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GENERALIZED k-MITTAG-LEFFLER FUNCTION AND ITS PROPERTIES

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ABSTRACT. Motivated essentially by the success of the applications of the Mittag-Leffler function and its generalizations in Science and Engineering and k-Calculus, we propose here a unification of certain k-generalizations of Mittag-Leffler function including k-generalization of Saxena-Nishimoto's function, Bessel Maitland function, Dotsenko function, Elliptic function, etc. We obtain the order and type, asymptotic estimate, a differential equation, Eigen function property and double series relation for the proposed unification. As a specialization, a generalized k-Konhauser polynomial is considered for which the series inequality relations and inverse series relations are obtained.

1. Introduction

In 1903, the Swedish mathematician Gosta Mittag-Leffler introduced the function [10]:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}, \tag{1.1}$$

where z is a complex variable and $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$.
Recently, Dave and Nathwani [12, 13] introduced the generalization in the form:

$$E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r) = \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s}{\Gamma(\alpha n+\beta)\ [(\lambda)_{\mu n}]^r} \frac{z^n}{n!}, \tag{1.2}$$

wherein the parameters $\alpha,\ \beta,\ \gamma,\ \lambda\ \in \mathbb{C}$ with $\Re(\alpha,\beta,\gamma,\lambda) > 0,\ \delta,\ \mu > 0,\ r \in \mathbb{N}\cup\{-1,0\}$ and $s \in \mathbb{N}\cup\{0\}$.
By assigning the appropriate value to the parameters in the function (1.2) gives the particular cases as follows([12, Eq.(1.10), Eq.(1.9), Eq.(1.8) p.381]):

$$E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;1,0) = E^{\gamma,q}_{\alpha,\beta}(z), \tag{1.3}$$

$$E^{\gamma,1}_{\alpha,\beta,\lambda,\mu}(z;1,0) = E^{\gamma}_{\alpha,\beta}(z), \tag{1.4}$$

$$E^{1,1}_{\alpha,\beta,\lambda,\mu}(z;1,0) = E_{\alpha,\beta}(z), \tag{1.5}$$

$$E^{1,1}_{\alpha,1,\lambda,\mu}(z;1,0) = E_{\alpha}(z), \tag{1.6}$$

$$E^{1,1}_{1,1,\lambda,\mu}(z;1,0) = \exp(z). \tag{1.7}$$

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In 2012, G. A. Dorrego and R. A. Cerutti [4] given k-generalization of the generalized Mittag-Leffler function of (1.4) as follows:

$$E^{\gamma}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n+\beta)} \frac{z^n}{n!}, \tag{1.8}$$

where $k \in \mathbb{R};\ \alpha,\beta,\gamma \in \mathbb{C};\ \Re(\alpha,\beta) > 0$. The function (1.3) is generalized by K. S. Gehlot [5] in the form:

$$GE^{\gamma,q}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k}}{\Gamma_k(\alpha n+\beta)} \frac{z^n}{n!}. \tag{1.9}$$

Here, $(\gamma)_{n,k}$ and Γ_k are the Pochhammer k-symbols and k-Gamma function which are introduced by R. Díaz and E. Pariguan [3]. The Pochhammer k-symbols is defined by

$$(x)_{n,k} = x(x+k)(x+2k)\ldots(x+(n-1)k), \tag{1.10}$$

where $x \in \mathbb{C},\ k \in \mathbb{R},\ n \in \mathbb{N}^+$.
With the help of the Pochhammer k-symbol, R. Díaz and E. Pariguan [3] has introduced k-Gamma function for $k > 0$ as follows:

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!\ k^n\ (nk)^{\frac{x}{k}} - 1}{(x)_{n,k}}, \quad x \in \mathbb{C}\backslash k\mathbb{Z}^-. \tag{1.11}$$

For $\Re(x) > 0$, Euler integral form of the k-Gamma function [3] is given by

$$\Gamma_k(x) = \int_0^{\infty} t^{x-1}\ e^{-\frac{t^k}{k}}\ dt. \tag{1.12}$$

The k-generalization of the Stirling's formula [3] for large z is given by

$$\Gamma_k(z+1) \sim (2\pi)^{\frac{1}{2}}\ (kz)^{\frac{-1}{2}}\ z^{\frac{z+1}{k}}\ e^{\frac{-z}{k}}. \tag{1.13}$$

The following properties[22] follow from (1.11) and (1.12)

$$\begin{aligned}
\Gamma_k(z+k) &= z\Gamma_k(z), \\
\Gamma_k(k) &= 1, \\
(z)_{n,k} &= \frac{\Gamma_k(z+nk)}{\Gamma_k(z)}, \\
(z)_{n-m,k} &= \frac{(-1)^m (z)_{n,k}}{(k-z-nk)_{m,k}}, \\
(z)_{mn,k} &= m^{mn} \prod_{j=1}^{m} \left(\frac{z+jk-k}{m}\right)_{n,k}.
\end{aligned}$$

When $k = 1$, these identities get reduced to the corresponding properties of the Gamma function and the Pochhammer symbol [19].

The generalized k-Write function due to K. Gehlot et al.[6] is defined by

$$ {}_q\Psi_r^k\left[\begin{array}{ll}(a_i,\alpha_i)_{1,q}; & z\\ (b_j,\beta_j)_{1,r}; & \end{array}\right] = \sum_{n=0}^{\infty}\frac{\left\{\prod_{i=1}^{q}\Gamma_k(a_i+\alpha_i n)\right\}}{\left\{\prod_{j=1}^{r}\Gamma_k(b_j+\beta_j n)\right\}n!}z^n, \tag{1.14}$$

where $z\in\mathbb{C},\ k>0,\ \alpha_i,\ \beta_j\in\mathbb{R}\setminus\{0\}$ and $a_i+\alpha_i n,\ b_j+\beta_j n\in\mathbb{C}\setminus k\mathbb{Z}^-$ for $1\le i\le q$ and $1\le j\le r$. In the notations

$$\begin{aligned}\Delta &= \sum_{j=1}^{r}\frac{\beta_j}{k}-\sum_{i=1}^{q}\frac{\alpha_i}{k}\ ;\ \delta=\left\{\prod_{j=1}^{r}\left|\frac{\beta_j}{k}\right|^{\frac{\beta_j}{k}}\right\}\left\{\prod_{i=1}^{q}\left|\frac{\alpha_i}{k}\right|^{-\frac{\alpha_i}{k}}\right\}\\ \mu &= \sum_{j=1}^{r}\frac{b_j}{k}-\sum_{i=1}^{q}\frac{a_i}{k}+\frac{q-r}{2},\end{aligned}$$

the series converges for all $z\in\mathbb{C}$ if $\Delta>-1$. If $\Delta=-1$ then the converges absolutely for $|z|<\delta$ and if $|z|=\delta$, then $\Re(\mu)>1/2$.

For $k>0,\ a\in\mathbb{C}$ and $|x|<\dfrac{1}{k}$, Diaz at el.[3] showed that

$$\sum_{n=0}^{\infty}\frac{(a)_{n,k}}{n!}x^n=(1-kx)^{-\frac{a}{k}}. \tag{1.15}$$

This may be regarded as the k-binomial series. It is noteworthy that the radius of convergence of this series can be enlarged or diminished by choosing k smaller or larger; unlike in the classical theory of radius of convergence of the binomial series which is fixed. This motivated us to study k-extension of certain Special functions[24].

The Mittag-Leffler function along with its various generalizations plays an important role in the fields such as Complex Analysis, and Mathematical Physics. It has applications in the modeling of anomalous diffusion, non-Markovian processes, and other phenomena that are not well described by exponential decay. Moreover, the Mittag-Leffler function and its generalization also appears in the solution of fractional differential equations, which are used to describe processes with memory and long-range correlations.

In view of k-calculus[3, 22, 23] and importance of generalizations of Mittag-Leffler function [11, 12, 14, 16, 17, 18, 21], we introduce here the generalized k-Mittag-Leffler function which is denoted and defined by

$$E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;k;s,r)=\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n,k}]^s}{\Gamma_k(\alpha n+\beta)\ [(\lambda)_{\mu n,k}]^r}\frac{z^n}{n!}, \tag{1.16}$$

wherein the parameters $\alpha,\ \beta,\ \gamma,\ \lambda\ \in\mathbb{C}$ with $\Re(\alpha,\beta,\gamma,\lambda)>0,\ \delta,\ \mu,\ k>0,\ r\in\mathbb{N}\cup\{-1,0\}$ and $s\in\mathbb{N}\cup\{0\}$. We shall refer to this function as *kgml*.

Note 1.1. $E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;1;s,r)=E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r)$.

The proposed *kgml* can be viewed as a special case of the k-Wright function(1.14), when the parameters $a_i = \gamma, \alpha_i = \delta k, b_1 = \beta, \beta_1 = \alpha, b_j = \lambda, \beta_j = \mu k$ for all $i = 1,2,\ldots,s$ and $j = 2,3,\ldots,q$. The function in (1.16), besides containing the above cited k-generalizations, also includes the k-generalization of exponential function, Bessel-Maitland function [7, Eq.(1.7.8), p.19], Dotsenko function [7, Eq.(1.8.9), p.24], a particular form ($m = 2$) of extension of Mittag-Leffler function due to Saxena and Nishimoto [25] and Elliptic function [9, Eq.(1), p.211].
The reducibility of the *kgml* to the above mentioned functions is tabulated below.

Table - 1

Function	r	s	α	β	γ	δ	λ	μ	z
k-Exponential (1.17)	0	1	k	k	k	1	-	-	z/k
k-Mittag-Leffler (1.18)	0	1	α	k	k	1	-	-	z/k
k-Wiman (1.19)	0	1	α	β	k	1	-	-	z/k
Dorrego and Cerutti (1.8)	0	1	α	β	γ	1	-	-	z
K. S. Gehlot (1.9)	0	1	α	β	γ	q	-	-	z
k-Bessel-Maitland (1.20)	0	0	μk	$\nu + k$	-	-	-	-	z
k- Dotsenko (1.21)	-1	1	$\omega k/\nu$	c	a	1	b	ω/ν	z
k-Saxena-Nishimoto (1.22)	1	1	$\alpha_1 k$	β_1	γ	K	β_2	α_2	z
k-Elliptic (1.23)	-1	1	k	k	$\frac{1}{2}$	1	$\frac{1}{2}$	1	z^2

The purpose of consideration of the parameters $'r'$ and $'s'$ is now clear from the above table (Table-1). Also, the parameter $'s'$ is special; as is seen in subsection 2.5.
The explicit forms of the functions mentioned in this table are as stated below.

(1) *k-Exponential function*:

$$\exp(z;k) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_k(nk+k)}, \tag{1.17}$$

Note 1.2. $\exp(z;k)\ \exp(-z;k) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma_i(ik+k)} \sum_{j=0}^{\infty} \frac{(-1)^j z^j}{\Gamma_k(jk+k)} = 1 + \sum_{i=1}^{\infty} \sum_{j=0}^{n} \frac{(-1)^j z^i}{k^i(i-j)!\,j!} = 1.$

(2) *k-Mittag-Leffler function*:

$$E_\alpha(z;k) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_k(\alpha n+k)}, \tag{1.18}$$

(3) *k-generalization of Wiman's function*:

$$E_{\alpha,\beta}(z;k)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma_k(\alpha n+\beta)},\quad \Re(\alpha,\beta)>0, \tag{1.19}$$

(4) *k-Bessel-Maitland function*

$$J_{\nu}^{\mu}(z;k)=\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma_k(\nu+\mu nk+k)}\frac{z^n}{n!}, \tag{1.20}$$

(5) *k-Dotsenko function*:

$${}_2R_1(a,b;c,\omega;\nu;z;k)=\frac{\Gamma_k(c)}{\Gamma_k(a)\,\Gamma_k(b)}\sum_{n=0}^{\infty}\frac{\Gamma_k(a+nk)\,\Gamma_k(b+\frac{\omega}{\nu}nk)}{\Gamma_k(c+\frac{\omega}{\nu}nk)}\frac{z^n}{n!}, \tag{1.21}$$

(6) k-generalization of a particular form ($m=2$) of extension of Mittag-Leffler function due to Saxena and Nishimoto given by

$$E_{\gamma,K}[(\alpha_j,\beta_j)_{1,2};z;k]=\sum_{n=0}^{\infty}\frac{(\gamma)_{Kn,k}}{\Gamma_k(\alpha_1 nk+\beta_1)\,\Gamma_k(\alpha_2 nk+\beta_2)}\frac{z^n}{n!}, \tag{1.22}$$

(7) *k-Elliptic function*:

$$K(z;k)=\frac{\pi}{2}\,{}_2F_{1,k}\left(\begin{array}{ccc}\frac{1}{2}, & \frac{1}{2}; & z^2\\ & 1; & \end{array}\right). \tag{1.23}$$

2. Main Results

In this section, we prove the convergence criteria, differential equation, Eigen function property, double series representation and the inverse series relation.

2.1. *Convergence.*

Theorem 2.1. *Let $\Re(\alpha,\beta,\gamma,\lambda)>0, \Re(\alpha/k+r\mu-s\delta+1)>0,\ \delta,\mu,k>0,\ r\in\mathbb{N}\cup\{-1,0\}$ and $s\in\mathbb{N}\cup\{0\}$, then $E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;k;s,r)$ is an entire function of order $\rho=1/(\Re(\alpha/k+r\mu-s\delta+1))$ and type $\sigma=(\delta k)^{s\delta\rho}/\left[\rho\left(\{\Re(\alpha)\}^{\Re(\alpha)/k}(\mu k)^{r\mu}\right)^{\rho}\right]$.*

Proof. Let us take

$$\begin{aligned}u_n &= \frac{[(\gamma)_{\delta n,k}]^s}{\Gamma_k(\alpha n+\beta)\,[(\lambda)_{\mu n,k}]^r\,n!}\\ &= \frac{[\Gamma_k(\gamma+\delta nk)]^s\,[\Gamma_k(\lambda)]^r}{[\Gamma_k(\gamma)]^s\,\Gamma_k(\alpha n+\beta)\,[\Gamma_k(\lambda+\mu nk)]^r\,\Gamma(n+1)}\end{aligned}$$

in (1.16) so that

$$E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;k;s,r)=\sum_{n=0}^{\infty}u_n\,z^n.$$

Now using the k-Stirling's asymptotic formula (1.13), we get

$$
\begin{aligned}
u_n \sim\ & \frac{\left[\sqrt{2\pi}\, e^{-(\gamma+\delta nk-1)/k}\, (\gamma+\delta nk-1)^{(\gamma+\delta nk)/k}\, (k(\gamma+\delta nk-1))^{-1/2}\right]^s}{\left[\sqrt{2\pi}\, e^{-(\alpha n+\beta-1)/k}(\alpha n+\beta-1)^{(\alpha n+\beta-1)/k}\, (k(\alpha n+\beta-1))^{-1/2}\right]} \\
& \times \frac{\left[\sqrt{2\pi}\, e^{-(\lambda-1)/k}(\lambda-1)^{\lambda/k}\, (k(\lambda-1))^{-1/2}\right]^r}{\left(\sqrt{2\pi}\, e^{-(\lambda+\mu nk-1)/k}(\lambda+\mu nk-1)^{(\lambda+\mu nk)/k}\, (k(\lambda+\mu nk-1))^{-1/2}\right)^r} \\
& \times \frac{1}{\left[\sqrt{2\pi}\, e^{-(\gamma-1)/k}(k(\gamma-1))^{-1/2}\, (\gamma-1)^{\gamma/k}\right]^s \left(\sqrt{2\pi}\, e^{-n}\, n^{n+1/2}\right)} \\
=\ & \frac{\left[e^{-\delta n}\, (\delta nk)^{(\gamma+\delta nk)/k}\, (1+(\gamma-1)/\delta nk)^{(\gamma+\delta nk)/k}\, k^{-1/2}\, (\delta nk)^{-1/2}\right]^s}{2\pi\, \left[(\gamma-1)^{\gamma/k}\, (k(\gamma-1))^{-1/2}\right]^s e^{-(\alpha n+\beta-1)/k}\, (\alpha n)^{(\alpha n+\beta)/k}\, (1+(\beta-1)/\alpha n)^{(\alpha n+\beta)/k}} \\
& \times \frac{(1+(\gamma-1)/\delta nk)^{-s/2}\left[(\lambda-1)^{\lambda/k}\, (k(\lambda-1)^{-1/2}\right]^r}{(\alpha n)^{-1/2}\, (k(1+(\beta-1)/\alpha n))^{-1/2}\, \left[(\mu nk)^{(\lambda+\mu nk)/k}(1+(\lambda-1)/\mu nk)^{(\lambda+\mu nk)/k}\right]^r} \\
& \times \frac{1}{\left[e^{-\mu n}\, (\mu nk)^{-1/2}(k(1+(\lambda-1)/\mu nk))^{-1/2}\right]^r e^{-n}\, n^{n+1/2}}.
\end{aligned}
$$

Hence, if R is radius of convergence of the series of $E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;k;s,r)$, then with the use of Cauchy-Hadamard formula,

$$
\begin{aligned}
\frac{1}{R} &= \limsup_{n\to\infty} \sqrt[n]{|u_n|} \\
&= \limsup_{n\to\infty} \left|\frac{e^{\alpha/k+r\mu-s\delta+1}(\delta k)^{s\delta}}{\alpha^{\alpha/k}(\mu k)^{r\mu}}\right| \left|n^{s\delta-\alpha/k-r\mu-1}\right| \\
&= \limsup_{n\to\infty} \frac{e^{\Re(\alpha/k+r\mu-s\delta+1)}(\delta k)^{s\delta}}{\{\Re(\alpha)\}^{\Re(\alpha/k)}(\mu k)^{r\mu}}\, n^{\Re(s\delta-\alpha/k-r\mu-1)} \\
&= 0,
\end{aligned}
$$

when $\Re(\alpha/k+r\mu-s\delta+1)>0$. Therefore, the function (1.16) turns out to be an *entire* function. In order to determine its order, we use the result [1, 13] which states that if $f(z)=\sum_{n=0}^{\infty} a_n z^n$ is an entire function then the order $\rho(f)$ of f is given by

$$
\rho(f) = \limsup_{r\to\infty} \frac{logM(r;f)}{logr} = \limsup_{n\to\infty} \frac{n\, \log n}{\log(1/|u_n|)}.
$$

By choosing $f(z)=E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;k;s,r)$, this particularizes to

$$
\rho = \rho(E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;k;s,r)) = \limsup_{n\to\infty} \frac{n\, \log n}{\log(1/|u_n|)}.
$$

Here,

$$
\begin{aligned}
\log\left(\frac{1}{|u_n|}\right) &= \log\left|\frac{[\Gamma_k(\gamma)]^s\,\Gamma_k(\alpha n+\beta)\,[\Gamma_k(\lambda+\mu nk)]^r\,\Gamma(n+1)}{[\Gamma_k(\gamma+\delta nk)]^s\,[\Gamma_k(\lambda)]^r}\right| \\
&\sim \log\left|\frac{\left[\sqrt{2\pi}\,e^{-(\alpha n+\beta-1)/k}(\alpha n+\beta-1)^{(\alpha n+\beta)/k}\,(k(\alpha n+\beta-1))^{-1/2}\right]}{\left[\sqrt{2\pi}\,e^{-(\gamma+\delta nk-1)/k}\,(\gamma+\delta nk-1)^{(\gamma+\delta nk)/k}\,(k(\gamma+\delta nk-1))^{-1/2}\right]^s}\right. \\
&\quad\times\frac{\left(\sqrt{2\pi}\,e^{-(\lambda+\mu nk-1)/k}(\lambda+\mu nk-1)^{(\lambda+\mu nk)/k}\,(k(\lambda+\mu nk-1))^{-1/2}\right)^r}{\left[\sqrt{2\pi}\,e^{-(\lambda-1)/k}(\lambda-1)^{\lambda/k}\,(k(\lambda-1))^{-1/2}\right]^r} \\
&\quad\left.\times\left[\sqrt{2\pi}\,e^{-(\gamma-1)/k}(k(\gamma-1))^{-1/2}\,(\gamma-1)^{\gamma/k}\right]^s\,\left(\sqrt{2\pi}\,e^{-n}n^{n+1/2}\right)\right| \\
&= \log\Big|(2\pi)\,(k(\gamma-1))^{-s/2}\,(\gamma-1)^{s\gamma/k}\,e^{s\delta n-\alpha n/k-\mu rn-n}\,(\alpha n+\beta-1)^{(\alpha n+\beta)/k} \\
&\quad\times(k(\alpha n+\beta-1))^{-1/2}\,(\lambda+\mu nk-1)^{r(\lambda+\mu nk)/k}\,(k(\lambda+\mu nk-1))^{-r/2}\,n^{n+1/2}\Big| \\
&\quad-\log\left|(\lambda-1)^{r\lambda/k}\,(k(\lambda-1))^{-r/2}\,(\gamma+\delta nk-1)^{s(\gamma+\delta nk)/k}\,(k(\gamma+\delta nk-1))^{-s/2}\right| \\
&= \log(2\pi)+\Re((s\gamma)/k)\log|\gamma-1|-s/2\log|k|-s/2\log|\gamma-1| \\
&\quad-\Re((\alpha n+\beta)/k)+\Re((\alpha n+\beta)/k)\log|\alpha n|-1/2\log|\alpha n| \\
&\quad+\Re((\alpha n+\beta)/k)\log|1+(\beta-1)/\alpha n|-1/2\log k \\
&\quad-1/2\log|1+\beta/\alpha n|-\Re(\mu nr)+\Re(r(\lambda+\mu nk)/k)\log|\mu nk| \\
&\quad-r/2\log|\mu nk|-r/2\log k+\Re(r(\lambda+\mu nk)/k)\log|1+(\lambda-1)/\mu nk| \\
&\quad-r/2\log|1+(\lambda-1)/\mu nk|-n+(n+1/2)\log n+\Re(s\delta n) \\
&\quad-\Re(s(\gamma+\delta nk)/k)\log|\delta nk|-\Re(s(\gamma+\delta nk)/k)\log|1+(\gamma-1)/\delta nk| \\
&\quad+s/2\log k+s/2\log|\delta nk|+s/2\log|1+(\gamma-1)/\delta nk| \\
&\quad-\Re(r\lambda/k)\log|\lambda-1|+r/2\log k+r/2\log|\lambda-1|.
\end{aligned}
$$

Hence,

$$\frac{1}{\rho}=\limsup_{n\to\infty}\left(\frac{\log(1/|u_n|)}{n\log n}\right)=\Re(\alpha/k+r\mu-s\delta+1).$$

Thus the order of *kgml* is

$$\rho=\frac{1}{\Re(\alpha/k+r\mu-s\delta+1)}. \tag{2.1}$$

The type σ of the *kgml* is given by [8]

$$\sigma(E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;k;s,r))=\frac{1}{e\,\rho}\limsup_{n\to\infty}\left(n\,|u_n|^{\rho/n}\right), \tag{2.2}$$

where

$$\begin{aligned}
|u_n| &= \left| \frac{[\Gamma_k(\gamma+\delta n)]^s\,[\Gamma_k(\lambda)]^r}{[\Gamma_k(\gamma)]^s\,\Gamma_k(\alpha n+\beta)\,[\Gamma_k(\lambda+\mu n)]^r\,\Gamma(n+1)} \right| \\
&\sim \left| \frac{\left[\sqrt{2\pi}\,e^{-(\gamma+\delta nk-1)/k}\,(\gamma+\delta nk-1)^{(\gamma+\delta nk)/k}\,(k(\gamma+\delta nk-1))^{-1/2}\right]^s}{\left[\sqrt{2\pi}\,e^{-(\alpha n+\beta-1)/k}(\alpha n+\beta-1)^{(\alpha n+\beta)/k}\,(k(\alpha n+\beta-1))^{-1/2}\right]} \right. \\
&\quad \times \frac{\left[\sqrt{2\pi}\,e^{-(\lambda-1)/k}(\lambda-1)^{\lambda/k}\,(k(\lambda-1))^{-1/2}\right]^r}{\left(\sqrt{2\pi}\,e^{-(\lambda+\mu nk-1)/k}(\lambda+\mu nk-1)^{(\lambda+\mu nk)/k}\,(k(\lambda+\mu nk-1))^{-1/2}\right)^r} \\
&\quad \left. \times \frac{1}{\left[\sqrt{2\pi}\,e^{-(\gamma-1)/k}(k(\gamma-1))^{-1/2}\,(\gamma-1)^{\gamma/k}\right]^s\,\left(\sqrt{2\pi}\,e^{-n}\,n^{n+1/2}\right)} \right| \\
&= \left| \frac{\left[e^{-\delta n}\,(\delta nk)^{(\gamma+\delta nk)/k}\,(1+\gamma/\delta nk)^{(\gamma+\delta nk)/k}\,k^{-1/2}\,(\delta nk)^{-1/2}\right]^s}{2\pi\,\left[(\gamma-1)^{\gamma/k}\,(k(\gamma-1))^{-1/2}\right]^s\,e^{-(\alpha n+\beta-1)/k}\,(\alpha n)^{(\alpha n+\beta)/k}\,(1+(\beta-1)/\alpha n)^{(\alpha n+\beta)/k}} \right. \\
&\quad \times \frac{(1+(\gamma-1)/\delta nk)^{-s/2}\,\left[(\lambda-1)^{\lambda/k}\,(k(\lambda-1))^{-1/2}\right]^r}{(\alpha n)^{-1/2}\,(k(1+(\beta-1)/\alpha n))^{-1/2}\,\left[(\mu nk)^{(\lambda+\mu nk)/k}(1+\lambda/\mu nk)^{(\lambda+\mu nk)/k}\right]^r} \\
&\quad \left. \times \frac{1}{\left[e^{-\mu n}\,(\mu nk)^{-1/2}(k(1+\lambda/\mu nk))^{-1/2}\right]^r\,e^{-n}\,n^{n+1/2}} \right|.
\end{aligned}$$

On substituting this on the right hand side of (2.2) and then using (2.1), we get

$$\begin{aligned}
\limsup_{n\to\infty}\left(n\,|u_n|^{\rho/n}\right) &= \left(\frac{(\delta k)^{s\delta}}{\{\Re(\alpha)\}^{\Re(\alpha/k)}\,(\mu k)^{r\mu}}\right)^{\rho} e^{\Re(\alpha/k+r\mu-s\delta+1)\rho} \\
&\quad \times \lim_{n\to\infty} n^{\Re(s\delta-\alpha/k-r\mu-1)\rho+1}.
\end{aligned}$$

This gives

$$\sigma(E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;k;s,r)) = \frac{1}{\rho}\left(\frac{(\delta k)^{s\delta}}{\{\Re(\alpha)\}^{\{\Re(\alpha/k)\}}\,(\mu k)^{r\mu}}\right)^{\rho}. \tag{2.3}$$

For every positive ε, the asymptotic estimate [8]

$$\left|E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;k;s,r)\right| < exp\left((\sigma+\varepsilon)\,|z|^{\rho}\right),\ |z|\geq r_0>0$$

holds with ρ,σ as in (2.1), (2.3) for $|z|\geq r_0(\varepsilon)$, and for sufficiently large $r_0(\varepsilon)$. □

2.2. *Differential Equation.* Let us take

$$\alpha/k=\chi,\ \frac{\delta^{s\delta}}{\chi^{\chi}\,\mu^{r\mu}}=p,\ \frac{d}{dz}=D,\ zD=\theta,$$

$$\prod_{j=0}^{a-1}\left[\left(\theta k+\frac{b+jk}{a}\right)\right]^m = {}_k\Delta_j^{(a,b;m)}, \tag{2.4}$$

$$\prod_{j=0}^{a-1}\left[\left(\theta k+\frac{b+jk}{a}-k\right)\right]^{m}={}_{k}\Upsilon_{j}^{(a,b;m)}, \tag{2.5}$$

$$\prod_{j=0}^{a-1}\left[\left(-\theta k+\frac{b+jk}{a}-k\right)\right]^{m}={}_{k}\Theta_{j}^{(a,b;m)}, \tag{2.6}$$

and

$$p^{-1}\,D\;{}_{k}\Theta_{m}^{(\delta,\gamma;-s)}\;{}_{k}\Upsilon_{i}^{(\mu,\lambda;r)}\;{}_{k}\Upsilon_{j}^{(\alpha,\beta;1)}={}_{k}\Omega_{\Theta;\Upsilon}. \tag{2.7}$$

Here the operators ${}_{k}\Theta_{m}^{(\delta,\gamma;-s)}$, ${}_{k}\Upsilon_{i}^{(\mu,\lambda;r)}$, ${}_{k}\Upsilon_{j}^{(\alpha,\beta;1)}$ in (2.7) are not commutative with the operator D. In these notations, we now derive the differential equation satisfied by (1.16).

Theorem 2.2. *Let $\alpha/k=\chi,\mu,\delta\in\mathbb{N}$, then $y=E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(z;k;s,r)$ satisfies the equation*

$$\left[{}_{k}\Upsilon_{i}^{(\mu,\lambda;r)}\;{}_{k}\Upsilon_{j}^{(\chi,\beta;1)}\;\theta-z\frac{\delta^{s\delta}}{\chi^{\chi}\,\mu^{r\mu}}\;{}_{k}\Delta_{m}^{(\delta,\gamma;s)}\right]y=0. \tag{2.8}$$

Proof. We have

$$\begin{aligned}
y &= \sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n,k}]^{s}\,z^{n}}{\Gamma_{k}(\alpha n+\beta)\,[(\lambda)_{\mu n,k}]^{r}\,n!}\\
&= \frac{1}{\Gamma_{k}(\beta)}\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n,k}]^{s}\,z^{n}}{(\beta)_{\frac{\alpha n}{k},k}\,[(\lambda)_{\mu n,k}]^{r}\,n!}\\
&= \frac{1}{\Gamma_{k}(\beta)}\sum_{n=0}^{\infty}\frac{[(\gamma)_{\delta n,k}]^{s}\,z^{n}}{(\beta)_{\chi n,k}\,[(\lambda)_{\mu n,k}]^{r}\,n!}\\
&= \frac{1}{\Gamma_{k}(\beta)}\sum_{n=0}^{\infty}\frac{\delta^{s\delta n}[(\frac{\gamma}{\delta})_{n,k}]^{s}\,[(\frac{\gamma+k}{\delta})_{n,k}]^{s}\ldots[(\frac{\gamma+(\delta-1)k}{\delta})_{n,k}]^{s}z^{n}}{(\chi)^{\chi n}\,(\frac{\beta}{\chi})_{n,k}\,(\frac{(\beta+k)}{\chi})_{n,k}\ldots(\frac{(\beta+(\chi-k))}{\chi})_{n,k}}\\
&\quad\times\frac{1}{\mu^{r\mu n}[(\frac{\lambda}{\mu})_{n,k}]^{r}\,[(\frac{\lambda+k}{\mu})_{n,k}]^{r}\ldots[(\frac{\lambda+(\mu-1)k}{\mu})_{n,k}]^{r}\,n!}\\
&= \frac{1}{\Gamma_{k}(\beta)}\sum_{n=0}^{\infty}\frac{\delta^{s\delta n}}{\chi^{\chi n}\,\mu^{r\mu n}}\frac{\left\{\prod_{m=0}^{\delta-1}[(\frac{\gamma+mk}{\delta})_{n,k}]^{s}\right\}}{\left\{\prod_{j=0}^{\chi-1}(\frac{\beta+jk}{\chi})_{n,k}\right\}\left\{\prod_{i=0}^{\mu-1}[(\frac{\lambda+ik}{\mu})_{n,k}]^{r}\right\}n!}\,z^{n}.
\end{aligned}$$

Now take

$$\frac{1}{\Gamma_{k}(\beta)}\prod_{m=0}^{\delta-1}\left[\left(\frac{\gamma+mk}{\delta}\right)_{n,k}\right]^{s}=A_{n},\;\prod_{j=0}^{\chi-1}\left(\frac{\beta+jk}{\chi}\right)_{n,k}=B_{n},\;\prod_{i=0}^{\mu-1}\left[\left(\frac{\lambda+ik}{\mu}\right)_{n,k}\right]^{r}=C_{n}\text{ and }\frac{\delta^{s\delta}}{\chi^{\chi}\,\mu^{r\mu}}=p,$$

then the *kgml* (1.16) takes the form

$$y=\sum_{n=0}^{\infty}\frac{A_{n}\,p^{n}}{B_{n}\,C_{n}\,n!}z^{n}. \tag{2.9}$$

Now,

$$\begin{aligned}\theta y &= \sum_{n=0}^{\infty} \frac{A_n\, p^n}{B_n\, C_n\, n!}\, \theta\, z^n \\ &= \sum_{n=1}^{\infty} \frac{A_n\, p^n}{B_n\, C_n\, (n-1)!}\, z^n.\end{aligned}$$

Further,

$$\begin{aligned}{}_k\Upsilon_j^{(\chi,\beta;1)}\, \theta\, y &= \sum_{n=1}^{\infty} \frac{A_n\, p^n}{B_n\, C_n\, (n-1)!} \prod_{j=0}^{\chi-1} \left(\theta k + \frac{\beta+jk}{\chi} - k\right) z^n \\ &= \sum_{n=1}^{\infty} \frac{A_n\, p^n}{B_n\, C_n\, (n-1)!} \prod_{j=0}^{\chi-1} \left(nk + \frac{\beta+jk}{\chi} - k\right) z^n \\ &= \sum_{n=1}^{\infty} \frac{A_n\, p^n}{B_{n-1}\, C_n\, (n-1)!}\, z^n.\end{aligned}$$

Finally,

$$\begin{aligned}{}_k\Upsilon_i^{(\mu,\lambda;r)}\ {}_k\Upsilon_j^{(\chi,\beta;1)}\, \theta\, y &= \sum_{n=1}^{\infty} \frac{A_n\, p^n}{B_{n-1}\, C_n\, (n-1)!} \prod_{i=0}^{\mu-1} \left[\left(\theta k + \frac{\lambda+ik}{\mu} - k\right)\right]^r z^n \\ &= \sum_{n=1}^{\infty} \frac{A_n\, p^n}{B_{n-1}\, C_n\, (n-1)!} \prod_{i=0}^{\mu-1} \left[\left(nk + \frac{\lambda+ik}{\mu} - k\right)\right]^r z^n \\ &= \sum_{n=1}^{\infty} \frac{A_n\, p^n}{B_{n-1}\, C_{n-1}\, (n-1)!}\, z^n.\end{aligned}$$

Thus,

$$(2.10)\qquad {}_k\Upsilon_i^{(\mu,\lambda;r)}\ {}_k\Upsilon_j^{(\chi,\beta;1)}\, \theta\, y = \sum_{n=0}^{\infty} \frac{A_{n+1}\, p^{n+1}}{B_n\, C_n\, n!}\, z^{n+1}.$$

On the other hand,

$$\begin{aligned}{}_k\Delta_m^{(\delta,\gamma;s)}\, y &= \sum_{n=0}^{\infty} \frac{A_n\, p^n}{B_n\, C_n\, n!} \prod_{m=0}^{\delta-1} \left[\left(\theta k + \frac{\gamma+mk}{\delta}\right)\right]^s z^n \\ &= \sum_{n=0}^{\infty} \frac{A_n\, p^n}{B_n\, C_n\, n!} \prod_{m=0}^{\delta-1} \left[\left(nk + \frac{\gamma+mk}{\delta}\right)\right]^s z^n \\ &= \sum_{n=0}^{\infty} \frac{A_{n+1}\, p^n}{B_n\, C_n\, n!}\, z^n,\end{aligned}$$

that is,

$$(2.11)\qquad p\, z\ {}_k\Delta_m^{(\delta,\gamma;s)}\, y = \sum_{n=0}^{\infty} \frac{A_{n+1}\, p^{n+1}}{B_n\, C_n\, n!}\, z^{n+1}.$$

On comparing (2.10) and (2.11), we get (2.8). □

2.3. *Eigen function property.*

Theorem 2.3. *Let $\alpha/k = \chi, \mu, \delta \in \mathbb{N}$ then $E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;k;s,r)$ is an eigen function with respect to the operator ${}_k\Omega_{\Theta;\Upsilon}$. That is,*

$$ {}_k\Omega_{\Theta;\Upsilon}\left(E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\zeta z;k;s,r)\right) = \zeta\, E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\zeta z;k;s,r). $$

Proof. We first note that

$$ \begin{aligned} w &= E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\zeta z;k;s,r) \\ &= \sum_{n=0}^{\infty} \frac{A_n\,(\zeta p)^n}{B_n\,C_n\,n!} z^n. \end{aligned} $$

Now in view of (2.5),

$$ \begin{aligned} {}_k\Upsilon_j^{(\chi,\beta;1)}\,w &= \sum_{n=0}^{\infty} \frac{A_n\,(\zeta p)^n}{B_n\,C_n\,n!} \prod_{j=0}^{\chi-1}\left(\theta k + \frac{\beta+jk}{\chi} - k\right) z^n \\ &= \sum_{n=0}^{\infty} \frac{A_n\,(\zeta p)^n}{B_n\,C_n\,n!} \prod_{j=0}^{\chi-1}\left(nk + \frac{\beta+jk}{\chi} - k\right) z^n \\ &= \sum_{n=0}^{\infty} \frac{A_n\,(\zeta p)^n}{B_{n-1}\,C_n\,n!}\, z^n. \end{aligned} $$

Next

$$ \begin{aligned} {}_k\Upsilon_i^{(\mu,\lambda;r)}\ {}_k\Upsilon_j^{(\chi,\beta;1)}\,w &= \sum_{n=0}^{\infty} \frac{A_n\,(\zeta p)^n}{B_{n-1}\,C_n\,n!} \prod_{i=0}^{\mu-1}\left[\left(\theta k + \frac{\lambda+ik}{\mu} - k\right)\right]^r z^n \\ &= \sum_{n=0}^{\infty} \frac{A_n\,(\zeta p)^n}{B_{n-1}\,C_n\,n!} \prod_{i=0}^{\mu-1}\left[\left(nk + \frac{\lambda+ik}{\mu} - k\right)\right]^r z^n \\ &= \sum_{n=0}^{\infty} \frac{A_n\,(\zeta p)^n}{B_{n-1}\,C_{n-1}\,n!}\, z^n. \end{aligned} $$

Further, from (2.6),

$$ \begin{aligned} &{}_k\Theta_m^{(\delta,\gamma;-s)}\ {}_k\Upsilon_i^{(\mu,\lambda;r)}\ {}_k\Upsilon_j^{(\chi,\beta;1)}\ w \\ &= \sum_{n=0}^{\infty} \frac{A_n\,(\zeta p)^n}{B_{n-1}\,C_{n-1}\,n!}\ \Theta_m^{(\delta,\gamma;-s)}\ z^n \\ &= \sum_{n=0}^{\infty} \frac{A_n\,(\zeta p)^n}{B_{n-1}\,C_{n-1}\,n!} \prod_{j=0}^{\delta-1}\left[\left(-\theta k + \frac{\gamma+jk}{\delta} - k\right)\right]^{-s} z^n \\ &= \sum_{n=0}^{\infty} \frac{A_n\,(\zeta p)^n}{B_{n-1}\,C_{n-1}\,n!} \prod_{j=0}^{\delta-1}\left[\left(-nk + \frac{\gamma+jk}{\delta} - k\right)\right]^{-s} z^n \\ &= \sum_{n=0}^{\infty} \frac{A_{n-1}\,(\zeta p)^n}{B_{n-1}\,C_{n-1}\,n!}\, z^n. \end{aligned} $$

Finally,

$$\begin{aligned}
{}_k\Omega^{(\delta,\gamma,s;\chi,\beta,\mu,\lambda,r)}_{\Theta;\Upsilon}\, w &= p^{-1}\, D\; {}_k\Theta^{(\delta,\gamma;-s)}_m\; {}_k\Upsilon^{(\mu,\lambda;r)}_i\; {}_k\Upsilon^{(\chi,\beta;1)}_j\, w \\
&= \sum_{n=0}^{\infty} \frac{A_{n-1}\, \zeta^n\, p^{n-1}}{B_{n-1}\, C_{n-1}\, n!}\, D\, z^n \\
&= \sum_{n=1}^{\infty} \frac{A_{n-1}\, \zeta^n\, p^{n-1}}{B_{n-1}\, C_{n-1}\, (n-1)!}\, z^{n-1} \\
&= \sum_{n=0}^{\infty} \frac{A_n\, \zeta^{n+1}\, p^n}{B_n\, C_n\, n!}\, z^n \\
&= \zeta \sum_{n=0}^{\infty} \frac{A_n\, \zeta^n\, p^n}{B_n\, C_n\, n!}\, z^n \\
&= \zeta\, E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\zeta z; k; s, r).
\end{aligned}$$

□

The above theorems may be specialized to yield the corresponding properties of the special cases stated in Table - 1.

2.4. *Double series representation.* As in Theorem 2.1, take

$$u_n = \frac{[(\gamma)_{\delta n,k}]^s}{\Gamma_k(\alpha n+\beta)\, [(\lambda)_{\mu n,k}]^r\, n!}.$$

and put

$$^*E^{\gamma,\delta,\rho}_{\alpha,\beta,\lambda,\mu}(z;k;s,r) = \sum_{n=0}^{\infty} u_n\, (\rho)_{n,k}\, z^n$$

which is valid under the condition $\Re(\alpha/k + r\mu - s\delta + 1) > 0$.

Theorem 2.4. *The function*

$$^*E^{\gamma,\delta,\rho}_{\alpha,\beta,\lambda,\mu}(z;k;s,r) = \sum_{i,j=0}^{\infty} (\rho)_{\frac{i}{k}+\frac{j}{k},k} \frac{(-1)^i}{\Gamma_k(ik+k)\Gamma_k(jk+k)}\; {}^*E^{\gamma,\delta,\rho+i+j}_{\alpha,\beta,\lambda,\mu}(z;k;s,r).$$

Proof. Here introducing the function (1.16) in the integrand of the integral (1.12) and applying this integral to the identity $\exp(z;k)\ \exp(-z;k) = 1$, one gets the integral

$$\int_0^\infty t^{\rho-1}\, e^{-\frac{t^k}{k}}\ \exp(-xt;k)\ \exp(xt;k)\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(zt^k;k;s,r)\mathrm{d}t$$

$$= \int_0^\infty e^{-\frac{t^k}{k}}\, t^{\rho-1}\, E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(zt^k;k;s,r)\,\mathrm{d}t. \tag{2.12}$$

Here substituting the corresponding series representations for $\exp(-xt;k)$ and $\exp(xt;k)$ on the left hand side, one gets

$$\begin{aligned} l.h.s. &= \int_0^\infty e^{-\frac{t^k}{k}}\, t^{\rho-1}\, \exp(-xt;k)\, \exp(xt;k)\, E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(zt^k;k;s,r)\,\mathrm{d}t \\ &= \int_0^\infty e^{-\frac{t^k}{k}} t^{\rho-1} \left\{\sum_{i=0}^\infty \frac{(-1)^i x^i t^i}{\Gamma_k(ik+k)}\right\} \left\{\sum_{j=0}^\infty \frac{x^j t^j}{\Gamma_k(jk+k)}\right\} \\ &\quad \times E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(zt^k;k;s,r)\,\mathrm{d}t. \end{aligned} \tag{2.13}$$

Here, the series in integrand of the above integral converges, we have

$$\begin{aligned} l.h.s. &= \sum_{i=0}^\infty \sum_{j=0}^\infty \frac{(-1)^i}{\Gamma_k(ik+k)\Gamma_k(jk+k)} x^{i+j} \int_0^\infty e^{-\frac{t^k}{k}} t^{\rho+i+j-1} \\ &\quad \times E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(zt^k;k;s,r)\mathrm{d}t \\ &= \sum_{i=0}^\infty \sum_{j=0}^\infty \frac{(-1)^i}{\Gamma_k(ik+k)\,\Gamma_k(jk+k)} x^{i+j} \\ &\quad \times \int_0^\infty e^{-\frac{t^k}{k}}\, t^{\rho+i+j-1} \left\{\sum_{n=0}^\infty \frac{[(\gamma)_{\delta n,k}]^s}{\Gamma_k(\alpha n+\beta)\,[(\lambda)_{\mu n,k}]^r} \frac{(zt^k)^n}{n!}\right\} \mathrm{d}t \\ &= \sum_{i=0}^\infty \sum_{j=0}^\infty \frac{(-1)^i}{\Gamma_k(ik+k)\,\Gamma_k(jk+k)} x^{i+j} \sum_{n=0}^\infty u_n z^n \int_0^\infty e^{-\frac{t^k}{k}} t^{\rho+i+j+nk-1}\,\mathrm{d}t \\ &= \sum_{i=0}^\infty \sum_{j=0}^\infty \frac{(-1)^i}{\Gamma_k(ik+k)\,\Gamma_k(jk+k)} x^{i+j} \sum_{n=0}^\infty u_n\, z^n\, \Gamma_k(\rho+i+j+nk) \\ &= \sum_{i=0}^\infty \sum_{j=0}^\infty \frac{(-1)^i}{\Gamma_k(ik+k)\Gamma_k(jk+k)} x^{i+j} \sum_{n=0}^\infty u_n z^n (\rho+i+j)_{n,k}\, \Gamma_k(\rho+i+j) \\ &= \sum_{i=0}^\infty \sum_{j=0}^\infty \Gamma_k(\rho+i+j) \frac{(-1)^i}{\Gamma_k(ik+k)\Gamma_k(jk+k)} x^{i+j}\; {}^{*}E^{\gamma,\delta,\rho+i+j}_{\alpha,\beta,\lambda,\mu}(z;k;s,r). \end{aligned} \tag{2.14}$$

Likewise,

$$\begin{aligned} r.h.s. &= \int_0^\infty e^{-\frac{t^k}{k}}\, t^{\rho-1} \left\{E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(zt^k;k;s,r)\right\} \mathrm{d}t \\ &= \sum_{n=0}^\infty u_n\, z^n \int_0^\infty e^{-\frac{t^k}{k}}\, t^{\rho+nk-1}\,\mathrm{d}t \\ &= \sum_{n=0}^\infty u_n\, \Gamma_k(\rho+nk)\, z^n \end{aligned}$$

$$\begin{aligned} &= \sum_{n=0}^{\infty} u_n(\rho)_{n,k}\,\Gamma_k(\rho)\,z^n \\ &= \Gamma_k(\rho)\,{}^*E^{\gamma,\delta,\rho}_{\alpha,\beta,\lambda,\mu}(z;k;s,r). \end{aligned} \tag{2.15}$$

Consequently, from (2.14) and (2.15), (2.12) yields

$$\begin{aligned} {}^*E^{\gamma,\delta,\rho}_{\alpha,\beta,\lambda,\mu}(z;k;s,r) &= \sum_{i,j=0}^{\infty} \frac{(-1)^i\Gamma_k(\rho+i+j)}{\Gamma_k(\rho)\Gamma_k(ik+k)\Gamma_k(jk+k)} \\ &\quad \times {}^*E^{\gamma,\delta,\rho+i+j}_{\alpha,\beta,\lambda,\mu}(z;k;s,r) \\ &= \sum_{i,j=0}^{\infty} (\rho)_{\frac{i}{k}+\frac{j}{k},k} \frac{(-1)^i}{\Gamma_k(ik+k)\Gamma_k(jk+k)}\,{}^*E^{\gamma,\delta,\rho+i+j}_{\alpha,\beta,\lambda,\mu}(z;k;s,r). \end{aligned}$$

□

The above theorems may be specialized to yield the corresponding properties of the special cases stated in Table - 1.

2.5. ***The generalized k-Konhauser polynomial.*** For $\alpha,\sigma,\lambda>0$, $m,\delta,\mu,p,s\in\mathbb{N}$, $r\in\mathbb{N}\cup\{0\}$, and $n^*=[n/m]$ denotes the integral part of $\frac{n}{m}$, the generalized Konhauser polynomial [13]

$$B^{(\alpha,\beta,\lambda,\mu)}_{n^*}(x^p;s,r)=\frac{\Gamma(\alpha n+\beta+1)}{(n!)^s}\sum_{j=0}^{[n/m]}\frac{[(-n)_{mj}]^s\,x^{pj}}{\Gamma(\alpha j+\beta+1)\,[(\lambda)_{\mu j}]^r\,j!},$$

admits a k-generalization by means of the *kgml* as follows.
Taking $\alpha,\beta,\lambda,k>0$, $\delta(=m),n,\mu,r,s\in\mathbb{N},\gamma=$ a negative integer: $-nk$, $n^*=[n/m]$ the greatest integer part, replacing β by $\beta+1$, and z by a real variable x^p and denoting the polynomial thus obtained by $B^{(\alpha,\beta,\lambda,\mu)}_{n^*}(x^p;k;s,r)$, we get

$$\begin{aligned} E^{-nk,m}_{\alpha,\beta+1,\lambda,\mu}(x^p;k;s,r) &= \sum_{j=0}^{n^*}\frac{\left[(-nk)_{mj,k}\right]^s x^{pj}}{\Gamma_k(\alpha j+\beta+1)\left[(\lambda)_{\mu n,k}\right]^r j!} \\ &= \frac{(n!)^s\,k^{sn}}{\Gamma_k(\alpha n+\beta+1)}\,B^{(\alpha,\beta,\lambda,\mu)}_{n^*}(x^p;k;s,r), \end{aligned}$$

say, where

$$B^{(\alpha,\beta,\lambda,\mu)}_{n^*}(x^p;k;s,r)=\frac{\Gamma_k(\alpha n+\beta+1)}{k^{sn}\,(n!)^s}\sum_{j=0}^{[n/m]}\frac{\left[(-nk)_{mj,k}\right]^s x^{pj}}{\Gamma_k(\alpha j+\beta+1)\left[(\lambda)_{\mu j,k}\right]^r j!}. \tag{2.16}$$

The presence of parameter $'s'$ yields the *unusual* inverse series relations involving the inequalities. In fact, for $s=1$ the usual inverse series relations occur whereas for other values of s the inverse inequality relations occur. This is shown in the following theorems.

Theorem 2.5. *Let $f(x,n;k;s)$ and $g(x,n;k;s)$ be real valued functions, $\alpha,\beta,\lambda>0$, and $\mu,p\in\mathbb{N}$, $r\in\mathbb{N}\cup\{0\}$, then*

$$f(x,n;k;s)<B^{(\alpha,\beta,\lambda,\mu)}_{n^*}(x^p;k;s,r) \tag{2.17}$$

implies

$$x^{pn} > \frac{\Gamma_k(\alpha n+\beta+1)\left[(\lambda)_{\mu n;k}\right]^r n!}{k^{smn}\,(mn!)^s}\sum_{j=0}^{mn}\frac{\left[(-kmn)_{j,k}\right]^s}{\Gamma_k(\alpha j+\beta+1)}\,f(x,j;k;s); \tag{2.18}$$

and

$$x^{pn} < \frac{\Gamma_k(\alpha n+\beta+1)\left[(\lambda)_{\mu n,k}\right]^r n!}{k^{smn}\,(mn!)^s}\sum_{j=0}^{mn}\frac{\left[(-kmn)_{j,k}\right]^s}{\Gamma_k(\alpha j+\beta+1)}\,g(x,j;k;s), \tag{2.19}$$

implies

$$g(x,n;k;s) > B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^p;k;s,r). \tag{2.20}$$

Proof. In order to prove (2.17) implies (2.18), assume that the inequality (2.17) holds. Denote the right hand side of (2.18) by ϕ_n then

$$\phi_n = \frac{\Gamma_k(\alpha n+\beta+1)\left[(\lambda)_{\mu n,k}\right]^r n!}{k^{smn}\,(mn!)^s}\sum_{j=0}^{mn}\frac{\left[(-kmn)_{j,k}\right]^s}{\Gamma_k(\alpha j+\beta+1)}\,f(x,j;k;s).$$

Now substituting for $f(x,j;k;s)$ from (2.17), we get

$$\begin{aligned}
\phi_n &< \frac{\Gamma_k(\alpha n+\beta+1)\left[(\lambda)_{\mu n,k}\right]^r n!}{k^{smn}\,(mn!)^s}\sum_{j=0}^{mn}\frac{\left[(-kmn)_{j,k}\right]^s}{\Gamma_k(\alpha j+\beta+1)}\\
&\quad\times\frac{\Gamma_k(\alpha j+\beta+1)}{k^{sj}\,(j!)^s}\sum_{i=0}^{[j/m]}\frac{\left[(-kj)_{mi,k}\right]^s x^{pi}}{\Gamma_k(\alpha i+\beta+1)\left[(\lambda)_{\mu i,k}\right]^r i!}\\
&= \frac{\Gamma_k(\alpha n+\beta+1)\left[(\lambda)_{\mu n,k}\right]^r n!}{k^{smn}\,(mn!)^s}\sum_{j=0}^{mn}\frac{(-k)^{sj}\,(mn!)^s}{k^{sj}\,(j!)^s\,\left[(mn-j)!\right]^s}\\
&\quad\times\sum_{i=0}^{[j/m]}\frac{(-k)^{smi}\,(j!)^s\,x^{pi}}{\left[(j-mi)!\right]^s\Gamma_k(\alpha i+\beta+1)\left[(\lambda)_{\mu i,k}\right]^r i!}\\
&= \sum_{j=0}^{mn}\sum_{i=0}^{[j/m]}\frac{(-1)^{sj}\,(-k)^{smi}\,\Gamma_k(\alpha n+\beta+1)\,\left[(\lambda)_{\mu n,k}\right]^r\,n!\,x^{pi}}{k^{smn}\,\left[(j-mi)!\right]^s\,\left[(mn-j)!\right]^s\,\Gamma_k(\alpha i+\beta+1)\,\left[(\lambda)_{\mu i,k}\right]^r\,i!}.
\end{aligned}$$

In view of the double series relation[27]:

$$\sum_{j=0}^{mn}\sum_{i=0}^{[j/m]}A(j,i) = \sum_{i=0}^{n}\sum_{j=0}^{mn-mi}A(j+mi,i), \tag{2.21}$$

we further have

$$\begin{aligned}
\phi_n &< \sum_{i=0}^{n}\sum_{j=0}^{mn-mi}\frac{(-1)^{sj}\,k^{smi}\,\Gamma_k(\alpha n+\beta+1)\,\left[(\lambda)_{\mu n,k}\right]^r\,n!\,x^{pi}}{k^{smn}\,\left[(mn-mi-j)!\right]^s\,\Gamma_k(\alpha i+\beta+1)\,\left[(\lambda)_{\mu i,k}\right]^r\,i!}\\
&= x^{pn}+\sum_{i=0}^{n-1}\frac{\Gamma_k(\alpha n+\beta+1)\,\left[(\lambda)_{\mu n,k}\right]^r\,n!\,x^{pi}}{k^{smn}\,\left[(mn-mi)!\right]^s\,\Gamma_k(\alpha i+\beta+1)\,\left[(\lambda)_{\mu i,k}\right]^r\,i!}
\end{aligned}$$

$$\times \sum_{j=0}^{mn-mi} (-1)^{sj} \binom{mn-mi}{j}^s$$
$$\leq \; x^{pn} + \sum_{i=0}^{n-1} \frac{\Gamma_k(\alpha n+\beta+1)\ \left[(\lambda)_{\mu n,k}\right]^r\ n!\ x^{pi}}{k^{smn}\ \left[(mn-mi)!\right]^s\ \Gamma_k(\alpha i+\beta+1)\ \left[(\lambda)_{\mu i,k}\right]^r\ i!}$$
$$\times \left(\sum_{j=0}^{mn-mi} (-1)^{j} \binom{mn-mi}{j} \right)^s.$$

Since the inner series on the right hand side vanishes, it follows that $\phi_n < x^{pn}$, furnishing the inequality (2.18). □

The proof of (2.19) implies (2.20) is similar and therefore omitted here for the sake of brevity. Towards the converse of these inequality relations, we have the following theorem.

Theorem 2.6. *Let $f(x,n;k;s)$ and $g(x,n;k;s)$ be real valued functions, $\alpha,\beta,\lambda,k>0$, and $\mu,p\in\mathbb{N}$, $r\in\mathbb{N}\cup\{0\}$, then*

$$x^{pn} > \frac{\Gamma_k(\alpha n+\beta+1)\left[(\lambda)_{\mu n,k}\right]^r n!}{k^{smn}\ (mn!)^s} \sum_{j=0}^{mn} \frac{\left[(-kmn)_{j,k}\right]^s}{\Gamma_k(\alpha j+\beta+1)}\ f(x,j;k;s);$$

implies

$$f(x,n;k;s) < B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^p;k;s,r)$$

and

$$g(x,n;k;s) > B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^p;k;s,r),$$

implies

$$x^{pn} < \frac{\Gamma_k(\alpha n+\beta+1)\left[(\lambda)_{\mu n,k}\right]^r n!}{k^{smn}\ (mn!)^s} \sum_{j=0}^{mn} \frac{\left[(-kmn)_{j,k}\right]^s}{\Gamma_k(\alpha j+\beta+1)}\ g(x,j;k;s).$$

The proof runs parallel to that of Theorem 2.5, hence is omitted.
Now, for $s=1$, we obtain the inverse series relations for the polynomial (2.16) which is stated as

Theorem 2.7. *For $\alpha,\beta,\lambda>0, m,\mu,p\in\mathbb{N}, r\in\mathbb{N}\cup\{0\}$,*

$$B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^p;k;1,r) = \frac{\Gamma_k(\alpha n+\beta+1)}{k^n\ n!} \sum_{j=0}^{[n/m]} \frac{(-nk)_{mj,k}\ x^{pj}}{\Gamma_k(\alpha j+\beta+1)\left[(\lambda)_{\mu j,k}\right]^r j!} \tag{2.22}$$

if and only if

$$\frac{x^{pn}}{n!} = \frac{\Gamma_k(\alpha n+\beta+1)\left[(\lambda)_{\mu n,k}\right]^r}{k^{mn}\ (mn)!} \sum_{j=0}^{mn} \frac{(-kmn)_{j,k}}{\Gamma_k(\alpha j+\beta+1)} B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^p;k;1,r), \tag{2.23}$$

and for $n\neq ml$, $l\in\mathbb{N}$,

$$\sum_{j=0}^{n} \frac{(-nk)_{j,k}}{\Gamma_k(\alpha j+\beta+1)} B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^p;k;1,r) = 0. \tag{2.24}$$

Proof. The proof of (2.22) implies (2.23) runs as follows.
Denoting the right hand side of (2.23) by Ω_n, and then substituting for $B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^p;k;1,r)$ from (2.22), we get

$$\begin{aligned}\Omega_n &= \frac{\Gamma_k(\alpha n+\beta+1)\left[(\lambda)_{\mu n,k}\right]^r}{k^{mn}\,(mn)!}\sum_{j=0}^{mn}\frac{(-kmn)_{j,k}}{\Gamma_k(\alpha j+\beta+1)}\,B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^p;k;1,r)\\ &= \frac{\Gamma_k(\alpha n+\beta+1)\left[(\lambda)_{\mu n,k}\right]^r}{k^{mn}(mn)!}\sum_{j=0}^{mn}\frac{(-kmn)_{j,k}}{k^j\,j!}\sum_{i=0}^{[j/m]}\frac{(-kj)_{mi,k}\,x^{pi}}{\Gamma_k(\alpha i+\beta+1)\left[(\lambda)_{\mu i,k}\right]^r i!}.\end{aligned}$$

This in view of the double series relation (2.21), further takes the form:

$$\begin{aligned}\Omega_n &= \sum_{j=0}^{mn}\sum_{i=0}^{[j/m]}\frac{(-k)^{j+mi}\,\Gamma_k(\alpha n+\beta+1)\ \left[(\lambda)_{\mu n,k}\right]^r\ x^{pi}}{k^{mn}\,k^j\,(mn-j)!\,(j-mi)!\,\Gamma_k(\alpha i+\beta+1)\ \left[(\lambda)_{\mu i,k}\right]^r\ i!}\\ &= \sum_{i=0}^{n}\sum_{j=0}^{mn-mi}\frac{(-1)^j\,\Gamma_k(\alpha n+\beta+1)\ \left[(\lambda)_{\mu n,k}\right]^r}{k^{mn}\,(mn-mi-j)!\,j!\,\Gamma_k(\alpha i+\beta+1)\ \left[(\lambda)_{\mu i,k}\right]^r\ i!}\,x^{pi}\\ &= \frac{x^{pn}}{n!}+\sum_{i=0}^{n-1}\frac{\Gamma_k(\alpha n+\beta+1)\ \left[(\lambda)_{\mu n,k}\right]^r\ x^{pi}}{k^{mn}\,\Gamma_k(\alpha i+\beta+1)\ \left[(\lambda)_{\mu i,k}\right]^r\ (mn-mi)!\,i!}\\ &\quad\times\sum_{j=0}^{mn-mi}(-1)^j\binom{mn-mi}{j}.\end{aligned}$$

Here the inner sum in the second term on the right hand side vanishes, consequently, we arrive at $\Omega_n=\frac{x^{pn}}{n!}$.
To show further that (2.22) also implies (2.24), Denoting the right hand side of (2.24) by Λ_n, and then substituting for $B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^p;k;1,r)$ from (2.22), we get

$$\begin{aligned}\Lambda_n &= \sum_{j=0}^{n}\frac{(-nk)_{j,k}}{\Gamma_k(\alpha j+\beta+1)}\,B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^p;k;1,r)\\ &= \sum_{j=0}^{n}\frac{(-1)^j\,n!}{(n-j)!}\sum_{i=0}^{[j/m]}\frac{(-k)^{mi}\,x^{pi}}{\Gamma_k(\alpha i+\beta+1)\,((\lambda)_{\mu i,k})^r\,(j-mi)!\,i!}\end{aligned}$$

In view of double series formula[27]:

$$\sum_{j=0}^{n}\sum_{i=0}^{[j/m]}A(i,j)=\sum_{i=0}^{[n/m]}\sum_{j=0}^{n-mi}A(i,j+mi),\qquad (n\neq ml),\tag{2.25}$$

we obtain

$$\begin{aligned}\Lambda_n &= \sum_{i=0}^{[n/m]}\frac{n!\,k^{mi}\,x^{pi}}{\Gamma_k(\alpha i+\beta+1)\,((\lambda)_{\mu i,k})^r\,(n-mi)!\,i!}\sum_{j=0}^{n-mi}(-1)^j\binom{n-mi}{j}\\ &= 0.\end{aligned}$$

If $n \neq ml,\ l \in \mathbb{N}$. Thus completing the first part. The proof of converse part runs as follows [2]. In order to show that both the series (2.23) and the condition (2.24) together imply the series (2.22), we first note the simplest inverse series relations [20, Eq.(1), p.43]:

$$\omega_n = \sum_{j=0}^{n} \frac{(-nk)_{j,k}}{k^j\ j!}\,\rho_j \Leftrightarrow \rho_n = \sum_{j=0}^{n} \frac{(-nk)_{j,k}}{k^j\ j!}\,\omega_j.$$

Here putting

$$\rho_j = \frac{k^j\ j!}{\Gamma_k(\alpha j+\beta+1)}\,B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^p;k;1,r),$$

and considering one sided relation that is, the series on the left hand side implies the series on the right side, we get

$$\omega_n = \sum_{j=0}^{n} \frac{(-nk)_{j,k}}{\Gamma_k(\alpha j+\beta+1)}\,B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^p;k;1,r), \tag{2.26}$$

and we obtain

$$B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^p;k;1,r) = \frac{\Gamma_k(\alpha n+\beta+1)}{k^n\ n!}\sum_{j=0}^{n} \frac{(-nk)_{j,k}}{k^j\ j!}\,\omega_j. \tag{2.27}$$

Since the condition (2.24) holds, $\omega_n = 0$ for $n \neq ml,\ l \in \mathbb{N}$, whereas

$$\omega_{mn} = \sum_{j=0}^{mn} \frac{(-mnk)_{j,k}}{\Gamma_k(\alpha j+\beta+1)}\,B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^p;k;1,r).$$

But since the series (2.23) also holds true,

$$\omega_{mn} = \frac{k^{mn}\ (mn)!\ x^{pn}}{n!\ \Gamma_k(\alpha n+\beta+1)\ ((\lambda)_{\mu n,k})^r}.$$

Consequently, the inverse pair (2.26) and (2.27) assume the form:

$$\begin{aligned}\frac{x^{pn}}{n!} &= \frac{\Gamma_k(\alpha n+\beta+1)((\lambda)_{\mu n,k})^r}{k^{mn}\ (mn)!}\sum_{j=0}^{mn} \frac{(-mnk)_{j,k}}{\Gamma_k(\alpha j+\beta+1)} \\ &\quad \times B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^p;k;1,r),\end{aligned}$$

and we obtain

$$\begin{aligned}B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^p;k;1,r) &= \frac{\Gamma_k(\alpha n+\beta+1)}{k^n\ n!}\sum_{j=0}^{[n/m]} \frac{(-nk)_{mj,k}}{(mj)!}\,\omega_{mj} \\ &= \frac{\Gamma_k(\alpha n+\beta+1)}{k^n\ n!}\sum_{j=0}^{[n/m]} \frac{(-nk)_{mj,k}\ x^{pj}}{\Gamma_k(\alpha j+\beta+1)\ ((\lambda)_{\mu j,k})^r\ j!},\end{aligned}$$

subject to the condition (2.24). □

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