MAKING CONTINUOUS FUNCTIONS LIPSCHITZ

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ABSTRACT. Let $\langle X, \tau \rangle$ be a metrizable topological space and let $\langle Y, \rho \rangle$ be a metric space. Let Ω be a family of bounded continuous functions from X to Y. We show that the family is Lipschitzian with respect to some compatible metric on X if and only if the family can be written as a countable union of pointwise equicontinuous subfamilies. From this, we easily characterize those families of continuous functions between metrizable spaces that are Lipschitzian with respect to appropriately chosen metrics on the domain and target space.

1. Introduction

Let $\langle X, \tau \rangle$ be a metrizable topological space and let $\langle Y, \rho \rangle$ be a metric space. We denote the continuous functions from X to Y by $C(X, Y)$ and the bounded members of $C(X, Y)$ by $C_b(X, Y)$. If X is equipped with a metric d that is compatible with τ, a function $f \in C(X, Y)$ is called *Lipschitz* (with respect to d and ρ) provided for some $\alpha > 0$, whenever $\{x, w\} \subseteq X$, we have $\rho(f(x), f(w)) \leq \alpha \cdot d(x, w)$ [6, 10]. In this case we say that f is α-*Lipschitz* with the metrics being understood. When f is 1-Lipschitz, we say that f is *nonexpansive* [5].

A basic question to be asked is this: given a subfamily of $C(X, Y)$, when does there exist a compatible metric on X such that the family is a Lipschitzian family with respect to it? Of course, one can also allow ρ to vary over equivalent metrics, but that is a different question (see [6, Section 3.2]). With respect to applications, there may be a commitment to a particular metric on the target space, e.g., the Euclidean metric on $\mathbb{R}^n$ or the ℓ_1-metric on the absolutely summable real sequences, but less so for the domain space.

We focus our attention here on subfamilies of $C_b(X, Y)$. As a warm-up, we answer this question for the full subfamily $C_b(X, Y)$ when $Y = \mathbb{R}$ equipped with the Euclidean metric. Let X' denote the (possibly empty) set of limit points of X.

Theorem 1.1. *Let $\langle X, \tau \rangle$ be a metrizable space and let $\mathbb{R}$ be equipped with the Euclidean metric. The following statements are equivalent:*

(1) *the topology of X is discrete, i.e., $X' = \emptyset$;*

(2) *there exists a compatible bounded metric d for X with respect to which each member of $C_b(X, \mathbb{R})$ is Lipschitz;*

(3) *there exists a compatible metric d for X with respect to which each member of $C_b(X, \mathbb{R})$ is Lipschitz.*

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Proof. (1) $\Rightarrow$ (2). Suppose $X' = \emptyset$; then the zero-one metric is a compatible metric, and for each $f \in C_b(X,Y)$, the diameter of the range of f serves as a Lipschitz constant for f.

(2) $\Rightarrow$ (3). This is trivial.

(3) $\Rightarrow$ (1). Suppose $X' \neq \emptyset$. Let x_0 be a limit point of X and let $\langle x_j \rangle$ be a sequence in $X \backslash \{x_0\}$ τ-convergent to x_0. Suppose d is a compatible metric for X. We produce $f \in C_b(X,\mathbb{R})$ that is not Lipschitz with respect to d. Put $f(x) := \min\{1, \sqrt{d(x,x_0)}\}$ whose range has diameter at most one. Choosing $k \in \mathbb{N}$ for which $d(x_j,x_0) \leq 1$ for all $j \geq k$, we compute

$$\sup_{j\geq k} \ \frac{|f(x_j) - f(x_0)|}{d(x_j,x_0)} = \sup_{j\geq k} \ \frac{1}{\sqrt{d(x_j,x_0)}} = \infty,$$

showing f is not Lipschitz with respect to d as promised. $\square$

Theorem 1.1 in particular shows that without additional assumptions, a family of bounded continuous functions from a metrizable space $\langle X, \tau \rangle$ to a metric space $\langle Y, \rho \rangle$ need not be a family of Lipschitz functions with respect to any compatible metric on X. It is a folk-theorem that this is so if Ω is countable, and we give a proof as a courtesy to the uninitiated (see Theorem 3.1 *infra*). But we do much better: we actually characterize those subfamilies Ω of $C_b(X,Y)$ whose members are all Lipschitz with respect to a compatible metric on X as those that can be written as a countable union of pointwise equicontinuous subfamilies. Furthermore, the metric can be chosen to be bounded if the listed criterion is met. The folk-theorem stated above immediately follows from our omnibus result. We provide an example that shows that such a nice compatible metric might not exist for a pointwise equicontinuous family of unbounded continuous functions. We also obtain two results that speak to when a common Lipschitz constant can be found for each member of Ω. Our results assume nothing additional about the target space $\langle Y, \rho \rangle$. Finally, we use our main result to completely resolve the basic question addressed in [6, Section 3.2].

2. Preliminaries

All metrizable topological spaces will consist of at least two points. If $\langle Y, \rho \rangle$ is a metric space and A is a nonempty subset, we put $\text{diam}(A) := \sup\{\rho(a_1,a_2) : \{a_1,a_2\} \subseteq A\}$. If $f : X \to Y$, we put $M(f) := \text{diam}(f(X))$; to say that f is bounded means that $M(f) < \infty$ [8, p. 327].

If Ω is a family of functions from a metrizable space $\langle X, \tau \rangle$ to a metric space $\langle Y, \rho \rangle$, we say that Ω is *equibounded* if $\sup\{M(f) : f \in \Omega\} < \infty$. This is a weaker requirement than *uniform boundedness*, which for us means that $\cup_{f\in\Omega} f(X)$ has finite diameter. The classical uniform boundedness principle of functional analysis [12, p. 169] in fact asserts that the uniform boundedness of a subfamily of X^* on the unit ball of a Banach space X in the above sense follows from its pointwise boundedness there.

The family is called *pointwise equicontinuous* [7, p. 266] if for each $\varepsilon > 0$ and each $x \in X$, there exists a neighborhood W of x such that for each $w \in W$ and $f \in \Omega$, we have $\rho(f(x), f(w)) < \varepsilon$. Notice that this is a property of the topology τ, and does not depend on the metric the space is equipped with. Indeed, our definition does not presume that X be equipped with a particular metric and indeed makes

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sense even if $\langle X, \tau \rangle$ is not metrizable. Of course, this concept arises in standard versions of the Arzela-Ascoli theorem (see, e.g., [7, p. 267]).

We need some additional terminology requiring that X be equipped with a particular compatible metric d. We say that Ω is *uniformly equicontinuous* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $\{x, w\} \subseteq X$ with $d(x, w) < \delta$ and whenever $f \in \Omega$, we have $\rho(f(x), f(w)) < \varepsilon$. Of course, uniform equicontinuity ensures pointwise equicontinuity. The family Ω is called *equi-Lipschitzian* if we can find $\alpha > 0$ such that each $f \in \Omega$ is α-Lipschitz with the respect to d and ρ. In this case, the family is uniformly equicontinuous; further, the family will be equibounded provided d is a bounded metric.

The following elementary result [10, p. 92] is most gracefully established using the fact that each open cover of a compact metric space has a Lebesgue number.

Proposition 2.1. *Let $\langle X, d \rangle$ be a compact metric space and let $\langle Y, \rho \rangle$ be a metric space. Then each pointwise equicontinuous subfamily of $C(X, Y)$ is uniformly equicontinuous.*

This property of pointwise equicontinuous families is in fact characteristic of the class of metric domains for which each open cover has a Lebesgue number. This well-studied class, properly trapped between the compact metric spaces and the complete metric spaces, is often called the *UC-spaces* in the literature, but it also goes by the *Atsuji spaces* and the *Lebesgue spaces* (see, e.g, [2, 3, 9, 11]). Usually, it is defined as the class of metric spaces $\langle X, d \rangle$ such that each continuous function defined on X with values in an arbitrary second metric space is uniformly continuous. It is known that a metrizable space $\langle X, \tau \rangle$ has a compatible UC-metric if and only if X' is compact [3, p. 59].

3. Results

Again, our major purpose is to find conditions under which a family of bounded continuous functions from a metrizable space $\langle X, \tau \rangle$ to a metric space $\langle Y, \rho \rangle$ is a Lipschitzian family with respect to a suitably chosen compatible metric on X. If we have a finite family of continuous functions $\Omega = \{f_1, f_2, f_3, \ldots, f_k\}$, this is easy: let d be any compatible metric, and define $d_1 : X \times X \to [0, \infty)$ by

$$d_1(x, w) := d(x, w) + \sum_{n=1}^{k} \rho(f_n(x), f_n(w)).$$

The metric d_1 is a compatible metric that does the job, even without the boundedness assumption on the functions. Actually, each f_n is nonexpansive with respect to d_1 and ρ. In the case that each function is bounded, we can make d_1 bounded by taking d to be bounded in the first place.

A similar result obtains when we have a countably infinite family of bounded continuous functions (cf. [6, Theorem 3.2.4] for a weaker result).

Theorem 3.1. *Let $\langle X, \tau \rangle$ be metrizable and let $\langle Y, \rho \rangle$ be a metric space. Suppose $\langle f_n \rangle$ is a sequence in $C_b(X, Y)$. Then there is a compatible bounded metric d_1 for X with respect to which each f_n is Lipschitz.*

Proof. Begin with a compatible bounded metric d for X. Noting that each constant function from X to Y is Lipschitz with respect to each compatible metric on X, we may assume without loss of generality that each f_n is non-constant, i.e., that $M(f_n) > 0$. Define $d_1 : X \times X \to [0, \infty)$ by

$$d_1(x, w) := d(x, w) + \sum_{n=1}^{\infty} 2^{-n} \cdot \frac{\rho(f_n(x), f_n(w))}{M(f_n)}.$$

Since $\sum_{n=1}^{\infty} 2^{-n} = 1$, the displayed series converges to a number in [0,1] and since it determines a pseudo-metric on X, when it is added to d, we get a metric for X. Also, d_1 is majorized by $d + 1$ so it is a bounded metric. We include a proof of its equivalence to d and thus its compatibility with τ.

Since d_1 majorizes d, it suffices to show what whenever $\langle x_j \rangle$ is convergent to $x \in X$ with respect to d, then it is d_1-convergent as well [8, p. 313]. Let $\varepsilon > 0$ be arbitrary and choose $k \in \mathbb{N}$ so large that $\sum_{n=k+1}^{\infty} 2^{-n} < \frac{\varepsilon}{2}$. Using the fact that $f_1, f_2, \ldots f_k$ are each continuous at x, we can find $j_0 \in \mathbb{N}$ such that

$$\forall j \geq j_0 \ \forall n \leq k, \text{ we have } \rho(f_n(x_j), f_n(x)) < \frac{2^n M(f_n)\varepsilon}{2k}.$$

With these estimates, we easily get $d_1(x_j, x) < d(x_j, x) + \varepsilon$ whenever $j \geq j_0$. As we are assuming that $\lim_{j\to\infty} d(x_j, x) = 0$, this yields $\lim_{j\to\infty} d_1(x_j, x) = 0$.

It remains to show that each f_n is Lipschitz with respect to d_1. Fix $n \in \mathbb{N}$ and let x and w be arbitrary points of X. We compute

$$\rho(f_n(x), f_n(w)) = 2^n \cdot M(f_n) \cdot 2^{-n} \cdot \frac{\rho(f_n(x), f_n(w))}{M(f_n)} \leq 2^n M(f_n) \cdot d_1(x, w),$$

and this shows that $2^n M(f_n)$ serves as a Lipschitz constant for f_n. □

Our next result ought to convince the reader that the notion of pointwise equicontinuity has a role to play outside the setting of compactness criteria for function spaces.

Theorem 3.2. *Let $\langle X, \tau \rangle$ be a metrizable space, and let $\langle Y, \rho \rangle$ be a metric space. Suppose Ω is a subfamily of $C_b(X, Y)$ that is pointwise equicontinuous. Then there is a bounded compatible metric d_1 on X with respect to which Ω is a Lipschitzian family.*

Proof. We recall that if p is a pseudo-metric on a set S, then so is $\min\{1, p\}$ and if $\mathcal{P}$ is a family of pseudo-metrics on S each bounded by 1, then $\sup\{p : p \in \mathcal{P}\}$ is also a pseudo-metric bounded by 1 [7, p. 198]. Let d be a bounded compatible metric for X and put

$$d_1(x, w) := d(x, w) + \sup\nolimits_{f\in\Omega} \min\{1, \rho(f(x), f(w))\} \quad \text{ for } \{x, w\} \subseteq X.$$

Then d_1 is a bounded metric on X. To show equivalence with d, since $d \leq d_1$, we again need only show that whenever $\langle x_j \rangle$ is a sequence in X with $\lim_{j\to\infty} d(x_j, x) = 0$, then $\lim_{j\to\infty} d_1(x_j, x) = 0$. Let $\varepsilon \in (0, 1)$ be arbitrary, and by the compatibility of d and pointwise equicontinuity of Ω, choose a neighborhood W of x such that whenever $w \in W$, (1) we have $d(w, x) < \frac{\varepsilon}{2}$, and (2) whenever f belongs to Ω, the inequality $\rho(f(x), f(w)) < \frac{\varepsilon}{2}$ holds. Eventually, x_j lies in W, and since $\frac{\varepsilon}{2} < \frac{1}{2}$, we have for all j sufficiently large

$$\sup_{f\in\Omega} \min\{1, \rho(f(x_j), f(x))\} \leq \frac{\varepsilon}{2}.$$

We conclude that eventually, $d_1(x_j, x) < \varepsilon$, establishing compatibility of d_1.

To exhibit a Lipschitz constant for an arbitrary $g \in \Omega$, let $x, w \in X$ be given. If $\rho(g(x), g(w)) < 1$ holds, we get

$$\rho(g(x), g(w)) \leq d(x, w)+\rho(g(x), g(w)) = d(x, w)+\min\{1, \rho(g(x), g(w))\} \leq d_1(x, w).$$

On the other hand, if $\rho(g(x), g(w)) \geq 1$ were true, then $d_1(x, w) = d(x, w) + 1 \geq 1$ holds, and since $\rho(g(x), g(w)) \leq M(g)$, we get

$$\frac{\rho(g(x), g(w))}{d_1(x, w)} \leq \frac{M(g)}{1} = M(g).$$

Since these two cases are exhaustive, we can state that $1 + M(g)$ is a Lipschitz constant for g with respect to d_1. $\square$

Theorem 3.1 does not follow from Theorem 3.2 since the sequence in the former may not be pointwise equicontinuous. An actual generalization will be achieved in our main result below. Meanwhile we address the following question.

One wonders if the set of terms of $\langle f_n\rangle$ can be made an equi-Lipschitzian family under an equivalent remetrization provided the set of terms is equibounded. But this may not be so, even if the set of terms is uniformly bounded.

Example 3.3. Let $X = Y = [0, 1]$ where X is equipped with its usual topology and Y is equipped with the Euclidean metric. Define $f_n \in C_b(X, Y)$ by

$$f_n(x) = \begin{cases} nx & \text{if } 0 \leq x < \frac{1}{n} \\ 1 & \text{if } \frac{1}{n} \leq x \leq 1. \end{cases}$$

Clearly, $\{f_n : n \in \mathbb{N}\}$ is uniformly bounded, as the range of each member is $[0, 1]$. Now suppose d is any compatible metric for the topology of X. As $\lim_{n\to\infty} d(\frac{1}{n}, 0) = 0$, we get

$$\sup_{n\in\mathbb{N}} \frac{|f_n(\frac{1}{n}) - f_n(0)|}{d(\frac{1}{n}, 0)} = \sup_{n\in\mathbb{N}} \frac{1}{d(\frac{1}{n}, 0)} = \infty,$$

showing the family is not equi-Lipschitzian with respect to d.

What goes wrong in this example is a failure of pointwise equicontinuity, because if a family of functions were equi-Lipschitzian with respect to metrics d and ρ on the domain and target space, then the family would have to be uniformly equicontinuous with respect to those metrics. With pointwise equicontinuity, we may use Theorem 3.2 to get the next result.

Corollary 3.4. *Let Ω be an equibounded family of functions from a metrizable space $\langle X, \tau\rangle$ to a metric space $\langle Y, \rho\rangle$. Then Ω is equi-Lipschitzian with respect to some compatible metric if and only if Ω is pointwise equicontinuous.*

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Proof. For sufficiency, if $M(f) \leq \alpha$ for each $f \in \Omega$, then $1 + \alpha$ will serve as a common Lipschitz constant with respect to the metric d_1 given in the proof of the last theorem. On the other hand, if Ω is equi-Lipschitzian with respect to some compatible metric, it is uniformly equicontinuous with respect to it, and thus pointwise equicontinuous with respect to τ. □

Corollary 3.5. *Let Ω be a pointwise equicontinuous family of functions from a metrizable space $\langle X, \tau \rangle$ to a metric space $\langle Y, \rho \rangle$. Then Ω is equibounded if and only if there is a compatible bounded metric d_1 on X with respect to which Ω is an equi-Lipschitzian family.*

Proof. Necessity follows the proof of Theorem 3.2. For sufficiency, if α is a Lipschitz constant for each $f \in \Omega$ with respect to d_1 and d_1 is bounded, then $\forall f \in \Omega \ \forall x, w \in X$, we have $\rho(f(x), f(w)) \leq \alpha \text{diam}(X)$. This means that Ω is equibounded. □

With respect to the second corollary, we remark that for an equibounded and pointwise equicontinuous family, we can scale our bounded metric d_1 so that the family becomes a family of nonexpansive maps with respect to it (if that serves some purpose).

We next give a pointwise equicontinuous family of unbounded real-valued functions on a certain metrizable space that is not a Lipschitzian family with respect to any compatible metric (where $\mathbb{R}$ is equipped with the Euclidean metric).

Example 3.6. Equip $\mathbb{N}$ with the discrete topology and equip $\mathbb{R}$ with the Euclidean metric. Then any family of functions from $\mathbb{N}$ to $\mathbb{R}$ is pointwise equicontinuous, as each point of the domain has a neighborhood consisting of just that point. Let Ω consist of $C(\mathbb{N}, \mathbb{R}) \backslash C_b(\mathbb{N}, \mathbb{R})$, that is to say, the family of all unbounded real-valued functions on $\mathbb{N}$. If d is a bounded compatible metric on $\mathbb{N}$, then no member of Ω is Lipschitz with respect to it. On the other hand, if d is unbounded, there exists a strictly increasing sequence of positive integers $\langle n_k \rangle$ such that for each k, $d(1, n_k) > k$ holds. Define $f : \mathbb{N} \to \mathbb{R}$ by

$$f(n) = \begin{cases} d(1, n_k)^2 & \text{if } \ n = n_k \text{ for some } k \\ 0 & \text{otherwise.} \end{cases}$$

Note that $f(1) = 0$, and as f is unbounded, it belongs to Ω. As

$$\frac{|f(1) - f(n_k)|}{d(1, n_k)} = d(1, n_k) \to \infty,$$

our function f is not Lipschitz with respect to d.

Our next result will be combined with Theorem 3.2 to get our characterization theorem.

Proposition 3.7. *Let $\langle X, \tau \rangle$ be a metrizable space and let $\langle Y, \rho \rangle$ be a metric space. Suppose for each $n \in \mathbb{N}$, Ω_n is a subfamily of $C_b(X, Y)$ for which there is a compatible bounded metric d_n on X such that Ω_n is a Lipschitzian family with respect to d_n and ρ. Then there is a compatible bounded metric d^* for X such that for each $n \in \mathbb{N}$ and $f \in \Omega_n$, the function f is Lipschitz with respect to d^* and ρ.*

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Proof. For each $n \in \mathbb{N}$ let $\text{diam}_n(X) \in (0, \infty)$ denote the diameter of X with respect to d_n. Then easily $d^* : X \times X \to [0, 1]$ defined by

$$d^*(x, w) := \sum_{n=1}^{\infty} 2^{-n} \cdot \frac{d_n(x, w)}{\text{diam}_n(X)}$$

is a compatible metric on X. To prove that $\cup_{n \in \mathbb{N}} \Omega_n$ consists of Lipschitz functions with respect to d^*, fix $f \in \Omega_n$ (where n is arbitrary) and let α be a Lipschitz constant for f with respect to d_n. For $x, w \in X$, we compute

$$\rho(f(x), f(w)) \leq \alpha d_n(x, w) = \alpha 2^n \text{diam}_n(X) \cdot 2^{-n} \cdot \frac{d_n(x, w)}{\text{diam}_n(X)}$$

$$\leq \alpha 2^n \text{diam}_n(X) \cdot d^*(x, w),$$

completing the proof. □

Notice that Theorem 3.1 is a consequence of Proposition 3.7 where $\Omega_n = \{f_n\}$ and $d_n(x, w) = d(x, w) + \rho(f_n(x), f_n(w))$, where d is a bounded compatible metric for X and f_n is bounded and continuous.

We now come to the main result of this paper.

Theorem 3.8. *Let $\langle X, \tau \rangle$ be a metrizable space and let $\langle Y, \rho \rangle$ be a metric space. Let Ω be a subfamily of $C_b(X, Y)$. The following statements are equivalent:*

(1) there is a compatible bounded metric d on X such that each $f \in \Omega$ is Lipschitz with respect to d and ρ;

(2) there is a compatible metric d on X such that each $f \in \Omega$ is Lipschitz with respect to d and ρ;

(3) Ω can be expressed as $\cup_{n=1}^{\infty} \Omega_n$ where each Ω_n is a pointwise equicontinuous family with respect to τ and ρ.

Proof. The implication (1) ⇒ (2) is trivial while (3) ⇒ (1) follows from Theorem 3.2 and Proposition 3.7. To prove (2) ⇒ (3), for each $n \in \mathbb{N}$ put

$$\Omega_n := \{f \in \Omega : \forall x, w \in X \text{ we have } \rho(f(x), f(w)) \leq n d(x, w)\}.$$

Each Ω_n is an equi-Lipschitzian family, and so is uniformly equicontinuous with respect to d and ρ. Thus, each Ω_n is pointwise equicontinuous with respect to τ and ρ. □

More information about the underlying domain space may lead to more refined consequences, such as the following corollary, which follows from Proposition 2.1.

Corollary 3.9. *Suppose $\langle X, \tau \rangle$ in Theorem 3.8 is compact. Then in item (3) of the theorem, pointwise equicontinuity can be replaced by uniform equicontinuity.*

The next corollary follows from Theorem 1.1 and Theorem 3.8.

Corollary 3.10. *Let $\langle X, \tau \rangle$ be a metrizable space and let $\mathbb{R}$ be equipped with the Euclidean metric. The following statements are equivalent.*

(1) The topology τ is discrete;

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(2) $C_b(X,\mathbb{R})$ *is a pointwise equicontinuous family;*

(3) $C_b(X,\mathbb{R})$ *can be written as a countable union of pointwise equicontinuous subfamilies.*

Cobzaş, Miculescu and Nicolae [6, p. 181] show that for $X = \mathbb{Q}$ and $Y = [0,1]$ equipped with their usual topologies, then given any pair of compatible metrics d and ρ on X and Y, there is a member of $C(X,Y)$ that fails to be Lipschitz with respect to them. In particular, it follows that if Ω is family of continuous functions between metrizable spaces, it may not be a Lipschitzian family with respect to whatever metrics we choose for the domain and target spaces. We can now state the result that they were after in Section 3.2 of their monograph [6].

Theorem 3.11. *Let $\langle X,\tau_1\rangle$ and $\langle Y,\tau_2\rangle$ be metrizable spaces and let Ω be a family of continuous functions from X to Y. The following statements are equivalent:*

(1) *there is a compatible bounded metric d on X and a compatible bounded metric ρ on Y such that each $f \in \Omega$ is Lipschitz with respect to d and ρ;*

(2) *there is a compatible metric d on X and a compatible metric ρ on Y such that each $f \in \Omega$ is Lipschitz with respect to d and ρ;*

(3) *there is a compatible metric ρ^* on Y such that Ω can be expressed as $\cup_{n=1}^{\infty} \Omega_n$ where each Ω_n is a pointwise equicontinuous family with respect to τ_1 and ρ^*.*

Proof. The implication (1) $\Rightarrow$ (2) is trivial, and (2) $\Rightarrow$ (3) is argued just as in the proof of Theorem 3.8, where ρ^* is taken to be ρ. For (3) $\Rightarrow$ (1), replace ρ^* by $\rho := \min\{1,\rho^*\}$. Each Ω_n remains pointwise equicontinuous, but now Ω consists of bounded continuous functions. Apply Theorem 3.8. $\square$

We next give an elementary application of Theorem 3.8 that nevertheless requires some delicate estimation.

Example 3.12. Let $X = [0,1]$ with its usual topology τ and equip $\mathbb{R}$ with the Euclidean metric. Our Ω is a 2-parameter family of functions of the form $f_{(x_0,\beta)}$ where $(x_0,\beta) \in [0,1] \times (0,\infty)$. Specifically,

$$f_{(x_0,\beta)}(x) := \beta\sqrt{|x-x_0|} \qquad \text{for } 0 \le x \le 1.$$

Each such function is uniformly continuous and bounded. Obviously, Ω is not a pointwise equicontinuous family, even if x_0 were to be held fixed.

We can write $\Omega = \cup_{n\in\mathbb{N}} \Omega_n$ where $\Omega_n := \{f_{(x_0,\beta)} \in \Omega : \beta \le n\}$. Each Ω_n is a uniformly bounded family. By Theorem 3.8, we can conclude that Ω is a Lipschitzian family with respect to some compatible (bounded) metric on $[0,1]$ if we can show that each Ω_n is pointwise equicontinuous. Actually, Ω_n is uniformly equicontinuous with respect to the Euclidean metric on $[0,1]$ (see Proposition 2.1 *supra*).

To see this, let $n \in \mathbb{N}$ and $\varepsilon > 0$ be fixed, and let $f_{(x_0,\beta)} \in \Omega_n$ be arbitrary. Choose $\delta > 0$ such that $\sqrt{\delta} < \frac{\varepsilon}{2n}$. We consider two mutually exclusive and exhaustive cases for $x, w \in [0,1]$: (1) both $|x-x_0| < \delta$ and $|w-x_0| < \delta$, or (2) either $|x-x_0| \ge \delta$ or $|w-x_0| \ge \delta$.

In case (1), we obtain this inequality string without any further control on $|x-w|$:

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$$|f_{(x_0,\beta)}(x) - f_{(x_0,\beta)}(w)| \leq n(\sqrt{|x-x_0|} + \sqrt{|w-x_0|}) < n \cdot (\frac{\varepsilon}{2n} + \frac{\varepsilon}{2n}) = \varepsilon.$$

In case (2), if $|x-w| < \frac{\varepsilon\sqrt{\delta}}{n}$, we compute

$$|f_{(x_0,\beta)}(x) - f_{(x_0,\beta)}(w)| \leq n \cdot \frac{||x-x_0| - |w-x_0||}{\sqrt{|x-x_0|} + \sqrt{|w-x_0|}} \leq n \cdot \frac{|x-w|}{\sqrt{\delta}} < \varepsilon.$$

Thus, wherever x and w may be relative to x_0, if $|x-w| < \frac{\varepsilon\sqrt{\delta}}{n}$, it follows that $|f_{(x_0,\beta)}(x) - f_{(x_0,\beta)}(w)| < \varepsilon$. This fact establishes uniform equicontinuity of Ω_n relative to the Euclidean metric on $[0,1]$, and pointwise equicontinuity relative to τ follows.

We next give a more abstract application of Theorem 3.8. Suppose $\langle X, d\rangle$ and $\langle Y, \rho\rangle$ are metric spaces and $f : X \to Y$. We recall that the *modulus of continuity of* f [1, 4] is the function $\omega_f : (0,\infty) \to [0,\infty]$ given by

$$\omega_f(t) := \sup\{\rho(f(x), f(w)) : d(x,w) \leq t\}.$$

Since ω_f is nondecreasing, both $\lim_{t\to 0^+}\omega_f(t)$ and $\lim_{t\to\infty}\omega_f(t)$ exist as extended nonnegative real numbers. The following properties are easy to verify:

(1) f is constant if and only if ω_f is the zero function;
(2) f is bounded if and only if $\lim_{t\to\infty} \omega_f(t)$ is finite;
(3) f is uniformly continuous if and only if $\lim_{t\to 0^+} \omega_f(t) = 0$;
(4) f is α-Lipschitz if and only if $\forall t > 0,$ we have $\omega_f(t) \leq \alpha t$.

Example 3.13. Suppose $\langle X, d\rangle$ and $\langle Y, \rho\rangle$ are metric spaces and $\Omega \subseteq C_b(X, Y)$. Suppose we can find a sequence of functions $\langle \phi_n \rangle$ from $(0,\infty)$ to $[0,\infty)$ and a positive sequence $\langle \delta_n \rangle$ such that

- $\forall n \in \mathbb{N}\ \ \lim_{t\to 0^+} \phi_n(t) = 0$, and
- $\forall f \in \Omega\ \exists n \in \mathbb{N}$ such that ϕ_n majorizes ω_f on $(0, \delta_n)$.

Then there exists a bounded metric d^* equivalent to d such that Ω is a Lipschitzian family with respect to d^* and ρ. Put differently, with respect to d^* and ρ, $\forall f \in \Omega\ \exists \alpha > 0$ such that for all $t > 0$ we have $\omega_f(t) \leq \alpha t$. This occurs because

$$\Omega = \cup_{n\in\mathbb{N}}\ \{f \in \Omega : \omega_f \leq \phi_n \text{ on } (0, \delta_n)\},$$

and so Ω is evidently a countable union of uniformly equicontinuous families.

We call $f : \mathbb{R} \to \mathbb{R}$ *quadratic* if $f(x) = ax^2 + bx + c$ where $a \neq 0$. Consider the family of quadratic functions from $\mathbb{R}$ to $\mathbb{R}$. If we equip the target space with the Euclidean metric and the domain with the equivalent metric $d(x,w) := |x-w| + |x^2 - w^2|$, then it is a routine exercise to verify that $|a| + |b|$ serves as a Lipschitz constant for $f(x) = ax^2 + bx + c$ with respect to these metrics.

It follows then from the equivalence of conditions (2) and (3) of Theorem 3.11 that the family Ω of quadratic functions must be decomposable into a countable

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union of pointwise equicontinuous families with respect to the Euclidean metric on the target space. We close by exhibiting such a decomposition without reference to the above remetrization of the domain.

The family Ω may be viewed as a 3-parameter family consisting of all functions of the form $f_{(x_0,a,y_0)}$ where $x_0 \in \mathbb{R},\ a \in \mathbb{R}\backslash\{0\}$ and $y_0 \in \mathbb{R}$, defined by

$$f_{(x_0,a,y_0)}(x) := a(x-x_0)^2 + y_0 \qquad \text{for } x \in \mathbb{R}.$$

Of course, none of these function is bounded with respect to the Euclidean metric on the target space. For each $n \in \mathbb{N}$, put $\Omega_n := \{f_{(x_0,a,y_0)} \in \Omega : |x_0| \leq n \text{ and } |a| \leq n\}$. Since $\Omega = \cup_{n\in\mathbb{N}}\ \Omega_n$, it suffices to show that each Ω_n is pointwise equicontinuous with respect to the Euclidean metric on the target space.

To this end, fix $n \in \mathbb{N},\ f_{(x_0,a,y_0)} \in \Omega_n,\ x \in \mathbb{R}$, and $\varepsilon > 0$. We intend to show that if

$$|x-w| < \min\{1, \frac{\varepsilon}{n(2|x|+1+2n)}\},$$

then $|f_{(x_0,a,y_0)}(x) - f_{(x_0,a,y_0)}(w)| < \varepsilon$. Factoring a difference of squares as a product of a sum and a difference, we compute

$$\begin{aligned}|f_{(x_0,a,y_0)}(x) - f_{(x_0,a,y_0)}(w)| &= |a| \cdot |(x-x_0)^2 - (w-x_0)^2| \leq n|(x-x_0)^2 - (w-x_0)^2| \\ &= n|x-w||x+w-2x_0| \leq n|x-w|(2|x|+1+2|x_0|) \\ &\leq n|x-w|(2|x|+1+2n) < \varepsilon,\end{aligned}$$

as required.

Note that if the quadratic functions were to be Lipschitzian with respect to compatible metrics d and ρ where d is bounded, then ρ is forced to be bounded as well. Such a scenario is guaranteed to exist by Theorem 3.11.

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