_{q,n}^{(m)}(x;\eta) < 0$ for $x > 0$. Hence, the function $x \mapsto -h_{q,n}(x;\eta)$ is completely monotonic on $(0,\infty)$.

Case 3: $1 = v_0 < \eta < v_\infty = n+1$. Since $(\eta - v_k) = v_k^*$ is decreasing for $k \geq 0$ with $v_1^* = \eta - v_0 > 0$ and $v_\infty^* = \eta - v_\infty < 0$, there is an integer k_0 such that

$v_k^* = (\eta - v_k) > 0$ for $1 \le k < k_0$ and $v_k^* = (\eta - v_k) < 0$ for $k > k_0$. This indicates that $\mathcal{H}(t)$ is a PN-type power series. Because

$$\begin{aligned} \lim_{t\to 1} \frac{\mathcal{H}(t)}{(-\ln q)^{n+1} q^{-x}\psi_{q,n}(x)} &= \lim_{x\to 0} \frac{h_{q,n}(x;\eta)}{q^{-x}\psi_{q,n}(x)} \\ &= \eta - \lim_{x\to 0}\left(\frac{q^x-1}{\ln q}\frac{\psi_{q,n+1}(x)}{\psi_{q,n}(x)}\right) = \eta - n - 1 < 0, \end{aligned}$$

by Lemma 3 we find that there is **a** $t_m \in (0,1)$ such that $\mathcal{H}(t) > 0$ for $t \in (0, t_m)$ and $\mathcal{H}(t) < 0$ for $t \in (t_m, 1)$. Therefore, there is a $x_m > 0$ such that $(-1)^m h_{q,n}^{(m)}(x;\eta) > 0$ for (x_m, ∞) and $(-1)^m h_{q,n}^{(m)}(x;\eta) < 0$ for $(0, x_m)$, where $x_m = \log_q t_m$. This completes the proof. □

5. Conclusions

In this paper, we proved that, for $q \in (0,1)$ and $n \in \mathbb{N}$, the function $x \mapsto F_{q,n}(x;\alpha)$ defined by (1.11) is increasing (decreasing) on $(0,\infty)$ if and only if $\alpha \le \alpha_0 = \log_q(2^n/(q+1))$, and is decreasing on $(0,\infty)$ if and only if $\alpha \ge 0$. This is similar to the monotonicity of the function $x \mapsto (x+r)\psi_{n+1}(x)/\psi_n(x)$. As a direct consequence, the function $x \mapsto (n+1)\psi_{q,n}(x)/\psi_{q,n+1}(x) - (q^x-1)/\ln q$ is increasing on $(0,\infty)$ for $q \in (0,1)$ and $n \in \mathbb{N}$, which yields the inequality (4.4). By means of the monotonicity of the $F_{q,n}(x;0)$ on $(0,\infty)$, we showed that the function $x \mapsto g_{q,n}(x;r)$ defined by (4.5) is increasing (decreasing) on $(0,\infty)$ if and only if $r \ge n$ $(r \le 1)$. Moreover, we found that the function $x \mapsto \pm h_{q,n}(x;\eta)$ is completely monotonic on $(0,\infty)$ if and only if $\eta \ge n+1$ $(\eta \le 1)$.

Finally, we list a problem and several remarks.

Remark 8. *It is difficult to compute the limit values involving q-gamma and q-polygamma functions when the independent variable tends to zero. Therefore, the limit relation (3.10) is significant. Moreover, it is checked that this limit relation is valid for all $q > 0$ and $n \in \mathbb{N}_0$ by employing the relation (1.4), L'Hospital rule and Lemma 4.*

Remark 9. *Noting that*

$$\frac{\psi_{q,n}(x)\psi_{q,n+2}(x)}{\psi_{q,n+1}(x)^2} = \left(\frac{q^x-1}{\ln q}\frac{\psi_{q,n+2}(x)}{\psi_{q,n+1}(x)}\right) \Big/ \left(\frac{q^x-1}{\ln q}\frac{\psi_{q,n+1}(x)}{\psi_{q,n}(x)}\right),$$

then utilizing the limit relation (3.10) gives

$$\lim_{x\to 0} \frac{\psi_{q,n}(x)\psi_{q,n+2}(x)}{\psi_{q,n+1}(x)^2} = \frac{n+2}{n+1}.$$

This together with inequality 4.4 inspires us to consider the following problem which is similar to the inequality

$$\frac{n+1}{n} > \frac{\psi_n(x)\psi_{n+2}(x)}{\psi_{n+1}(x)^2} > \frac{n+2}{n+1}$$

for $x > 0$ and $n \in \mathbb{N}$ (see [46, Theorem 2.1], [10, Corollary 2]*).*

Problem 2. *Let $q > 0$ with $q \ne 1$ and $n \in \mathbb{N}$. What are the conditions such that the inequalities*

$$\frac{\psi_{q,n}(x)\psi_{q,n+2}(x)}{\psi_{q,n+1}(x)^2} > (<) \frac{n+2}{n+1}$$

hold for all $x > 0$?

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