QUASI-REVERSIBILITY METHOD FOR THE ONE-DIMENSIONAL BACKWARD TIME-SPACE FRACTIONAL DIFFUSION PROBLEM

JIN WEN[1], SHAN-SHAN WANG, ZHUAN-XIA LIU AND FEN-QIN WANG[2]

ABSTRACT. In this paper, a backward problem for one-dimensional time-space fractional diffusion process is considered. That is to determine the initial data from a noisy final data. In general, the inverse problem is ill-posed and the quasi-reversibility regularization method is used to solve it. Based on a priori and a posteriori regularization parameter selection rules, the corresponding convergence error estimates of the proposed regularization method are obtained, respectively. Finally, several numerical examples are presented to verify that our proposed scheme works well.

1. Introduction. In the past few decades, the fractional diffusion equations have attracted wide attentions due to their vast practical applications. Fractional derivative calculus and fractional differential equations have been used to describe a range of problems in viscoelastic material mechanics, hydrology, random walk, biomedicine, physics, medicine, and finance [1,2,6,7,11]. Because the fractional-order derivatives and integrals can describe the memory and hereditary properties of different substances, fractional diffusion (diffusion-wave) equations can characterize abnormal diffusion phenomenon more accurately than standard diffusion equations.

Some space or time fractional diffusion equations, which are obtained by replacing the first-order time derivative or second-order space derivative in the standard diffusion equation by a generalized derivative

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of fractional order, respectively, were successfully used for modelling relevant physical processes. These fractional diffusion equations arise quite naturally in continuous-time random walks. For the backward problem of time-fractional diffusion equation, several researchers put forward different schemes, such as quasi-reversibility method [12, 29], modified quasi-boundary method [23], Tikhonov regularization method [18, 19, 25], truncation method [22, 26], simplified Tikhonov regularization [20] and so on. As well as the problem of identifying the source term, it has also been fully discussed in [21, 24, 27, 31]. In addition, many scholars discussed the inverse problem of space-fractional diffusion equation. In [5, 34, 35], the authors considered the Riesz-Feller space-fractional backward diffusion problem, and they used the spectral regularization method, generalized Tikhonov method, Fourier transform spectral method and mollification method, respectively. In [17], Tian et al. took both the Fourier and wavelet dual least squares regularization methods to determine unknown source in space-fractional diffusion equation.

Due to the memory property of fractional derivatives, time-fractional diffusion equations have advantages in describing hereditary diffusions. However, in many practical engineering applications, when simulating anomalous diffusion, the time-space fractional diffusion equation needs to be considered. The fractional derivative in time can be used to describe particle adhesion and capture phenomena as well as fractional spatial derivative models for long particle hopping. The combined effect produces a concentration profile with sharper peaks and heavier tails. Up to now, not too much research has been done on the inverse problems of time-space fractional diffusion equations. In [8], Jia et al. studied backward problem for a time-space fractional diffusion equation, and they constructed the initial function by minimizing data residual error in Fourier space domain with variable total variation (TV) regularizing term which can protect the edges as TV regularizing term and reduce staircasing effect. In [32], Yang et al. used Landweber iterative method to identify the initial value problem of the time-space fractional diffusion-wave equation. In [9], Karapinar et al. studied the space source term problem for time-space fractional diffusion equation by quasi-reversibility method. In [33], Yang et al. utilized quasi-boundary value method for identifying the initial value of the time-space fractional diffusion equation. In [10], Kirane et al. intro-

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duced maximum principle for space and time-space fractional partial differential equations. In [28], Yang et al. used fractional Landweber method to consider the identification of the space source term problem for time-space fractional diffusion equation. In [3], Djennadi et al. considered a fractional Tikhonov regularization method for an inverse backward and source problems in the time-space fractional diffusion equations. In [30], Yang et al. discussed the inverse source problem of time-space fractional equations by two regularization methods, i. e., modified quasi-boundary regularization method and the Landweber iterative regularization method, and compared their advantages. As far as we know, the results about applying the quasi-reversibility regularization method to solve the backward problem for the time-space fractional diffusion equation is still limited.

In this paper, we consider the following backward problem: to find a function $u(x,0)$, which satisfies the time-space fractional diffusion equation as follows:

$$\begin{cases} \partial_t^\alpha u(x,t) = {}_xD_\theta^\beta u(x,t), & x \in [-C,C], \quad t \in [0,T], \\ u(-C,t) = u(C,t) = 0, & t \in [0,T], \\ u(x,0) = h(x), & x \in [-C,C], \\ u(x,T) = g(x), & x \in [-C,C], \end{cases} \tag{1.1}$$

where $\alpha \in (0,1]$, $\beta \in (0,2]$. In the process of solving this problem, we try to extend the problem (1.1) from $[-C,C]$ to $\mathbb{R}$ by zero extension, and the above problem is transformed into the following problem

$$\begin{cases} \partial_t^\alpha u(x,t) = {}_xD_\theta^\beta u(x,t), & x \in \mathbb{R}, \quad t \in [0,T], \\ u(x,0) = h(x), & x \in \mathbb{R}, \\ u(x,T) = g(x), & x \in \mathbb{R}, \\ u(x,t)|_{|x|\to\infty} bounded, & t \in [0,T], \end{cases} \tag{1.2}$$

and the time derivative is the Caputo fractional derivative of order α defined by

$$\partial_t^\alpha u(x,t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u_\tau(x,\tau)}{(t-\tau)^\alpha} d\tau, & 0 < \alpha < 1, \\ u_t(x,t), & \alpha = 1, \end{cases} \tag{1.3}$$

in which $\Gamma(\cdot)$ is the Gamma function. Then the space-fractional derivative ${}_xD_\theta^\beta$ is the Riesz-Feller fractional derivation of order β $(0 < \beta \leq 2)$ and skewness θ ($|\theta| \leq \min\{\beta, 2-\beta\}$, $\theta \neq \pm 1$), its Fourier transform is defined in [13] as

$$\mathcal{F}\{ {}_xD_\theta^\beta f(x);\xi\} = -\psi_\beta^\theta(\xi)\widehat{f}(\xi), \tag{1.4}$$

with

$$\psi_\beta^\theta(\xi) = |\xi|^\beta e^{i(sign(\xi))\theta\pi/2} = |\xi|^\beta\big(\cos(\frac{\theta\pi}{2}) + isign(\xi)\sin(\frac{\theta\pi}{2})\big). \tag{1.5}$$

The Riesz-Feller fractional derivative is defined as

$$\begin{aligned} {}_xD_\theta^\beta f(x) &= \frac{\Gamma(1+\beta)}{\pi}\sin\frac{(\beta+\theta)\pi}{2}\int_0^\infty \frac{f(x+\zeta)-f(x)}{\zeta^{1+\beta}}d\zeta \\ &+ \frac{\Gamma(1+\beta)}{\pi}\sin\frac{(\beta-\theta)\pi}{2}\int_0^\infty \frac{f(x-\zeta)-f(x)}{\zeta^{1+\beta}}d\zeta, \quad 0<\beta<2, \end{aligned} \tag{1.6}$$

$${}_xD_\theta^2 f(x) = \frac{d^2 f(x)}{dx^2}, \quad \beta = 2. \tag{1.7}$$

For $\theta = 0$ we have a symmetric operator with respect to x, which can be interpreted as

$${}_xD_0^\beta = -(-\frac{d^2}{dx^2})^{\beta/2}, \tag{1.8}$$

which can be formally deduced by writing $-|\xi|^\beta = -(\xi^2)^{\beta/2}$. More detailed explanations can be found in [4,13]. In this paper, we consider symmetric Riesz-Feller space fractional derivative operator.

Denote $g^\delta(x)$ in Ω to be the measurement data, our backward problem is to approximate the temperature $u(x,t)$ for $t\in[0,T]$ from the measurement value $g^\delta(x)$, which is noise-contaminated data for the exact data $u(x,T)$:

$$\|g^\delta(\cdot) - g(\cdot)\|_{L^2(\Omega)} \le \delta. \tag{1.9}$$

The rest of this paper is organized as follows. In Section 2, some useful notations and auxiliary lemmas are introduced. In Section 3, the exact solution of the proposed problem is derived by simple calculation combined with Fourier transform, and the ill-posed analysis of the problem is also given. In Section 4, we construct the quasi-reversibility method for the proposed problem, and give the convergent rates between the exact solution and the regularized solution under the a priori and the a posteriori parameter choice rules. Some numerical experiments are presented to verify the efficiency of our theoretical results in Section 5. The last section concludes our work.

2. Preliminary results. Throughout this paper, we use the following definition and lemmas.

Definition 2.1. *[15] The Mittag-Leffler function is*

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\quad z\in\mathbb{C}, \tag{2.1}$$

where $\alpha>0$ and $\beta\in\mathbb{R}$ are arbitrary constants.

Lemma 2.2. *[15] Let $\lambda>0$, we have*

$$\frac{d^\alpha}{dt^\alpha}E_{\alpha,1}(-\lambda t^\alpha)=-\lambda E_{\alpha,1}(-\lambda t^\alpha),\quad t>0,\quad 0<\alpha<1. \tag{2.2}$$

Lemma 2.3. *[16] For $0<\alpha<1$, $\eta>0$, we have $0<E_{\alpha,1}(-\eta)<1$, and $E_{\alpha,1}(-\eta)$ is a completely monotonic function, that is,*

$$(-1)^n\frac{d^n}{d\eta^n}E_{\alpha,1}(-\eta)\geq 0. \tag{2.3}$$

Lemma 2.4. *[15] Assume that $\alpha\in(0,1)$. Then the Mittag-Leffler functions have the asymptotic*

$$E_{\alpha,1}(x)=\frac{1}{\alpha}e^{x^{1/\alpha}}-\frac{1}{x\Gamma(1-\alpha)}+O(\frac{1}{x^2}),\quad 0<x\to+\infty,$$
$$E_{\alpha,1}(x)=-\frac{1}{x\Gamma(1-\alpha)}+O(\frac{1}{x^2}),\quad -\infty\leftarrow x<0.$$

Lemma 2.5. *[12] Assume that $0<\alpha_0<\alpha_1<1$. Then there exist constants $C_{\pm}>0$ depending only on α_0,α_1 such that*

$$\frac{C_-}{\Gamma(1-\alpha)}\frac{1}{1-x}\leq E_{\alpha,1}(x)\leq\frac{C_+}{\Gamma(1-\alpha)}\frac{1}{1-x}\quad for\quad all\quad x\leq 0, \tag{2.4}$$

This estimates are uniform for all $\alpha\in[\alpha_0,\alpha_1]$.

Lemma 2.6. *For $\xi \in \mathbb{R}$, $0 < \mu < 1$, the following inequality holds:*

$$\frac{E_{\alpha,1}(-|\xi|^{\beta}T^{\alpha})}{E_{\alpha,1}[(-|\xi|^{\beta}-\mu\xi^{4})T^{\alpha}]} \leq 1+\mu|\xi|^{4-\beta}. \tag{2.5}$$

Proof. Since the Mittag-Leffler function $E_{\alpha,1}(-zT^{\alpha})$ about variable $z>0$ is monotonic decreasing function, and $(|\xi|^{\beta}+\mu\xi^{4})T^{\alpha} \gg |\xi|^{\beta}T^{\alpha}$ as $\xi \to \infty$ and μ is fixed, we obtain that $\frac{E_{\alpha,1}(-|\xi|^{\beta}T^{\alpha})}{E_{\alpha,1}[(-|\xi|^{\beta}-\mu\xi^{4})T^{\alpha}]}$ is a monotonic increasing function.

Moreover, according to Lemma 2.4, we have the following

$$\begin{aligned}\frac{E_{\alpha,1}(-|\xi|^{\beta}T^{\alpha})}{E_{\alpha,1}[(-|\xi|^{\beta}-\mu\xi^{4})T^{\alpha}]} &\leq \lim_{|\xi|\to\infty}\frac{E_{\alpha,1}(-|\xi|^{\beta}T^{\alpha})}{E_{\alpha,1}[(-|\xi|^{\beta}-\mu\xi^{4})T^{\alpha}]}\\ &= \frac{(|\xi|^{\beta}+\mu\xi^{4})T^{\alpha}}{|\xi|^{\beta}T^{\alpha}}\\ &= 1+\mu|\xi|^{4-\beta}.\end{aligned}$$

So, we complete the proof about this lemma. □

Lemma 2.7. *For $0<\mu<1$, $p>4-\beta$, the following inequality holds:*

$$\sup_{\xi\in\mathbb{R}}\left|(1+\xi^{2})^{-\frac{p}{2}}\left[1-(1+\mu|\xi|^{4-\beta})\right]\right| \leq \max\{\mu^{\frac{p}{4-\beta}},\mu\}. \tag{2.6}$$

Proof. Let

$$A(\xi) := (1+\xi^{2})^{-\frac{p}{2}}\left[(1+\mu|\xi|^{4-\beta})-1\right].$$

The proof is separated into three cases:
Case 1. $|\xi| \geq \xi_0 = \mu^{-\frac{1}{4-\beta}}$; we get

$$A(\xi) \leq |\xi|^{-p}\cdot\mu|\xi|^{4-\beta} = \mu|\xi|^{4-\beta-p} \leq \mu|\xi_0|^{4-\beta-p} = \mu^{\frac{p}{4-\beta}}. \tag{2.7}$$

Case 2. $1<|\xi|<\xi_0$; we obtain

$$A(\xi) \leq |\xi|^{-p}\cdot\mu|\xi|^{4-\beta} = \mu|\xi|^{4-\beta-p} \leq \mu. \tag{2.8}$$

Case 3. $|\xi| \leq 1$; we get

$$A(\xi) = \mu|\xi|^{4-\beta} \leq \mu. \tag{2.9}$$

Combining (2.7)-(2.9), we obtain

$$\sup_{\xi\in\mathbb{R}}\big|A(\xi)\big| \le \max\{\mu^{\frac{p}{4-\beta}},\mu\}.$$

Hence, we complete the proof. □

3. Ill-posedness for the backward problem. If $u(.,t)\in L^2(\mathbb{R})$, then the Fourier transform operator $\mathcal{F}: L^2(\mathbb{R})\to L^2(\mathbb{R})$ about x is given by

$$\widehat{u}(\xi,t)=\mathcal{F}(u(x,t))=\int_{-\infty}^{\infty}u(x,t)e^{-i\xi x}dx,\quad \xi\in\mathbb{R}, \tag{3.1}$$

the corresponding inverse Fourier transform is defined by

$$u(x,t)=\mathcal{F}^{-1}(u(\xi,t))=\frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{u}(\xi,t)e^{i\xi x}d\xi,\quad x\in\mathbb{R}. \tag{3.2}$$

Taking the Fourier transform of problem (1.2) with respect to x, then for $\xi\in\mathbb{R}$, the problem (1.2) in the frequency domain can be expressed as follows

$$\begin{cases} \partial_t^\alpha\widehat{u}(\xi,t)=-|\xi|^\beta\widehat{u}(\xi,t), & \xi\in\mathbb{R}, t\in[0,T],\\ \widehat{u}(\xi,0)=\widehat{h}(\xi), & \xi\in\mathbb{R},\\ \widehat{u}(\xi,T)=\widehat{g}(\xi), & \xi\in\mathbb{R},\\ \widehat{u}(\xi,t)|_{|\xi|\to\infty} bounded, & t\in[0,T]. \end{cases} \tag{3.3}$$

By using the Laplace transform with respect to t in (3.3), we obtain the exact solution of problem (3.3) as follows

$$\widehat{u}(\xi,t)=\widehat{h}(\xi)E_{\alpha,1}(-|\xi|^\beta t^\alpha), \tag{3.4}$$

or equivalently,

$$u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\xi x}\widehat{h}(\xi)E_{\alpha,1}(-|\xi|^\beta t^\alpha)d\xi. \tag{3.5}$$

Using $\widehat{u}(\xi,T)=\widehat{g}(\xi)$ in (3.3),we have

$$\widehat{g}(\xi)=\widehat{h}(\xi)E_{\alpha,1}(-|\xi|^\beta T^\alpha), \tag{3.6}$$

so

$$g(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\xi x}\widehat{h}(\xi)E_{\alpha,1}(-|\xi|^\beta T^\alpha)d\xi. \tag{3.7}$$

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Thus, we have

$$\widehat{u}(\xi,t) = \frac{\widehat{g}(\xi)}{E_{\alpha,1}(-|\xi|^{\beta}T^{\alpha})}E_{\alpha,1}(-|\xi|^{\beta}t^{\alpha}), \tag{3.8}$$

and

$$\widehat{u}(\xi,0) = \widehat{h}(\xi) = \frac{\widehat{g}(\xi)}{E_{\alpha,1}(-|\xi|^{\beta}T^{\alpha})}. \tag{3.9}$$

Then

$$h(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}\frac{\widehat{g}(\xi)}{E_{\alpha,1}(-|\xi|^{\beta}T^{\alpha})}d\xi. \tag{3.10}$$

Because it is the interval after zero extension of $[-C,C]$, it is equivalent to

$$h(x) = \frac{1}{2\pi}\int_{-C}^{C} e^{i\xi x}\frac{\widehat{g}(\xi)}{E_{\alpha,1}(-|\xi|^{\beta}T^{\alpha})}d\xi. \tag{3.11}$$

Due to $E_{\alpha,1}(-|\xi|^{\beta}T^{\alpha}) \neq 0$, ie, $\widehat{h}(\xi) = 0$ in this case $\widehat{g}(\xi) = 0$, the problem (1.1) has unique solution. According to the literature [8], recovering $u(x,0)$ from the noisy measurement of exact $g(x)$ based on relation (3.7) in the frequency domain is ill posed due to the rapid decay of the forward process.

It is well known that for any ill-posed problem some a priori assumptions on the exact solution are needed. Suppose that there exists a constant $E > 0$ such that the following a priori bound holds

$$\|u(\cdot,0)\|_{H^p(\mathbb{R})} \leq E, \quad p > 0. \tag{3.12}$$

Here, $\|\cdot\|_{H^p(\mathbb{R})}$ denotes the norm in the Sobolev space $H^p(\mathbb{R})$ defined by

$$H^p(\mathbb{R}) := \{\psi \in L^2(\mathbb{R}) : \|\psi\|_{H^p(\mathbb{R})} < \infty\},$$

and

$$\|\psi(\cdot)\|_{H^p(\mathbb{R})} := (\int_{\mathbb{R}}(1+|\xi|^2)^p|\widehat{\psi}(\xi)|^2 d\xi)^{\frac{1}{2}}.$$

Moreover, we know that it is the L^2-norm when $p = 0$. Throughout this paper, we denote the L^2-norm by $\|\cdot\|$.

4. Quasi-reversibility method and convergence rates. In this section, we will use the quasi-reversibility regularization method to obtain the regularization solution of the problem (1.1). The main idea of the quasi-reversibility method is to add a perturbation term to the right side of the equation of the original ill-posed problem, so that the disturbed problem becomes a well-posed problem, and then use the solution of this new problem to construct the regularization solution of the original ill-posed problem. In nature, we will investigate the following problem

$$\begin{cases} \partial_t^\alpha u(x,t) = {}_xD_\theta^\beta u(x,t) - \mu u_{xxxx}(x,t), & x \in [-C,C], \quad t \in [0,T], \\ u(-C,t) = u(C,t) = U_{xx}(-C,t) = u_{xx}(C,t) = 0, & t \in [0,T], \\ u(x,0) = h(x), & x \in [-C,C], \\ u(x,T) = g(x), & x \in [-C,C], \end{cases} \tag{4.1}$$

where $\mu > 0$ is a regularization parameter. After zero extension for the domain of the above equation to $\mathbb{R}$. Taking Fourier transform with respect to x for the above equation, we obtain

$$\partial_t^\alpha \widehat{u}(\xi,t) = -|\xi|^\beta \widehat{u}(\xi,t) - \mu\xi^4 \widehat{u}(\xi,t). \tag{4.2}$$

Thus, by using the separation of variables, we know $\widehat{u}_\mu(\xi,t)$ has the following form

$$\widehat{u}_\mu(\xi,t) = \widehat{h}(\xi) E_{\alpha,1}[(-|\xi|^\beta - \mu\xi^4)t^\alpha]. \tag{4.3}$$

From $\widehat{u}_\mu^\delta(\xi,T) = \widehat{g}^\delta(\xi)$, we can obtain

$$\widehat{u}_\mu^\delta(\xi,t) = \frac{\widehat{g}^\delta(\xi)}{E_{\alpha,1}[(-|\xi|^\beta - \mu\xi^4)T^\alpha]} E_{\alpha,1}[(-|\xi|^\beta - \mu\xi^4)t^\alpha], \tag{4.4}$$

and

$$\widehat{u}_\mu^\delta(\xi,0) = \widehat{h}_\mu^\delta(\xi) = \frac{\widehat{g}^\delta(\xi)}{E_{\alpha,1}[(-|\xi|^\beta - \mu\xi^4)T^\alpha]}. \tag{4.5}$$

Applying inverse Fourier transform, we obtain

$$h_\mu^\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x} \frac{\widehat{g}^\delta(\xi)}{E_{\alpha,1}[(-|\xi|^\beta - \mu\xi^4)T^\alpha]} d\xi. \tag{4.6}$$

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Because it is the interval after zero extension of $[-C, C]$, it is equivalent to

$$h_\mu^\delta(x) = \frac{1}{2\pi}\int_{-C}^{C} e^{i\xi x}\frac{\widehat{g}^\delta(\xi)}{E_{\alpha,1}[(-|\xi|^\beta - \mu\xi^4)T^\alpha]}d\xi. \tag{4.7}$$

Denote

$$\widehat{u}_\mu(\xi, 0) = \frac{\widehat{g}(\xi)}{E_{\alpha,1}[(-|\xi|^\beta - \mu\xi^4)T^\alpha]}. \tag{4.8}$$

In the following, we give two convergence rate estimates for $\|\widehat{h}_\mu^\delta(\xi) - \widehat{h}(\xi)\|$ by using a priori and a posteriori choice rules.

4.1. Convergence estimate under an a priori regularization parameter choice rule.

Theorem 4.1. *Let $h(x)$ given by (3.11) be the exact solution of problem (1.1) and $h_\mu^\delta(x)$ be its regularization approximation given by (4.7). Let assumption (1.9) and the priori condition (3.12) hold. If we choose the regularization parameter*

$$\mu = (\frac{\delta}{E})^{\frac{4-\beta}{p+(4-\beta)}}, \tag{4.9}$$

then the following error estimate holds

$$\|u_\mu^\delta(x, 0) - u(x, 0)\| \le E^{\frac{4-\beta}{(4-\beta)+p}}\delta^{\frac{p}{(4-\beta)+p}}\big[C_1 + \max\{1, (\frac{\delta}{E})^{\frac{4-\beta-p}{4-\beta+p}}\}\big], \tag{4.10}$$

where $C_1 = 2T^\alpha\Gamma(1-\alpha)\max\{C^\beta, C^4\}$.

Proof. By the triangle inequality and the Parseval formula, we have

$$\begin{aligned}\|u_\mu^\delta(x, 0) - u(x, 0)\| &= \|\widehat{u}_\mu^\delta(\xi, 0) - \widehat{u}(\xi, 0)\| \\ &\le \|\widehat{u}_\mu^\delta(\xi, 0) - \widehat{u}_\mu(\xi, 0)\| + \|\widehat{u}_\mu(\xi, 0) - \widehat{u}(\xi, 0)\|. \end{aligned} \tag{4.11}$$

We firstly give an estimate for the first term. From (1.9) and Lemma 2.4, we have

$$\begin{aligned}
\|\widehat{u}^{\delta}_{\mu}(\xi,0)-\widehat{u}_{\mu}(\xi,0)\| &= \|\frac{\widehat{g}^{\delta}(\xi)-\widehat{g}(\xi)}{E_{\alpha,1}[(-|\xi|^{\beta}-\mu\xi^4)T^{\alpha}]}\| \\
&\le \|\widehat{g}^{\delta}(\xi)-\widehat{g}(\xi)\|\|\frac{1}{E_{\alpha,1}[(-|\xi|^{\beta}-\mu\xi^4)T^{\alpha}]}\| \\
&\le \delta\|\frac{1}{E_{\alpha,1}[(-|\xi|^{\beta}-\mu\xi^4)T^{\alpha}]}\| \\
&\le \delta\sup_{\xi\in\mathbb{R}}|(|\xi|^{\beta}+\mu\xi^4)T^{\alpha}\Gamma(1-\alpha)| \\
&\le \delta T^{\alpha}\Gamma(1-\alpha)(1+\mu)\max\{C^{\beta},C^4\} \\
&\le C_1\frac{\delta}{\mu}.
\end{aligned}$$

Now we estimate the second term of the right side of (4.11). Applying the a priori bound condition (3.12), Lemma 2.6 and Lemma 2.7, we obtain

$$\begin{aligned}
\|\widehat{u}_{\mu}(\xi,0)-\widehat{u}(\xi,0)\| &= \|\frac{\widehat{g}(\xi)}{E_{\alpha,1}[(-|\xi|^{\beta}-\mu\xi^4)T^{\alpha}]}-\frac{\widehat{g}(\xi)}{E_{\alpha,1}(-|\xi|^{\beta}T^{\alpha})}\| \\
&= \|\frac{\widehat{g}(\xi)[E_{\alpha,1}(-|\xi|^{\beta}T^{\alpha})-E_{\alpha,1}[(-|\xi|^{\beta}-\mu\xi^4)T^{\alpha}]]}{E_{\alpha,1}[(-|\xi|^{\beta}-\mu\xi^4)T^{\alpha}]E_{\alpha,1}(-|\xi|^{\beta}T^{\alpha})}\| \\
&= \|\frac{\widehat{g}(\xi)}{E_{\alpha,1}(-|\xi|^{\beta}T^{\alpha})}(1+\xi^2)^{\frac{p}{2}}(1+\xi^2)^{-\frac{p}{2}} \\
&\times\frac{E_{\alpha,1}(-|\xi|^{\beta}T^{\alpha})-E_{\alpha,1}[(-|\xi|^{\beta}-\mu\xi^4)T^{\alpha}]}{E_{\alpha,1}[(-|\xi|^{\beta}-\mu\xi^4)T^{\alpha}]}\| \\
&\le \|\frac{\widehat{g}(\xi)}{E_{\alpha,1}(-|\xi|^{\beta}T^{\alpha})}(1+\xi^2)^{\frac{p}{2}}\| \\
&\times\|(1+\xi^2)^{-\frac{p}{2}}(\frac{E_{\alpha,1}(-|\xi|^{\beta}T^{\alpha})}{E_{\alpha,1}[(-|\xi|^{\beta}-\mu\xi^4)T^{\alpha}}-1)\| \\
&\le E\sup_{\xi\in\mathbb{R}}|(1+\xi^2)^{-\frac{p}{2}}(\frac{E_{\alpha,1}(-|\xi|^{\beta}T^{\alpha})}{E_{\alpha,1}[(-|\xi|^{\beta}-\mu\xi^4)T^{\alpha}}-1)| \\
&\le E\sup_{\xi\in\mathbb{R}}|(1+\xi^2)^{-\frac{p}{2}}[(1+\mu|\xi|^{4-\beta})-1]| \\
&\le E\max\{\mu^{\frac{p}{4-\beta}},\mu\}.
\end{aligned}$$

Combining the above two inequalities, we have

$$\|u_\mu^\delta(x,0)-u(x,0)\| \le C_1\frac{\delta}{\mu}+E\max\{\mu^{\frac{p}{4-\beta}},\mu\}.$$

Choosing the regularization parameter $\mu=(\frac{\delta}{E})^{\frac{4-\beta}{(4-\beta)+p}}$, we obtain

$$\|u_\mu^\delta(x,0)-u(x,0)\| \le E^{\frac{4-\beta}{(4-\beta)+p}}\delta^{\frac{p}{(4-\beta)+p}}[C_1+\max\{1,(\frac{\delta}{E})^{\frac{4-\beta-p}{4-\beta+p}}\}].$$

The proof is completed. □

4.2. Convergence estimate under an a posteriori regularization parameter choice rule. In this subsection, we use an a posteriori regularization parameter choice rule. The most general a posteriori rule is Morozov's discrepancy principle [14]. We seek the regularization parameter μ by the equation

(4.12)
$$\rho(\mu):=\|\frac{E_{\alpha,1}[(-|\xi|^\beta-\mu\xi^4)T^\alpha]}{E_{\alpha,1}(-|\xi|^\beta T^\alpha)}\widehat{g}^\delta(\cdot)-\widehat{g}^\delta(\cdot)\|=\delta+\tau(\ln\ln(\frac{1}{\delta}))^{-1},$$

where $\tau>1$ is a constant. According to the following lemma, we know there exists a unique solution for (4.12) if $\|\widehat{g}^\delta(\cdot)\|>\delta+\tau(\ln\ln(\frac{1}{\delta}))^{-1}>0$.

Lemma 4.2. *If $\delta>0$, then the following results hold*
(a) $\rho(\mu)$ is a continuous function;
(b) $\lim_{\mu\to0^+}\rho(\mu)=0$;
(c) $\lim_{\mu\to+\infty}\rho(\mu)=\|\widehat{g}^\delta\|$;
(d) $\rho(\mu)$ is a strictly increasing function.

Proof. From Lemma 2.4, we can get

$$\begin{aligned}
\rho(\mu) :&= \|(\frac{E_{\alpha,1}[(-|\xi|^{\beta}-\mu\xi^4)T^{\alpha}]}{E_{\alpha,1}(-|\xi|^{\beta}T^{\alpha})}-1)\widehat{g}^{\delta}(\xi)\| \\
&\le \|(\frac{-1}{E_{\alpha,1}(-|\xi|^{\beta}T^{\alpha})}\frac{1}{(|\xi|^{\beta}+\mu\xi^4)T^{\alpha}\Gamma(1-\alpha)}-1)\widehat{g}^{\delta}(\xi)\|, \\
\lim_{\mu\to\infty}\rho(\mu) &= \lim_{\mu\to\infty}\|(\frac{1}{E_{\alpha,1}(-|\xi|^{\beta}T^{\alpha})}\frac{1}{(|\xi|^{\beta}+\mu\xi^4)T^{\alpha}\Gamma(1-\alpha)}-1)\widehat{g}^{\delta}(\xi)\| \\
&= \|\widehat{g}^{\delta}(\xi)\|.
\end{aligned}$$

The above results (a), (b) and (d) can be easily proved and we omit the detailed procedure here. □

Lemma 4.3. *The following inequality holds*

$$\|\widehat{g}^{\delta}(\xi)-\frac{E_{\alpha,1}[(-|\xi|^{\beta}-\mu\xi^4)T^{\alpha}]}{E_{\alpha,1}(-|\xi|^{\beta}T^{\alpha})}\widehat{g}(\xi)\| \le 2\delta+\tau(\log\log(\frac{1}{\delta}))^{-1}. \tag{4.13}$$

Proof. According to triangle inequality and (4.12), there holds

$$\begin{aligned}
&\|\widehat{g}^{\delta}(\xi)-\frac{E_{\alpha,1}[(-|\xi|^{\beta}-\mu\xi^4)T^{\alpha}]}{E_{\alpha,1}(-|\xi|^{\beta}T^{\alpha})}\widehat{g}(\xi)\| \\
&\le \|\widehat{g}^{\delta}(\xi)-\frac{E_{\alpha,1}[(-|\xi|^{\beta}-\mu\xi^4)T^{\alpha}]}{E_{\alpha,1}(-|\xi|^{\beta}T^{\alpha})}\widehat{g}^{\delta}(\xi)\| \\
&+\|\frac{E_{\alpha,1}[(-|\xi|^{\beta}-\mu\xi^4)T^{\alpha}]}{E_{\alpha,1}(-|\xi|^{\beta}T^{\alpha})}\big(\widehat{g}^{\delta}(\xi)-\widehat{g}(\xi)\big)\| \\
&\le \delta+\tau(\log\log(\frac{1}{\delta}))^{-1}+\delta \\
&\le 2\delta+\tau(\log\log(\frac{1}{\delta}))^{-1}.
\end{aligned}$$

□

Lemma 4.4. *The following inequality also holds*

$$\mu \le C_2\frac{E}{\tau}(\log\log(\frac{1}{\delta})). \tag{4.14}$$

Proof. From (4.12) and Lemma 2.5, there holds

$$
\begin{aligned}
&\delta+\tau(loglog(\frac{1}{\delta}))^{-1}\\
&=\|\frac{E_{\alpha,1}[(-|\xi|^{\beta}-\mu\xi^{4})T^{\alpha}]}{E_{\alpha,1}(-|\xi|^{\beta}T^{\alpha})}\widehat{g}^{\delta}(\xi)-\widehat{g}^{\delta}(\xi)\|\\
&=\|(\frac{E_{\alpha,1}[(-|\xi|^{\beta}-\mu\xi^{4})T^{\alpha}]}{E_{\alpha,1}(-|\xi|^{\beta}T^{\alpha})}-1)\widehat{g}^{\delta}(\xi)\|\\
&\le\|(\frac{E_{\alpha,1}[(-|\xi|^{\beta}-\mu\xi^{4})T^{\alpha}]}{E_{\alpha,1}(-|\xi|^{\beta}T^{\alpha})}-1)(\widehat{g}^{\delta}(\xi)-\widehat{g}(\xi)\|\\
&\quad+\|(\frac{E_{\alpha,1}[(-|\xi|^{\beta}-\mu\xi^{4})T^{\alpha}]}{E_{\alpha,1}(-|\xi|^{\beta}T^{\alpha})}-1)\widehat{g}(\xi)\|\\
&\le\delta+\|(\frac{E_{\alpha,1}[(-|\xi|^{\beta}-\mu\xi^{4})T^{\alpha}]}{E_{\alpha,1}(-|\xi|^{\beta}T^{\alpha})}-1)\frac{\widehat{g}^{\delta}(\xi)}{E_{\alpha,1}(-|\xi|^{\beta}T^{\alpha})}\\
&\times E_{\alpha,1}(-|\xi|^{\beta}T^{\alpha})(1+\xi^{2})^{-\frac{p}{2}}(1+\xi^{2})^{\frac{p}{2}}\|\\
&\le\delta+E\sup_{\xi\in\mathbb{R}}|(\frac{E_{\alpha,1}[(-|\xi|^{\beta}-\mu\xi^{4})T^{\alpha}]}{E_{\alpha,1}(-|\xi|^{\beta}T^{\alpha})}-1)E_{\alpha,1}(-|\xi|^{\beta}T^{\alpha})(1+\xi^{2})^{-\frac{p}{2}}|\\
&\le\delta+E\sup_{\xi\in\mathbb{R}}|E_{\alpha,1}[(-|\xi|^{\beta}-\mu\xi^{4})T^{\alpha}]|\\
&\le\delta+E\frac{C_{+}}{\Gamma(1-\alpha)}\frac{1}{1+(|\xi|^{\beta}+\mu\xi^{4})T^{\alpha}}\\
&\le\delta+C_{2}E\frac{1}{\mu}.
\end{aligned}
$$

Hence

$$\mu\le C_{2}\frac{E}{\tau}(\log\log(\frac{1}{\delta})), \tag{4.15}$$

where $C_2=\frac{C_+}{\Gamma(1-\alpha)}\frac{1}{\xi^4T^\alpha}$. □

Now we give the main result of this section.

Theorem 4.5. *Suppose the a priori condition (3.12) and the noise assumption (1.9) hold, and there exists $\tau>1$ such that $\|g^{\delta}\|>\delta+\tau(\log\log(\frac{1}{\delta}))^{-1}>0$. The regularization parameter $\mu>0$ is chosen* *by discrepancy principle (4.12). Then*

$$\|u_\mu^\delta(\cdot,0)-u(\cdot,0)\| \le \Big(o(1)+(2C_1)^{\frac{p+(4-\beta)}{4-\beta}}\Big)^{\frac{4-\beta}{p+(4-\beta)}} E^{\frac{4-\beta}{p+(4-\beta)}}\Big(2\delta+\tau\big(\log\log(\frac{1}{\delta})\big)^{-1}\Big)^{\frac{p}{p+(4-\beta)}},$$

where $C_1 = T^\alpha\Gamma(1-\alpha)\delta\max\{C^\beta, C^4\}$, $C_2=\frac{C_+}{\Gamma(1-\alpha)}\frac{1}{\xi^4T^\alpha}$.

Proof. By the Parseval formula, the Hölder inequality, (3.9), (4.5), (1.9), (3.12), (4.13) and (4.14), we obtain

$$\begin{aligned}
&\|u_\mu^\delta(\cdot,0)-u(\cdot,0)\|^2 = \|\widehat{u}_\mu^\delta(\xi,0)-\widehat{u}(\xi,0)\|^2\\
&\quad= \Big\|\frac{\widehat{g}^\delta(\xi)}{E_{\alpha,1}[(-|\xi|^\beta-\mu\xi^4)T^\alpha]}-\frac{\widehat{g}(\xi)}{E_{\alpha,1}(-|\xi|^\beta T^\alpha)}\Big\|^2\\
&\quad= \Big\|\frac{1}{E_{\alpha,1}[(-|\xi|^\beta-\mu\xi^4)T^\alpha]}\Big(\widehat{g}^\delta(\xi)-\frac{E_{\alpha,1}[(-|\xi|^\beta-\mu\xi^4)T^\alpha]}{E_{\alpha,1}(-|\xi|^\beta T^\alpha)}\widehat{g}(\xi)\Big)\Big\|^2\\
&\quad= \int_{\mathbb{R}}\Big(\frac{1}{E_{\alpha,1}[(-|\xi|^\beta-\mu\xi^4)T^\alpha]}\Big)^2\Big[\widehat{g}^\delta(\xi)-\frac{E_{\alpha,1}[(-|\xi|^\beta-\mu\xi^4)T^\alpha]}{E_{\alpha,1}(-|\xi|^\beta T^\alpha)}\widehat{g}(\xi)\Big]^{\frac{2(4-\beta)}{p+(4-\beta)}}\\
&\qquad\Big[\widehat{g}^\delta(\xi)-\frac{E_{\alpha,1}[(-|\xi|^\beta-\mu\xi^4)T^\alpha]}{E_{\alpha,1}(-|\xi|^\beta T^\alpha)}\widehat{g}(\xi)\Big]^{\frac{2p}{p+(4-\beta)}}d\xi\\
&\quad\le \Bigg[\int_{\mathbb{R}}\Bigg(\Big(\frac{1}{E_{\alpha,1}[(-|\xi|^\beta-\mu\xi^4)T^\alpha]}\Big)^2\Big(\widehat{g}^\delta(\xi)-\frac{E_{\alpha,1}[(-|\xi|^\beta-\mu\xi^4)T^\alpha]}{E_{\alpha,1}(-|\xi|^\beta T^\alpha)}\widehat{g}(\xi)\Big)^{\frac{2(4-\beta)}{P+(4-\beta)}}\Bigg)^{\frac{p+(4-\beta)}{4-\beta}}d\xi\Bigg]^{\frac{4-\beta}{p+(4-\beta)}}\\
&\times\Bigg[\int_{\mathbb{R}}\Bigg(\widehat{g}^\delta(\xi)-\frac{E_{\alpha,1}[(-|\xi|^\beta-\mu\xi^4)T^\alpha]}{E_{\alpha,1}(-|\xi|^\beta T^\alpha)}\widehat{g}(\xi)\Big)^{\frac{2(4-\beta)}{P+(4-\beta)}}\Bigg)^{\frac{p+(4-\beta)}{4-\beta}}d\xi\Bigg]^{\frac{p}{p+(4-\beta)}}\\
&\quad= \Bigg[\int_{\mathbb{R}}\Big(\frac{1}{E_{\alpha,1}[(-|\xi|^\beta-\mu\xi^4)T^\alpha]}\Big)^{\frac{2(p+4-\beta)}{4-\beta}}\Big(\widehat{g}^\delta(\xi)-\frac{E_{\alpha,1}[(-|\xi|^\beta-\mu\xi^4)T^\alpha]}{E_{\alpha,1}(-|\xi|^\beta T^\alpha)}\widehat{g}(\xi)\Big)^2d\xi\Bigg]^{\frac{4-\beta}{p+(4-\beta)}}\\
&\qquad\Bigg[\int_{\mathbb{R}}\Big(\widehat{g}^\delta(\xi)-\frac{E_{\alpha,1}[(-|\xi|^\beta-\mu\xi^4)T^\alpha]}{E_{\alpha,1}(-|\xi|^\beta T^\alpha)}\widehat{g}(\xi)\Big)^2d\xi\Bigg]^{\frac{p}{p+(4-\beta)}}\\
&\quad= \Big\|\Big(\frac{1}{E_{\alpha,1}[(-|\xi|^\beta-\mu\xi^4)T^\alpha]}\Big)^{\frac{p+(4-\beta)}{4-\beta}}\Big(\widehat{g}^\delta(\xi)-\frac{E_{\alpha,1}[(-|\xi|^\beta-\mu\xi^4)T^\alpha]}{E_{\alpha,1}(-|\xi|^\beta T^\alpha)}\widehat{g}(\xi)\Big)\Big\|^{\frac{2(4-\beta)}{p+(4-\beta)}}
\end{aligned}$$

$$
\begin{aligned}
&\times \Big\|\widehat{g}^{\delta}(\xi)-\frac{E_{\alpha,1}[(-|\xi|^{\beta}-\mu\xi^{4})T^{\alpha}]}{E_{\alpha,1}(-|\xi|^{\beta}T^{\alpha})}\widehat{g}(\xi)\Big\|^{\frac{2p}{p+(4-\beta)}}\\
&\le \left(\Big\|\Big(\frac{1}{E_{\alpha,1}[(-|\xi|^{\beta}-\mu\xi^{4})T^{\alpha}]}\Big)^{\frac{p+(4-\beta)}{4-\beta}}\Big(\widehat{g}^{\delta}(\xi)-\widehat{g}(\xi)\Big)\Big\|\right.\\
&\left.+\Big\|\Big(\frac{1}{E_{\alpha,1}[(-|\xi|^{\beta}-\mu\xi^{4})T^{\alpha}]}\Big)^{\frac{p+(4-\beta)}{4-\beta}}\Big(1-\frac{E_{\alpha,1}[(-|\xi|^{\beta}-\mu\xi^{4})T^{\alpha}]}{E_{\alpha,1}(-|\xi|^{\beta}T^{\alpha})}\Big)\widehat{g}(\xi)\Big\|\right)^{\frac{2(4-\beta)}{p+(4-\beta)}}\\
&\times\left(2\delta+\tau(\log\log(\frac{1}{\delta}))^{-1}\right)^{\frac{2p}{p+(4-\beta)}}\\
&\le\left(\sup_{\xi\in\mathbb{R}}\left|\Big(\frac{1}{E_{\alpha,1}[(-|\xi|^{\beta}-\mu\xi^{4})T^{\alpha}]}\Big)^{\frac{p+(4-\beta)}{4-\beta}}\right|\delta+E\sup_{\xi\in\mathbb{R}}\left|\Big(\frac{1}{E_{\alpha,1}[(-|\xi|^{\beta}-\mu\xi^{4})T^{\alpha}]}\Big)^{\frac{p+(4-\beta)}{4-\beta}}\right.\right.\\
&\left.\left.\times E_{\alpha,1}(-|\xi|^{\beta}T^{\alpha})(1+\xi^{2})^{-\frac{p}{2}}\right|\right)^{\frac{2(4-\beta)}{p+(4-\beta)}}\left(2\delta+\tau(\log\log(\frac{1}{\delta}))^{-1}\right)^{\frac{2p}{p+(4-\beta)}}\\
&\le\left(\Big[C_{1}(1+\mu)\Big]^{\frac{p+(4-\beta)}{4-\beta}}\delta+E(2C_{1})^{\frac{p+(4-\beta)}{4-\beta}}\right)^{\frac{2(4-\beta)}{p+(4-\beta)}}\left(2\delta+\tau(\log\log(\frac{1}{\delta}))^{-1}\right)^{\frac{2p}{p+(4-\beta)}}\\
&\le\left(\Big[C_{1}C_{2}\frac{E}{\tau}(\log\log(\frac{1}{\delta}))\Big(\frac{\tau}{C_{2}E}(\log\log(\frac{1}{\delta}))^{-1}+1\Big)\Big]^{\frac{p+(4-\beta)}{4-\beta}}\delta+E(2C_{1})^{\frac{p+(4-\beta)}{4-\beta}}\right)^{\frac{2(4-\beta)}{p+(4-\beta)}}\\
&\times\left(2\delta+\tau(\log\log(\frac{1}{\delta}))^{-1}\right)^{\frac{2p}{p+(4-\beta)}}\\
&\le\left(\Big[C_{1}C_{2}\frac{1}{\tau}(\log\log(\frac{1}{\delta}))\Big(\frac{\tau}{C_{2}E}(\log\log(\frac{1}{\delta}))^{-1}+1\Big)\Big]^{\frac{p+(4-\beta)}{4-\beta}}E^{\frac{p}{4-\beta}}\delta+(2C_{1})^{\frac{p+(4-\beta)}{4-\beta}}\right)^{\frac{2(4-\beta)}{p+(4-\beta)}}\\
&\times E^{\frac{2(4-\beta)}{p+(4-\beta)}}\left(2\delta+\tau\big(\log\log(\frac{1}{\delta})\big)^{-1}\right)^{\frac{2p}{p+(4-\beta)}}.
\end{aligned}
$$

Hence

$$\|u_\mu^\delta(\cdot,0)-u(\cdot,0)\| \le \Bigg(\Big[C_1C_2\frac{1}{\tau}(\log\log(\frac{1}{\delta}))\Big(\frac{\tau}{C_2E}(\log\log(\frac{1}{\delta}))^{-1}+1\Big)\Big]^{\frac{p+(4-\beta)}{4-\beta}}E^{\frac{p}{4-\beta}}\delta+ (2C_1)^{\frac{p+(4-\beta)}{4-\beta}}\Bigg)^{\frac{4-\beta}{p+(4-\beta)}}E^{\frac{4-\beta}{p+(4-\beta)}}\Bigg(2\delta+\tau\big(\log\log(\frac{1}{\delta})\big)^{-1}\Bigg)^{\frac{p}{p+(4-\beta)}}.$$

Noting that

$$\lim_{\delta\to 0}\big(\log\log(\frac{1}{\delta})\big)^{\frac{p+(4-\beta)}{4-\beta}}\delta=0,$$

we obtain

(4.16)

$$\|u_\mu^\delta(\cdot,0)-u(\cdot,0)\| \le \Big(o(1)+(2C_1)^{\frac{p+(4-\beta)}{4-\beta}}\Big)^{\frac{4-\beta}{p+(4-\beta)}}E^{\frac{4-\beta}{p+(4-\beta)}}\Big(2\delta+\tau\big(\log\log(\frac{1}{\delta})\big)^{-1}\Big)^{\frac{p}{p+(4-\beta)}},$$

as $\delta\to 0$.

The proof of Theorem 4.5 is completed. □

5. Numerical examples. In this section, we show some numerical results obtained by the regularization method in three examples. We use the discrete Fourier transform to complete our numerical experiment. Since the analytic solution of the problem (1.1) is difficult to obtain, we construct the final data $g(x)$ by solving the following forward problem

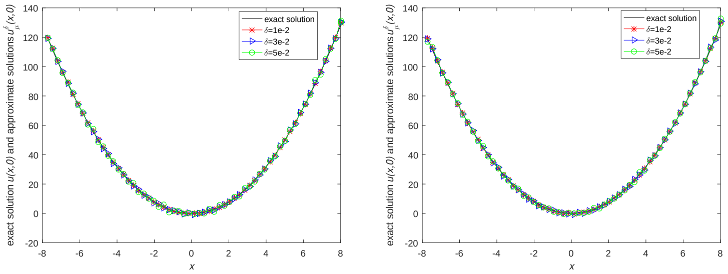
$$\begin{cases} \partial_t^\alpha u(x,t) = {}_xD_\theta^\beta u(x,t), & x\in[-C,C], \quad t\in[0,T],\\ u(-C,t)=u(C,t)=0, & t\in[0,T],\\ u(x,0)=h(x), & x\in[-C,C], \end{cases} \tag{5.1}$$

with the given data $h(x)$.

The noise data is generated by adding a random perturbation, i.e.,

$$g^\delta = g+\delta\cdot randn(size(g)), \tag{5.2}$$

where the function "$randn(\cdot)$" generates arrays of random numbers whose elements are normally distributed with mean 0, variance σ^2, and standard deviation $\sigma=1$. "$Randn(size(g))$" returns an array of random entries that is the same size as g. In order to make the



(a) a priori (b) a posteriori

FIGURE 1. The exact solution and approximate solution for Example 1 with $\alpha = 0.3$, $\beta = 0.8$, and $T = 0.3$.

sensitivity analysis for numerical results, we calculate the relative error by

$$\epsilon(u) = \|u(x,t) - u_\mu^\delta(x,t)\|/\|u(x,t)\|. \tag{5.3}$$

The numerical examples are constructed in the following way: First, we select the exact solution $h(x)$ and perform discrete Fourier transform, and obtain the exact data function $g(x)$ by using (3.7). Then we add a normally distributed perturbation to each data function giving vectors g^δ after inverse Fourier transform. Finally we obtain the regularization solutions using (4.7).

Example 1. We consider the following case of smooth initial value

$$u(x,0) = 2x^2. \tag{5.4}$$

In this case, the exact solution has the following form

$$\widehat{u}(\xi,T) = \widehat{u}(\xi,0)E_{\alpha,1}(-|\xi|^\beta T^\alpha). \tag{5.5}$$

Example 2. Consider a non-smooth function

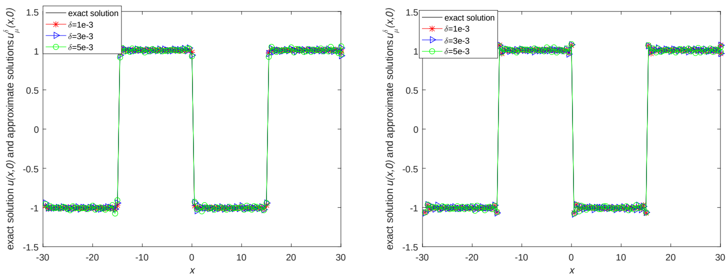
$$u(x,0) = \begin{cases} -1, & -30 \le x < -15, \\ 1, & -15 \le x < 0, \\ -1, & 0 \le x < 15, \\ 1, & 15 \le x \le 30. \end{cases}$$

TABLE 1. The relative error between the regularized solutions and exact solution under the a priori regularization parameter for Example 1 at $\beta = 0.8$.

$\delta \setminus \alpha$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$
$\delta = 0.001$	0.0078	0.0066	0.0051	0.0040
$\delta = 0.01$	0.0090	0.0077	0.0069	0.0106
$\delta = 0.1$	0.0379	0.0333	0.0438	0.1344

TABLE 2. The relative error between the regularized solutions and exact solution under the a posteriori regularization parameter for Example 1 at $\beta = 0.8$.

$\delta \setminus \alpha$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$
$\delta = 0.001$	0.0090	0.0076	0.0061	0.0040
$\delta = 0.01$	0.0101	0.0089	0.0072	0.0047
$\delta = 0.1$	0.0370	0.0367	0.0389	0.0278



(a) a priori (b) a posteriori

FIGURE 2. The exact solution and approximate solution for Example 2 with $\alpha = 0.3$, $\beta = 0.8$, and $T = 0.3$.

Example 3. Consider a piecewise continuous function

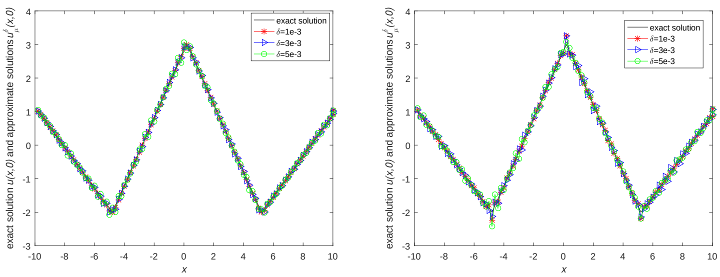
$$u(x,0) = \begin{cases} -0.6x - 5, & -10 \le x < -5, \\ x + 3, & -5 \le x < 0, \\ -x + 3, & 0 \le x < 5, \\ 0.6x - 5, & 5 \le x \le 10. \end{cases}$$

TABLE 3. The relative error between the regularized solutions and exact solution under the a priori regularization parameter for Example 2 at $\beta = 0.8$.

$\delta \setminus \alpha$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$
$\delta = 0.001$	0.0111	0.0102	0.0112	0.0304
$\delta = 0.01$	0.0431	0.0425	0.0408	0.0785
$\delta = 0.1$	0.2182	0.1990	0.1977	0.1766

TABLE 4. The relative error between the regularized solutions and exact solution under the a posteriori regularization parameter for Example 2 at $\beta = 0.8$.

$\delta \setminus \alpha$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$
$\delta = 0.001$	0.0217	0.0213	0.0266	0.0755
$\delta = 0.01$	0.0394	0.0379	0.0417	0.0866
$\delta = 0.1$	0.3973	0.3115	0.3097	0.2779



(a) a priori (b) a posteriori

FIGURE 3. The exact solution and approximate solution for Example 3 with $\alpha = 0.3$, $\beta = 0.8$, and $T = 0.3$.

Figs. 1-3 show the comparisons between the exact solution and its regularized solution of Examples 1-3 for various noise levels δ in the case of $\alpha = 0.3$, $\beta = 0.8$. Tables 1-4 show the comparison of relative error under the a priori and a posteriori regularization parameter of Examples 1-2 for different α with $\delta = 0.1, 0.01, 0.001$. Tables 5-6 show

TABLE 5. The relative error between the regularized solutions and exact solution under the a priori regularization parameter for Example 3 at $\beta = 0.8$.

$\delta \setminus \alpha$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$
$\delta = 0.0001$	0.0210	0.0175	0.0133	0.0100
$\delta = 0.001$	0.0233	0.0203	0.0171	0.0144
$\delta = 0.01$	0.0941	0.0862	0.0232	0.0610

TABLE 6. The relative error between the regularized solutions and exact solution under the a posteriori regularization parameter for Example 3 at $\beta = 0.8$.

$\delta \setminus \alpha$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$
$\delta = 0.0001$	0.0394	0.0371	0.0414	0.0771
$\delta = 0.001$	0.0438	0.0467	0.0442	0.0845
$\delta = 0.01$	0.1740	0.1668	0.2466	0.6434

the comparisons of relative error under the a priori and a posteriori regularization parameter of Example 3 for different α with $\delta = 0.01$, 0.001, 0.0001.

From Figs. 1-3, we find that the smaller the error levels are, the better the fitting effect is. From Table 1, we can notice that with the increase of α and δ, the relative error gradually decreases, but when α approaches 1, the relative error gradually increases with the increase of δ. The other Tables have similar change rules, which show the change of the relative error under different α. From Tables 1-6, we can see that the smaller δ is, the smaller the relative error between exact solution and regularization solution is, that is, the better the fitting effect is. The noise data in the table are affected by $randn(\cdot)$ function, and some of them do not strictly follow the law, but this does not affect our conclusion. At the same time, numerical examples in three different cases verify the validity and accuracy of the quasi-reversibility regularization method. From these tables, we can also see that our proposed method is stable and efficient for different inverse problems.

6. Conclusion.

In this paper, a time-space fractional backward diffusion equation is investigated. We use quasi-reversibility regularization method to deal with the ill-posed problem and obtain the regularization solution. Moreover, under the a priori and the a posteriori parameter choice rules, we obtain the error estimates. Finally, different types of numerical experiments show that the proposed method works effectively.

We can observe that the regularity of the initial function $u(x, 0)$ is required to be $H^p(\mathbb{R})$, and $p > 4 - \beta$. This condition is very strict. If one can reduce the regularity condition, this is also an open problem. And in the future, we will consider the two-dimensional case which can be applied to the image processing.

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Department of Mathematics, Northwest Normal University, Lanzhou 730070, P. R. China
Email address: **wenjin0421@163.com.**

Department of Mathematics, Northwest Normal University, Lanzhou 730070, P. R. China
Email address: **wang1518668181@163.com**

Department of Mathematics, Northwest Normal University, Lanzhou 730070, P. R. China
Email address: **liu11200810@163.com**

School of Telecommunication Engineering, Lanzhou Bowen College of Science and Technology, Lanzhou 730000, Gansu, P. R. China
Email address: **wangfenqin@163.com**

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