DOUBLY WEIGHTED SHARP WIRTINGER INEQUALITIES ON $\mathbb{R}_+$[1]

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Abstract We consider the ρ-weighted p-norm of functions $f:\mathbb{R}_+\to\mathbb{R}$ with finite $\|f^{(n)}\psi\|_q, 1\le q\le+\infty$. A sharp Wirtinger type inequality

$$\|f\rho\|_p\le C_{n,p,q}\|f^{(n)}\psi\|_q \text{ for all } 1\le p,q\le+\infty$$

is established for function f such that $f^{(j)}(x_i)=0$ for all $0\le j\le\alpha_i-1, i=1,\ldots,r, n=\sum_{i=1}^r\alpha_i$, where ψ, ρ and $\omega=\rho/\psi$ are non-increasing on $\mathbb{R}_+$, and $\omega^{1/\alpha}$ is integrable for $\alpha=n-1/q+1/p$. Using Hermite interpolation, we express $C_{n,p,q}$ in terms of the norm of a certain integral type operator. Then we calculate $C_{n,1,1}$ and $C_{n,\infty,\infty}$ in two specific cases.

Keywords: Hermite interpolation, weighted L_p-norm, Wirtinger inequality.

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1 Introduction

Let $\mathbb{N},\mathbb{R}$ and $\mathbb{R}_+$ be respectively the sets of all positive integers, all real numbers and all non-negative real numbers. For $1\le p\le+\infty$ and weight function ρ^p on $D\subset\mathbb{R}$, let $L_{p,\rho}(D)$ be the space of weighted p-power Lebesgue integrable functions $f:D\to\mathbb{R}$ with the corresponding weighted $L_{p,\rho}$-norms $\|\cdot\|_{p,\rho}$; i.e.,

$$\|f\|_{\infty,\rho}=\text{ess sup}_{x\in D}|f(x)\rho(x)|<+\infty, \tag{1.1}$$

and

$$\|f\|_{p,\rho}=\left(\int_D|f(x)\rho(x)|^p\mathrm{d}x\right)^{1/p}<+\infty, 1\le p<+\infty. \tag{1.2}$$

For $\rho(x)=1$, we simply write $\|\cdot\|_{p,\rho}$ as $\|\cdot\|_p$. Denote by $W_p^n(D), n\in\mathbb{N}$, the class of all continuous real-valued functions f defined on D such that $f^{(n-1)}$ (with $f^{(0)}=f$) is absolutely continuous and $\|f^{(n)}\|_p<+\infty$.

The relationships among the norms of a function and its derivatives play an important role in the study of harmonic analysis and function approximation theory. There are many well known inequalities in this area, for example, Landau-Kolmogorov inequality, Gorny inequality, Wirtinger inequality, Schmidt inequality, Sobolev inequality, Bernstein inequality and Markov inequality. Wirtinger type inequality is one kind of the most important inequalities in this aspect. The first result appeared in [2, p.105]. It says that for any locally absolutely continuous and 2π-periodic function f with the first-order derivative $f'\in L_2([0,2\pi])$ and $\int_0^{2\pi}f(x)dx=0$, we have

$$\|f\|_2\le\|f'\|_2,$$

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where the equality is valid if and only if $f \in \text{span}\{\cos, \sin\}$.

Since then there have been many results of Wirtinger type inequality for $\rho(x) = \psi(x) = 1$ and $D = [a, b]$. For example, for the case $n = 1$, E. Schmidt [17] proved the following two results.

Let $0 < p \leq +\infty, 1 \leq q \leq +\infty$. Then for an arbitrary function $f \in W_q^1([a, b])$ satisfying $f(a) = 0$ (or equivalently $f(b) = 0$) there is the sharp inequality

$$\|f\|_p \leq \frac{(1/p + 1/q')^{-1/p-1/q'}(1/p)^{1/p}(1/q')^{1/q'}\Gamma(1 + 1/p + 1/q')}{\Gamma(1 + 1/p)\Gamma(1 + 1/q')}(b - a)^{1+1/p-1/q}\|f'\|_q, \quad (1.3)$$

here q' is the conjugate exponent of q, and $1/\infty$ is to be interpreted (in the usual way) as 0. At the same time, for an arbitrary function $f \in W_q^1([a, b])$ satisfying $f(a) = f(b) = 0$ there is the sharp inequality

$$\|f\|_p \leq \frac{1}{2}\frac{(1/p + 1/q')^{-1/p-1/q'}(1/p)^{1/p}(1/q')^{1/q'}\Gamma(1 + 1/p + 1/q')}{\Gamma(1 + 1/p)\Gamma(1 + 1/q')}(b - a)^{1+1/p-1/q}\|f'\|_q. \quad (1.4)$$

Further generalizations and applications of (1.3) and (1.4) can be found in [1, 3, 9, 19].

For the case $n > 1$, the most important result is that if $f \in W_q^n([a, b])$ with j multiple zeros a and $n - j$ multiple zeros b, $0 \leq j \leq n, 1 \leq p, q \leq +\infty$, then we have the following inequality

$$\|f\|_p \leq C(n, j, p, q)(b - a)^{n+1/p-1/q}\|f^{(n)}\|_q, \quad (1.5)$$

and the best constants $C(n, j, p, q)$ can be found in [22]. Some authors such as A. Shadrin [18], S. Waldron [20] and S.N. Kudryavtsev [5] obtained the inequalities in the form

$$\|f - H_\Theta(f)\|_p \leq C(n, p, q)(b - a)^{n+1/p-1/q}\|f^{(n)}\|_q \text{ for all } f \in W_q^n([a, b]),$$

where $H_\Theta(f)$ is the Hermite interpolation to f at some multiset of n points in $[a, b]$. Further generalizations and the best constants $C(n, p, q)$ can be found in [11]. If $H_\Theta(f) = 0$, then above relationship becomes the Wirtinger inequality,

$$\|f\|_p \leq C(n, p, q)(b - a)^{n+1/p-1/q}\|f^{(n)}\|_q \text{ for all } f \in W_q^n([a, b]). \quad (1.6)$$

Recently, F.Y. Kuo, L. Plaskota, G.W. Wasilkowski [6] and P. Kritzer, F. Pillichshammer, L. Plaskota, G.W. Wasilkowski [4] considered doubly weighted approximation problems for piecewise Lagrange interpolation and piecewise Taylor interpolation on $\mathbb{R}_+$, respectively, I. Kh. Musin [12] considered the approximation of infinitely differentiable functions by polynomials in weighted spaces on $\mathbb{R}$. It is noticed that G.W. Wasilkowski and H. Woźniakowski [21] considered doubly weighted approximation problems on $\mathbb{R}$ based on Hermite data. In this paper we consider doubly weighted Wirtinger type inequality on $\mathbb{R}_+$ based on Hermite data, which is equivalent to doubly weighted approximation by Hermite interpolation, but we obtain sharp estimates other than weak asymptotic results in [4, 6].

The paper is organized as follows. Section 2 contains our main theorem and its proof. Section 3 gives two examples to show our method.

2 Basic concepts and our main result

Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a given positive and measurable function. For a positive integer n and $q \in [1, +\infty]$, let $F = F(n, q, \psi)$ be the linear space of functions $f : \mathbb{R}_+ \to \mathbb{R}$ with (locally) absolutely continuous derivative $f^{(n-1)}$ and $\|f^{(n)}\psi\|_q < +\infty$. The function spaces $F(n, q, \psi)$ were introduced in [21], see also [7, 8, 10, 13–16]. Let $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ be a function such that ρ^p is a weight function for all $1 \le p \le +\infty$.

Similar to [6], in this paper we assume that $\psi, \rho : \mathbb{R}_+ \to \mathbb{R}_+$ and $\omega = \rho/\psi$ are non-increasing on $\mathbb{R}_+$. For $n \in \mathbb{N}, 1 \le p, q \le +\infty$, let $\alpha = n - 1/q + 1/p$. We assume that

$$\|\omega^{1/\alpha}\|_1 = \int_0^{+\infty} \omega^{1/\alpha}(x)\mathrm{d}x < +\infty. \tag{2.1}$$

Recall that

$$\|v_n\rho\|_p < +\infty \tag{2.2}$$

is a necessary and sufficient condition for $\|f\rho\|_p < +\infty$ for all $f \in F(n, q, \psi)$, where $v_n(x) = x^{n-1}$, see Proposition 1 in [6].

Corresponding to [4, 6], we will consider doubly weighted Wirtinger inequality. Now we introduce the Hermite interpolation.

For r distinct points

$$0 \le z_1 < z_2 < \cdots < z_r < +\infty$$

and numbers $\alpha_k \in \mathbb{N}$ with $n = \sum_{k=1}^r \alpha_k$, we denote by $\Delta := \{z_k, \alpha_k, 1 \le k \le r\}$ a Hermite interpolation nodes with their multiplicities. Then, the Hermite interpolation polynomial $H_\Delta(g)$ of a function $g \in C^{n-1}(\mathbb{R}_+)$ based on nodes Δ is defined as

$$H_\Delta(g) \in \mathcal{P}_{n-1}, \quad \text{and} \quad H_\Delta^{(j)}(g, z_k) = g^{(j)}(z_k), \quad 0 \le j \le \alpha_k - 1, 1 \le k \le r, \tag{2.3}$$

where and in what follows, $\mathcal{P}_n$ represents the space of all algebraic polynomials of a degree at most n. The classical Hermite interpolation formula is given by

$$\begin{aligned} H_\Delta(g, x) &= \sum_{k=1}^r \frac{W_\Delta(x)}{(x - z_k)^{\alpha_k}} \sum_{s=0}^{\alpha_k - 1} g^{(s)}(z_k) \frac{(x - z_k)^s}{s!} \left\{ \frac{(x - z_k)^{\alpha_k}}{W_\Delta(x)} \right\}_{(z_k)}^{(\alpha_k - s - 1)} \\ &= \sum_{k=1}^r \sum_{s=0}^{\alpha_k - 1} g^{(s)}(z_k) h_{k,s}(x), \end{aligned} \tag{2.4}$$

where and in what follows,

$$h_{k,s}(x) = \frac{W_\Delta(x)}{(x - z_k)^{\alpha_k}} \frac{(x - z_k)^s}{s!} \left\{ \frac{(x - z_k)^{\alpha_k}}{W_\Delta(x)} \right\}_{(z_k)}^{(\alpha_k - s - 1)}, \quad W_\Delta(x) = \prod_{k=1}^r (x - z_k)^{\alpha_k}, \tag{2.5}$$

and $\{g(x)\}_{(z_k)}^{(s)}$ is the s-th degree Taylor polynomial of g at z_k.

For $g \in F(n, q, \psi)$, by the Taylor expansion of g at 0 with integral remainder, we obtain that

$$g(x)=\{g(x)\}_{(0)}^{(n-1)}+\frac{1}{(n-1)!}\int_0^{+\infty}(x-t)_+^{n-1}g^{(n)}(t)\mathrm{d}t, \tag{2.6}$$

where $x_+=x$ for $x\geq 0$ and $x_+=0$ for $x<0$. Here, corresponding to the Hermite interpolation nodes Δ, it would be convenient to define the classes of functions

$$F_\Delta=F_\Delta(n,q,\psi)=\{f\in F(n,q,\psi): f^{(j)}(z_k)=0, 0\leq j\leq\alpha_k-1, 1\leq k\leq r\}.$$

If $f\in F_\Delta(n,q,\psi)$, then from (2.4), (2.6) and $f^{(j)}(z_k)=0, 0\leq j\leq\alpha_k-1, 1\leq k\leq r$, it follows that

$$\begin{aligned} f(x) &= f(x)-H_\Delta(f,x)=\{f(x)\}_{(0)}^{(n-1)}-H_\Delta(\{f\}_{(0)}^{(n-1)},x)\\ &\quad+\frac{1}{(n-1)!}\int_0^{+\infty}(x-t)_+^{n-1}f^{(n)}(t)\mathrm{d}t-\frac{1}{(n-1)!}H_\Delta\left(\int_0^{+\infty}(\cdot-t)_+^{n-1}f^{(n)}(t)\mathrm{d}t,x\right)\\ &= \frac{1}{(n-1)!}\int_0^{+\infty}\left[(x-t)_+^{n-1}-\sum_{k=1}^{r}\sum_{s=0}^{\alpha_k-1}\frac{(n-1)!}{(n-1-s)!}(z_k-t)_+^{n-1-s}h_{k,s}(x)\right]f^{(n)}(t)\mathrm{d}t\\ &= \frac{1}{(n-1)!}\int_0^{+\infty}\left[(x-t)_+^{n-1}-H_\Delta((\cdot-t)_+^{n-1},x)\right]f^{(n)}(t)\mathrm{d}t\\ &= \int_0^{+\infty}K_\Delta(x,t)f^{(n)}(t)\mathrm{d}t, \end{aligned} \tag{2.7}$$

where we used that $\{f(x)\}_{(0)}^{(n-1)}$ is an algebraic polynomial of a degree at most $n-1$, and

$$K_\Delta(x,t)=\frac{(x-t)_+^{n-1}-H_\Delta((\cdot-t)_+^{n-1},x)}{(n-1)!}. \tag{2.8}$$

Now we introduce some information about the norms of integral operators. Let $K(x,t)$ be a continuous function on $\mathbb{R}_+\times\mathbb{R}_+$, and let

$$S(f,x)=\int_0^{+\infty}K(x,t)f(t)\mathrm{d}t. \tag{2.9}$$

Let $\|S\|_{q,p}$ be the norm of S treated as a linear operator from $L_q(\mathbb{R}_+)$ to $L_p(\mathbb{R}_+)$; i.e.,

$$\|S\|_{q,p}=\sup_{f\in L_q(\mathbb{R}_+),f\neq 0}\frac{\|S(f)\|_p}{\|f\|_q}. \tag{2.10}$$

Obviously, S is a linear and continuous operator from $L_q(\mathbb{R}_+)$ to $L_p(\mathbb{R}_+)$ if and only if $\|S\|_{q,p}$ is finite. In particular, it is known that

$$\|S\|_{1,1}=\sup_{t\in\mathbb{R}_+}\int_0^{+\infty}|K(x,t)|\mathrm{d}x, \tag{2.11}$$

$$\|S\|_{\infty,\infty}=\sup_{x\in\mathbb{R}_+}\int_0^{+\infty}|K(x,t)|\mathrm{d}t. \tag{2.12}$$

For given $\Delta:=\{0\leq z_1<z_2<\cdots<z_r<+\infty, \alpha_i\in\mathbb{N}, n=\sum_{i=1}^r\alpha_i\geq 2\}$ with $n\geq 2$, let

$$K_{\Delta,\psi,\rho}(x,t)=\frac{\rho(x)}{\psi(t)}K_\Delta(x,t), \tag{2.13}$$

and for $f \in L_q(\mathbb{R}_+)$, let

$$S_{\Delta,\psi,\rho}(f,x) = \int_0^{+\infty} K_{\Delta,\psi,\rho}(x,t) f(t) \mathrm{d}t, \ \ x \in \mathbb{R}_+. \tag{2.14}$$

Lemma 2.1. Assume that ψ and ρ are such that (2.1) and (2.2) hold. Then for any Δ with $n \geq 2$ and $1 \leq p, q \leq +\infty$, $S_{\Delta,\psi,\rho}$ is a linear and continuous operator from $L_q(\mathbb{R}_+)$ to $L_p(\mathbb{R}_+)$; i.e.,

$$\|S_{\Delta,\psi,\rho}\|_{q,p} < +\infty, \ \ 1 \leq p, q \leq +\infty. \tag{2.15}$$

Proof. First, we consider the case $p = q = +\infty$. For $f \in L_\infty(\mathbb{R}_+)$, it follows from (2.14) that for any $x \in \mathbb{R}_+$, we have that

$$|S_{\Delta,\psi,\rho}(f,x)| \leq \|f\|_\infty \cdot \int_0^{+\infty} |K_{\Delta,\psi,\rho}(x,t)| \mathrm{d}t. \tag{2.16}$$

From (2.8) it is easy to verify that

$$K_\Delta(x,t) = 0 \ \text{ for } t \notin [\min(x,z_1), \max(x,z_r)], \tag{2.17}$$

$$K_\Delta(x,t) = \frac{(x-t)_+^{n-1}}{(n-1)!}, \ t > z_r. \tag{2.18}$$

For $0 \leq x \leq z_r$, it follows from the monotonicity of ψ, ρ and (2.17) that

$$\int_0^{+\infty} |K_{\Delta,\psi,\rho}(x,t)| \mathrm{d}t = \int_0^{z_r} |K_{\Delta,\psi,\rho}(x,t)| \mathrm{d}t \leq \frac{\rho(0)}{\psi(z_r)} \int_0^{z_r} |K_\Delta(x,t)| \mathrm{d}t. \tag{2.19}$$

By (2.8) we know that $|K_\Delta(x,t)|$ is continuous on $[0,z_r]^2$. Therefore there exists an M such that $|K_\Delta(x,t)| \leq M$ for all $(x,t) \in [0,z_r]^2$. This and (2.19) imply that for all $x \in [0,z_r]$, we have that

$$\int_0^{+\infty} |K_{\Delta,\psi,\rho}(x,t)| \mathrm{d}t \leq \frac{M z_r \rho(0)}{\psi(z_r)}. \tag{2.20}$$

For $x > z_r$, denote $\Delta' = \{0 \leq x_1 < x_2 < \cdots < x_n < +\infty, \alpha_i = 1, 1 \leq i \leq n\}$. Then it follows from Lemma 1 in [6] that for $t \in [x_1, x_n]$, we have that

$$|K_{\Delta'}(x,t)| \leq \frac{(x-t)^{n-1}}{(n-1)!}. \tag{2.21}$$

By the continuity of Hermite interpolation on nodes Δ and (2.21), we obtain that

$$|K_\Delta(x,t)| = \lim_{\substack{\lim x_i = z_k, \\ \sum_{s=1}^{k-1} \alpha_s < i \leq \sum_{s=1}^{k} \alpha_s}} |K_{\Delta'}(x,t)| \leq \frac{(x-t)^{n-1}}{(n-1)!}. \tag{2.22}$$

From (2.17), (2.18) and (2.22) it follows that

$$\begin{aligned} \int_0^{+\infty} |K_{\Delta,\psi,\rho}(x,t)| \mathrm{d}t &\leq \frac{1}{(n-1)!} \int_{z_1}^{x} \frac{\rho(x)}{\psi(t)} (x-t)^{n-1} \mathrm{d}t \\ &\leq \frac{1}{(n-1)!} \int_{z_1}^{x} \omega(x)(x-t)^{n-1} \mathrm{d}t \\ &= \frac{\omega(x)(x-z_1)^n}{n!} \leq \frac{\omega(x) x^n}{n!}. \end{aligned} \tag{2.23}$$

Since ω is non-increasing on $\mathbb{R}_+$, by $\alpha = n$ and (2.1), it is easy to see that

$$\omega^{1/\alpha}(x)x \le \int_0^x \omega(u)^{1/\alpha}\mathrm{d}u \le \int_0^{+\infty} \omega(u)^{1/\alpha}\mathrm{d}u = \|\omega^{1/\alpha}\|_1. \tag{2.24}$$

From (2.23) and (2.24) it follows that

$$\int_0^{+\infty} |K_{\Delta,\psi,\rho}(x,t)|\mathrm{d}t \le \frac{\|\omega^{1/\alpha}\|_1^\alpha}{n!}. \tag{2.25}$$

It follows from (2.16), (2.20) and (2.25) that $\|S_{\Delta,\psi,\rho}\|_{\infty,\infty} < +\infty$.

Now we consider the case $1 \le p < +\infty, q = +\infty$. From (2.16), (2.20) and (2.23) it follows that

$$\begin{aligned}\|S_{\Delta,\psi,\rho}(f)\|_p^p &\le \|f\|_\infty^p \left(\int_0^{z_r}\left(\int_{\mathbb{R}_+}|K_{\Delta,\psi,\rho}(x,t)|\mathrm{d}t\right)^p\mathrm{d}x + \int_{z_r}^{+\infty}\left(\int_{\mathbb{R}_+}|K_{\Delta,\psi,\rho}(x,t)|\mathrm{d}t\right)^p\mathrm{d}x\right)\\ &\le \|f\|_\infty^p\left(z_r\left(\frac{Mz_r\rho(0)}{\psi(z_r)}\right)^p + \frac{1}{(n!)^p}\int_0^{+\infty}\omega^p(x)x^{pn}\mathrm{d}x\right).\end{aligned} \tag{2.26}$$

Note that $\alpha = n + 1/p$. Hence, similar to (2.24), we obtain that

$$\int_0^{+\infty}\omega^p(x)x^{pn}\mathrm{d}x \le \|\omega^{1/\alpha}\|_1^{pn}\int_0^{+\infty}\omega^{1/\alpha}(x)\mathrm{d}x = \|\omega^{1/\alpha}\|_1^{p\alpha}. \tag{2.27}$$

From (2.26) and (2.27) it follows that

$$\|S_{\Delta,\psi,\rho}(f)\|_p \le \|f\|_\infty\left(z_r\left(\frac{Mz_r\rho(0)}{\psi(z_r)}\right)^p + \frac{\|\omega^{1/\alpha}\|_1^{p\alpha}}{(n!)^p}\right)^{1/p};$$

i.e., $\|S_{\Delta,\psi,\rho}\|_{\infty,p} < +\infty$ for all $1 \le p < +\infty$.

Next, we consider the case $p = +\infty, 1 \le q < +\infty$. In this case, for $q = 1$, from (2.14) it follows that for any $x \in \mathbb{R}_+$, we have that

$$|S_{\Delta,\psi,\rho}(f,x)| \le \|f\|_1 \max_{t\in\mathbb{R}_+}|K_{\Delta,\psi,\rho}(x,t)|. \tag{2.28}$$

For $0 \le x \le z_r$, from $K_\Delta(x,t) = 0$ for $t > z_r$, it follows that

$$\max_{t\in\mathbb{R}_+}|K_{\Delta,\psi,\rho}(x,t)| \le \frac{M\rho(0)}{\psi(z_r)}, \tag{2.29}$$

where M is given in (2.20). For $x > z_r$, it follows from (2.17), (2.18) and (2.22) that

$$\max_{t\in\mathbb{R}_+}|K_{\Delta,\psi,\rho}(x,t)| \le \max_{t\in\mathbb{R}_+}\frac{\rho(x)(x-t)_+^{n-1}}{(n-1)!\psi(t)} \le \frac{\omega(x)x^{n-1}}{(n-1)!}. \tag{2.30}$$

Note that $\alpha = n-1$. Hence it follows from (2.30) and (2.1) that

$$\max_{t\in\mathbb{R}_+}|K_{\Delta,\psi,\rho}(x,t)| \le \frac{\omega(x)x^\alpha}{(n-1)!} \le \frac{1}{(n-1)!}\left(\int_0^x\omega^{1/\alpha}(u)\mathrm{d}u\right)^\alpha \le \frac{\|\omega^{1/\alpha}\|_1^\alpha}{(n-1)!}. \tag{2.31}$$

From (2.28), (2.29) and (2.31), it follows that

$$\|S_{\Delta,\psi,\rho}(f)\|_\infty \le \left(\frac{M\rho(0)}{\psi(z_r)} + \frac{\|\omega^{1/\alpha}\|_1^\alpha}{(n-1)!}\right)\|f\|_1. \tag{2.32}$$

For $1 < q < +\infty$, from (2.14) and Hölder inequality it follows that for any $x \in \mathbb{R}_+$, we have that

$$|S_{\Delta,\psi,\rho}(f,x)| \leq \|f\|_q \left(\int_0^{+\infty} |K_{\Delta,\psi,\rho}(x,t)|^{q/(q-1)} \mathrm{d}t \right)^{1-1/q}. \tag{2.33}$$

For $0 \leq x \leq z_r$, similar to the proof of (2.20), we obtain that

$$\int_0^{+\infty} |K_{\Delta,\psi,\rho}(x,t)|^{q/(q-1)} \mathrm{d}t \leq z_r \left(\frac{M\rho(0)}{\psi(z_r)} \right)^{q/(q-1)}. \tag{2.34}$$

For $x > z_r$, similar to the proof of (2.25), from (2.30) and $\alpha = n - 1/q$ it follows that

$$\int_0^{+\infty} |K_{\Delta,\psi,\rho}(x,t)|^{q/(q-1)} \mathrm{d}t \leq \frac{\omega^{q/(q-1)}(x) x^{1+(n-1)q/(q-1)}}{((n-1)!)^{q/(q-1)}} \leq \frac{\|\omega^{1/\alpha}\|_1^{q\alpha/(q-1)}}{((n-1)!)^{q/(q-1)}}. \tag{2.35}$$

From (2.33)-(2.35) it follows that

$$\|S_{\Delta,\psi,\rho}(f)\|_\infty \leq \left(z_r \left(\frac{M\rho(0)}{\psi(z_r)} \right)^{q/(q-1)} + \frac{\|\omega^{1/\alpha}\|_1^{q\alpha/(q-1)}}{((n-1)!)^{q/(q-1)}} \right)^{1-1/q} \|f\|_q. \tag{2.36}$$

From (2.32) and (2.36) it follows that $\|S_{\Delta,\psi,\rho}\|_{q,\infty} < +\infty$ for all $1 \leq q < +\infty$.

At last we consider the case $1 \leq p, q < +\infty$. For $q = 1$, from (2.28) it follows that

$$\begin{aligned} \|S_{\Delta,\psi,\rho}(f)\|_p^p &\leq \|f\|_1^p \int_0^{+\infty} \max_{t\in\mathbb{R}_+} |K_{\Delta,\psi,\rho}(x,t)|^p \mathrm{d}x \\ &= \|f\|_1^p \left(\int_0^{z_r} \max_{t\in\mathbb{R}_+} |K_{\Delta,\psi,\rho}(x,t)|^p \mathrm{d}x + \int_{z_r}^{+\infty} \max_{t\in\mathbb{R}_+} |K_{\Delta,\psi,\rho}(x,t)|^p \mathrm{d}x \right) \end{aligned} \tag{2.37}$$

By (2.29) we obtain that

$$\int_0^{z_r} \max_{t\in\mathbb{R}_+} |K_{\Delta,\psi,\rho}(x,t)|^p \mathrm{d}x \leq z_r \left(\frac{M\rho(0)}{\psi(z_r)} \right)^p. \tag{2.38}$$

Similar to the proof of (2.25), from (2.30) and $\alpha = n - 1 + 1/p$ it follows that

$$\int_{z_r}^{+\infty} \max_{t\in\mathbb{R}_+} |K_{\Delta,\psi,\rho}(x,t)|^p \mathrm{d}x \leq \frac{1}{((n-1)!)^p} \int_{z_r}^{+\infty} \omega^p(x) x^{(n-1)p} \mathrm{d}x \leq \frac{\|\omega^{1/\alpha}\|_1^{p\alpha}}{((n-1)!)^p}. \tag{2.39}$$

It follows from (2.37)-(2.39) that

$$\|S_{\Delta,\psi,\rho}(f)\|_p \leq \|f\|_1 \left(z_r \left(\frac{M\rho(0)}{\psi(z_r)} \right)^p + \frac{\|\omega^{1/\alpha}\|_1^{p\alpha}}{((n-1)!)^p} \right)^{1/p}. \tag{2.40}$$

For $1 < q < +\infty$, from (2.33) and Minkowski inequality it follows that

$$\begin{aligned}\|S_{\Delta,\psi,\rho}(f)\|_p &\leq \|f\|_q\left(\int_0^{+\infty}\left(\int_0^{+\infty}|K_{\Delta,\psi,\rho}(x,t)|^{q/(q-1)}\mathrm{d}t\right)^{p-p/q}\mathrm{d}x\right)^{1/p}\\ &\leq \|f\|_q\left(\int_0^{z_r}\left(\int_0^{z_r}|K_{\Delta,\psi,\rho}(x,t)|^{q/(q-1)}\mathrm{d}t\right)^{p-p/q}\mathrm{d}x\right)^{1/p}\\ &+ \|f\|_q\left(\int_{z_r}^{+\infty}\left(\int_{z_1}^{+\infty}|K_{\Delta,\psi,\rho}(x,t)|^{q/(q-1)}\mathrm{d}t\right)^{p-p/q}\mathrm{d}x\right)^{1/p}. \qquad (2.41)\end{aligned}$$

From (2.29) and $\alpha = n - 1/q + 1/p$ it follows that

$$\left(\int_0^{z_r}\left(\int_0^{z_r}|K_{\Delta,\psi,\rho}(x,t)|^{q/(q-1)}\mathrm{d}t\right)^{p-p/q}\mathrm{d}x\right)^{1/p} \leq \frac{z_r^{\alpha-n+1}M\rho(0)}{\psi(z_r)}. \qquad (2.42)$$

Similar to the proof of (2.25), from (2.30) it follows that

$$\begin{aligned}&\int_{z_r}^{+\infty}\left(\int_{z_1}^{+\infty}|K_{\Delta,\psi,\rho}(x,t)|^{q/(q-1)}\mathrm{d}t\right)^{p-p/q}\mathrm{d}x\\ \leq\ &\frac{1}{((n-1)!)^p}\int_0^{+\infty}\omega^p(x)x^{np-p/q}\mathrm{d}x \leq \frac{\|\omega^{1/\alpha}\|_1^{p\alpha}}{((n-1)!)^p}. \qquad (2.43)\end{aligned}$$

From (2.41)-(2.43) it follows that

$$\|S_{\Delta,\psi,\rho}(f)\|_p \leq \|f\|_q\left(\frac{z_r^{\alpha-n+1}M\rho(0)}{\psi(z_r)} + \frac{\|\omega^{1/\alpha}\|_1^{\alpha}}{(n-1)!}\right). \qquad (2.44)$$

From (2.40) and (2.44) it follows that $\|S_{\Delta,\psi,\rho}\|_{q,p} < +\infty$ for all $1 \leq p, q < +\infty$. The proof is completed. □

Theorem 2.2. Assume that ψ and ρ are such that (2.1) and (2.2) hold. Then for any Δ with $n \geq 2$ and for all $f \in F_\Delta(n,q,\psi)$, we have the sharp inequality

$$\|f\rho\|_p \leq \|S_{\Delta,\psi,\rho}\|_{q,p}\|f^{(n)}\psi\|_q \text{ for all } 1 \leq p, q \leq +\infty. \qquad (2.45)$$

Furthermore, the following relations hold.

$$\|S_{\Delta,\psi,\rho}\|_{1,1} = \sup_{t\in\mathbb{R}_+}\int_0^{+\infty}|K_{\Delta,\psi,\rho}(x,t)|\mathrm{d}x, \qquad (2.46)$$

$$\|S_{\Delta,\psi,\rho}\|_{\infty,\infty} = \sup_{x\in\mathbb{R}_+}\int_0^{+\infty}|K_{\Delta,\psi,\rho}(x,t)|\mathrm{d}t. \qquad (2.47)$$

Proof. If $f \in F_\Delta(n,q,\psi)$, then it follows from (2.7), (2.8), (2.13) and (2.14) that

$$f(x)\rho(x) = \int_0^{+\infty}\frac{\rho(x)}{\psi(t)}K_\Delta(x,t)\cdot f^{(n)}(t)\psi(t)\mathrm{d}t = S_{\Delta,\psi,\rho}(f^{(n)}\psi, x). \qquad (2.48)$$

From (2.48), (2.15) and (2.10) it follows that

$$\|f\rho\|_p = \|S_{\Delta,\psi,\rho}(f^{(n)}\psi)\|_p \leq \|S_{\Delta,\psi,\rho}\|_{q,p}\|f^{(n)}\psi\|_q \text{ for all } 1 \leq p, q \leq +\infty; \qquad (2.49)$$

i.e., (2.45) holds true.

We now show that (2.45) is sharp. For $g \in L_q(\mathbb{R}_+)$, let

$$\bar{f}(x) = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1}\frac{g(t)}{\psi(t)}\mathrm{d}t,\ \forall x \in [0,+\infty)$$

and let

$$f(x) = \bar{f}(x) - H_\Delta(\bar{f}, x).$$

It is known that $H_\Delta(\bar{f})$ is an algebraic polynomial of a degree at most $n-1$. Then we easily check that $f^{(n)} = \frac{g}{\psi}$ and from (2.3) it follows that $f^{(j)}(z_k) = 0,\, ,\ 0 \le j \le \alpha_k - 1, k = 1, 2, \ldots, r$. Hence $f^{(n)}\psi = g$ and $f \in F_\Delta(n, q, \psi)$. Therefore (2.48) turns into

$$f(x)\rho(x) = S_{\Delta,\psi,\rho}(f^{(n)}\psi, x) = S_{\Delta,\psi,\rho}(g, x). \tag{2.50}$$

From (2.50) and (2.10) it follows that

$$\sup_{f\in F_\Delta(n,q,\psi)} \frac{\|f\rho\|_p}{\|f^{(n)}\psi\|_q} \ge \sup_{g\in L_q(\mathbb{R}_+),g\ne 0} \frac{\|S_{\Delta,\psi,\rho}(g)\|_p}{\|g\|_q} = \|S_{\Delta,\psi,\rho}\|_{q,p}. \tag{2.51}$$

From (2.49) and (2.51) it follows that (2.45) holds true and is sharp. Besides, from (2.11) and (2.12) we respectively obtain (2.46) and (2.47). The proof is completed. □

It is obvious that $F(n, q, \psi)$ includes all polynomials p of a degree at most $n-1$ due to $p^{(n)} = 0$. Hence, for all $f \in F(n, q, \psi)$, from (2.3) we obtain $f - H_\Delta(f) \in F_\Delta(n, q, \psi)$. Combining this fact with Theorem 2.2, we obtain the following result.

Corollary 2.3. Assume that ψ and ρ are such that (2.1) and (2.2) hold. Then for any Δ with $n \ge 2$ and for all $f \in F(n, q, \psi)$, we have the sharp inequality

$$\|(f - H_\Delta(f))\rho\|_p \le \|S_{\Delta,\psi,\rho}\|_{q,p}\|f^{(n)}\psi\|_q \text{ for all } 1 \le p, q \le +\infty. \tag{2.52}$$

3 Two examples

In this section we give two examples to show how to calculate the values of $\|S_{\Delta,\psi,\rho}\|_{1,1}$ and $\|S_{\Delta,\psi,\rho}\|_{\infty,\infty}$.

Example 3.1. Let $\psi(x) = e^{-\beta_1 x}, \rho(x) = e^{-\beta_2 x}, 0 < \beta_1 < \beta_2 < +\infty, 0 \le x < +\infty$. Then for any Δ with $n \ge 2, 1 \le p, q \le +\infty$, $\alpha = n - 1/q + 1/p$, we calculate that

$$\|\omega^{1/\alpha}\|_1 = \int_0^{+\infty} e^{-(\beta_2-\beta_1)x/\alpha}\mathrm{d}x = \frac{\alpha}{\beta_2 - \beta_1} < +\infty;$$

i.e., (2.1) holds true. At the same time, we calculate that

$$\|v_n\rho\|_p = \begin{cases} \left(\frac{n-1}{\beta_2}\right)^{n-1} e^{1-n}, & p = +\infty; \\ \frac{\Gamma^{1/p}(1+(n-1)p)}{(p\beta_2)^{n-1+1/p}}, & 1 \le p < +\infty; \end{cases}$$

i.e., (2.2) holds true. Hence, the sharp inequality (2.15) holds for all $f \in F_\Delta(n, q, \psi)$. Furthermore, the best constants $\|S_{\Delta,\psi,\rho}\|_{1,1}$ and $\|S_{\Delta,\psi,\rho}\|_{\infty,\infty}$ can be calculated by the explicit formulas (2.46) and (2.47), respectively. We list two Hermite interpolation nodes to show it.

(1) If $r=1, z_1=0, \alpha_1=2, n=2, \beta_1=1, \beta_2=2$, then from (2.18) it follows that

$$K_\Delta(x,t)=(x-t)_+,\ (x,t)\in\mathbb{R}_+^2, \tag{3.1}$$

and hence

$$K_{\Delta,\psi,\rho}(x,t)=e^{-2x+t}(x-t)_+,\ (x,t)\in\mathbb{R}_+^2. \tag{3.2}$$

Let $p=q=1$. Then for any $t\in\mathbb{R}_+$, by (3.2) we calculate that

$$F_1(t)=\int_0^{+\infty}|K_{\Delta,\psi,\rho}(x,t)|\mathrm{d}x=e^t\int_t^{+\infty}(x-t)e^{-2x}\mathrm{d}x=\frac{1}{4}e^{-t}. \tag{3.3}$$

From (2.46) and (3.3), it follows that

$$\|S_{\Delta,\psi,\rho}\|_{1,1}=\max_{t\in\mathbb{R}_+}F_1(t)=F_1(0)=\frac{1}{4}.$$

Let $p=q=+\infty$. Then for any $x\in\mathbb{R}_+$, by (3.2) we calculate that

$$F_2(x)=\int_0^{+\infty}|K_{\Delta,\psi,\rho}(x,t)|\mathrm{d}t=e^{-2x}\int_0^x(x-t)e^t\mathrm{d}t=e^{-x}-e^{-2x}(x+1). \tag{3.4}$$

Using (2.47) and (3.4), we get (by using Mathematica) that

$$\|S_{\Delta,\psi,\rho}\|_{\infty,\infty}=\max_{t\in\mathbb{R}_+}F_2(x)=F_2(1.2564)=0.1018.$$

(2) If $r=2, z_1=0, z_2=1, \alpha_1=\alpha_2=1, n=2, \beta_1=1, \beta_2=2$, then from (2.8) it follows that

$$K_\Delta(x,t)=\begin{cases}(t-1)x, & 0\le x\le t\le 1;\\ (x-1)t, & 0\le t\le 1, t<x<+\infty;\\ (x-t)_+, & 1<t, 0\le x<+\infty,\end{cases} \tag{3.5}$$

and hence

$$K_{\Delta,\psi,\rho}(x,t)=\begin{cases}(t-1)xe^{-2x+t}, & 0\le x\le t\le 1;\\ (x-1)te^{-2x+t}, & 0\le t\le 1, t<x<+\infty;\\ e^{-2x+t}(x-t)_+, & 1\le t, 0\le x<+\infty.\end{cases} \tag{3.6}$$

Let $p=q=1$. Then for any $t\in\mathbb{R}_+$, by (3.6) we calculate that

$$\begin{aligned}F_1(t)&=\int_0^{+\infty}|K_{\Delta,\psi,\rho}(x,t)|\mathrm{d}x\\ &=\begin{cases}e^t(1-t)\int_0^t xe^{-2x}\mathrm{d}x+te^t\int_t^{+\infty}|x-1|e^{-2x}\mathrm{d}x, & 0\le t\le 1;\\ e^t\int_t^{+\infty}(x-t)e^{-2x}\mathrm{d}x, & 1<t<+\infty,\end{cases}\\ &=\begin{cases}\frac{1}{4}(2te^{t-2}-e^{-t}+e^t-te^t), & 0\le t\le 1;\\ \frac{1}{4}e^{-t}, & 1<t<+\infty.\end{cases}\end{aligned} \tag{3.7}$$

Using (2.46) and (3.7), we get (by using Mathematica) that

$$\|S_{\Delta,\psi,\rho}\|_{1,1} = \max_{t\in\mathbb{R}_+} F_1(t) = F_1(0.7055) = 0.1223.$$

Let $p = q = +\infty$. Then for any $x \in \mathbb{R}_+$, by (3.6) we calculate that

$$\begin{aligned} F_2(x) &= \int_0^{+\infty} |K_{\Delta,\psi,\rho}(x,t)|\mathrm{d}t \\ &= \begin{cases} e^{-2x}(1-x)\int_0^x te^t\mathrm{d}t + xe^{-2x}\int_x^1 (1-t)e^t\mathrm{d}t, & 0 \le x \le 1; \\ e^{-2x}(x-1)\int_0^1 te^t\mathrm{d}t + e^{-2x}\int_1^x (x-t)e^t\mathrm{d}t, & 1 < x < +\infty, \end{cases} \\ &= \begin{cases} e^{-2x}(-e^x + 1 - x + ex), & 0 \le x \le 1; \\ e^{-2x}(x - 1 + e^x - ex), & 1 < x < +\infty. \end{cases} \end{aligned} \tag{3.8}$$

Using (2.47) and (3.8), we get (by using Mathematica) that

$$\|S_{\Delta,\psi,\rho}\|_{\infty,\infty} = \max_{x\in\mathbb{R}_+} F_2(x) = F_2(0.3179) = 0.0911.$$

Example 3.2. Let $\psi(x) = e^{-\beta_1 x^2}, \rho(x) = e^{-\beta_2 x^2}, 0 < \beta_1 < \beta_2 < +\infty, 0 \le x < +\infty$. Then for any Δ with $n \ge 2, 1 \le p, q \le +\infty$, $\alpha = n - 1/q + 1/p$, we calculate that

$$\|\omega^{1/\alpha}\|_1 = \int_0^{+\infty} e^{-(\beta_2-\beta_1)x^2/\alpha}\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi\alpha}{\beta_2-\beta_1}} < +\infty; \tag{3.9}$$

i.e., (2.1) holds true. At the same time, we calculate that

$$\|v_n\rho\|_\infty = \max_{x\in\mathbb{R}_+} \frac{x^{n-1}}{e^{\beta_2 x^2}} = \left(\frac{n-1}{2\beta_2}\right)^{\frac{n-1}{2}} e^{\frac{1-n}{2}}, \tag{3.10}$$

and for $1 \le p < +\infty$, we calculate that

$$\|v_n\rho\|_p = \frac{\Gamma^{1/p}(\frac{1+(n-1)p}{2})}{2^{1/p}(p\beta_2)^{\frac{1+(n-1)p}{2p}}}; \tag{3.11}$$

i.e., (2.2) holds true. Hence, the sharp inequality (2.15) holds true for all $f \in F_\Delta(n, q, \psi)$. Furthermore, the best constants $\|S_{\Delta,\psi,\rho}\|_{1,1}$ and $\|S_{\Delta,\psi,\rho}\|_{\infty,\infty}$ can be calculated by the explicit formulas (2.46) and (2.47), respectively. We list two Hermite interpolation nodes to show it.

(1) If $r = 1, z_1 = 0, \alpha_1 = 2, n = 2, \beta_1 = 1, \beta_2 = 2$, then from (3.1) it follows that

$$K_{\Delta,\psi,\rho}(x,t) = e^{-2x^2+t^2}(x-t)_+,\ (x,t) \in \mathbb{R}_+^2. \tag{3.12}$$

Let $p = q = 1$. Then for any $t \in \mathbb{R}_+$, by (3.12) we calculate that

$$F_1(t) = \int_0^{+\infty} |K_{\Delta,\psi,\rho}(x,t)|\mathrm{d}x = e^{t^2}\int_t^{+\infty}(x-t)e^{-2x^2}\mathrm{d}x = \frac{1}{4}e^{-t^2} - te^{t^2}\int_t^{+\infty} e^{-2x^2}\mathrm{d}x. \tag{3.13}$$

By a direct calculation we obtain that $F_1'(t) < 0$, and from (2.46) and (3.13) it follows that

$$\|S_{\Delta,\psi,\rho}\|_{1,1} = \max_{t\in\mathbb{R}_+} F_1(t) = F_1(0) = \frac{1}{4}.$$

Let $p=q=+\infty$. Then for any $x\in\mathbb{R}_+$, by (3.12) we calculate that

$$\begin{aligned} F_2(x) &= \int_0^{+\infty} |K_{\Delta,\psi,\rho}(x,t)|\mathrm{d}t = e^{-2x^2}\int_0^x (x-t)e^{t^2}\mathrm{d}t \\ &= 2xe^{-2x^2}\left(e^{x^2}-1\right) + \left(1-4x^2\right)e^{-2x^2}\int_0^x e^{t^2}\mathrm{d}t. \end{aligned} \tag{3.14}$$

Using (2.47) and (3.14), we get (by using Mathematica) that

$$\|S_{\Delta,\psi,\rho}\|_{\infty,\infty} = \max_{x\in\mathbb{R}_+} F_2(x) = F_2(0.7433) = 0.1010.$$

(2) If $r=2, z_1=0, z_2=1, \alpha_1=\alpha_2=1, n=2, \beta_1=1, \beta_2=2$, then from (3.5) it follows that

$$K_{\Delta,\psi,\rho}(x,t) = \begin{cases} (t-1)xe^{-2x^2+t^2}, & 0\le x\le t\le 1; \\ (x-1)te^{-2x^2+t^2}, & 0\le t\le 1, t<x<+\infty; \\ e^{-2x^2+t^2}(x-t)_+, & 1\le t, 0\le x<+\infty. \end{cases} \tag{3.15}$$

Let $p=q=1$. Then for any $t\in[0,1]$, by (3.15) we calculate that

$$\begin{aligned} F_1(t) &= (1-t)e^{t^2}\int_0^t xe^{-2x^2}\mathrm{d}x + te^{t^2}\int_t^{+\infty}|x-1|e^{-2x^2}\mathrm{d}x \\ &= -\frac{1}{4}e^{-t^2} + \frac{1}{4}(1-t)e^{t^2} + \frac{1}{2}te^{t^2-2} + te^{t^2}\left(\int_t^1 e^{-2x^2}\mathrm{d}x - \int_1^{+\infty} e^{-2x^2}\mathrm{d}x\right). \end{aligned} \tag{3.16}$$

For any $t\in(1,+\infty)$, by (3.15) we calculate that

$$F_1(t) = e^{t^2}\int_t^{+\infty}(x-t)e^{-2x^2}\mathrm{d}x = \frac{1}{4}e^{-t^2} - te^{t^2}\int_t^{+\infty} e^{-2x^2}\mathrm{d}x. \tag{3.17}$$

Using (2.46), (3.16) and (3.17), we get (by using Mathematica) that

$$\|S_{\Delta,\psi,\rho}\|_{1,1} = \max_{t\in\mathbb{R}_+} F_1(t) = \max\{\max_{0\le t\le 1} F_1(t), \max_{1<t<+\infty} F_1(t)\} = F_1(0.5790) = 0.1022.$$

Let $p=q=+\infty$. Then for any $x\in[0,1]$, by (3.15) we calculate that

$$\begin{aligned} F_2(x) &= e^{-2x^2}(1-x)\int_0^x te^{t^2}\mathrm{d}t + xe^{-2x^2}\int_x^1 (1-t)e^{t^2}\mathrm{d}t \\ &= \frac{1}{2}e^{-2x^2}\left(e^{x^2}+x-1-xe\right) + xe^{-2x^2}\int_x^1 e^{t^2}\mathrm{d}t. \end{aligned} \tag{3.18}$$

For any $x\in(1,+\infty)$, by (3.15) we calculate that

$$\begin{aligned} F_2(x) &= e^{-2x^2}(x-1)\int_0^1 te^{t^2}\mathrm{d}t + e^{-2x^2}\int_1^x (x-t)e^{t^2}\mathrm{d}t \\ &= \frac{1}{2}e^{-2x^2}\left(ex-x+1-e^{x^2}\right) + xe^{-2x^2}\int_1^x e^{t^2}\mathrm{d}t. \end{aligned} \tag{3.19}$$

Using (2.47), (3.18) and (3.19), we get (by using Mathematica) that

$$\|S_{\Delta,\psi,\rho}\|_{\infty,\infty} = \max_{x\in\mathbb{R}_+} F_2(x) = \max\{\max_{0\le x\le 1} F_2(x), \max_{1<x<+\infty} F_2(x)\} = F_2(0.3648) = 0.1166.$$

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